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INTRODUCTION

to the abstract book for the Oberwolfach meeting on research into belief

In the industrialized countries, everybody seems to know what mathematics is. But when the question is put forward, one gets different answers depending on the individual in question. School children understand mathematics differently from their mathematics teachers, and teachers of other subjects will explain it again differently. Still another description will be given by the man in the street. And mathematics professors have their own view of mathematics.

This large variety of answers to the question "What is mathematics?" hints at the fact that there is not only one understanding of mathematics, but several different views of mathematics. And not in the sense that there is only *one correct* view of mathematics, and the other ones are wrong. Philosophers of mathematics have introduced several *correct* views of mathematics which are also accepted among mathematicians. This state of art with the constructivist view of learning has led researchers of mathematics education to investigate teachers' and pupils' views of mathematics and the implication of their views for mathematics teaching and learning. This strand was supported by the constructivist view of learning.

The influence of a teacher's philosophy of mathematics

Lerman (1983) pointed out that a teacher's problem-centered view of mathematics resulted in different teaching than a knowledge-centered view. Similar results have been achieved in research again and again, e.g. by Lloyd & Wilson (1998). The same is valid for pupils and students. If they think that mathematics means merely solving textbook tasks, they might have difficulties in solving real-life problem situations.

Within research in school, the understanding of learning has focussed above all on following cognitive academic achievement. Affective by-results, however, which stand in connection with an individual's metacognitions, however, determine the quality of learning. According to recent psychological research, these both aspects are central to learning (e.g. Bereiter & Scardamalia 1996). During the last decade, researchers throughout the world have paid more and more attention to learning of mathematics from the viewpoint of metacognitions, especially in the form of pupils' and teachers' beliefs. Beliefs seem to be situated in the "twilight zone" between the cognitive and affective domain and thus have a component in each domain.

Behind the above-mentioned active understanding of learning, one finds the view of learning which is compatible with constructivism. According to that view, it is essential that a learner is actively working, in order to be able to elaborate his knowledge structure (e.g. Davis & al. 1990). Thus, the significance of the pupils' own beliefs (subjective knowledge) concerning mathematics and its learning is emphazised as being a regulating system of their knowledge structure. Since the teacher is the central influential factor as an organizer of learning environments, his beliefs are also essential. Therefore, teachers' and pupils' mathematical beliefs play a key role when trying to understand their mathematical behavior (Noddings 1990, 14).

Beliefs and belief systems

Twenthieth century research into beliefs and belief systems carried out especially in the early years of this century, concentrated on social psychology (Thompson 1992). But shortly after that, behaviorism spread into research in the psychological domain. Then the focus shifted to the observational parts of human behavior, and beliefs were nearly forgotten. New interest in beliefs and belief systems emerged mainly in the 1970s, through the developments in cognitive science (Abelson 1979).

An individual continuously receives perceptions from the world around him. According to his experiences and perceptions, he draws conclusions about different phenomena and their nature. The individual's personal knowledge, i.e. his beliefs, are a compound of these conclusions. Furthermore, he compares these beliefs with his new experiences and with the beliefs of other individuals, and thus his beliefs are under continuous evaluation and change. When he adopts a new belief, this will automatically form a part of the larger structure of his personal knowledge, of his belief system, since beliefs never appear in isolation. Thus, the individual's belief system is a compound of his conscious or unconscious beliefs, hypotheses or expectations and their combinations. (Green 1971)

In the literature, there are several overviews of belief research (e.g. Underhill 1988a, Underhill 1988b, Thompson 1992, Pehkonen 1994, Pehkonen & Törner 1996). A quick search (on Oct 18, 1999) in the ERIC Database[1] confirms that research around this topic is very vivid. Table 1 shows the number of items found in the search of several combinations, e.g. "mathematics & beliefs & 1998" where the description word "belief" has varied within the belief-related expressions, and the period considered encompasses the last ten years.

