Effects of the Theory of Relativity in the GPS

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Abstract

The Global Positioning System (GPS) is a world wide navigation system run by the United State Department of Defense. The GPS is based on a constellation of NAVSTAR satellites and provides reliable positioning and timing data at any time all over the world. Since the services of the GPS are open to the public due to the availability of receiver hardware, the importance of the GPS has increased enormously.

To obtain this level of accuracy a number of impacts have to be taken into account. In this paper we will discuss the relevance of the theory of relativity first formulated by A. Einstein in [6] as a source of error for the GPS.

1 Introduction

We will show that the velocity of the GPS satellites in an earth inertial reference frame is high enough to lead to some effects not expected by classical mechanics, which are significant for the precision of position determination. If these effects were neglected, an error of 12 km per day for position determination or 39 μs per day for time determination would occur.

The first two sections will introduce the basics of the GPS and the theory of relativity. Afterwards the relativistic effects influencing the GPS are considered. Due to the complexity of theory of relativity we will only consider major effects.

2 Basics of the GPS

The GPS consists of three parts called the space segment, the control segment and the user segment.

The space segment consists of a formation of 31 navigation satellites[1] with atomic clocks on board. These clocks are synchronised within 20 ns to each other and within 100 ns to UTC [10]. The satellites are placed in nearly circular medium earth orbits with an orbital period half of a sidereal day.[2] Navigation signals are continuously broadcasted by the satellites which contain the data needed for navigation purpose. Satellites were chosen as navigation base because medium earth orbits are highly predictable. Residual perturbations through the moon, the sun or the solar system gas giants influencing the orbits, are measured and taken into account for further calculations of the orbital elements.

These measurements are done by the control segment. This global network of monitoring stations tracks the satellites' orbits, fulfils updates to the orbital data and synchronises the satellites' clocks.

The user segment is represented by the GPS receivers which recognize the navigation signals transmitted by the satellites. Based on navigation data and signal timing information, the GPS receiver is able to estimate positioning and timing information.

[1] December 2008. See `http://www.navcen.uscg.gov/GPS/status_and_outage_info.htm`

[2] The time the earth needs for a full revolution w.r.t. distant stars (approx. 23h56m4.09s)

2.1 Satellite communication

The navigation signals are broadcasted via L-Band on two frequencies denoted as L1 and L2,[3] Whereas L1 is modulated with the C/A-Code (1.023 MHz) and the P(Y)-Code (10.23 MHz). L2 is modulated only with the P(Y)-Code. Both codes are based on pseudorandom noise sequences (PRN), whereas the P(Y)-Code can be encrypted. Due to the shorter wave length of the P(Y)-Code and the transmission on two separate carriers, a more precise position calculation is possible. The PRN is used to measure the signal propagation delay of several signals simultaneously.[4]

Each code carries the actual navigation message with a 50 Hz data rate. The messages are repeated twice a minute and one data frame contains 1500 bits. Every frame consists of 6 subframes and contains the following data

- Time of transmission
- Satellite ephemeris data (orbital elements)
- Satellite clock information (clock offset)
- Ephemeris data of the other satellites (split to several frames)
- Signal propagation information
- System health status.

For more information about the technical background of the GPS see [10, 9].

2.2 Reference frames

Reference frames, which map every point of space to an unique tuple of coordinates are essential for position determination. The GPS uses two basic reference frames, the ECEF (Earth-Centered, Earth-Fixed) and the ECI (Earth Centered Inertial). Both reference frames are Cartesian coordinate systems with the origin located in the barycenter of the earth and the z-axis pointing to the terrestrial north pole. Hence the z-axis equals the rotational axis of the earth.

[3] L1: 1575.42 MHz, L2: 1227.60 MHz

[4] Within 1 μs for the C/A-Code and 100 ns for the P(Y)-Code.

The x-axis of the ECEF crosses the equator at the zero meridian. Hence the coordinates of a location on the earth measured in the ECEF are time independent, this frame therefore is convenient for terrestrial navigation. The WGS84 [11] describes a reference ellipsoid in the ECEF which approximates the shape of the earth. The equatorial radius r_e and the polar radius r_p of this ellipsoid are given by the WGS84 with:

$$\begin{aligned} r_e &\approx 6378\,\mathrm{km} \\ r_p &\approx 6357\,\mathrm{km} \end{aligned} \tag{1}$$

The ECEF reference frame with the WGS84 reference ellipsoid, which is used as reference plane by the GPS, is shown in figure 1.

