Quasi-Frobenius functors with application to corings

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Introduction

Müller generalized in [12] the notion of a Frobenius extension to left (right) quasi-Frobenius extension and proved the endomorphism ring theorem for these extensions. Recently, Guo observed in [9] that for a ring homomorphism $\varphi : R \to S$, the restriction of scalars functor has to induction functor $S \otimes_R - : {}_R\mathcal{M} \to {}_S\mathcal{M}$ as right "quasi" adjoint if and only if φ is a left quasi-Frobenius extension. In this paper we shall give a further generalization of the notion of quasi-Frobenius extension in a more general setting from the viewpoint of an adjoint triple of functors. Using the definition of *quasi-strongly adjoint pair* for module categories given by K. Morita [11], we introduce the notion of *quasi-Frobenius triple of functors* for Grothendieck categories. A *triple of functors* $(\mathbf{L}, \mathbf{F}, \mathbf{R})$, where $\mathbf{F} : \mathcal{A} \to \mathcal{B}$ has a left adjoint $\mathbf{L} : \mathcal{B} \to \mathcal{A}$ and a right adjoint $\mathbf{R} : \mathcal{B} \to \mathcal{A}$ is said to be *quasi-Frobenius* whenever the functors $\mathbf{L}$ and $\mathbf{R}$ are *similar* in sense functorial. In this case, the functor $\mathbf{F}$ is called *quasi-Frobenius functor*. $\mathbf{F}$ is a *Frobenius functor* in the case it has the same right and left adjoint, i.e., $\mathbf{L} \cong \mathbf{R}$ (cf. [5]). Clearly, the class of quasi-Frobenius functors include to the class of Frobenius functors. First we study basic properties of quasi-Frobenius functors for Grothendieck categories. This concept generalizes la notion of left quasi-Frobenius pair of functors given in Guo [9]. In Section 2 we give an easy and natural proof of the characterization of quasi-Frobenius functors between module categories. In particular, a bijective correspondence between quasi-Frobenius triple of functors is presented (in fact, a duality). Another interesting case is given by graded rings and modules and this is considered in Section 3. The notion of Frobenius extension for coalgebras over fields was

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introduced by Castaño Iglesias et al., [5] and extended recently to corings by Zarouali-Darkaoui [17]. In Section 4 we characterize the quasi-Frobenius functors between categories of comodules over corings. In the last section we introduce the notion de *quasi-Frobenius* homomorphism of corings. Next we focus on corings for which the forgetful functor from the category of right comodules of an A-coring to the category of right A-modules is quasi-Frobenius. We term such corings *quasi-Frobenius corings*.

1 Quasi-Frobenius functors and properties

Let $\mathcal{A}$ be a Grothendieck category and consider an object X of $\mathcal{A}$. For any positive integer n, we denote by X^n the direct sum of n copies of X in the category $\mathcal{A}$. Given additive and covariant functors $\mathbf{L}, \mathbf{R} : \mathcal{B} \to \mathcal{A}$ between Grothendieck categories, we say that $\mathbf{L}$ *divide* to $\mathbf{R}$, denoted by $\mathbf{L} \mid \mathbf{R}$, if for some positive integer n there are natural transformations

$$\mathbf{L}(X) \xrightarrow{\phi(X)} \mathbf{R}(X)^n \xrightarrow{\psi(X)} \mathbf{L}(X)$$

such that $\psi(X) \circ \phi(X) = 1_{\mathbf{L}(X)}$ for every $X \in \mathcal{B}$. Analogously, $\mathbf{R} \mid \mathbf{L}$ if for some positive integer m there are natural transformations

$$\mathbf{R}(X) \xrightarrow{\phi'(X)} \mathbf{L}(X)^m \xrightarrow{\psi'(X)} \mathbf{R}(X)$$

such that $\psi'(X) \circ \phi'(X) = 1_{\mathbf{R}(X)}$ for every $X \in \mathcal{B}$. The functor $\mathbf{L}$ is said to be *similar* to $\mathbf{R}$, denoted by $\mathbf{L} \sim \mathbf{R}$, whenever $\mathbf{L} \mid \mathbf{R}$ and $\mathbf{R} \mid \mathbf{L}$.

Consider now a *triple of functors* $\Gamma = (\mathbf{L}, \mathbf{F}, \mathbf{R})$, where $\mathbf{F} : \mathcal{A} \to \mathcal{B}$ has a left adjoint $\mathbf{L} : \mathcal{B} \to \mathcal{A}$ and also a right adjoint $\mathbf{R} : \mathcal{B} \to \mathcal{A}$. Notice that $\mathbf{F}$ is exact and preserves inverse and direct limits, $\mathbf{L}$ always preserves projective objects and $\mathbf{R}$ injective objects.

1.1 Definition. A *quasi-Frobenius triple* for the categories $\mathcal{A}$ and $\mathcal{B}$ consists of a triple of functors $\Gamma = (\mathbf{L}, \mathbf{F}, \mathbf{R})$ where $\mathbf{L}$ and $\mathbf{R}$ are similar functors.

The following are some nice properties of a quasi-Frobenius triple.

1.2 Lemma. *Consider Grothendieck categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. If $(\mathbf{L}, \mathbf{F}, \mathbf{R})$ is a quasi-Frobenius triple for $\mathcal{A}$ and $\mathcal{B}$, then*

(a) The functors $\mathbf{L}, \mathbf{F}$ and $\mathbf{R}$ are exact, preserve injective and projective objects, direct sums and finitely generated objects.

(b) If $(\mathbf{L}', \mathbf{F}', \mathbf{R}')$ is a quasi-Frobenius triple for $\mathcal{B}$ and $\mathcal{C}$, then

$$(\mathbf{L} \circ \mathbf{L}', \mathbf{F}' \circ \mathbf{F}, \mathbf{R} \circ \mathbf{R}')$$

is also a quasi-Frobenius triple for $\mathcal{A}$ and $\mathcal{C}$.