[1] The www address for the ERIC Database is as follows: http://ericir.syr.edu/Eric/

Table 1. The number of publications in belief in the ERIC Database in the years 1989–98.

expression	*1989*	*1990*	*1991*	*1992*	*1993*	*1994*	*1995*	*1996*	*1997*	*1998*
belief	42	54	49	67	70	72	83	72	52	36
conception	559	574	472	501	472	376	412	394	527	354
view	103	130	139	108	101	94	100	128	149	56
understanding	178	252	205	259	248	236	246	210	232	179
perception	74	116	66	118	117	68	76	64	70	31
perspective	80	81	84	94	99	93	121	87	87	69

The figures in the column 1998 are smaller than in other ones, since coding publications in the database takes time.

The meaning of mathematical beliefs

The central role of beliefs for the successful learning of mathematics has been pointed out again and again by several educators in mathematics (e.g. Schoenfeld 1985, Silver 1985, Borasi 1990, Schoenfeld 1992). In this regard, the following reasons are given as an explanation for these effects: Beliefs may have a powerful impact on how children will learn and use mathematics, and therefore they may also form an obstacle for the effective learning of mathematics. Pupils who have rigid and negative beliefs of mathematics and its learning easily become passive learners, whose emphasis will be placed more on memorizing than on understanding in the process of learning.

Beliefs and learning seem to form a circle: Pupils' experiences in learning mathematics influence and form their beliefs. On the other hand, beliefs have consequences on how pupils will behave in situations of learning mathematics, and therefore, on the way in which they are able to learn mathematics. (Spangler 1992) Thus, pupils' beliefs as revealed in research reflect teaching practices in the classroom. The way mathematics is taught in the classroom will little by little form pupils' view of mathematics.

The connection between a teacher's beliefs and his teaching practice is well-documented. If a teacher thinks that the learning of mathematics happens at its best by doing calculation tasks, his teaching will concentrate on doing as many calculations as possible. This phenomenon has already been observed more than ten years ago: Teachers' different teaching philosophies (belief systems) will lead to different teaching practices in classrooms (Lerman 1983; also Ernest 1991).

When discussing teacher change and the possibilities of influencing it, the question arises of which is the primary cause: On the one hand, Ernest (1989) states that beliefs wholly regulate a teacher's teaching practice in classroom. On the other hand, there is evidence that changes in classroom practices may also change a teacher's beliefs (e.g. Guskey 1986). This might be a similar circle as is formed by beliefs and learning in the case of pupils, since teacher change can be considered as learning.

In an earlier paper (Pehkonen & Törner 1996), we elaborated on four groups of meaning of beliefs:

(1) Mathematical beliefs as a regulating system.

(2) Mathematical beliefs as an indicator.

(3) Mathematical beliefs as an force of inertia.

(4) The prognostic character of mathematical belief systems.

Mathematical beliefs as a regulating system are easily understood when we remember that an individual's mathematical beliefs form a frame for his knowledge structure. Within this frame, on the one hand, the individual may act and think. On the other hand, this frame broadly influences his mathematical performance. Let's take an example: There is a pupil who understands mathematics merely as calculations. His understanding is often resulting from a one-sided, teaching at the primary level with special emphasis on calculations. In that case such tasks which require thinking and where mere calculation does not lead to an answer might be difficult or even impossible for him.

Mathematical beliefs may be a practical indicator in a situation which one is not otherwise able to observe. Since the view of mathematics transmitted through beliefs, expressed by an individual, allows a good estimation of his experiences in learning and teaching mathematics, we will thus have a method for indirectly evaluating the instruction he has received or has given. It thus seems obvious that beliefs of pupils as well as of teachers present condensed information on personally experienced "meetings" with cognitive elements in the past.

If we aim at changing the teaching of mathematics in schools, we are compelled to take into account teachers' beliefs – and also pupils‘ beliefs as a possible force of inertia. Usually, it is experienced teachers' rigid attitudes and their steady teaching styles that will hinder change trials. Experienced teachers know through their long practice which ways of teaching mathematics have proved a success, and this subjective knowledge (beliefs) is usually deeply rooted.

As a consequence of the arguments mentioned, it should be stressed that mathematical belief systems also have a prognostic aspect. Pupils who consider mathematics as a manipulative calculation system only have an ignorant attitude toward problem solving, and therefore their opportunities to learn effectively in school are restricted, since studying mathematics in upper grades, e.g. in secondary schools or at university, demands more components of mathematics than mere calculations.