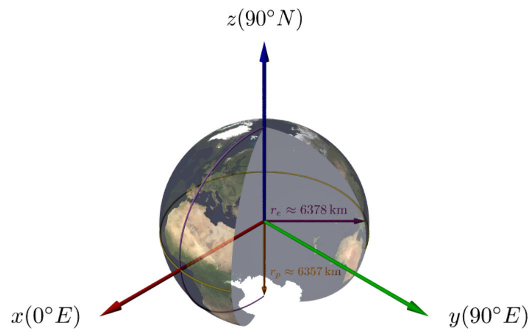


Fig. 1: The WGS84 reference ellipsoid in the ECEF

Every point (x, y, z) in the ECEF can be expressed in a more practical notation by the spherical angles λ (longitude) and φ (latitude) and the height h above the reference ellipsoid by the equations

$$\begin{aligned} \lambda &= \arctan \frac{y}{x} \\ \varphi &= \arctan \frac{z + \tilde{\varepsilon}^2 \cdot r_p \cdot \sin^3 \theta}{\sqrt{x^2 + y^2} - \varepsilon^2 \cdot r_e \cdot \cos^3 \theta} \\ h &= \frac{\sqrt{x^2 + y^2}}{\cos \varphi} - \frac{r_p}{\sqrt{1 - \varepsilon^2 \sin^2 \varphi}} \end{aligned} \tag{2}$$

with

$$\theta = \arctan\left(\frac{z}{\sqrt{x^2+y^2}} \cdot \frac{r_e}{r_p}\right).$$
$$\varepsilon^2 = 1 - r_p^2/r_e^2$$
$$\tilde{\varepsilon}^2 = r_e^2/r_p^2 - 1$$

Whereas the ECEF is fixed w.r.t. the earth, the ECI reference frame is fixed w.r.t. distant stars and hence non rotating. The x-axis points to the vernal point[5] located in the formation Aries (♈). Therefore the ECEF rotates with the angular velocity $\omega_♁$ around the z-axis in the ECI. The WGS84 defines the value of $\omega_♁$ with

$$\omega_♁ = 7.292115 \cdot 10^{-5}\,\mathrm{rad} \cdot \mathrm{s}^{-1} \qquad (3)$$

Due to the fact that many physical processes are simpler to describe in non rotating reference frames, the ECI is used by the GPS for position determination [2, 4]. Afterwards a final rotation to the ECEF has to be performed.

The y-axis of the ECEF and the ECI is perpendicular to the x- and z-axis so that the coordinate system is right handed.

2.3 Orbital mechanics

The orbit of a satellite moving periodically around a massive celestial body can be regarded as an ellipse. The barycenter of the central body is located in one focus of this ellipse. The orbit is unambiguously defined by a tuple of six parameters $(a, e, \omega, i, \Omega, \nu)$. The length of semi-major axis a and the numerical eccentricity e describing the shape of the orbit. The meaning of the argument of periapsis ω, the inclination to the equatorial plane i and the argument of ascending node Ω, which are describing the orientation of the orbit, is shown in figure 2. ν defines the true anomaly of the satellite at the begin of epoch. ν is measured as the angle enclosed between the satellite and the periapsis at the focus located in the center of mass.

With the orbital elements given, it is possible to calculate the position of the satellite at any specific

[5] The point at which the sun crosses the celestial equator from south to north (around every march 20th).

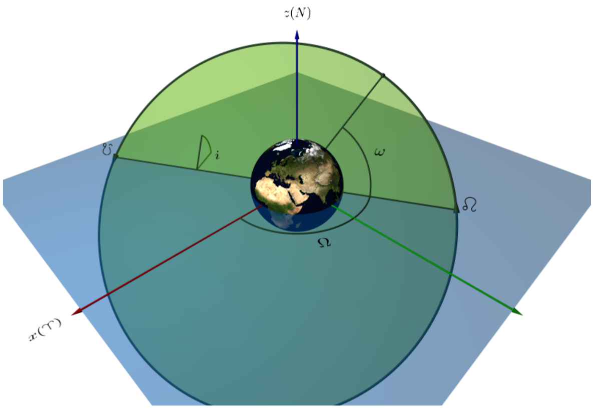


Fig. 2: Orbital elements

time. Moreover, these orbital elements are invariant over time. They are only perturbed slowly by influence of other massive bodies like the moon, the sun or the Jovian solar system bodies.