Proof. (b) is straightforward.
(a) Since $\mathbf{F}$ is exact, $\mathbf{L}$ preserves projective objects and $\mathbf{R}$ injective objects. Suppose that $\mathbf{L} \sim \mathbf{R}$. Then there are natural transformations

$$\mathbf{L} \xrightarrow{\phi} \mathbf{R}^n \xrightarrow{\psi} \mathbf{L} \tag{1}$$

such that $\psi \circ \phi = 1_{\mathbf{L}}$ and

$$\mathbf{R} \xrightarrow{\phi'} \mathbf{L}^m \xrightarrow{\psi'} \mathbf{R} \tag{2}$$

such that $\psi' \circ \phi' = 1_{\mathbf{R}}$. From (1) and (2), we easily deduce that $\mathbf{R}$ preserve projective objects and $\mathbf{L}$ injective objects. Consider now any short exact sequence in $\mathcal{B}$

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \tag{3}$$

Applying $\mathbf{R}^n$ to (3), we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbf{R}^n(X) & \xrightarrow{\mathbf{R}^n(f)} & \mathbf{R}^n(Y) & \xrightarrow{\mathbf{R}^n(g)} & \mathbf{R}^n(Z) & \\
& \phi_X \uparrow & & \phi_Y \uparrow & & \phi_Z \uparrow & \\
& \mathbf{L}(X) & \xrightarrow{\mathbf{L}(f)} & \mathbf{L}(Y) & \xrightarrow{\mathbf{L}(g)} & \mathbf{L}(Z) & \longrightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

from which it follows that $\mathbf{L}(f)$ is monic and therefore, $\mathbf{L}$ exact. A similar argument with $\mathbf{L}^m$ shows that $\mathbf{R}$ is exact.

To prove that $\mathbf{R}$ preserves direct sums, suppose that $(M_\alpha)_{\alpha \in I}$ is an indexed set of objects of $\mathcal{B}$. From the inclusion maps $i_\alpha : M_\alpha \to \oplus_{\alpha \in I} M_\alpha$, we have the monomorphisms $\mathbf{R}(i_\alpha) : \mathbf{R}(M_\alpha) \to \mathbf{R}(\oplus_{\alpha \in I} M_\alpha)$. Then the direct sum map $i = \oplus i_\alpha : \oplus_\alpha \mathbf{R}(M_\alpha) \to \mathbf{R}(\oplus_{\alpha \in I} M_\alpha)$ is also a monomorphism. Now, from commutative diagram (right square)

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbf{R}(\oplus M_\alpha) & \xrightarrow{\phi_{\oplus M_\alpha}} & \mathbf{L}^n(\oplus M_\alpha) & \xrightarrow{\psi_{\oplus M_\alpha}} & \mathbf{R}(\oplus M_\alpha) & \longrightarrow 0 \\
& i \uparrow & & \cong \uparrow & & i \uparrow & \\
0 \longrightarrow & \oplus \mathbf{R}(M_\alpha) & \xrightarrow{\oplus \phi_{M_\alpha}} & \oplus \mathbf{L}^n(M_\alpha) & \xrightarrow{\oplus \psi_{M_\alpha}} & \oplus \mathbf{R}(M_\alpha) & \longrightarrow 0
\end{array}$$

we have that i is also epic which implies that the functor $\mathbf{R}$ preserves direct sums. Similar proof being that $\mathbf{L}$ preserves also direct sums. Recalling that an object M of $\mathcal{A}$ is finitely generated if the functor $\mathrm{Hom}_{\mathcal{A}}(M, -)$ preserves direct unions, the last affirmation is straightforward. □

1.3 Definition. A functor $\mathbf{F} : \mathcal{A} \to \mathcal{B}$ is said to be *quasi-Frobenius functor* if $(\mathbf{L}, \mathbf{F}, \mathbf{R})$ is a quasi-Frobenius triple for some functors $\mathbf{L}, \mathbf{R} : \mathcal{B} \to \mathcal{A}$.

The notion of a *left quasi-Frobenius pair of functors* appeared in Guo [9] where he proved that a ring extension $\varphi : R \to S$ is left quasi-Frobenius if and only if $(\varphi_*, - \otimes_R S)$ is a left quasi-Frobenius pair of functors, where φ_* is the restriction of scalars functor.
In general, to categories $\mathcal{A}$ and $\mathcal{B}$ with finite direct sums, the pair de functors $(\mathbf{F}, \mathbf{L})$ is called a *left quasi-Frobenius pair of functors* if $\mathbf{F} : \mathcal{A} \to \mathcal{B}$ is a right adjoint of $\mathbf{L} : \mathcal{B} \to \mathcal{A}$ and for some positive integer n, there are natural transformations

$$\alpha : 1_{\mathcal{A}} \to (\mathbf{LF})^n \quad \text{and} \quad \overline{\alpha} : (\mathbf{FL})^n \to 1_{\mathcal{B}}$$

such that

$$\overline{\alpha}_{\mathbf{F}(X)} \circ \mathbf{F}(\alpha_X) = 1_{\mathbf{F}(X)}$$

for all $X \in \mathcal{A}$.

On the other hand, if the functor $\mathbf{R} : \mathcal{B} \to \mathcal{A}$ is a right adjoint to $\mathbf{F}$, then the unity $\overline{\eta} : 1_{\mathcal{A}} \to \mathbf{RF}$ and the counit $\overline{\rho} : \mathbf{FR} \to 1_{\mathcal{B}}$ satisfied the identities $\overline{\rho}_{\mathbf{F}(X)} \circ \mathbf{F}(\overline{\eta}_X) = 1_{\mathbf{F}(X)}$ and $\mathbf{R}(\overline{\rho}_Y) \circ \overline{\eta}_{\mathbf{R}(Y)} = 1_{\mathbf{R}(Y)}$, for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

The next proposition implies, in particular, that if $\Gamma = (\mathbf{L}, \mathbf{F}, \mathbf{R})$ is a quasi-Frobenius triple of functors, then $(\mathbf{F}, \mathbf{L})$ is a left quasi-Frobenius pair.

1.4 Proposition. *If* $\Gamma = (\mathbf{L}, \mathbf{F}, \mathbf{R})$ *is a quasi-Frobenius triple of functors for* $\mathcal{A}$ *and* $\mathcal{B}$*, then* $(\mathbf{F}, \mathbf{L})$ *and* $(\mathbf{R}, \mathbf{F})$ *are left quasi-Frobenius pair.*

Proof. Assume $\mathbf{R}|\mathbf{L}$. Then there exist morphisms

$$\mathbf{R} \xrightarrow{\phi'} \mathbf{L}^n \xrightarrow{\psi'} \mathbf{R}$$

such that $\psi' \circ \phi' = 1_R$.

