

On the Exactness of Convex Relaxation for Optimal Power Flow in Tree Networks

Lingwen Gan, Na Li, Ufuk Topcu, and Steven Low

Abstract—The optimal power flow problem is nonconvex, and a convex relaxation has been proposed to solve it. We prove that the relaxation is exact, if there are no upper bounds on the voltage, and any one of some conditions holds. One of these conditions requires that there is no reverse real power flow, and that the resistance to reactance ratio is non-decreasing as transmission lines spread out from the substation to the branch buses. This condition is likely to hold if there are no distributed generators. Besides, avoiding reverse real power flow can be used as rule of thumb for placing distributed generators.

I. INTRODUCTION

The optimal power flow (OPF) problem seeks to minimize a certain objective function, such as power loss or generation cost, subject to various constraints including power flow constraints, voltage regulation constraints, and load constraints. There has been extensive research on OPF since Carpentier's first formulation in 1962 [1], and surveys can be found in, e.g., [2]–[6]. The OPF problem is in general nonconvex, and a lot of algorithms have been proposed to approximate OPF or relax OPF to a convex optimization problem. Relaxation methods have the potential of providing exact solutions, and are the focus of this work. A relaxation is exact if every of its solutions also solves the original problem.

In transmission networks, which are usually mesh networks, a semi-definite relaxation (SDR) has been proposed to solve OPF [7]–[9], but whether or when the SDR is exact can only be checked after solving the SDR. In distribution networks, which are usually tree networks, different convex relaxations [10]–[12] for OPF have been proposed. Reference [10] proposes a second-order-cone relaxation (SOCR) for OPF, and proves that the SOCR is exact if there are no upper bounds on the loads. This condition can be checked before solving the SOCR. However, upper bounds on the loads are important for various applications including demand response [13] and Volt/VAR control [14]. Motivated by this shortcoming, this paper seeks sufficient conditions for the exactness of SOCR, in the presence of load upper bounds.

In this paper, we prove that the SOCR is exact, if there are no upper bounds on the voltage, and any one of the following conditions holds.

This work was supported by Bell Labs of Alcatel-Lucent, NSF NetSE grant CNS 0911041, ARPA-E grant DE-AR0000226, Southern California Edison, National Science Council of Taiwan, R.O.C, grant NSC 101-3113-P-008-001, Resnick Institute, Okawa Foundation, Boeing Corporation, Cisco, and AFOSR award number FA9550-12-1-0302.

Lingwen Gan, Na Li, and Steven Low are with the Division of Engineering and Applied Sciences, California Institute of Technology, e-mails: lgan, nali, slow@caltech.edu.

Ufuk Topcu is with the Electrical and Systems Engineering Department, University of Pennsylvania, e-mail: utopcu@seas.upenn.edu.

- (1) There is no reverse real or reactive power flow, i.e., on every transmission line, both real and reactive power flow from the bus that is closer to the substation to the bus that is farther from the substation. This condition holds when there are no shunt capacitors and no distributed generators in the network.
- (2) There is no reverse real power flow, and the resistance to reactance ratio is non-decreasing as transmission lines spread out from the substation to the branch buses. As transmission lines spread out, they are in general getting thinner, which implies that the resistance to reactance ratio is non-decreasing. Hence, this condition mostly requires no reverse real power flow.
- (3) There is no reverse reactive power flow, and the resistance to reactance ratio is non-increasing as the transmission lines spread out from the substation.
- (4) The resistance to reactance ratio is identical throughout the network. This condition holds if transmission lines are uniform throughout the network.

We ignore voltage upper bounds in this paper. It is reasonable in at least two cases: 1) there are no distributed generators; 2) the load is heavy. (The voltage tends to be low in both cases.) Besides, accepting that the resistance to reactance ratio is non-decreasing for practical distribution networks, condition (2) holds if there are no distributed generators (there are no sources of reverse real power flow). Moreover, avoiding reverse real power flow can be used as rule of thumb for placing distributed generators.

In the rest of the paper, we introduce the OPF problem and its convex relaxation SOCR in Section II, provide sufficient conditions for the exactness of SOCR in Section III, and use a practical network to illustrate how to place distributed generators to avoid reverse real power flow in Section IV.

II. THE OPTIMAL POWER FLOW PROBLEM AND ITS CONVEX RELAXATION

A. Notation

Consider a tree distribution network that consists of $N + 1$ buses. Index the substation bus by 0 and the branch buses by $1, \dots, N$. Let $\mathcal{N} := \{0, \dots, N\}$ denote the set of buses. Each transmission line in the network connects an ordered pair (i, j) of buses, where bus i is on the path from bus 0 to bus j . Let \mathcal{L} denote the set of transmission lines. For each bus $i \in \mathcal{N}$, let p_i and q_i denote its real and reactive load respectively (distributed generation can be considered as negative loads), let V_i denote its voltage, and define $v_i := |V_i|^2$. Bus 0, the substation, has fixed voltage magnitude,

TABLE I
LIST OF SYMBOLS

\mathcal{N}	Set of buses.
\mathcal{L}	Set of transmission lines.
p_i	Real load at bus i .
q_i	Reactive load at bus i .
V_i	Voltage at bus i .
v_i	$v_i = V_i ^2$.
r_{ij}	Resistance of (i, j) .
x_{ij}	Reactance of (i, j) .
P_{ij}	Real power flow from i to j .
Q_{ij}	Reactive power flow from i to j .
I_{ij}	Current flowing from i to j .
ℓ_{ij}	$\ell_{ij} = I_{ij} ^2$.