Different conceptions of beliefs

Although beliefs are popular as a topic of study, the theoretical concept of “belief” has not yet been dealt with thoroughly. The main difficulty has been the inability to distinguish beliefs from knowledge, and the problem is still unclarified (e.g. Abelson 1979, Thompson 1992).

An implication of this fuzziness in the definition of the concept is that beliefs are conceived as denoting different matters, depending on the discipline and the researchers involved. For example, beliefs are considered equal to concepts, meanings, propositions, rules, preferences or mental images (Thompson 1992). An educationalist (Pajares 1992) uses the following explanation: "*Beliefs and conceptions are regarded as part of knowledge. Beliefs are the incontrovertible personal "truths" held by everyone, deriving from experience or from fantasy, with a strong affective and evaluative component.*" Both Furinghetti (1996) and Ponte (1994) accept Pajares' definition (1992) for their studies. In social psychology, for example, the impressions of and reactions to other people are typically divided into beliefs, expectations and attitudes. For researchers in this field, beliefs are statements thought to be true, whether they are or not. Expectations are explicit or implicit predictions about people's future behaviors, and attitudes are emotional reactions to them. (Brophy & Evertson 1981; 8, 25)

There are many variations of the concepts of "belief" and "belief system" used in studies in the field of education in mathematics. As a consequence of the vague definition of the concept, researchers have often formulated their own definition for "belief", which may even be in contradiction with others. For example, Schoenfeld (1985, 44) states that in order to give a first rough impression "*belief systems are one's mathematical world view*". He later modifies his definition, interpreting beliefs as an individual's understanding and feelings that shape the way the individual conceptualizes and engages in mathematical behavior (Schoenfeld 1992). Hart (1989, 44) – under the influence of Schoenfeld's (1985) and Silver's (1985) ideas – uses the word *belief* "*to reflect certain types of judgments about a set of objects*". Törner & Grigutsch (1994) named their research object "mathematical world view", according to Schoenfeld (1985), elaborated it in a recent paper (Grigutsch & al. 1998) and anchored the concept within the theory of attitudes.

Some researchers think that beliefs are some kind of attitudes (e.g. Underhill 1988a), whereas Lester & al. (1989, 77) explain that "*beliefs constitute the individual's subjective knowledge about self, mathema-tics, problem solving, and the topics dealt with in problem statements*". On the other hand, Thompson (1992) understands beliefs as a subclass of conceptions. Furinghetti (1996, 20) also explains an individual's conception of mathematics as being a set of certain beliefs. Equally, Lloyd & Wilson (1998, 249) connect beliefs with conceptions saying: "*We use the word conceptions to refer to a person's general mental structures that encompass knowledge, beliefs, understandings, preferences, and views*". Yet another different explanation is given by Bassarear (1989) who sees attitudes and beliefs on the opposite poles of a bipolar dimension.

When looking at these different, many times even contradicting, "definitions" of beliefs, one observes that most of most researchers (Underhill 1988a, Lester & al. 1989, Thompson 1992, Furinghetti 1996, Lloyd & Wilson 1998) give a "real definition" in the sense that they refer to the static part of beliefs phrasing their ideas in the following way: beliefs are, constitute, are contained etc. By contrast the definition of Schoenfeld (1992) stresses the dynamic part of beliefs, how beliefs function. And the definition of

Hart (1989) puts forward the aspect of justification which distinguishes beliefs from knowledge.

The focus of the meeting

The Oberwolfach meeting MATHEMATICAL BELIEFS AND THEIR IMPACT ON TEACHING AND LEARNING will focus on the current state of the research regarding beliefs in the area of mathematics which will be presented by the participants. The purpose of this meeting is, among others, to draw attention to theoretical deficiencies of belief research: Firstly, the concept of belief (and other related concepts) are often left undefined, since they are thought to be self-explanatory (e.g. Cooney & al. 1998), or researchers give their own definitions which might differ drastically from each other or even be in contradiction with each other (e.g. Bassarear 1989 and Underhill 1988). The second important problem is the inability to distinguish between belief and knowledge. Additionally, the purpose of the meeting is to chart research on belief at large, and to chart its impact on teaching and learning in mathematics.