2.4 Trilateration

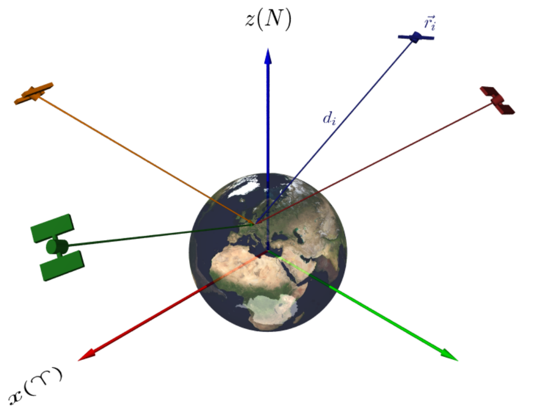


Fig. 3: Constellation of 4 GPS-satellites with ranges d_i to Chemnitz, Germany

Consider a constellation of four GPS satellites as shown in figure 3. The navigation signals broadcasted by the GPS satellites are propagating with speed of light c. Assume the signal of satellite i is transmitted at time t_i. Let $\vec{r}_i$ be the position of satellite i at this time. Assume further all these signals are received by the observer located at $\vec{r}$ simultaneously at time t. Now the ranges d_i between satellite i and the observer can be estimated by equation (4).

$$\|\vec{r} - \vec{r}_i\| = d_i = c \cdot (t - t_i) \tag{4}$$

$\vec{r}_i$ can be calculated by t_i and the orbital elements of satellite i, which are transmitted within every navigation message. To estimate t_i simultaneously, the pseudorandom sequence is needed, because the GPS-time is only signaled at start of each subframe in the navigation message. Due to the PRN sequences the offsets of the signals to each other can be measured and only one subframe start is needed to get all t_i. t is the time of reception and can be measured by the observer's clock.

There are three degrees of freedom in equation (4) (the three components of $\vec{r}$). Therefore the measurements of three different satellites are needed to solve the given system of equations.

At this point we assumed that the observer's clock is synchronised to the GPS clocks. This cannot be guaranteed. But it can be guaranteed, that the observer's clock is stable enough to allow a correct measurement of short term time differences. Simple crystal oscillators comply this requirement. Assume the observers clock reads τ at the time of reception with an unknown offset Δ between the local clock and the GPS clocks, expressed by

$$t = \tau + \Delta.$$

Therefore equation (4) reads now

$$\|\vec{r} - \vec{r}_i\| - c \cdot \Delta = c \cdot (\tau - t_i) \tag{5}$$

with 4 values unknown (three components of $\vec{r}$ and Δ). Analogously this system of equations is solvable by four independent measurements. Therefore each GPS receiver needs at least four satellites to determine the observer's position. The term $c \cdot (\tau - t_i)$ is called "the pseudorange to satellite i". Apart from position information the GPS time can be determined through the term $\tau + \Delta$. For special applications where one parameter is known (e.g. the height of a ship's GPS receiver above main sea level) only three satellites are needed.

A second approach is to fix the time of transmission (t') and to measure the time of reception (t'_i). Due to the rotation of the earth the position of the observer in the ECI changes between the subsequent measurements, so this fact must to be taken into account.

2.5 Determining the position

This section shall give a brief overview of the steps which are performed by the user segment to determine the positioning and timing data as described in [3]. For more detailed informations see [4] as cited in [2].

1. Measure the pseudoranges to the satellites using the PRN at a chosen time and retrieve the transmission times and orbital elements.
2. Apply some corrections to the transmission times (e.g. clock offset, ionospheric propagation effects, eccentricity correction[6]).
3. Calculate the satellites' positions in the ECEF from the orbital elements.
4. Choose an ECI frame and convert the satellites' positions to it.
5. Solve the propagation delay equations (5).
6. Convert back the observers position to the ECEF.
7. Apply equations (2) to get the geographical coordinates.

3 Theory of Relativity

3.1 Special Relativity

The special theory of relativity (STR or SR) was first introduced by Albert Einstein 1905 in [6] to explain some results of experiments related to the propagation of electromagnetic waves (e.g. light).

The theory of special relativity is based on the two fundamental postulates mentioned in proposition 1.

[6] Described in section 4.4.

Proposition 1

i) The speed of light in vacuum c is the same, regardless of the relative motion between the observer and the light source.

ii) The laws of physics are the same for observer's in uniform motion to another.