i.e., v_0 is a constant. For each line $(i, j) \in \mathcal{L}$, let r_{ij} and x_{ij} denote its resistance and reactance respectively, let P_{ij} and Q_{ij} denote the real and reactive power flow from bus i to bus j respectively, let I_{ij} denote the current flowing from bus i to bus j , and define $\ell_{ij} := |I_{ij}|^2$. These notations are summarized in Table I. We use the letter without subscript to denote a column vector of the corresponding quantity, e.g.,

$$\begin{aligned} r &:= (r_{ij}, (i, j) \in \mathcal{L})^T, & p &:= (p_1, \dots, p_N)^T, \\ x &:= (x_{ij}, (i, j) \in \mathcal{L})^T, & q &:= (q_1, \dots, q_N)^T, \\ P &:= (P_{ij}, (i, j) \in \mathcal{L})^T, & v &:= (v_1, \dots, v_N)^T, \\ Q &:= (Q_{ij}, (i, j) \in \mathcal{L})^T, & \ell &:= (\ell_{ij}, (i, j) \in \mathcal{L})^T. \end{aligned}$$

For $k \in \{1, 2, \dots\}$, $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, and $b = (b_1, \dots, b_k) \in \mathbb{R}^k$, define the relations \succ , \succeq , \prec , and \preceq as

$$\begin{aligned} a \succ b &\stackrel{\text{def}}{\iff} a_i > b_i \text{ for } i = 1, \dots, k, \\ a \succeq b &\stackrel{\text{def}}{\iff} a_i \geq b_i \text{ for } i = 1, \dots, k, \\ a \prec b &\stackrel{\text{def}}{\iff} a_i < b_i \text{ for } i = 1, \dots, k, \\ a \preceq b &\stackrel{\text{def}}{\iff} a_i \leq b_i \text{ for } i = 1, \dots, k. \end{aligned}$$

B. The Optimal Power Flow Problem

In tree networks, power flow is characterized by (see [15])

$$P_{ij} = r_{ij}\ell_{ij} + p_j + \sum_{k:(j,k) \in \mathcal{L}} P_{jk}, \quad (i, j) \in \mathcal{L}; \quad (1)$$

$$Q_{ij} = x_{ij}\ell_{ij} + q_j + \sum_{k:(j,k) \in \mathcal{L}} Q_{jk}, \quad (i, j) \in \mathcal{L}; \quad (2)$$

$$v_j = v_i - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (r_{ij}^2 + x_{ij}^2)\ell_{ij}, \quad (i, j) \in \mathcal{L}; \quad (3)$$

$$\ell_{ij} = \frac{P_{ij}^2 + Q_{ij}^2}{v_i}, \quad (i, j) \in \mathcal{L} \quad (4)$$

The optimal power flow (OPF) problem is to minimize the power loss, subject to power flow constraints (1)–(4), voltage regulation constraint (6), and load constraint (7).

OPF:

$$\min_{p, q, P, Q, \ell, v} \sum_{(i, j) \in \mathcal{L}} r_{ij}\ell_{ij} \quad (5)$$

$$\text{s.t.} \quad (1)–(4);$$

$$\underline{v} \preceq v; \quad (6)$$

$$\underline{p} \preceq p \preceq \bar{p}, \quad \underline{q} \preceq q \preceq \bar{q} \quad (7)$$

where v_0 , r , x , \underline{v} , \underline{p} , \bar{p} , \underline{q} , and \bar{q} are given constants.

Remark 1: The objective function in (5) is the power loss, and can be generalized to any function of the type

$$f \left(\sum_{(i, j) \in \mathcal{L}} r_{ij}\ell_{ij} \right) + g(p, q)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing and $g : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ is arbitrary.

Remark 2: It is claimed in [16] that given substation voltage v_0 and load p , q , there exists a unique practical P , Q , ℓ , v satisfying power flow equations (1)–(4). In practice, the control is to find the optimal load p and q , and let other quantities P , Q , ℓ and v be determined by physical laws (1)–(4). Loads p_i and q_i at bus i can be either positive or negative, depending on whether it represents a load, a distributed generator, or a shunt capacitor.

Remark 3: The voltage regulation constraint is $\underline{v} \preceq v \preceq \bar{v}$ rather than (6), i.e., there is an upper bound \bar{v} on v as well as a lower bound \underline{v} . We ignore \bar{v} in the current paper. This is reasonable in at least two cases: 1) there are no distributed generators; 2) the load is heavy. (The voltage tends to be low in both cases.)

Remark 4: In applications including demand response and Volt/VAR control, load constraint is usually considered to be box constraint as in (7). Results in this paper extend to arbitrary load constraint (it does not even need to be convex).

C. Second-Order-Cone Relaxation

Problem OPF is nonconvex due to the non-affine equality constraint in (4). An approach (see [10]) to convexify OPF is relaxing (4) to the inequality constraint

$$\ell_{ij} \geq \frac{P_{ij}^2 + Q_{ij}^2}{v_i}, \quad (i, j) \in \mathcal{L}. \quad (8)$$

Constraint (8) can be transformed to a second-order-cone constraint. Therefore, we call the following convex problem second-order-cone relaxation (SOCR).

SOCR:

$$\begin{aligned} \min_{p, q, P, Q, \ell, v} & \sum_{(i, j) \in \mathcal{L}} r_{ij}\ell_{ij} \\ \text{s.t.} & (1)–(3), (6)–(8). \end{aligned}$$

Problem SOCR is convex and can be solved efficiently. If its solution $w^{\text{opt}} := (p^{\text{opt}}, q^{\text{opt}}, P^{\text{opt}}, Q^{\text{opt}}, \ell^{\text{opt}}, v^{\text{opt}})$ satisfies (4), then w^{opt} also solves Problem OPF.

Definition 1: The relaxation SOCR is exact if every solution of SOCR also solves OPF.