Organization of the meeting

We were successful in organizing this meeting on research into belief in "a mathematician's Mecca", i.e. at the Research Institute Oberwolfach which is renowned for its high quality meetings. Our schedule for the meeting was as follows: 25 persons can be invited and the time slot was given (Nov 21 – 27, 1999). Since we wanted to have top-level researchers in belief from all parts of the world, the first step to organize the meeting was to seek advice from a group of remarkable researchers from different continents whom we had met beforehand. The following five persons were our advisers: Prof. Tom Cooney (Athens/GA, USA), Prof. Fulvia Furinghetti (Genoa, Italy), Prof. Reuben Hersh (New Mexico, USA), Prof. Gilah Leder (Melbourne, Australia) and Prof. Steve Lerman (London, UK). Here we want to thank them very warmly for their help.

Thus, we asked these five persons to name the researchers whom they considered to be qualified for such a meeting. We also wanted to cover all possible countries of the world, in order to get a representative and high-level group of researchers to discuss problems of research on belief in Oberwolfach. Additionally, we wanted to have some mathematicians who are sensitive to the problems of teaching and who bring this topic into the discussions. Based on the lists of our advisers and our own considerations we invited 25 persons all over the world. But some of the persons invited had to cancel their participation for different reasons. Nevertheless, we consider the group participating in the meeting as the "cream of researchers on belief".

References

Abelson, R. 1979. Differences between belief systems and knowledge systems. Cognitive Science 3, 355–366.

Bassarear, T.J. 1989. The interactive nature of cognition and affect in the learning of mathematics: two case studies. Proceedings of the PME-NA-8 (eds. C.A. Maher, G.A. Goldin & R.B. Davis). Volume 1, 3–10. Piscataway, NJ.

Bereiter, C. & Scardamalia, M. 1996. Rethinking learning. In: The handbook of education and learning. New models of learning, teaching and schooling (eds. D.R. Olson & N. Torrance). Cambridge (MA): Blackwell.

Borasi, R. 1990. The Invisible Hand Operating in Mathematics Instruction: Students Conceptions and Expectations. In: Teaching and Learn-ing Mathema-tics in the 1990s. Yearbook 1990 (ed. T. J. Cooney), 174–182. Reston: NCTM.

Brophy, J.E. & Evertson, C.M. 1981. Student Characteristics and Teaching. New York (NY): Longman.

Cooney, T.J., Shealy, B.E. & Arvold, B. 1998. Conceptualizing Belief Structures of Preservice Secondary Mathematics Teachers. Journal for Research in Mathematics Education 29 (3), 306–333.

Davis, R.B., Maher, C.A. & Noddings, N. (eds.) 1990. Constructivist Views on the Teaching and Learning of Mathematics. JRME Mono-graph Number 4. Reston (Va): NCTM.

Ernest, P. 1991. The Philosophy of Mathematics Education. Hampshire (U.K.): The Falmer Press.

Furinghetti, F. 1996. A theoretical framework for teachers' conceptions. In: Current State of Research on Mathematical Beliefs III (ed. E. Pehkonen). Proceedings of the MAVI-3 Workshop, 19–25. University of Helsinki. Department of Teacher Education. Research Report 170.

Green, T.F. 1971. The Activities of Teaching. Tokyo: McGraw-Hill Kogakusha.

Grigutsch, S., Raatz, U. & Törner, G. 1998. Einstellungen gegenüber Mathematik bei Mathematiklehrern. Journal für Mathematik-Didaktik 19 (1), 3-45.

Hart, L.E. 1989. Describing the Affective Domain: Saying What We Mean. In: Affect and Mathematical Problem Solving (eds. McLeod & Adams), 37-45. New York: Springer-Verlag.

Lerman, S. 1983. Problem-solving or knowledge-centred: the influence of philo-sophy on mathematics teaching. International Journal of Mathematical Education in Science and Tech-nology 14 (1), 59–66.

Lester, F.K., Garofalo, J. & Kroll, D.L. 1989. Self-Confidence, Interest, Beliefs, and Metacognition: Key Influences on Problem-Solving Behavior. In: Affects and

Mathematical Problem Solving (eds. D.B. McLeod & V.M. Adams), 75-88. New York: Springer-Verlag.

Lloyd, G.M. & Wilson, M. 1998. Supporting Innovation: The Impact of a Teacher's Conceptions on Functions on His Implementation of a Reform Curriculum. Journal for Research in Mathematics Education 29 (3), 248–274.