Whereas item ii complies with practical experience, item i seems to be counterintuitive at first. This fact is demonstrated in figure 4. Let there be a light source which emits light pulses. Now the blue observer will measure the velocity at which these light pulses are propagating. The blue observer rests w.r.t. the light source and will measure c. Now the red observer which moves with velocity v towards the light source will perform the same measurement. According to classical mechanics the red observer would measure $c + v$ but in conformance with item i he would measure c, too. Direction and magnitude of this relative motion is immaterial for this result.

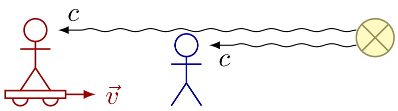


Fig. 4: Constancy of c

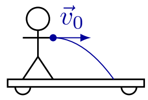
To illustrate item ii for clarity, look at figure 5. A body is pitched with velocity v_0 horizontally by the thrower. Neglecting air drag, the trajectory will equal a parabola. This process is observed from two different points of view. At first, the thrower is the observer and would measure the trajectory shown in figure 5a. Next, the thrower is in uniform motion to a second observer by $\vec{v}$. This observer would do the same measurement and get the results illustrated in 5b.

Both trajectories are solutions of the equation given by[7]

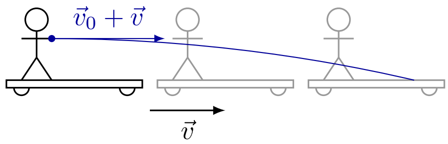
$$\frac{\partial^2 \vec{x}(t)}{\partial t^2} = m \cdot \vec{F}(t)$$

with $\vec{x}(t)$ being the position of the body at time t w.r.t. the observer's reference frame, m being the mass of the body, $F(t)$ being the force applied to

[7] This equation is only legal in classical mechanics, but should merely illustrate the principle of relativity.



(a) Resting Observer



(b) Moving observer

Fig. 5: Principle of relativity

it and the following boundary condition:

$$\left.\frac{\partial \vec{x}(t)}{\partial t}\right|_{t=0} = \begin{cases} \vec{v}_0 & \text{case (a)} \\ \vec{v}_0 + \vec{v} & \text{case (b)} \end{cases}$$

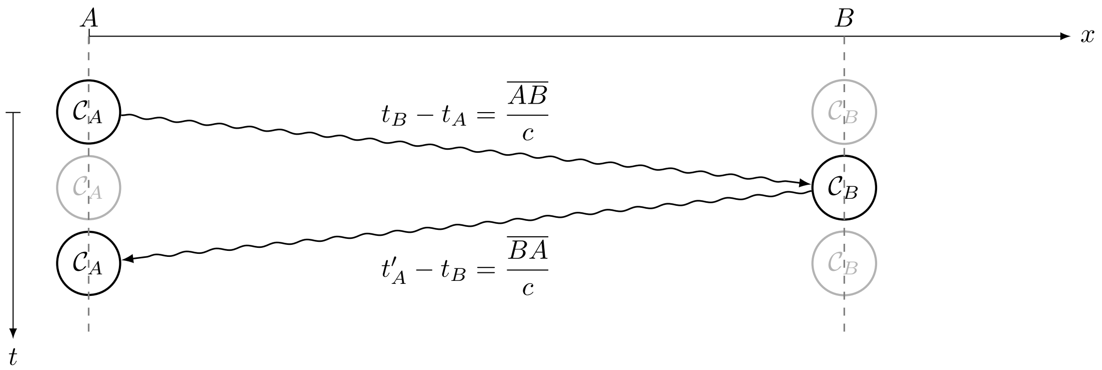
3.1.1 Relativity of simultaneity

Now let us consider the consequences of proposition 1 by the following gedanken experiment. There are two clocks $\mathcal{C}_A$ and $\mathcal{C}_B$ located in A and B. $\mathcal{C}_A$ emits a light pulse, when it reads t_A. This pulse propagates to B with velocity c and is reflected to A instantaneously. At this moment $\mathcal{C}_B$ reads t_B. When $\mathcal{C}_A$ reads t'_A the reflected pulse arrives A. Now we can say "$\mathcal{C}_A$ and $\mathcal{C}_B$ are synchronous" if and only if