When SOCR is exact, we can solve the nonconvex problem OPF by solving the convex problem SOCR. Reference [10] proves that SOCR is exact if there are no upper bounds \bar{p} and \bar{q} in (7) (but reference [10] considers upper bounds on the voltage while the current paper does not). Upper bounds \bar{p} and \bar{q} are important for various applications including demand response and Volt/VAR control. Therefore, the current paper works on sufficient conditions for the exactness of SOCR, in the presence of \bar{p} and \bar{q} .

III. EXACTNESS OF SOCR

Let $\text{OPF}(p, q)$ denote the OPF problem with $p = \bar{p} = p$ and $q = \bar{q} = q$, and $\text{SOCR}(p, q)$ denote the SOCR problem with $\underline{p} = \bar{p} = p$ and $\underline{q} = \bar{q} = q$. If $\text{SOCR}(p, q)$ is an exact relaxation of $\text{OPF}(p, q)$ for every feasible (p, q) , then SOCR is an exact relaxation of OPF. Hence, we start by exploring the exactness of $\text{SOCR}(p, q)$.

Let $P^{\text{lin}}(p)$ and $Q^{\text{lin}}(q)$ satisfy

$$P_{ij} = p_j + \sum_{k:(j,k) \in \mathcal{L}} P_{jk}, \quad (i, j) \in \mathcal{L}; \quad (9)$$

$$Q_{ij} = q_j + \sum_{k:(j,k) \in \mathcal{L}} Q_{jk}, \quad (i, j) \in \mathcal{L}. \quad (10)$$

Note that equations (9)–(10) ignore the ℓ terms (which bring nonlinearity) in equations (1)–(2), and $P^{\text{lin}}(p)$, $Q^{\text{lin}}(q)$ are linear functions of p , q respectively.

Theorem 1: The relaxation $\text{SOCR}(p, q)$ is exact, if any of the following conditions holds:

- (i) $P^{\text{lin}}(p) \succeq 0$, $Q^{\text{lin}}(q) \succeq 0$.
- (ii) $P^{\text{lin}}(p) \succeq 0$, $r_{ij}/x_{ij} \leq r_{jk}/x_{jk}$ for $(i, j), (j, k) \in \mathcal{L}$.
- (iii) $Q^{\text{lin}}(q) \succeq 0$, $r_{ij}/x_{ij} \geq r_{jk}/x_{jk}$ for $(i, j), (j, k) \in \mathcal{L}$.
- (iv) $r_{ij}/x_{ij} = r_{jk}/x_{jk}$ for $(i, j), (j, k) \in \mathcal{L}$.

Since the values $P^{\text{lin}}(p)$ and $Q^{\text{lin}}(q)$ can be computed before solving the $\text{SOCR}(p, q)$, all four conditions in Theorem 1 can be checked before solving the $\text{SOCR}(p, q)$.

Condition (i) requires $P_{ij}^{\text{lin}} \geq 0$ and $Q_{ij}^{\text{lin}} \geq 0$ for $(i, j) \in \mathcal{L}$, i.e., both real and reactive power flow from bus i to bus j for every transmission line (i, j) . Since bus i is closer to the substation, this condition can be interpreted as no “reverse” real or reactive power flow. If there are no distributed generators (such as photovoltaic on rooftops) injecting real power, or their real power injection is smaller than the downstream loads, then there is no reverse real power flow, i.e., $P^{\text{lin}} \succeq 0$. If there are no shunt capacitors injecting reactive power, or their reactive power injection is smaller than the downstream reactive loads, then there is no reverse reactive power flow, i.e., $Q^{\text{lin}} \succeq 0$. Though distributed generation is insignificant in current distribution networks, shunt capacitors are widely used for voltage regulation, and $Q^{\text{lin}}(q) \succeq 0$ is likely to be violated.

Condition (ii) removes the requirement $Q^{\text{lin}}(q) \succeq 0$, but imposes a new requirement that the resistance to reactance ratio r/x is non-decreasing as transmission lines spread out from the substation to the branch buses. The new requirement on r/x is satisfied in most practical distribution networks since the transmission lines usually get thinner as they spread out from the substation. Therefore, condition (ii) holds if there are no distributed generators in the network, for most practical distribution networks.

When there is significant distributed generation causing reverse real power flow, the requirement $P^{\text{lin}}(p) \succeq 0$ in condition (ii) does not hold. Condition (iv) further removes the requirement $P^{\text{lin}}(p) \succeq 0$, but imposes a new requirement that the ratio r/x should be identical throughout the network. It follows that SOCR is exact if transmission lines are uniform throughout the network.

Corollary 1: The relaxation SOCR is exact, if any of the following conditions holds:

- (i') $P^{\text{lin}}(\underline{p}) \succeq 0$, $Q^{\text{lin}}(\underline{q}) \succeq 0$.
- (ii') $P^{\text{lin}}(\underline{p}) \succeq 0$, $r_{ij}/x_{ij} \leq r_{jk}/x_{jk}$ for $(i, j), (j, k) \in \mathcal{L}$.
- (iii') $Q^{\text{lin}}(\underline{q}) \succeq 0$, $r_{ij}/x_{ij} \geq r_{jk}/x_{jk}$ for $(i, j), (j, k) \in \mathcal{L}$.
- (iv') $r_{ij}/x_{ij} = r_{jk}/x_{jk}$ for $(i, j), (j, k) \in \mathcal{L}$.

Proof: When conditions (i')–(iv') hold, conditions (i)–(iv) hold respectively, for all (p, q) satisfying (7). Let $w^{\text{opt}} := (p^{\text{opt}}, q^{\text{opt}}, P^{\text{opt}}, Q^{\text{opt}}, \ell^{\text{opt}}, v^{\text{opt}})$ denote an arbitrary solution to SOCR. It follows from $\text{SOCR}(p^{\text{opt}}, q^{\text{opt}})$ being exact that w^{opt} satisfies (4). Therefore SOCR is exact. ■

Condition (ii') can be used for, e.g., Volt/VAR control when there are no distributed generators. In this case, the voltage v tends to be low, and we can ignore its upper bound \bar{v} to obtain Problem OPF. Besides, $P^{\text{lin}}(\underline{p}) \succeq 0$ holds since there are no distributed generators. We assume that the r/x ratio is non-decreasing as transmission lines spread out from the substation (this is likely to be true in practical distribution networks), then condition (ii') holds. It follows that SOCR is an exact relaxation of OPF.