We define the functorial morphism $\alpha : 1_{\mathcal{A}} \to (\mathbf{LF})^n$ by the composition of morphisms

$$X \xrightarrow{\overline{\eta}_X} \mathbf{RF}(X) \xrightarrow{\phi'_{\mathbf{F}(X)}} \mathbf{L}^n(\mathbf{F}(X)) \cong (\mathbf{LF})^n(X)$$

for every $X \in \mathcal{A}$. Similarly, for any $Y \in \mathcal{B}$, the composition

$$(\mathbf{FL})^n(Y) \cong \mathbf{F}(\mathbf{L}^n(Y)) \xrightarrow{\mathbf{F}(\psi'_Y)} \mathbf{FR}(Y) \xrightarrow{\overline{\rho}_Y} Y$$

define the functorial morphism $\overline{\alpha} : (\mathbf{FL})^n \to 1_{\mathcal{B}}$. Then

$$\begin{aligned} \overline{\alpha}_{\mathbf{F}(X)} \circ \mathbf{F}(\alpha_X) &= \overline{\rho}_{\mathbf{F}(X)} \circ \mathbf{F}(\psi'_{\mathbf{F}(X)} \circ \phi'_{\mathbf{F}(X)} \circ \overline{\eta}_X) \\ &= \overline{\rho}_{\mathbf{F}(X)} \circ \mathbf{F}(1_{\mathbf{RF}(X)} \circ \overline{\eta}_X) \\ &= \overline{\rho}_{\mathbf{F}(X)} \circ \mathbf{F}(\overline{\eta}_X) = 1_{\mathbf{F}(X)} \end{aligned} .$$

This means that $(\mathbf{F}, \mathbf{L})$ is a left quasi-Frobenius pair. Likewise, using that $\mathbf{L}|\mathbf{R}$, we obtain the other afirmation. □

2 Quasi-Frobenius functors for module categories

Consider associative and unital rings R and S. We use the standard module theory notation, for example a (R,S)-bimodule M is denoted by ${}_RM_S$, $\mathrm{Hom}_R(-,-)$ denotes the Abelian group of R-module maps. The dual of a left R-module M is denoted by $({}_RM)^*$. Finally, ${}_R\mathcal{M}_S$ denotes the category of all (R,S)-bimodules. Bimodules $M \in {}_R\mathcal{M}_S$ and $N \in {}_R\mathcal{M}_S$ are called *similar*, abbreviated ${}_RM_S \sim {}_RN_S$, if there are $m,n \in \mathbb{N}$ and (R,S)-bimodules P and Q such that $M \oplus P \cong N^{(m)}$ and $N \oplus Q \cong M^{(n)}$ as bimodules (cf. [2]). It is easy see that "$\sim$" defines an equivalence relation on the class of (R,S)-bimodules.

2.1 Lemma. *Let M and N be (R,S)-bimodules. Then ${}_RM_S \sim {}_RN_S$ if and only if the tensor functors $M \otimes_S -$ and $N \otimes_S -$ are similar.*

Proof. It is clear because the tensor product preserves direct sums. □

The following characterization could be deduced from results in the Morita's paper [11], but we shall give here its proof for the sake of completeness.

2.2 Theorem. *For functors $\mathbf{F} : {}_R\mathcal{M} \to {}_S\mathcal{M}$ and $\mathbf{L}, \mathbf{R} : {}_S\mathcal{M} \to {}_R\mathcal{M}$, the following statements are equivalent.*

(a) $(\mathbf{L}, \mathbf{F}, \mathbf{R})$ *is a quasi-Frobenius triple.*

(b) There exist bimodules ${}_SM_R$, ${}_RN_S$ and ${}_R\overline{N}_S$ with the following properties:

(1) ${}_SM_R$ and ${}_R\overline{N}_S$ are finitely generated and projectives on both sides.

(2) $\mathbf{F} \cong M \otimes_R -$, $\mathbf{L} \cong N \otimes_S -$ *and* $\mathbf{R} \cong \overline{N} \otimes_S -$.

(3) $({}_RN)^* \cong {}_SM_R$ *and* $({}_SM)^* \cong {}_R\overline{N}_S$ *as bimodules and* ${}_RN_S \sim {}_R\overline{N}_S$.

Proof. Assume (a) and write $M = \mathbf{F}({}_RR)$, $N = \mathbf{L}({}_SS)$ and $\overline{N} = \mathbf{R}({}_SS)$. It is well known that M is an (S,R)-bimodule and $N, \overline{N}$ are (R,S)-bimodules. By Lemma 1.2, ${}_SM$, ${}_RN$ and ${}_R\overline{N}$ are finitely generated and projectives. Also, from Lemma 1.2 and [15, Theorem 3.7.5] , we have $\mathbf{F} \cong M \otimes_R -$, $\mathbf{L} \cong N \otimes_S -$ and $\mathbf{R} \cong \overline{N} \otimes_S -$. Now from the adjoint pair $(\mathbf{L}, \mathbf{F})$ we have the functorial isomorphism

$$\mathrm{Hom}_R(\mathbf{L}(-), -) \cong \mathrm{Hom}_S(-, \mathbf{F}(-))$$

and, in particular, the isomorphism of (R,S)-bimodules

$$\mathrm{Hom}_R(\mathbf{L}({}_SS), {}_RR) \cong \mathrm{Hom}_S({}_SS, \mathbf{F}({}_RR))$$

So, $({}_RN)^* \cong {}_SM_R$ as (S,R)-bimodules and this implies that M_R is finitely generated and projective as right R-module. Similarly, from adjoint pair $(\mathbf{F}, \mathbf{R})$ we have the isomorphism of (R,S)-bimodules $({}_SM)^* \cong {}_R\overline{N}_S$, and, hence, $\overline{N}_S$ is also finitely generated and projective. Finally, by Lemma 2.1, ${}_RN_S \sim {}_R\overline{N}_S$ since $\mathbf{L} \sim \mathbf{R}$.

Assume (b). We want to prove that $(N \otimes_S -, M \otimes_R -, \overline{N} \otimes_S -)$ is a quasi-Frobenius triple. By [2], the functor $\mathbf{F} \cong M \otimes_R -$ is left adjoint to $\mathrm{Hom}_S(M, -)$. But

$$\mathrm{Hom}_S(M, -) \cong ({}_SM)^* \otimes_S - \cong \overline{N} \otimes_S -,$$

which implies that

$$\mathrm{Hom}_S(M, -) \cong \overline{N} \otimes_S -$$

Hence, the functor $\overline{N} \otimes_S -$ is right adjoint to the functor $M \otimes_R -$. Analogously, the functor $N \otimes_S -$ is left adjoint to $M \otimes_R -$. The similarity of the functors $N \otimes_S -$ and $\overline{N} \otimes_S -$ it follows of Lemma 2.1, since ${}_RN_S \sim {}_R\overline{N}_S$.