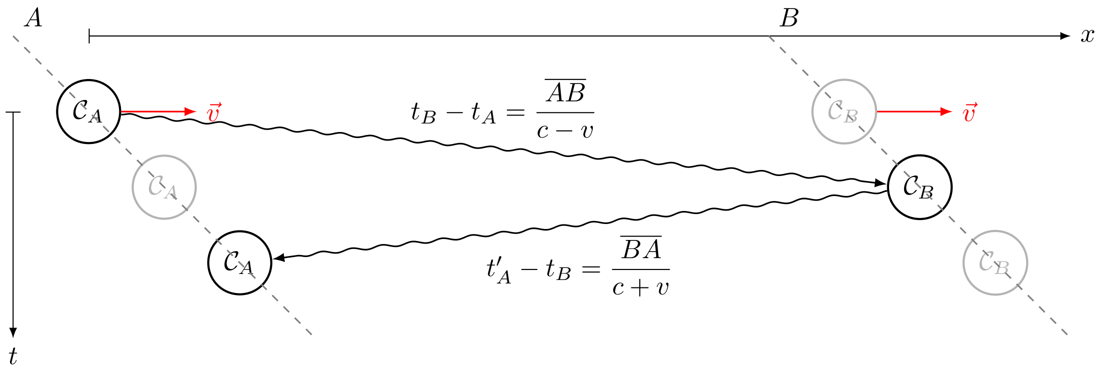
$$t_B - t_A = t'_A - t_B. \tag{6}$$

This experiment is illustrated in figure 6. Suppose $\mathcal{C}_A$ and $\mathcal{C}_B$ are justified in a manner that they are synchronous for the observer which is stationary to the clocks (fig. 6a). Now, the same clock ensemble is observed by a second observer moving with velocity v w.r.t. the clocks (fig. 6b). The light pulses still propagate with velocity c w.r.t. the observer due to item i of proposition 1. As a consequence the propagation delay of the pulse from A to B will be observed larger than the propagation delay from B to A. This leads to the conclusion that the second observer will evaluate the clocks to be asynchronous.

Therefore the term "simultaneity" depends to the reference frame there the measurements are made. Further it is not possible to define a global time.



(a) Stationary clocks w.r.t. the observer



(b) Clocks in motion w.r.t. the observer

Fig. 6: The relativity of simultaneity

3.1.2 Time dilatation and length contraction

Due to lack of global time, as seen in the last section, we have to extend the common three-dimensional space to the four-dimensional space time. For further considerations we define two reference frames

$$\mathcal{I}(x, y, z, t) \quad \text{and} \quad \mathcal{I}'(x', y', z', t').$$

Let (x, y, z, t) and (x', y', z', t') denote the space time coordinates of the same event measured in both reference frames. Assume the axes of $\mathcal{I}$ and $\mathcal{I}'$ match pairwise at $t = t' = 0$ and the origin of $\mathcal{I}'$ moves with velocity v along the x-axis of $\mathcal{I}$. An observer resting in $\mathcal{I}'$ would assign (x', y', z', t') to (x, y, z, t) according to the LORENTZ transformation

$$\begin{aligned} x' &= \gamma \cdot (x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma \cdot \left(t - \frac{v}{c^2} x\right) \end{aligned} \tag{7}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{8}$$

called the LORENTZ factor. As consequence of item ii of proposition 1 the equations (7) hold for (x, y, z, t) and (x', y', z', t') interchanged, too, regarding the change of the sign of v. Applying the LORENTZ transformation to distances Δx, $\Delta x'$ and time differences Δt, $\Delta t'$ measured in $\mathcal{I}$ and $\mathcal{I}'$ we get:

$$\left.\begin{aligned}\Delta x' &= \gamma \cdot (\Delta x - v\Delta t) \\ \Delta t' &= \gamma \cdot \left(\Delta t - \frac{v}{c^2}\Delta x\right)\end{aligned}\right\} \text{observer in } \mathcal{I}'$$

$$\left.\begin{aligned}\Delta x &= \gamma \cdot (\Delta x' + v\Delta t') \\ \Delta t &= \gamma \cdot \left(\Delta t' + \frac{v}{c^2}\Delta x'\right)\end{aligned}\right\} \text{observer in } \mathcal{I}$$

Consider a clock resting in $\mathcal{I}'$ ($\Delta x' = 0$). Then an observer in $\mathcal{I}$ would get

$$\Delta t = \gamma \cdot \Delta t'. \tag{9a}$$

Due to the principle of relativity the observer in $\mathcal{I}$ would realize the time difference of a clock resting in $\mathcal{I}'$ as following

$$\Delta t' = \gamma \cdot \Delta t. \tag{9b}$$

Keeping in mind that $\gamma \geq 1$, this leads to the following corollary.

Corollary 2 (Time dilatation)
Moving clocks seems to tick slower than identical stationary clocks.