Noddings, N. 1990. Constructivism in Mathematics Education. In: Constructivist Views on the Teaching and Learning of Mathema-tics (eds. R.B. Davis, C.A. Maher & N. Noddings), 7–18. JRME Mono-graph Number 4. Reston (Va): NCTM.

Pajares, M.F. 1992. Teachers' beliefs and educational research: cleaning up a messy construct. Review of educational research 62, 307–332.

Pehkonen, E. 1994. On Teachers' Beliefs and Changing Mathematics Teaching. Journal für Mathematik-Didaktik 15 (3/4), 177–209.

Pehkonen, E. & Törner, G. 1996. Mathematical Beliefs and Different Aspects of Their Meaning. International Reviews on Mathematical Education [=ZDM] 28 (4), 101–108.

Ponte, J.P. 1994. Mathematics teachers' professional knowledge. In: Proceedings of the 18th PME conference (eds. J.P. Ponte & J.F. Matos). Vol. 1, 195–210. University of Lisbon.

Schoenfeld, A.H. 1985. Mathematical Problem Solving. Orlando (FL.): Academic Press.

Schoenfeld, A.H. 1992. Learning to think mathematically: problem solving, meta-cognition, and sense making in mathematics. In: Handbook of research on mathematics learning and teaching (ed. D.A. Grouws), 334–370. New York: Macmillan.

Silver, E.A. 1985. Research on Teaching Mathematical Problem Solving: Some underrepresented Themes and Directions. In: Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives (ed. E.A. Silver) 1985, 247–266. Hillsdale (NJ): Lawrence Erlbaum Associates.

Spangler, D.A. 1992. Assessing Students' Beliefs About Mathematics. Arithmetic Teacher 40 (3), 148–152.

Thompson, A.G. 1992. Teachers' Beliefs and Conceptions: A Synthesis of the Research. In: Handbook of research on mathematics learning and teaching (ed. D.A. Grouws), 127–146. New York: Macmillan.

Törner, G. & Grigutsch, S. 1994. "Mathematische Weltbilder" bei Studienanfängern – eine Erhebung. Journal für Mathematik-Didaktik 15 (3/4), 211–251.

Underhill, R.G. 1988a. Mathematics Learners' Beliefs: A Review. Focus on Learning Problems in Mathematics 10 (1), 55-69.

Underhill, R.G. 1988b. Mathematics Teachers' Beliefs: Review and Reflections. Focus on Learning Problems in Mathematics 10 (3), 43–58.

How to attract students into programs in mathematics and keep them there?

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Various studies in recent years have documented the decline of students in programs in mathematics (e.g. S. A. Garfunkel, G. S. Young; The sky is falling; Notices of the AMS 45(2) (1998) 256-257). H. Bass (in Notices of the AMS 44(1) (1997) 18-21) analyses the transitional period in which we live and makes several suggestions how to cope, in particular he insists that University professors have to become more professional in their teaching.

My own department has introduced new programs, in particular in MathematicalFinance, the number of graduate students has increased, however the number of students in the undergraduate programs has decreased, and our honors program, which in the past produced a number of highly trained and motivated students, will have to be reorganized in order to survive.

Some staff members have conducted very creative Saturday afternoon sessions for Elementary and Junior High students, which , together with mathematical competitions, have attracted some students to our programs. After surveying several hundred students that graduated in mathematics, we also visited High Schools and provided information about mathematics and related careers. A fair number of students would be interested, but many nevertheless considered careers in engineering and computing.

A mentor program for graduate students has been introduced in order to improve their teaching skills. However the fact that the administration takes the teaching evaluation, done by the students for every course, very seriously, has probably had the greatest impact on teaching. The students tell us that they want to see many examples, and I am not sure whether this information is sufficient to understand how our students actally think; maybe the evaluation could be used to help in the study of the mental representation of students (see: R. B. Davis in the Handbook of Research on Mathematics Teaching and Learning; D. A. Grouws, Editor, New York 1992, 724-734) To me it seems that teaching could be guided by what I. Kant (in: The critique of judgement) sees as central to aesthetic delight: The free play into which imagination and understanding enter.