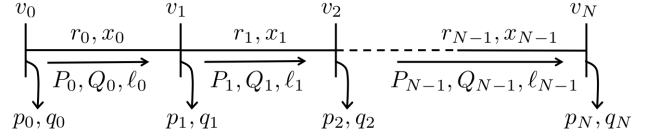


Fig. 1. A one-line distribution network with simplified notations.

For ease of presentation, we prove Theorem 1 in a one-line network, and the proof can be extended to general tree networks. In a one-line network, we can abbreviate r_{ij} , x_{ij} , P_{ij} , Q_{ij} , and ℓ_{ij} by r_i , x_i , P_i , Q_i and ℓ_i respectively, as shown in Figure 1. We re-state Problem $\text{OPF}(p, q)$ and $\text{SOCR}(p, q)$ with the simplified notations as follows.

OPF-line:

$$\begin{aligned} \min_{P, Q, \ell, v} \quad & \sum_{i=0}^{N-1} r_i \ell_i \\ \text{s.t.} \quad & P_i = r_i \ell_i + p_{i+1} + P_{i+1}, \\ & i = 0, \dots, N-1; \end{aligned} \quad (11)$$

$$\begin{aligned} Q_i = x_i \ell_i + q_{i+1} + Q_{i+1}, \\ i = 0, \dots, N-1; \end{aligned} \quad (12)$$

$$\begin{aligned} v_i = v_{i+1} + 2(r_i P_i + x_i Q_i) - (r_i^2 + x_i^2) \ell_i, \\ i = 0, \dots, N-1; \end{aligned} \quad (13)$$

$$\ell_i = \frac{P_i^2 + Q_i^2}{v_i}, \quad i = 0, \dots, N-1; \quad (14)$$

$$v_i \geq \underline{v}_i, \quad i = 1, \dots, N \quad (15)$$

where $P_N := 0$, $Q_N := 0$.

SOCR-line:

$$\begin{aligned} \min_{P, Q, \ell, v} \quad & \sum_{i=0}^{N-1} r_i \ell_i \\ \text{s.t.} \quad & (11)–(13), (15); \\ & \ell_i \geq \frac{P_i^2 + Q_i^2}{v_i}, i = 0, \dots, N-1. \end{aligned} \quad (16)$$

Note that p and q are given constants in OPF-line and SOCR-line. Quantities $P^{\text{lin}}(p)$ and $Q^{\text{lin}}(q)$ can be calculated by

$$P_i^{\text{lin}}(p) = \sum_{j=i+1}^N p_j, \quad i = 0, \dots, N-1;$$

$$Q_i^{\text{lin}}(q) = \sum_{j=i+1}^N q_j, \quad i = 0, \dots, N-1.$$

Lemma 1: The relaxation SOCR-line is exact, if any of the following conditions holds:

- $P^{\text{lin}}(p) \succeq 0, Q^{\text{lin}}(q) \succeq 0.$
- $P^{\text{lin}}(p) \succeq 0, r_i/x_i \leq r_{i+1}/x_{i+1}$ for $i = 0, \dots, N-2.$
- $Q^{\text{lin}}(q) \succeq 0, r_i/x_i \geq r_{i+1}/x_{i+1}$ for $i = 0, \dots, N-2.$
- $r_i/x_i = r_{i+1}/x_{i+1}$ for $i = 0, \dots, N-2.$

Proof: According to (11)–(13), P, Q and v are linear functions of ℓ . Therefore, we can rewrite SOCR-line as

$$\begin{aligned} \min_{\ell} \quad & \sum_{i=0}^{N-1} r_i \ell_i \\ \text{s.t.} \quad & \ell_i \geq \frac{P_i(\ell)^2 + Q_i(\ell)^2}{v_i(\ell)}, \quad i = 0, \dots, N-1; \quad (17) \\ & v_i(\ell) \geq \underline{v}_i, \quad i = 1, \dots, N. \quad (18) \end{aligned}$$

Associate the Lagrangian multipliers $\lambda_i \geq 0$ with (17) and $u_i \geq 0$ with (18). Then the Lagrangian of SOCR-line is

$$\begin{aligned} L(\ell, \lambda, u) = & \sum_{i=0}^{N-1} r_i \ell_i + \sum_{i=0}^{N-1} \lambda_i \left(\frac{P_i^2 + Q_i^2}{v_i} - \ell_i \right) \\ & + \sum_{i=1}^N u_i (\underline{v}_i - v_i). \quad (19) \end{aligned}$$

If SOCR-line is infeasible, then OPF-line is also infeasible. If SOCR-line is feasible, then its optimal solution ℓ^{opt} exists according to Lemma 2 in the appendix. Hence, there exists dual variable $(\lambda^{\text{opt}}, u^{\text{opt}}) \succeq 0$ such that $w^{\text{opt}} := (\ell^{\text{opt}}, \lambda^{\text{opt}}, u^{\text{opt}})$ is primal dual optimal for SOCR-line, therefore w^{opt} satisfies the KKT conditions [17, Chap. 5]. If any of the four conditions in Lemma 1 holds, then $\lambda^{\text{opt}} \succ 0$ according to Lemma 3-6 in the appendix. It follows from complementary slackness (one of the KKT conditions) that the equality in (17) is attained, therefore ℓ^{opt} is feasible for OPF-line. Furthermore, ℓ^{opt} is optimal for OPF-line since ℓ^{opt} solves the relaxed problem SOCR-line. Therefore SOCR-line is exact. ■

IV. CASE STUDY

As already stated, the sufficient condition (ii) in Theorem 1 is likely to be satisfied in practical distribution networks if there are no distributed generators. In this section, we demonstrate through a practical network, that condition (ii) may be used as rule of thumb to place distributed generators (so that SOCR is guaranteed to be exact). The network we study is a distribution network in the service area of Southern California Edison [18], with high penetration of distributed generation. The network is shown in Figure 2, and its line impedances, peak spot loads, and nameplate ratings of shunt

capacitors and distributed generators are shown in Table II. Bus 1 represents the substation, and there are 5 photovoltaic (PV) generators located at bus 13, 17, 19, 23 and 24.