□

2.3 Proposition. *For each quasi-Frobenius triple* $(\mathbf{L}, \mathbf{F}, \mathbf{R})$ *for* ${}_R\mathcal{M}$ *and* ${}_S\mathcal{M}$ *there is a quasi-Frobenius triple* $(\overline{\mathbf{L}}, \overline{\mathbf{F}}, \overline{\mathbf{R}})$ *for* $\mathcal{M}_R$ *and* $\mathcal{M}_S$ *such that the correspondence* $(\mathbf{L}, \mathbf{F}, \mathbf{R}) \mapsto (\overline{\mathbf{L}}, \overline{\mathbf{F}}, \overline{\mathbf{R}})$ *between quasi-Frobenius triples is bijective up to natural isomorphisms. In fact, is a duality between the categories of quasi-Frobenius functors.*

Proof. By Theorem 2.5, there are finitely generated projective bimodules ${}_SM_R$, ${}_RN_S$ and ${}_R\overline{N}_S$ with the property $({}_RN)^* \cong {}_SM_R$, $({}_SM)^* \cong {}_R\overline{N}_S$ and ${}_RN_S \sim {}_R\overline{N}_S$. Moreover, $\mathbf{F} \cong M \otimes_R -$, $\mathbf{L} \cong N \otimes_S -$ and $\mathbf{R} \cong \overline{N} \otimes_S -$. Put ${}_RM'_S = ({}_SM)^*$; ${}_SN'_R = ({}_R\overline{N})^*$ and ${}_S\overline{N}'_R = ({}_RN)^*$. The bimodules ${}_RM'_S$ and ${}_S\overline{N}'_R$ are finitely generated and projective on both side. Define the functors $\overline{\mathbf{F}} : {}_R\mathcal{M} \to {}_S\mathcal{M}$ and $\overline{\mathbf{L}}, \overline{\mathbf{R}} : {}_S\mathcal{M} \to {}_R\mathcal{M}$ as

$$\overline{\mathbf{F}} = - \otimes_R \overline{N};\ \overline{\mathbf{L}} = - \otimes_S M;\ \overline{\mathbf{R}} = - \otimes_S ({}_R\overline{N})^*.$$

Clearly $(M'_S)^* \cong {}_S\overline{N}'_R$ and $(N'_R)^* \cong {}_RM'_S$. Finally, from ${}_RN_S \sim {}_R\overline{N}_S$, we get $({}_RN)^* \sim ({}_R\overline{N})^*$. But

$$({}_RN)^* \cong {}_SM_R = {}_S\overline{N}'_R \text{ and } ({}_R\overline{N}^* \cong {}_SN'_R.$$

Therefore, ${}_S\overline{N}'_R \sim_S N'_R$. Now, by Theorem 2.5, $(\overline{\mathbf{L}}, \overline{\mathbf{F}}, \overline{\mathbf{R}})$ is a quasi-Frobenius triple for $\mathcal{M}_R$ and $\mathcal{M}_S$. Clearly, this correspondence is bijective. □

Let $\varphi : R \to S$ be a ring homomorphism. Then S can be regarded as a two-sided R-module by φ in the natural way. Thus $\mathrm{Hom}_R({}_RS, R)$ has a structure of (R, S)-bimodule: $(rfs)(x) = rf(sx)$ for $f \in \mathrm{Hom}_R({}_RS, R), s, x \in S$ and $r \in R$. We can associate to φ the restriction of scalars functor ${}^l\varphi_* : {}_S\mathcal{M} \to {}_R\mathcal{M}$, the induction functor $S \otimes_R - : {}_R\mathcal{M} \to {}_S\mathcal{M}$ and the coinduction functor $\mathrm{Hom}_R({}_RS_S, -) : {}_R\mathcal{M} \to {}_S\mathcal{M}$. It is well known that $S \otimes_R -$ is left adjoint to ${}^l\varphi_*$ and that $\mathrm{Hom}_R({}_RS_S, -)$ is right adjoint to ${}^l\varphi_*$. From Theorem 2.5, ${}^l\varphi_*$ is a quasi-Frobenius functor if and only if ${}_RS$ is finitely generated projective and ${}_SS_R \sim \mathrm{Hom}_R(S, R)$. By symmetry, we can consider also the restriction functor $\varphi^r_* : \mathcal{M}_S \to \mathcal{M}_R$ and its adjoint functors $- \otimes_R S$ and $\mathrm{Hom}_R(S_R, -)$.

2.4 Corollary. *If* $(S \otimes_R -, {}^l\varphi_*, \mathrm{Hom}_R({}_RS, -))$ *is a quasi-Frobenius triple for* ${}_R\mathcal{M}$ *and* ${}_S\mathcal{M}$*, then* $(- \otimes_R S, \varphi^r_*, \mathrm{Hom}_R(S_R, -))$ *is a quasi-Frobenius triple for* $\mathcal{M}_R$ *and* $\mathcal{M}_S$.

Proof. Straightforward from Proposition 2.3, since $\overline{F} \cong - \otimes_S S = \varphi_*^r$. □

The rest of this section will study the notion of quasi-Frobenius extension (left and right extension) in terms of quasi-Frobenius functors. A ring homomorphism $\varphi : R \to S$ is called *left quasi-Frobenius extension* if ${}_R S$ is finitely generated and projective and ${}_S S_R \mid \mathrm{Hom}_R({}_R S, R)$. Equivalently, S_R is finitely generated projective and $\mathrm{Hom}_R(S_R, R) \mid {}_R S_S$. Similarly, φ is called *right quasi-Frobenius extension* if S_R is finitely generated and projective and ${}_R S_S \mid \mathrm{Hom}_R(S_R, R)$ (or $\mathrm{Hom}_R({}_R S, R) \mid {}_S S_R$). Then φ is a *quasi-Frobenius extension* (left and right extension) if ${}_R S$ is a finitely generated projective and ${}_S S_R \sim \mathrm{Hom}_R({}_R S, R)$. Equivalently, S_R is finitely generated projective and ${}_R S_S \sim \mathrm{Hom}_R(S_R, R)$. Therefore, the ring extension φ is quasi-Frobenius if and only $\varphi_* : {}_S\mathcal{M} \to {}_R\mathcal{M}$ is a quasi-Frobenius functor. The next lemma it follows from our remarks above.

2.5 Lemma. *Let $\varphi : R \to S$ be a ring extension. Then the following statements are equivalent.*

(a) φ is a quasi-Frobenius extension.

(b) ${}^l\varphi_*$ is a quasi-Frobenius functor.

(c) φ_*^r is a quasi-Frobenius functor.

Recall of [2] that a k-algebra R is called quasi-Frobenius if ${}_R R_k \sim \mathrm{Hom}_k(R, k)$. This is equivalent to say that the functor $(-)_\varepsilon : \mathcal{M}_R \to \mathcal{M}_k$ is quasi-Frobenius where $\varepsilon : k \to R$ is the canonical ring morphism.

2.6 Lemma. *Let $\varphi : R \to S$ be and $\psi : S \to T$ be two quasi-Frobenius extensions of k-algebras. Then the composition $\psi \circ \varphi$ is also a quasi-Frobenius extension. In particular, if R is a quasi-Frobenius algebra, then S too.*

Proof. This follows from Lemma 1.2 (b). □

2.7 Remark. An example of finite-dimensional quasi-Frobenius algebra which is not a Frobenius algebra is gives in [13]. This shows that there are quasi-Frobenius functors which are not Frobenius functors.