Beliefs as generative metaphors in mathematics teachers' growth

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A primary concern for mathematics teacher educators, particularly when reforms to mathematics education are being advocated, is how to effectively facilitate meaningful teacher change or growth in teaching mathematics. Facilitating or accomplishing fundamental changes in mathematics teaching can be a challenging endeavor for both teachers and teacher educators given the fact that mathematics high school classrooms, in particular, tend to look the same as those of the teachers, their teachers, and their teachers' teachers. But some teachers do make significant, positive shifts in their teaching that are self-motivated and self-determined. This paper reports on a study that investigated a sample of these teachers in terms of the role beliefs in the form of metaphors played in facilitating these shifts in their teaching. The intent is to offer insights of teacher change or growth that are meaningful in understanding and conceptualizing mathematics teacher development.

Metaphor seems to be receiving growing attention in the mathematics education literature. Recently, it has been used to account for the learning or construction of mathematical knowledge in terms of aiding learners and mathematicians in making sense of mathematical ideas (English, 1997; Presmeg, 1998). It has also been used in studies involving preservice teachers (Chapman, 1998, in press; Cooney et al, 1998) and inservice teachers (Chapman, 1997). In the work with teachers, metaphors provided a way to look at teachers' beliefs, explicitly and implicitly. For example, the two preservice teachers who perceived mathematical problem solving as eating broccoli and as downhill skiing, respectively, (Chapman, in press) did not have the same set of beliefs about problem solving. The study being reported in this paper focused on teachers' beliefs in the form of metaphors and the relationship to growth in the teaching of high school mathematics. In the context of this study, belief and metaphor were interpreted in the same way, i.e., from the perspective that they are conceptual systems that provide a frame through which one gives meaning to and receives meaning from the "things" of the world. This perspective is reflected in the works of, for e.g., Bogdon, (1986), Lakoff and Johnson (1980), and Pajares (1987).

Research process

The study discussed here [metaphor-change study] evolved unintentionally from an ongoing 3-year study [problem-solving project] on mathematics teachers' thinking in teaching mathematical word problems. The problem-solving project involves a total of eighteen participants consisting of experienced and beginning teachers at the elementary, junior, and senior high school levels. During preliminary analysis of data for the problem-solving study, there appeared to be a unique relationship between specific beliefs about mathematics held by the participants and specific changes in their teaching. I decided to explore this relationship as a study by itself [the metaphor-change study] and as a possible relevant context for the problem-solving project. The data already collected for the problem-solving project was found to be adequate for the metaphor-change study.

The first year (1998-1999) of the problem-solving project focused on four experienced high school mathematics teachers. These teachers were recommended to me as outstanding teachers and I was curious about how they conceptualized and experienced the teaching of mathematical word problems – a topic that causes distress for many students. Specific to the metaphor-change study, the teachers' teaching, in general, evolved from being very teacher-centered, as they started their practice, to eventually being more student-centered. This shift did not occur as a result of any specific professional development program.

Data collection and analysis for the problem-solving project followed a humanistic approach (Chapman, 1999). Data collection involved extensive interviews focusing on narrative accounts of the teachers' past, present, and possible future teaching behaviors and their thinking in relation to word problems, problem solving, and mathematics. There were also classroom observations and role-play of scenarios related to the teachers' classrooms. The analysis for the problem-solving project involved identifying themes in the data that conveyed the meaning underlying the teachers' thinking. One stage of this process involved coding the data for explicit beliefs (Chapman, 1999). The analysis for the metaphor-change study involved identifying all beliefs in the form of a metaphor. These beliefs were then examined in the contexts in which they occurred and the relationships among contexts.

Findings

The findings indicated a transformative connection between a particular belief/metaphor held by the teachers and the teachers' teaching, i.e., the former played a significant role in influencing changes to the latter. This connection I later associated with Schon's (1979) "generative metaphor" and Lakoff and Johnson's (1980) "structural metaphor". These metaphors facilitate a process by which we gain new perspective on the world, i.e., a process that involves generating or structuring one concept in terms of another.

This particular belief of each teacher was stated explicitly as the nature of mathematics but embodied beliefs about its learning and teaching. The belief emerged in the teachers' talk without any prompting to provide one in the form of a metaphor.

No similar belief (i.e., explicitly stated in the form of a metaphor) emerged for learning or teaching. These beliefs for the four participants were:

Mathematics is experience.