In applying condition (ii) to place distributed generators, we ignore the requirement on r/x , and try to satisfy $P^{\text{lin}}(p) \succeq 0$. The load p is chosen assuming that every bus is drawing its peak spot load at power factor 1, and every PV generator is generating real power at its nameplate capacity, e.g., load at bus 22 is absorbing 2.23MW real power ($p_{22} = 2.23\text{MW}$), and PV generator at bus 24 is generating 2MW real power ($p_{24} = -2\text{MW}$). Noting that line (2, 13) has zero impedance, we consider bus 2 and 13 as a single bus. Similarly, bus 16 and 17, bus 18 and 19, bus 21 and 24, and bus 22 and 23 can be considered as a single bus.

It can be checked that $P^{\text{lin}}(p) \succeq 0$ does not hold, e.g.,

$$\begin{aligned} P_{20,21}^{\text{lin}}(p) &= p_{21} + p_{22} + p_{23} + p_{24} \\ &= 0.45 + 2.23 - 1 - 2 = -0.32\text{MW} < 0. \end{aligned}$$

However, if we move the PV generators at bus 17, 19, 24 to bus 15, 3, 20 respectively, then $P^{\text{lin}}(p) \succeq 0$ holds.¹ Therefore SOCR (with $\underline{p} = \bar{p} = p$) will be an exact relaxation of OPF (with $\underline{p} = \bar{p} = p$), if the r/x ratio is non-decreasing.

The way we change the placement of PV generators is as follows. Starting from the leaves to the root (substation) of the network, whenever $P_{ij}^{\text{lin}} < 0$ for some $(i, j) \in \mathcal{L}$, move the distributed generator at bus j to bus i . For example, since $P_{20,21}^{\text{lin}}(p) < 0$, we move the PV generator at bus 24 (recall that bus 24 and 21 are considered as a single bus) to bus 20, then $P_{20,21}^{\text{lin}}(p)$ becomes

$$\begin{aligned} P_{20,21}^{\text{lin}}(p) &= p_{21} + p_{22} + p_{23} \\ &= 0.45 + 2.23 - 1 = 1.68\text{MW} > 0. \end{aligned}$$

Iterate this procedure for all $(i, j) \in \mathcal{L}$, from the leaves to the root, and we will end up moving the PV generators at bus 17, 19, and 24 to bus 15, 3, and 20 respectively.

In summary, condition (ii) is likely to hold in practical distribution networks if there are no distributed generators. Besides, the requirement $P^{\text{lin}}(p) \succeq 0$ in condition (ii) offers a simple rule of thumb for placing distributed generators.

V. CONCLUSION

We have studied the exactness of convex relaxation SOCR for an OPF problem in tree networks. We proved that the SOCR is exact, if there are no upper bounds on the voltage and any one of some conditions holds. One of these conditions requires that there is no reverse real power flow, and that the resistance to reactance ratio is non-decreasing as transmission lines spread out from the substation. This condition is likely to hold in practical distributed networks if there are no distributed generators. Besides, avoiding reverse real power flow can be used as rule of thumb for placing distributed generators.

¹ Actually, $P_{ij}^{\text{lin}} \geq 0$ holds for (i, j) where r_{ij} and x_{ij} are not both zero.

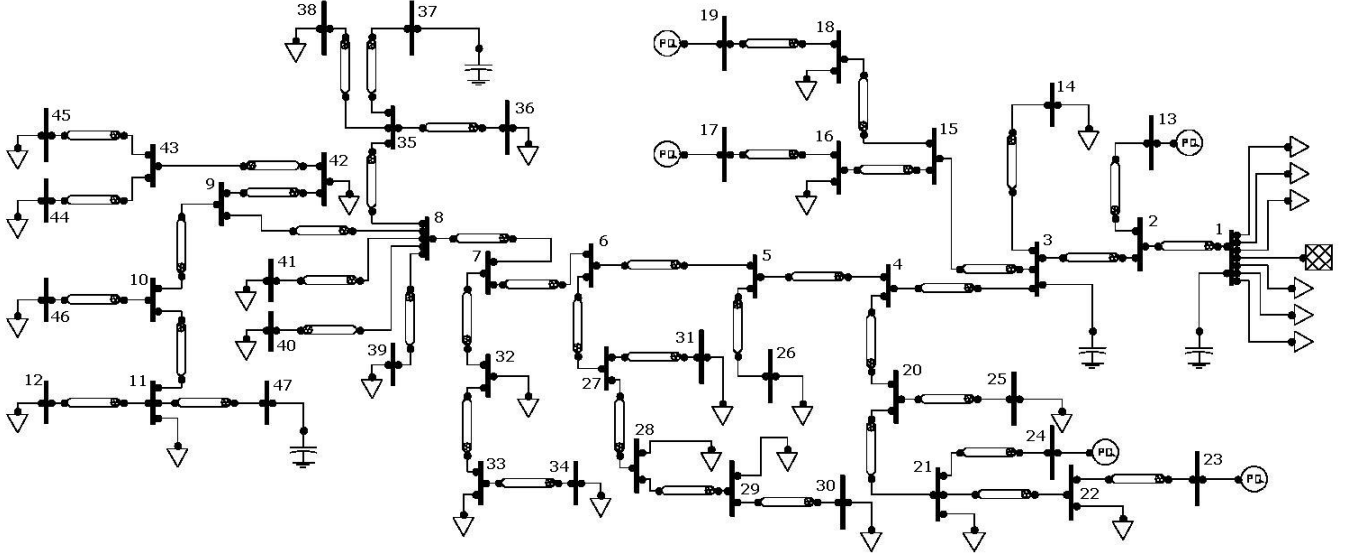


Fig. 2. A schematic diagram of a distribution feeder with high penetration of distributed generation (photovoltaics). Bus 1 is the substation and the 6 loads attached to it model other feeders on this substation.