3 Quasi-Frobenius functors in graded rings

Let G be a group with neutral element e. A ring R is said to be G-graded if there is a family $\{R_x; x \in G\}$ of additive subgroups of R such that $R = \bigoplus_{x \in G} R_x$, and the multiplication in R is such that, for all x and y in G, $R_x R_y \subseteq R_{xy}$. Similarly, a left R-module M is graded by G if there is a family $\{M_x; x \in G\}$ of additive subgroups of M such that $M = \bigoplus_{x \in G} M_x$, and for all x and y in G, $R_x M_y \subseteq M_{xy}$. We will denote by R-gr the category of all G-graded left R-modules over the unital group-graded ring R. It is well known (see e.g. [14]) that associated to ring homomorphism $\varphi : R_e \to R$ we have always the adjoint pair

$$Ind(-) : {}_{R_e}\mathcal{M} \rightleftarrows R\text{-gr} : (-)_e$$

where $(-)_e$ is the exact *restriction at* e functor give by $M \mapsto M_e$, for every left graded R-module $M = \bigoplus_{x\in G} M_x$ and $Ind(-)$ the *induction* functor give by $Ind(N) = R \otimes_{R_e} N$, for every left R_e-module N. This left R-module can be graded by putting $(Ind(N))_y = R_y \otimes_{R_e} N$ for every $y \in G$. It was shown in [14] that the functor $Ind(-)$ is a left adjoint of the functor $(-)_e$ and the unity of the adjunction $\eta : 1_{_{R_e}\mathcal{M}} \to (-)_e \circ Ind(-)$ is a functorial isomorphism.
The functor $(-)_e$ has also a right adjoint called the *e-th coinduced* functor

$$Coind(-) : {}_{R_e}\mathcal{M} \to R\text{-gr},$$

where for every left R_e-module N, $Coind(N)$ is an object of R-gr, with gradation

$$Coind(N)_y = \{f \in \mathrm{Hom}_{R_e}(R, N) \mid f(R_x) = 0, \forall x \neq y^{-1}\}$$

Moreover, the counity of this adjunction $\tau : (-)_e \circ Coind(-) \to 1_{_{R_e}\mathcal{M}}$ is a functorial isomorphism. Clearly, $(-)_e$ is a quasi-Frobenius functor if $(Ind(-), (-)_e, Coind(-))$ is a quasi-Frobenius triple of functors.

3.1 Theorem. *Let R be a G-graded ring. The following assertions are equivalent.*

(a) $(-)_e$ is a quasi-Frobenius functor.

(b) $Ind(-) \sim Coind(-)$.

(c) $\forall\, x \in G$, R_x is finitely generated and projective in ${}_{R_e}\mathcal{M}$ and $R \sim Coind(R_e)$.

Proof. $(a) \Leftrightarrow (b)$ is clear.
$(b) \Rightarrow (c)$. Assume that $Ind(-) \sim Coind(-)$. Then $Ind(R_e) \sim Coind(R_e)$. But $Ind(R_e) \cong R$ which implies that $R \sim Coind(R_e)$ as (R, R_e)-bimodules. On the other hand, by the properties of adjoint functors it follows that the functor $(-)_e$ has the following property: "If $M \in R$-gr is a finitely generated projective object, then $M_e \in {}_{R_e}\mathcal{M}$ is finitely generated projective R_e-module". In particular, if $M = R(x)$ for any $x \in G$, then R_x is a projective and finitely generated R_e-module.
$(c) \Rightarrow (b)$. Assume that ${}_R R_{R_e} \mid Coind(R_e)$. Then there exists morphisms in ${}_{R_e}\mathcal{M}$

$$R \xrightarrow{f} Coind(R_e)^n \xrightarrow{g} R \tag{4}$$

with $g \circ f = 1_R$. For any left R_e-module X, apply the functor $- \otimes_{R_e} X$ to (4) and we obtain

$$R \otimes_{R_e} X \xrightarrow{f\otimes X} Coind(R_e)^n \otimes_{R_e} X \cong (Coind(R_e) \otimes_{R_e} X)^n \xrightarrow{g\otimes X} R \otimes_{R_e} X \tag{5}$$

By assumption R_x is finitely generated and projective, whence $\mathrm{Hom}_{R_e}(R_x, R_e) \otimes_{R_e} X \cong \mathrm{Hom}_{R_e}(R_x, X)$. In particular, $Coind(R_e) \otimes_{R_e} X \cong Coind(X)$. Then the sequence (5) is given by

$$R \otimes_{R_e} X \xrightarrow{f\otimes X} Coind(X)^n \xrightarrow{g\otimes X} R \otimes_{R_e} X$$

Since $(g \otimes X) \circ (f \otimes X) = (g \circ f) \otimes X = 1_R \otimes X$, this implies that $Ind(-) \mid Coind(-)$. Analogously, we can prove that $Coind(-) \mid Ind(-)$. Therefore, $Ind(-) \sim Coind(-)$. □

3.2 Remark. Let $R = \bigoplus_{x \in G} R_x$ be a k-algebra graded by a group G. We consider the forgetful functor $U : R\text{-gr} \to {}_R\mathcal{M}$, where R-gr is the category of G-graded modules. It is well know that U has a right adjoint functor $F : {}_R\mathcal{M} \to R\text{-gr}$. If U is a quasi-Frobenius functor, then U commutes with direct products and by [5, Corollary 4.4], G is finite. This implies that U is a Frobenius functor (see [6, Proposition 2.5]).