Mathematics is play.

Mathematics is language.

These metaphors grew out of the teachers' personal experiences with doing and teaching mathematics. They were metaphors from the teachers' perspectives in that mathematics was being understood in terms of something else. This "something" had unique meanings for the teachers. For example, two of the teachers thought of mathematics as language, but had very different interpretations of language. These metaphors behaved as generative metaphors (Schon, 1979) in that they facilitated shifts in the teachers' thinking about mathematics that resulted in significant changes to their teaching. The teachers used characteristics directly appropriate for one domain as a lens for seeing another. Thus, as the characteristics they perceived for experience, play, and language grew/changed, so did their views of mathematics and their teaching. The generative process was triggered whenever there were significant conflicts for the teachers between their prevailing interpretations of the metaphors and their lived experiences in the classroom. However, this was not a conscious process for the teachers. For them, the connection to the metaphor was unconscious. For example, the teachers consciously viewed each characteristic added to the "something" mathematics was compared to as an isolated event, a reaction to a new problematic situation in their teaching. As Schon (1979) noted, "Generative metaphors are ordinarily tacit."

One unique feature of the teachers' situation was that although the metaphor was stated only in relation to mathematics, it provided a way for the teachers to reshape their perception and understanding of teaching and learning mathematics and their classroom behaviors. This dependence of teaching behavior on the metaphor for mathematics was reflected by the situation that every significant shift in the teachers' teaching generally followed a shift in interpreting "mathematics is ----." This supports the view that changing or broadening beliefs about mathematics is primary to changing teaching of mathematics.

Only two of the teachers, Elise and Mark (pseudonyms), will be discussed here in order to illustrate the nature of these metaphors in the teachers' thinking and the effect on the teachers' growth. The summaries that follow highlight only a small sample of each teacher's situation.

Elise

Elise has been a high school teacher for about 16 years. As a student, Elise believed, "Mathematics is play". At this point she articulated play only as fun. Thus, mathematics was fun. As a beginning teacher, Elise was mentored to use a traditional teaching approach that emphasized memorization of facts/procedures through drill and practice. This experience conflicted with her experience of mathematics teaching as a student of mathematics. She expected and wanted teaching to reflect mathematics as play. But her mentor led her to believe that this was unrealistic. To deal with the conflict while being mentored during the first two years of her practice, she decided that doing mathematics was different from teaching mathematics.

Once on her own, Elise still wanted to make her teaching more like mathematics, i.e., to reflect mathematics as play, but she could not see how to do this. Play as fun was incompatible with high school mathematics based on how she was led to believe that it must be taught and she could not see any alternative on her own. She started to occasionally give students a "fun" activity as an add-on, but it still did not seem to work in transforming her classroom. She eventually realized that, based on how she did mathematics, "strategies" were critical. This attribute became a further understanding of play, i.e., play now involved fun and strategies. Strategies made sense to Elise as a basis for teaching high school mathematics and became the focus of her teaching. She later added "reflection on strategies" as an aspect of play in order to satisfy a new expectation for her students. Thus, it was not until after Elise expanded her interpretation of play that a different understanding of teaching high school mathematics was generated for her. The result was a shift in her teaching to a student-centered approach that focused on investigations of strategies and self-reflection.

Elise's metaphor, mathematics is play, is creating ongoing dilemmas for her because her understanding of it is biased to "pure mathematics". Recently, Elise discovered that her students were more motivated when she introduced a concept through applications. This has created a new conflict for her and she claims that once again, doing mathematics and teaching mathematics are moving apart, i.e., this focus on application is making her think of doing mathematics as being different from teaching mathematics and the two will never be the same. For her, play is not applications, therefore "real" mathematics is not applications. She continues to focus on strategies and wishes there was an emphasis on proofs instead of applications.