TABLE II

NETWORK OF FIGURE 2: LINE IMPEDANCES, PEAK SPOT LOAD KVA, CAPACITORS AND PV GENERATION'S NAMEPLATE RATINGS.

Network Data																	
Line Data				Line Data				Line Data				Load Data		Load Data		PV Generators	
From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	Bus No.	Peak MVA	Bus No.	Peak MVA	Bus No.	Nameplate Capacity
1	2	0.259	0.808	8	41	0.107	0.031	21	22	0.198	0.046	1	30	34	0.2		
2	13	0	0	8	35	0.076	0.015	22	23	0	0	11	0.67	36	0.27	13	1.5MW
2	3	0.031	0.092	8	9	0.031	0.031	27	31	0.046	0.015	12	0.45	38	0.45	17	0.4MW
3	4	0.046	0.092	9	10	0.015	0.015	27	28	0.107	0.031	14	0.89	39	1.34	19	1.5 MW
3	14	0.092	0.031	9	42	0.153	0.046	28	29	0.107	0.031	16	0.07	40	0.13	23	1 MW
3	15	0.214	0.046	10	11	0.107	0.076	29	30	0.061	0.015	18	0.67	41	0.67	24	2 MW
4	20	0.336	0.061	10	46	0.229	0.122	32	33	0.046	0.015	21	0.45	42	0.13		
4	5	0.107	0.183	11	47	0.031	0.015	33	34	0.031	0	22	2.23	44	0.45		
5	26	0.061	0.015	11	12	0.076	0.046	35	36	0.076	0.015	25	0.45	45	0.2		
5	6	0.015	0.031	15	18	0.046	0.015	35	37	0.076	0.046	26	0.2	46	0.45		
6	27	0.168	0.061	15	16	0.107	0.015	35	38	0.107	0.015	28	0.13				
6	7	0.031	0.046	16	17	0	0	42	43	0.061	0.015	29	0.13				
7	32	0.076	0.015	18	19	0	0	43	44	0.061	0.015	30	0.2				
7	8	0.015	0.015	20	21	0.122	0.092	43	45	0.061	0.015	31	0.07				
8	40	0.046	0.015	20	25	0.214	0.046					32	0.13				
8	39	0.244	0.046	21	24	0	0					33	0.27				

APPENDIX

$\partial L / \partial \ell_{N-1} = 0$ at $(\ell^{\text{opt}}, \lambda^{\text{opt}}, u^{\text{opt}})$. It follows that

$$r_j = \lambda_j^{\text{opt}} - \sum_{i=0}^j \frac{2r_j P_i^{\text{opt}} + 2x_j Q_i^{\text{opt}}}{v_i^{\text{opt}}} \lambda_i^{\text{opt}} + \sum_{i=0}^{N-1} \frac{(P_i^{\text{opt}})^2 + (Q_i^{\text{opt}})^2}{v_i^{\text{opt}2}} \frac{\partial v_i}{\partial \ell_j} \lambda_i^{\text{opt}} + \sum_{i=1}^N u_i^{\text{opt}} \frac{\partial v_i}{\partial \ell_j} \quad (20)$$

Lemma 2: If SOCR-line is feasible, then there exists an optimal solution $(\ell^{\text{opt}}, P^{\text{opt}}, Q^{\text{opt}}, v^{\text{opt}})$ to SOCR-line.

Proof: The set \mathcal{F} of feasible ℓ for problem SOCR-line is closed, and lies in the non-negative orthant ($\mathcal{F} \subseteq \mathbb{R}_+^N$). If SOCR-line is feasible, then \mathcal{F} is non-empty. Pick an arbitrary feasible $\hat{\ell} \in \mathcal{F}$, and consider the set $\mathcal{O} := \{\ell \in \mathcal{F} : r^T \ell \leq r^T \hat{\ell}\}$. The set \mathcal{O} is non-empty (since $\hat{\ell} \in \mathcal{O}$), closed, and bounded (since $r \succ 0$), therefore a compact set. Define $\ell^{\text{opt}} := \arg \min_{\ell \in \mathcal{O}} r^T \ell$, then $(\ell^{\text{opt}}, P(\ell^{\text{opt}}), Q(\ell^{\text{opt}}), v(\ell^{\text{opt}}))$ is an optimal solution to SOCR-line. ■

The Lagrangian $L(\ell, \lambda, u)$ for Problem SOCR-line is given in (19). Its partial derivatives satisfy $\partial L / \partial \ell_0 = \dots =$

for $j = 0, \dots, N-1$. We are now going to prove that $\lambda^{\text{opt}} \succ 0$ if either condition in Lemma 1 holds. Define $\bar{r}_k := \sum_{i=0}^k r_i$, $\bar{x}_k := \sum_{i=0}^k x_i$ for $k = 0, \dots, N-1$. It can be checked that

$$\frac{\partial P_i}{\partial \ell_j} = \begin{cases} r_j & i \leq j \\ 0 & i \geq j+1 \end{cases}, \quad \frac{\partial Q_i}{\partial \ell_j} = \begin{cases} x_j & i \leq j \\ 0 & i \geq j+1 \end{cases},$$

$$\frac{\partial v_i}{\partial \ell_j} = \begin{cases} -2r_j \bar{r}_{j-1} - r_j^2 - 2x_j \bar{x}_{j-1} - x_j^2 & i \geq j+1 \\ -2r_j \bar{r}_{i-1} - 2x_j \bar{x}_{i-1} & i \leq j \end{cases}$$

for $i = 0, \dots, N$ and $j = 0, \dots, N - 1$. For brevity, we assume that SOCR-line is feasible, and drop the superscript ‘‘opt’’ (which stands for optimal) if there is no confusion. Lemma 3–6 show that $\lambda^{\text{opt}} > 0$ under each of the conditions in Lemma 1 respectively.