4 Quasi-Frobenius functors between categories of comodules over corings

Let A be an associative and unitary algebra over a commutative ring (with unit) k. We recall from [16] that an A-coring $\mathfrak{C}$ consists of an A-bimodule $\mathfrak{C}$ with two A-bimodule maps

$$\Delta : \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C}, \ \epsilon : \mathfrak{C} \to A$$

such that $(\mathfrak{C} \otimes_A \Delta) \circ \Delta = (\Delta \otimes_A \mathfrak{C}) \circ \Delta$ and $(\mathfrak{C} \otimes_A \epsilon) \circ \Delta = (\epsilon \otimes_A \mathfrak{C}) \circ \Delta = 1_{\mathfrak{C}}$. A right $\mathfrak{C}$-comodule is a pair (M, ρ_M) consisting of a right A-module M and an A-linear map $\rho_M : M \to M \otimes_A \mathfrak{C}$ satisfying $(M \otimes_A \Delta) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M$ and $(M \otimes_A \epsilon) \circ \rho_M = 1_M$. The right $\mathfrak{C}$-comodules together with their morphisms form the additive category $\mathcal{M}^{\mathfrak{C}}$. If ${}_A\mathfrak{C}$ is flat, then $\mathcal{M}^{\mathfrak{C}}$ is a Grothendieck category.
Throughout this section, $\mathfrak{C}$ and $\mathfrak{D}$ denote corings over the associative and unitary k-algebras A and B, respectively. Following [8, 2.3 and 2.4], the $(\mathfrak{C}, \mathfrak{D})$-bicomodules are the objects of the k-linear category ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ whose morphism are those (A, B)-bimodule maps, which are morphisms of $\mathfrak{C}$-comodules and of $\mathfrak{D}$-comodules. Furthermore, given bicomodules $N \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$ and $\overline{N} \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}'}$ with $\mathfrak{D}'$ any B'-coring, we can consider the *cotensor product* $N \square_{\mathfrak{C}} \overline{N}$. If $\mathfrak{D}_B$ and ${}_{B'}\mathfrak{D}$ are flat modules, then $N \square_{\mathfrak{C}} \overline{N}$ is a $(\mathfrak{D}, \mathfrak{D}')$-bicomodule. In particular, $N \square_{\mathfrak{C}} \mathfrak{C} \cong N$ as $(\mathfrak{D}, \mathfrak{C})$-bicomodule.

4.1 Definition. A bicomodule ${}_{\mathfrak{D}}N_{\mathfrak{C}}$ will be said to be *similar* to bicomodule ${}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$, abbreviated ${}_{\mathfrak{D}}N_{\mathfrak{C}} \sim {}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$, if there are $m, n \in \mathbb{N}$ and $(\mathfrak{D}, \mathfrak{C})$-bicomodules P and Q such that $N \oplus P \cong \overline{N}^{(m)}$ and $\overline{N} \oplus Q \cong N^{(n)}$ as bicomodules.

It is easy see that "$\sim$" defines an equivalence relation on the class of $(\mathfrak{D}, \mathfrak{C})$-bicomodules. In this section we will characterize quasi-Frobenius functors between categories of comodules over corings. We first prove the following result.

4.2 Lemma. *Suppose that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat and $N, \overline{N} \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. Then ${}_{\mathfrak{D}}N_{\mathfrak{C}} \sim {}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$ if and only if $-\square_{\mathfrak{D}} N \sim -\square_{\mathfrak{D}} \overline{N}$.*

Proof. If $-\square_{\mathfrak{D}} N \sim -\square_{\mathfrak{D}} \overline{N}$, then $\mathfrak{D} \square_{\mathfrak{D}} N \sim \mathfrak{D} \square_{\mathfrak{D}} \overline{N}$ and hence ${}_{\mathfrak{D}}N_{\mathfrak{C}} \sim {}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$. Assume now that ${}_{\mathfrak{D}}N_{\mathfrak{C}} \mid {}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$. This condition establish that for some positive integer n there are bicomodule morphisms

$$N \xrightarrow{f} \overline{N}^{\,n} \xrightarrow{g} N$$

such that $g \circ f = 1_N$. For any right $\mathfrak{D}$-comodule X we apply the cotensor functor $X\square_{\mathfrak{D}}-$ to the above sequence and we obtain

$$X\square_{\mathfrak{D}}N \xrightarrow{1_X\square f} (X\square_{\mathfrak{D}}\overline{N})^n \xrightarrow{1_X\square g} X\square_{\mathfrak{D}}N$$

Clearly, $(1_X\square g)\circ(1_X\square f) = 1_X\square(g\circ f) = 1_{X\square_{\mathfrak{D}}N}$. This implies that $-\square_{\mathfrak{D}}N \mid -\square_{\mathfrak{D}}\overline{N}$. Analogously, from ${}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}} \mid {}_{\mathfrak{D}}N_{\mathfrak{C}}$ we get $-\square_{\mathfrak{D}}\overline{N} \mid -\square_{\mathfrak{D}}N$. Thus $-\square_{\mathfrak{D}}N \sim -\square_{\mathfrak{D}}\overline{N}$. □

Recall of [1] that a bicomodule $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ is called an *injector* as right $\mathfrak{D}$-comodule if the functor $-\otimes_A N : \mathcal{M} \to \mathcal{M}^{\mathfrak{D}}$ preserves injective objects.

4.3 Theorem. *Suppose that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat. For k-linear functors $\mathbf{F} : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ and $\mathbf{L}, \mathbf{R} : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$, the following statements are equivalent.*

(a) $\Gamma = (\mathbf{L}, \mathbf{F}, \mathbf{R})$ *is a quasi-Frobenius triple.*

(b) There exist bicomodules ${}_{\mathfrak{C}}M_{\mathfrak{D}}$, ${}_{\mathfrak{D}}N_{\mathfrak{C}}$ and ${}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$ with the following properties.

- *(1) $M_{\mathfrak{D}}$ and $\overline{N}_{\mathfrak{C}}$ are quasi-finite and injector comodules.*
- *(2)* $\mathbf{F} \cong \mathrm{h}_{\mathfrak{C}}(\overline{N}, -)$, $\mathbf{L} \cong \mathrm{h}_{\mathfrak{D}}(M, -)$ *and* $\mathbf{R} \cong -\square_{\mathfrak{D}}\overline{N}$.
- *(3)* $\mathrm{h}_{\mathfrak{C}}(\overline{N}, \mathfrak{C}) \cong {}_{\mathfrak{C}}M_{\mathfrak{D}}$, $\mathrm{h}_{\mathfrak{D}}(M, \mathfrak{D}) \cong {}_{\mathfrak{D}}N_{\mathfrak{C}}$ *and* $\mathrm{h}_{\mathfrak{D}}(M, \mathfrak{D}) \sim {}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$ *as bicomodules.*