Now consider a measuring rod of length $\Delta x'$ resting along the x'-axis of $\mathcal{I}'$. The observer in $\mathcal{I}$ doing an instantaneous measurement ($\Delta t = 0$) of the rod would get Δx as following

$$\Delta x = \frac{1}{\gamma} \cdot \Delta x'. \tag{10a}$$

Analogous the observer in $\mathcal{I}'$ would measure $\Delta x'$ to

$$\Delta x' = \frac{1}{\gamma} \cdot \Delta x. \tag{10b}$$

Corollary 3 (Length contraction)
Moving bodies seems to be shorter then identical stationary bodies in direction of relative movement.

3.2 General Relativity

In contrast to the special theory of relativity the general theory of relativity (GR or GTR) takes the effects of mass and gravity into account [7]. The laws of GR are expressed by methods of differential geometry. These methods allow to migrate from classical coordinate systems to curvilinear coordinate systems. The core statement of GR is, that all forms of energy[8] curve the 4-dimensional space time. Light propagates with velocity c along geodesic lines in this curved manifold. Thus gravity could be explained as geometric feature.

Figure 7 illustrates a curved 2-dimensional manifold embedded into a 3-dimensional space. This curved manifold shows two spatial dimensions of an equatorial plane of the space time curved by a massive non-rotating spherical body with homogeneous density and no charge.[9] The green and red lines are the coordinate lines. The third spatial dimension and the dimension in time are not shown.

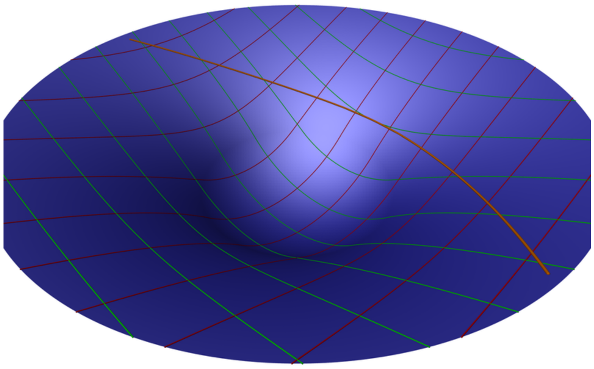


Fig. 7: Curved 2-dimensional manifold in 3-dimensional space with coordinate lines and a geodesic line.

Due to the complex structure of general relativity, we will only focus on the effects relevant for the GPS, namely the gravitational time dilatation and gravitational blue- or redshift, respectively.

3.2.1 Gravitational time dilatation

One effect of GR is the gravitational time dilatation. This effect is simply demonstrated in figure 8. A light ray emitted towards a massive body will gain energy due to the stronger gravitational potential and be shifted to higher frequencies. In

[8] As shown by Einstein in [5] mass can be regarded as a form of energy.
[9] The so called Schwarzschild solution.

contrast a light ray propagating to a weaker gravitational potential will lose energy and therefore be shifted to lower frequencies.

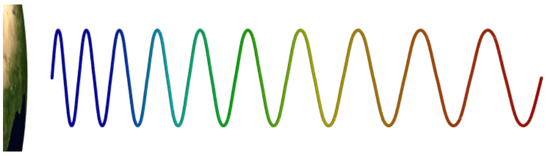


Fig. 8: Gravitational time dilatation

As summarized by corollary 4 it should be paid attention that this effect is not symmetric like the time dilatation due to relative movement.

Corollary 4
Gravitational stronger bounded clocks tick slower. Gravitational weaker bounded clocks tick faster.

The amount of time dilatation $\frac{\Delta t}{t}$ for an observer resting at gravitational potential Φ_0 to a clock at gravitational potential Φ is given by equation (11).

$$\Delta t = \frac{\Phi - \Phi_0}{c^2} \cdot t \tag{11}$$

4 Relativity in the GPS

Now let us apply the laws of the theory of relativity to the global positioning system. For GPS it is suitable to neglect terms of order $o(c^{-2})$ [2]. Further we assume that the GPS receiver moves slowly across the surface of the earth and is located in low altitudes. For receivers in high altitudes and/or with high ground speed, special considerations have to be done.