Lemma 3: If $P^{\text{lin}}(p) \geq 0$ and $Q^{\text{lin}}(q) \geq 0$, then $\lambda > 0$.

Proof: It follows from (11)–(13) and (16) that

- P, Q, v are affine functions of ℓ and $\frac{\partial P_i}{\partial \ell_j} \geq 0, \frac{\partial Q_i}{\partial \ell_j} \geq 0, \frac{\partial v_i}{\partial \ell_j} \leq 0$ for $i = 0, \dots, N$ and $j = 0, \dots, N - 1$;
- $(P^{\text{lin}}, Q^{\text{lin}})$ is the (P, Q) corresponding to $\ell = 0$;
- $\ell \geq 0$.

Therefore $P \geq P^{\text{lin}} \geq 0$ and $Q \geq Q^{\text{lin}} \geq 0$. Since $\lambda \geq 0$ and $u \geq 0$, it follows from (20) that $r_j \leq \lambda_j$ for $j = 0, \dots, N - 1$, therefore $\lambda > 0$. ■

Lemma 4: If $P^{\text{lin}}(p) \geq 0$ and $r_i/x_i \leq r_{i+1}/x_{i+1}$ for $i = 0, \dots, N - 2$, then $\lambda > 0$.

Proof: If $\lambda > 0$ does not hold, then there exists indices i such that $\lambda_i = 0$. Define $k := \min\{i \geq 0 : \lambda_i = 0\}$ as the smallest one of such indices. Then $k \geq 1$ since by substituting $j = 0$ into (20), we have

$$r_0 \leq \lambda_0 \left[1 - \frac{2(r_0 P_0 + x_0 Q_0)}{v_0} \right] \leq \lambda_0 \frac{v_1}{v_0} \Rightarrow \lambda_0 > 0.$$

Define $\eta_i := r_i/x_i$, and note that η_i is non-decreasing in i . It follows from (20) that

$$\begin{aligned} \frac{r_k}{x_k} &= - \sum_{i=0}^{k-1} \frac{2\eta_k P_i + Q_i}{v_i} \lambda_i \\ &\quad + \sum_{i=1}^{N-1} \frac{P_i^2 + Q_i^2}{v_i^2} \lambda_i \frac{\partial v_i}{x_k \partial \ell_k} + \sum_{i=1}^N u_i \frac{\partial v_i}{x_k \partial \ell_k}, \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{r_{k-1}}{x_{k-1}} &= \frac{\lambda_{k-1}}{x_{k-1}} - \sum_{i=0}^{k-1} \frac{2\eta_{k-1} P_i + Q_i}{v_i} \lambda_i \\ &\quad + \sum_{i=1}^{N-1} \frac{P_i^2 + Q_i^2}{v_i^2} \lambda_i \frac{\partial v_i}{x_{k-1} \partial \ell_{k-1}} + \sum_{i=1}^N u_i \frac{\partial v_i}{x_{k-1} \partial \ell_{k-1}}. \end{aligned} \quad (22)$$

Since $\eta_k \geq \eta_{k-1} > 0$ and $P \geq P^{\text{lin}} \geq 0$, we have

$$- \sum_{i=0}^{k-1} \frac{2\eta_k P_i + Q_i}{v_i} \lambda_i \leq - \sum_{i=0}^{k-1} \frac{2\eta_{k-1} P_i + Q_i}{v_i} \lambda_i. \quad (23)$$

It is not difficult to verify that

$$\frac{\partial v_i}{x_k \partial \ell_k} \leq \frac{\partial v_i}{x_{k-1} \partial \ell_{k-1}}$$

for $i = 0, \dots, N$ and $k = 1, \dots, N - 1$. Hence,

$$\begin{aligned} &\sum_{i=1}^{N-1} \frac{P_i^2 + Q_i^2}{v_i^2} \lambda_i \frac{\partial v_i}{x_k \partial \ell_k} + \sum_{i=1}^N u_i \frac{\partial v_i}{x_k \partial \ell_k} \\ &\leq \sum_{i=1}^{N-1} \frac{P_i^2 + Q_i^2}{v_i^2} \lambda_i \frac{\partial v_i}{x_{k-1} \partial \ell_{k-1}} + \sum_{i=1}^N u_i \frac{\partial v_i}{x_{k-1} \partial \ell_{k-1}}. \end{aligned}$$

Then, it follows from (21) and (22) that

$$\frac{r_k}{x_k} \leq \frac{r_{k-1}}{x_{k-1}} - \frac{\lambda_{k-1}}{x_{k-1}} < \frac{r_{k-1}}{x_{k-1}},$$

which contradicts with the condition $r_k/x_k \geq r_{k-1}/x_{k-1}$. Hence, we must have $\lambda > 0$. ■

Lemma 5: If $Q^{\text{lin}}(q) \geq 0$ and $r_i/x_i \geq r_{i+1}/x_{i+1}$ for $i = 0, \dots, N - 2$, then $\lambda > 0$.