Proof. $(a) \Rightarrow (b)$. $\mathbf{F}$ is exact and preserves inductive limits. By [8, Theorem 3.5], the functor $\mathbf{F}$ is naturally equivalent to $-\square_{\mathfrak{C}}M$ where ${}_{\mathfrak{C}}M_{\mathfrak{D}}$ is a bicomodule such that $M_{\mathfrak{D}} = \mathbf{F}(\mathfrak{C})$. From Lemma 1.2, $\mathbf{L}$ and $\mathbf{R}$ are also exact. Then $\mathbf{L} \cong -\square_{\mathfrak{D}}N$ and $\mathbf{R} \cong -\square_{\mathfrak{D}}\overline{N}$ where ${}_{\mathfrak{D}}N_{\mathfrak{C}}$ and ${}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$ are bicomodules such that $N_{\mathfrak{C}} = \mathbf{L}(\mathfrak{D})$ and $\overline{N}_{\mathfrak{C}} = \mathbf{R}(\mathfrak{D})$. Now taking $X = {}_{\mathfrak{C}}M_{\mathfrak{D}}$ and $\Lambda = {}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$, the condition (i) in [17, Proposition 2.9] hold. This implies that $\overline{N}_{\mathfrak{C}}$ is quasi-finite and injector as a right $\mathfrak{C}$-comodule and $\mathrm{h}_{\mathfrak{D}}(\overline{N}, \mathfrak{C}) \cong {}_{\mathfrak{C}}M_{\mathfrak{D}}$. Since $\mathbf{F}$ is a left adjoint to $\mathbf{R}$ and $\overline{N}_{\mathfrak{C}}$ is quasi-finite, we obtain that $\mathbf{F} \cong \mathrm{h}_{\mathfrak{C}}(\overline{N}, -)$. By a similar argument, taking $X = {}_{\mathfrak{D}}N_{\mathfrak{C}}$ and $\Lambda = {}_{\mathfrak{C}}M_{\mathfrak{D}}$, we find that $M_{\mathfrak{D}}$ is quasi-finite and injector as a right $\mathfrak{D}$-comodule and $\mathrm{h}_{\mathfrak{D}}(M, \mathfrak{D}) \cong {}_{\mathfrak{D}}N_{\mathfrak{C}}$. Moreover, $\mathbf{L} \cong \mathrm{h}_{\mathfrak{D}}(M, -)$. Finally, from $\mathbf{L} \sim \mathbf{R}$ it follows that $\mathrm{h}_{\mathfrak{D}}(M, \mathfrak{D}) \sim {}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$.
$(b) \Rightarrow (a)$. Assume that there exist bicomodules ${}_{\mathfrak{C}}M_{\mathfrak{D}}$, ${}_{\mathfrak{D}}N_{\mathfrak{C}}$ and ${}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$ satisfying (1), (2) and (3). From the condition $\mathrm{h}_{\mathfrak{D}}(M, \mathfrak{D}) \sim {}_{\mathfrak{D}}\overline{N}_{\mathfrak{C}}$ of (3) and Lemma 4.2, $-\square_{\mathfrak{D}}\mathrm{h}_{\mathfrak{D}}(M, \mathfrak{D}) \sim -\square_{\mathfrak{D}}\overline{N}$. But $\mathrm{h}_{\mathfrak{D}}(M, -) \cong -\square_{\mathfrak{D}}\mathrm{h}_{\mathfrak{D}}(M, \mathfrak{D})$, since $M_{\mathfrak{D}}$ is quasi-finite and injector (see [1, Corollary 3.12]). Therefore, $\mathbf{L} \sim \mathbf{R}$. Now, [8, Proposition 4.2] establish that the functors $\mathrm{h}_{\mathfrak{C}}(\overline{N}, -)$ and $\mathrm{h}_{\mathfrak{D}}(M, -)$ are left adjoint to $-\square_{\mathfrak{D}}\overline{N}$ and $-\square_{\mathfrak{C}}M$, respectively. Hence Γ is a quasi-Frobenius triple. □

5 Quasi-Frobenius corings versus quasi-Frobenius extension of corings

Following [8], a coring homomorphism from the A-coring $\mathfrak{C}$ into the B-coring $\mathfrak{D}$ is a pair (φ, ρ), where $\rho : A \to B$ is a homomorphism of k-algebras and $\varphi : \mathfrak{C} \to \mathfrak{D}$ is a homomorphism of A-bimodules such that

$$\epsilon_{\mathfrak{D}} \circ \varphi = \rho \circ \epsilon_{\mathfrak{C}} \text{ and } \Delta_{\mathfrak{D}} \circ \varphi = \omega_{\mathfrak{D},\mathfrak{D}} \circ (\varphi \otimes_A \varphi) \circ \Delta_{\mathfrak{C}},$$

where $\omega_{\mathfrak{D},\mathfrak{D}} : \mathfrak{D} \otimes_A \mathfrak{D} \to \mathfrak{D} \otimes_B \mathfrak{D}$ is the canonical map induce by $\rho : A \to B$. The functor $- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ has a right adjoint $-\Box_{\mathfrak{D}}(B \otimes_A \mathfrak{C}) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$ [8, Proposition 5.4]. Moreover, if $(\mathfrak{C} \otimes_A B)_{\mathfrak{D}}$ is a quasi-finite D-comodules, then the functor $\mathrm{h}_{\mathfrak{D}}(\mathfrak{C} \otimes_A B, -) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$ is a left adjoint to $-\Box_{\mathfrak{C}}(\mathfrak{C} \otimes_A B) \cong (-\Box_{\mathfrak{C}}\mathfrak{C}) \otimes_A B \cong - \otimes_A B$. In this case we have a tripe de functors

$$\Gamma = (\mathrm{h}_{\mathfrak{D}}(\mathfrak{C} \otimes_A B, -), - \otimes_A B, -\Box_{\mathfrak{D}}(B \otimes_A \mathfrak{C}))$$

between the Grothendieck categories $\mathcal{M}^{\mathfrak{C}}$ and $\mathcal{M}^{\mathfrak{D}}$ where $(B \otimes_A \mathfrak{C})_{\mathfrak{C}}$ is quasi-finite by [8, Proposition 5.4]. Moreover, $(B \otimes_A \mathfrak{C})_{\mathfrak{C}}$ is an injector because the functor $-\Box_{\mathfrak{D}}(B \otimes_A \mathfrak{C})$ is right adjoint to the exact functor $- \otimes_A B$. From Theorem 4.3 we have the following

5.1 Theorem. *Let $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of corings such that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat. The following statements are equivalent*

(a) $- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ *is a quasi-Frobenius functor;*

(b) $\mathfrak{C} \otimes_A B$ *is quasi-finite and injector as a right* $\mathfrak{D}$*-comodule and* $\mathrm{h}_{\mathfrak{D}}(\mathfrak{C} \otimes_A B, \mathfrak{D}) \sim B \otimes_A \mathfrak{C}$ *as* $(\mathfrak{D}, \mathfrak{C})$*-bicomodules.*

5.2 Remark. 1. When applied to the case where $\mathfrak{C} = A$ and $\mathfrak{D} = B$ are the trivial corings, Theorem 5.1 recover Lemma 2.5 where is gives a functorial characterization of quasi-Frobenius ring extensions. In this case we have that $A \otimes_A B \cong B$ is quasi-finite as a right B-comodule if and only ${}_AB$ is finitely generated and projective. Moreover, $\mathrm{h}_B(B, -) \simeq - \otimes_B \mathrm{Hom}_A({}_AB, A)$.