Figure 9 illustrates the state vectors of a satellite vehicle. The satellite is accelerated towards the center of earth due to effects of gravity resulting in acceleration. The vector of acceleration $\vec{a}$ is perpendicular to the velocity vector $\vec{v}$ at any time. So only the direction of $\vec{v}$ changes over time. The magnitude $v_S = \|\vec{v}\|$ is constant and can be estimated by equation (12).

$$v_S = \sqrt{\frac{GM_{\earth}}{r_S}} \tag{12}$$

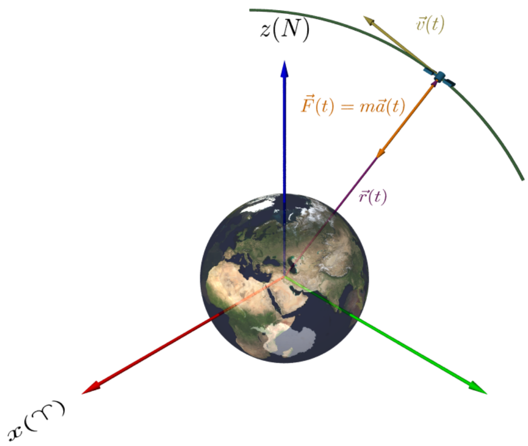


Fig. 9: State vectors of a GPS satellite

The magnitude of the range vector in circular orbits is constant too and for GPS

$$\|\vec{r}\| = r_S = a = 26562\,\mathrm{km}.$$

The factor $GM_{\earth}$ is defined to

$$GM_{\earth} = 3.986005 \cdot 10^{14}\,\mathrm{m}^3\mathrm{s}^{-2} \tag{13}$$

by WGS84 [11]. So v_S can be evaluated to

$$v_S = 3874\,\mathrm{m} \cdot \mathrm{s}^{-1}.$$

4.1 Special Relativity

As seen in the last section the satellite vehicles move with a speed of $3874\,\mathrm{m} \cdot \mathrm{s}^{-1}$ w.r.t. the ECI. Although this motion is not uniform, it is in compliance with item ii of proposition 1, because the acceleration[10] applied to the satellite vehicles only affects the direction of movement. Due to this motion an observer resting in the ECI will see the satellites' clocks beat with a slower rate by the factor

$$\begin{aligned} 1 - \frac{1}{\gamma'} &= 1 - \sqrt{1 - \frac{v_S^2}{c^2}} \\ &\approx -\frac{1}{2} \cdot \frac{v_S^2}{c^2} \\ &\approx -8.349 \cdot 10^{-11}. \end{aligned}$$

[10] The effects of gravity which causes this acceleration has still to be discussed separately.

We have to keep in mind that an observer resting on the surface of the earth moves w.r.t. the ECI too, unless he is located at the North or South Pole. The magnitude of this relative speed is maximal at the equator, namely

$$\omega_{\oplus} \cdot r_e \approx 465\,\mathrm{m} \cdot \mathrm{s}^{-1}.$$

Consequently the time dilatation between clocks resting in the ECEF and clocks resting in the ECI has to be taken into account. At the equator the amount of time dilatation will read

$$\begin{aligned} 1 - \frac{1}{\gamma''} &= 1 - \sqrt{1 - \frac{(\omega_{\oplus} r_e)^2}{c^2}} \\ &\approx -1.203 \cdot 10^{-12}. \end{aligned}$$

Therefore at the equator, the amount of time dilatation between satellite clocks and terrestrial clocks is given by

$$\begin{aligned} 1 - \frac{1}{\gamma} &= \frac{1}{\gamma'} - \frac{1}{\gamma''} \\ &\approx -8.229 \cdot 10^{-11} \\ &= -7109\,\mathrm{ns} \cdot \mathrm{d}^{-1}. \end{aligned} \tag{14}$$

We will see in the next section that clocks on the surface of the earth (more precise "on the geoid") beat at the same rate despite their relative motion w.r.t. the ECI. Because the terrestrial time scale is realised at the equator [8, 2] the factor γ is legal for the whole geoid.

4.2 General Relativity

As mentioned in the last section we will discuss the influence of the gravitational field of the earth now. The effective gravitational potential Φ at a specific point of the surface of the earth is approximately given by equation (15).

$$\begin{aligned} \Phi_0 = & \underbrace{-\frac{GM_{\oplus}}{r(\theta)} \left(1 - \frac{J_2 \cdot r_e^2}{r^2(\theta)} \cdot \frac{1}{2}(3\sin^2\theta - 1)\right)}_{\Phi_{\text{static}}} \\ & \underbrace{-\frac{1}{2}\left(\omega_{\oplus} r(\theta) \cos\theta\right)^2}_{\Phi_{\text{centripetal}}} \end{aligned} \tag{15}$$

The angle θ is measured from the equator to the north or south and $r(\theta)$ is the radius of the earth at this latitude. J_2 denotes the quadrupole moment coefficient of the earth which is defined by WGS84 [11] as

$$J_2 = 1.08263 \cdot 10^{-3}. \tag{16}$$

Φ_{static} is the part of the potential caused by the mass of the earth, whereas $\Phi_{\text{centripetal}}$ arises from centripetal forces due to the earths rotation. Now we see the last term will compensate the effects of time dilatation due to the motion of resting clocks on the surface of the earth in the ECI.