Proof: The proof of Lemma 5 is similar to that of Lemma 4, and omitted for brevity. ■

Lemma 6: If $r_i/x_i = r_{i+1}/x_{i+1}$ for $i = 0, \dots, N - 2$, then $\lambda > 0$.

If $r_i/x_i = r_{i+1}/x_{i+1}$ for $i = 0, \dots, N - 2$, then the inequality in (23) attains equality, and we can apply the proof of Lemma 4. We present another proof here.

Proof: Write (20) in a vector form to obtain

$$r = \lambda - A\lambda - BCN\lambda - Bu, \quad (24)$$

where $r = (r_0, \dots, r_{N-1})^T$, $\lambda = (\lambda_0, \dots, \lambda_{N-1})^T$, $u = (u_1, \dots, u_N)^T$,

$$A = \begin{pmatrix} \frac{2(r_0 P_0 + x_0 Q_0)}{v_0} & & & \\ \vdots & \ddots & & \\ \frac{2(r_{N-1} P_{N-1} + x_{N-1} Q_{N-1})}{v_0} & \dots & \frac{2(r_{N-1} P_{N-1} + x_{N-1} Q_{N-1})}{v_{N-1}} & \end{pmatrix},$$

$$B = - \begin{pmatrix} \frac{\partial v_1}{\partial \ell_0} & \dots & \frac{\partial v_N}{\partial \ell_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_1}{\partial \ell_{N-1}} & \dots & \frac{\partial v_N}{\partial \ell_{N-1}} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

$$\text{and } C = \begin{pmatrix} \frac{P_1^2 + Q_1^2}{v_1^2} & & \\ & \ddots & \\ & & \frac{P_N^2 + Q_N^2}{v_N^2} \end{pmatrix}.$$

Define

$$w := CN\lambda + u \geq 0, \quad \eta := x_0/r_0,$$

and note that $\eta = x_i/r_i$ for $i = 0, \dots, N - 1$. Left multiply both sides of (24) by the inverse of the diagonal matrix

$$R = \begin{pmatrix} r_0 & & & \\ & r_1 & & \\ & & \ddots & \\ & & & r_{N-1} \end{pmatrix},$$

and then use (13) to simplify, we obtain

$$\mathbf{1} = LR^{-1}\lambda - E\lambda - R^{-1}Bw,$$

where $\mathbf{1} = (1, \dots, 1)^T$,

$$L = \begin{pmatrix} \frac{v_1}{v_0} & & & \\ \frac{v_1 - v_0}{v_0} & \frac{v_2}{v_1} & & \\ \vdots & \vdots & \ddots & \\ \frac{v_1 - v_0}{v_0} & \frac{v_2 - v_1}{v_1} & \dots & \frac{v_N}{v_{N-1}} \end{pmatrix},$$

$$\text{and } E = (1 + \eta^2) \begin{pmatrix} \frac{r_0 \ell_0}{v_0} & \frac{r_1 \ell_1}{v_1} & & \\ \vdots & \vdots & \ddots & \\ \frac{r_0 \ell_0}{v_0} & \frac{r_1 \ell_1}{v_1} & \dots & \frac{r_{N-1} \ell_{N-1}}{v_{N-1}} \end{pmatrix}.$$

Hence,

$$R^{-1}\lambda = L^{-1}\mathbf{1} + L^{-1}E\lambda + L^{-1}R^{-1}Bw. \quad (25)$$

By Claim 1-3 proven below, it follows from (25) that $R^{-1}\lambda \succ 0$. Consequently, $\lambda = R(R^{-1}\lambda) \succ 0$. ■

We first give an important lemma. Define

$$J := \begin{pmatrix} 1 & & & \\ \vdots & \ddots & & \\ 1 & \cdots & 1 & \end{pmatrix},$$

$$\hat{L} := \begin{pmatrix} a_1 & & & \\ a_1 - \Delta_1 & a_2 & & \\ \vdots & \vdots & \ddots & \\ a_1 - \Delta_1 & a_2 - \Delta_2 & \cdots & a_N \end{pmatrix},$$

where $a_i \neq 0$, $\Delta_i \in \mathbb{R}$ for $i = 1, \dots, N$.

Lemma 7: The matrix $\hat{D} := \hat{L}^{-1}J$ is given by

$$\hat{D}_{ij} = \begin{cases} \frac{1}{a_i} \prod_{k=j}^{i-1} \frac{\Delta_k}{a_k} & i \geq j \\ 0 & i < j \end{cases}$$

for $i, j = 1, \dots, N$.

Proof: The proof is based on Gaussian elimination. ■

It follows from Lemma 7 that

$$D := L^{-1}J = \begin{pmatrix} \frac{v_0}{v_1} & & & \\ \frac{v_0}{v_2} & \frac{v_1}{v_2} & & \\ \vdots & \vdots & \ddots & \\ \frac{v_0}{v_N} & \frac{v_1}{v_N} & \cdots & \frac{v_{N-1}}{v_N} \end{pmatrix}.$$

Claim 1: $L^{-1}\mathbf{1} \succ 0$.

Proof: $L^{-1}\mathbf{1}$ is the first column of the matrix D. ■

Claim 2: $L^{-1}E$ is pointwise nonnegative.

Proof: Since

$$L^{-1}E = (1 + \eta^2)D \begin{pmatrix} \frac{r_0 \ell_0}{v_0} & & & \\ & \ddots & & \\ & & \frac{r_{N-1} \ell_{N-1}}{v_{N-1}} & \end{pmatrix},$$

it is pointwise nonnegative. ■

Claim 3: $L^{-1}R^{-1}B$ is point-wise nonnegative.

Proof: Since $L^{-1}R^{-1}B = (1 + \eta^2)L^{-1}J(I + N)RJT = (1 + \eta^2)D(I + N)RJT$, it is pointwise nonnegative. ■

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