2. When $A = B$, the corectriction functor $(-)_\varphi : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is quasi-Frobenius if and only if $\mathfrak{C}_{\mathfrak{D}}$ is a quasi-finite and injector and $\mathrm{h}_{\mathfrak{D}}(\mathfrak{C}, \mathfrak{D}) \sim \mathfrak{C}$ as $(\mathfrak{D}, \mathfrak{C})$-bicomodules.

3. When $A = B = k$, Theorem 5.1 establish that the corectriction functor $(-)_\varphi : \mathcal{M}^C \to \mathcal{M}^D$ is quasi-Frobenius if and only if C_D is a quasi-finite and injector and $\mathrm{h}_D(C, D) \sim C$ as (D, C)-bicomodules. If k is a field, then the "injector" condition is equivalent to the "injectivity" condition (cf. [1]).

It is then reasonable to give the following definition.

5.3 Definition. Let $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of corings such that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat. It is said to be a *right quasi-Frobenius* morphism of corings if $- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is a quasi-Frobenius functor.

Quasi-Frobenius corings. Let $\mathfrak{C}$ be an A-coring. The forgetful functor $\mathbf{U} : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ has the right adjoint $- \otimes_A \mathfrak{C}$ (see [3, Lemma 3.1]). When ${}_A\mathfrak{C}$ is finitely generated and projective, the functor $\mathrm{Hom}_A(\mathfrak{C}, -) : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{C}}$ is a left adjoint to $\mathbf{U}$. We study corings for which $- \otimes_A \mathfrak{C}$ and $\mathrm{Hom}_A(\mathfrak{C}, -)$ are similar, i.e., those for which $\mathbf{U}$ is a quasi-Frobenius functor.

5.4 Definition. An A-coring $\mathfrak{C}$ is called *quasi-Frobenius coring* provided the forgetful functor $\mathbf{U} : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is a quasi-Frobenius functor.

A characterization of such corings is the following that generalizes [9, Theorem 4.2] for left quasi-Frobenius corings. Before, we recall of [16, Proposition 3.2(a)] that, $T =_A \mathrm{Hom}(\mathfrak{C}, A)$ is a ring with unit $\epsilon_{\mathfrak{C}}$. T is a left A-module via $(a \cdot t)(c) = t(c \cdot a)$, for all $a \in A, c \in \mathfrak{C}, t \in T$. Furthermore the map $i : A \to T$ given by $i(a)(c) = \epsilon_{\mathfrak{C}}(c)a$ is a ring map.

5.5 Theorem. *Let $\mathfrak{C}$ be an A-coring wiht ${}_A\mathfrak{C}$ flat and T be the opposite algebra of ${}^*\mathfrak{C}$. Then the following assertions are equivalent.*

(a) $\mathfrak{C}$ is a quasi-Frobenius coring;

(b) ${}_A\mathfrak{C}$ is finitely generated projective module and $i : A \to T$ is a quasi-Frobenius extension.

(c) ${}_A\mathfrak{C}$ is finitely generated projective module and $\mathfrak{C} \sim T$ as (A, T)-bimodules where $\mathfrak{C}$ is a right T-module via $c \cdot t = c_{(1)} \cdot t(c_{(2)})$, for all $c \in \mathfrak{C}$, and $t \in T$.

Proof. By [4, Example 2.6], A can be viewed as a trivial A-coring and the category of comodules of A, $\mathcal{M}^A$ is isomorphic to $\mathcal{M}_A$. Then $(a) \Leftrightarrow (c)$ follows from letting $\mathfrak{D} = A$ in the Theorem 5.1 together with [17, Lemma 4.4 (1)].
$(a) \Leftrightarrow (b)$. It follows from [3, Lemma 4.3]. Indeed, if ${}_A\mathfrak{C}$ is finitely generated projective module, then the categories $\mathcal{M}^{\mathfrak{C}}$ and $\mathcal{M}_T$ are isomorphic. This implies that the functor $i_* : \mathcal{M}_T \to \mathcal{M}_A$ is quasi-Frobenius if and only if the forgetful functor $\mathbf{U} : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is quasi-Frobenius. □

5.6 Corollary. *Let $\varphi : \mathfrak{C} \to \mathfrak{D}$ be a right quasi-Frobenius homorphism of A-corings. If $\mathfrak{D}$ is a quasi-Frobenius A-coring, then $\mathfrak{C}$ is also a quasi-Frobenius A-coring.*

Proof. Obvious from Lemma 1.2 (b), since the forgetful functor $\mathbf{U}_{\mathfrak{C}}$ is the composition of the quasi-Frobenius functors

$$\mathcal{M}^{\mathfrak{C}} \xrightarrow{-\otimes_A A} \mathcal{M}^{\mathfrak{D}} \xrightarrow{\mathbf{U}_{\mathfrak{D}}} \mathcal{M}_A$$

□

5.7 Remark. A categorical description of *quasi-co-Frobenius corings* was initiated recently by Iovanov and Vercruysse [10]. Clearly, our concept of *quasi-Frobenius coring* is different.

From Sweedler [16], given a ring extension $\rho : R \to S$ one can view $\mathfrak{C} = S \otimes_R S$ as an S-coring. $\mathfrak{C}$ is known as *Sweedler's coring* associated to ρ. The following result generalizes [4, Theorem 2.7] and [9, Proposition 4.3] and can be viewed as the endomorphism ring theorem for quasi-Frobenius extension in terms of corings.

5.8 Proposition. *Let $\mathfrak{C} = S \otimes_R S$ be the Sweedler's coring associated to a ring extension $\rho : R \to S$. If S is a quasi-Frobenius extension of R, then $\mathfrak{C}$ is a quasi-Frobenius S-coring.*

Proof. Assume that ρ_* is a quasi-Frobenius functor. Then ${}_RS$ is finitely generated projective and ${}_SS_R \sim ({}_RS)^*$ as (S,R)-bimodules. By Lemma 1.2, the functor $-\otimes_R S$ preserves finitely generated and projective modules. Hence, $(S\otimes_R S)_S$ is finitely generated and projective as right S-module. Applying $-\otimes_R S$ to ${}_SS_R \sim ({}_RS)^*$ we obtain

$$_R(S\otimes_R S)_S \sim ({}_RS)^* \otimes_R S \cong \mathrm{End}_R(S).$$

Now from [7, Proposition 2.1], $\mathrm{End}_R(S) \cong \overline{T}$, where $\overline{T}$ is the opposite algebra of $((S\otimes_R S)_S)^*$. Therefore $S\otimes_R S \sim \overline{T}$, and $S\otimes_R S$ is a quasi-Frobenius S-coring by Theorem 5.5. □

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