At the equator ($\theta = 0$) equation (15) reads

$$\Phi_0 = -\frac{GM_{\oplus}}{r_e}\left(1 + \frac{J_2}{2}\right) - \frac{1}{2}\left(\omega_{\oplus} r_e\right)^2. \tag{17}$$

For the satellites the quadrupole moment could be neglected. Additional the satellite vehicles are in free fall and so there is no centripetal gravitational field. So the gravitational potential Φ_S for the satellites is given by

$$\Phi_S = -\frac{GM_{\oplus}}{r_S}. \tag{18}$$

Combining equations (17) and (18) with (11) we will get

$$\begin{aligned} \frac{\Delta t}{t} &= \frac{\Phi_S - \Phi_0}{c^2} \\ &\approx 5.288 \cdot 10^{-10} \\ &\approx 45685\,\mathrm{ns} \cdot \mathrm{d}^{-1}. \end{aligned} \tag{19}$$

4.3 Sagnac delay

Whereas the speed of light is constant in local inertial frames (proposition 1 item i) the constancy of c is not longer given for rotating reference frames. This phenomenon is called SAGNAC effect. The SAGNAC effect makes it possible to determine the absolute rotation to an inertial frame and is used by laser navigation instruments for example.

For measurements in the ECI (e.g. the solution of the propagation delay equations (5)), which is non rotating by convention, the SAGNAC effect becomes irrelevant. However the SAGNAC effect must be accounted for when synchronising GPS clocks from a fixed point on earth or comparing clocks resting on the earth using GPS signals.

In 1984 four GPS satellites were used for a world wide experiment ([1] as cited in [2]). The pairwise signal offsets of these satellites in common view were measured by three ground stations all over the world simultaneously. In a non rotating reference frame these differences would cancel out, but due to the rotation of the earth SAGNAC delays up to 350 ns were measured.

4.4 Eccentricity correction

Until now we have assumed that the satellite vehicles move in circular orbits ($e = 0$). Due to slight perturbations these orbits are merely near circular in reality, with eccentricity $e < 0.02$.[11] On eccentrical orbits the satellite's velocity v_S and the gravitational potential Φ_S varies periodically due to changes of r_S. The error applying to the effects discussed in sections 4.1 and 4.2 is given by

$$\Delta_e(t) = 2 \cdot \frac{\vec{v}(t) \cdot \vec{r}(t)}{c^2} \tag{20}$$

as shown in [2], with satellites velocity $\vec{v}$ and position $\vec{r}$ measured in the ECI at time of transmission t. These vectors are computable by the broadcasted orbital elements. $\Delta_e(t)$ could be approximated by

$$\Delta_e(t) \approx 2 \cdot \frac{\sqrt{GM_{\oplus} a}}{c^2} e \cdot \sin E(t) \tag{21}$$

with eccentric anomaly E at GPS time t and semi-major axis a of the orbit. For $e = 0.02$ this would lead to a maximum error of 46 ns.

The correction of this error must be done by the user segment, because the computing power of the first GPS satellites was strongly limited. Both, the user segment and the control segment use the value defined by WGS84 for $GM_{\oplus}$ (see equation (13)) to get consistent results [11].

4.5 Conclusion

Combining the results of sections 4.1 and 4.2 for an observer on the earth's surface the clocks on board the GPS satellites would gain approx. 39 μs per day. When using three satellites for position determination (equation (4)) this corresponds to an error up to 12 km per day. For measurements based on four satellites, only the term Δ of equation (5) is affected by clock drifts. To correct this error, the time base of the GPS satellites is modified.

GPS clocks are based on a 10.23 MHz reference signal [9]. To account for relativistic effects, the reference oscillators are adjusted by the so called "factory offset" given by

$$4.465 \cdot 10^{-10} \approx 39\,\mu\text{s} \cdot \text{d}^{-1}.$$

The oscillators are therefore tuned to

$$10.22999999543\,\text{MHz}$$

thus the GPS clocks beat the correct rate for observers on the earth's surface. Hence precise time determination is possible with GPS receivers. Other effects like the eccentricity correction or non relativistic corrections are applied by the receiver software.

[11] http://www.navcen.uscg.gov/gps/current/current.alm

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