Mark

Mark has been a high school teacher for about 33 years, the latter half of which has been at a large private school with grades 1 to 12. For most of this time Mark believed that mathematics was "a series of skills and processes". About 7 years ago, Mark accepted a request to teach a Grade 4 mathematics class for a year to provide release time for the teacher while she was assigned to an administrative position. Discussions with friends who were "outstanding" elementary teachers influenced Mark to think of elementary mathematics as a concrete activity. Thus, as the year progressed, Mark skeptically tried using small-groups and concrete materials in the Grade 4 class. He was impressed with the nature of the students' learning and their newly found enthusiasm for mathematics. The following year he agreed to teach Grade 6 mathematics in addition to Grades 10 to 12. By now, mathematics was becoming something different for Mark but he was only able to articulate it in terms of the hands-on activities and group work based on the elementary classroom experience.

Mark started to experience a serious conflict between his teaching of elementary mathematics and high school mathematics in that he wanted to teach the latter like the former but did not know how. After two years of elementary teaching, Mark was able to articulate a different view of mathematics, i.e., “Mathematics is experience”, but he could not interpret this in the context of high school mathematics. For him, “experience” was still working with hands-on activities, which conflicted with his view of high school mathematics that was still “a series of skills and processes.” After being exposed to the NCTM standards (NCTM 1989), Mark realized that what was happening in his elementary classroom could be described in terms of “communication, reasoning, connection, and problem solving.” These four attributes became a further understanding of “experience” for him. However, his interpretations of these attributes were based on his elementary teaching and not a literal adoption of the NCTM standards’ description of them. For example, he understood communication as having students talk more than him and the talk must be based on past and current experiences.

Based on his interpretation, these four attributes made sense to Mark as a basis for teaching high school mathematics and became the focus of his teaching. Thus, it was only after Mark expanded his interpretation of experience that a different understanding of high school mathematics was generated for him. He was then able to make significant shifts to his teaching, engaging his students more meaningfully in learning mathematics.

Conclusion

For the teachers in the study, the belief, “mathematics is X”, behaved like a generative metaphor that facilitated the development of new perceptions, explanations, and inventions in their teaching. This resulted from a restructuring of their perception of X that allowed them to see high school mathematics in terms of X. The study brings to light the importance of generative metaphors that may underlie mathematics teachers’ personal story of growth and the possible significance of consciously attending to such metaphors to assist teachers in achieving desired changes and choices in their teaching.

References

Bogdon, R.J. (Ed.) (1986). *Belief: Form, content, and function.* Oxford: Clarendon Press.

Chapman, O. (in press) Mathematics teachers’ beliefs about problem solving and teaching problem solving. In J. Carrillo (Ed.) *Problem solving in the beginning of the 21st century: An international overview from multiple perspectives and educational levels*. Madrid, Spain: Herque.

Chapman, O. (1999). Researching mathematics teachers’ thinking. In O. Zaslavsky (Ed.) *Proceedings of the 23rd conference of the International Group for the Psychology of Mathematics Education (PME)*. Haifa, Israel: The Technion - Israel Institute of Technology, Vol. 2, 185-192.

Chapman, O. (1998). Metaphor as a tool in facilitating preservice teacher development in mathematical problem solving. In A. Oliver & K. Newstead (Eds.) *Proceedings of the*

22nd conference of the International Group for the Psychology of Mathematics Education (PME). Stellenbosch, SA: University of Stellenbosch, vol. 2, 176-183.

Chapman, O. (1997). Metaphors in the teaching of mathematical problem solving. *Educational Studies in Mathematics,* 32, 201-228.

Cooney, T.J., Shealy, B.A., & Arvold, B. (1998). Conceptualizing beliefs structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education,* 29(3), 306-333.

English, L. (Ed.) (1997). *Mathematical Reasoning: Analogies, metaphors, and images*. Mahwah, NJ: Erlbaum.

Lakoff, G. and Johnson, M. (1980). *Metaphors we live by*. Chicago: University of Chicago Press.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards*. Reston, VA: Author.

Pajares, M.F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research,* 62, 307-332.

Presmeg, N. (1998). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behavior*, 17 (1), 25-32.

Schon, D. (1979). Generative metaphor: A perspective on problem solving in social policy. In A. Ortony (Ed.) *Metaphor and thought.* New York: Cambridge University Press.

Examining what we belief about beliefs

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Research on the teaching and learning of mathematics has taken many twists and turns through the years. During the late 1960s and early 1970s the primary emphasis of research was on students' learning, particularly that of young children, and often evoked a strong Piagetian flavor. In the United States, for example, there was considerable research on Piaget's notion of conservation and how educators could influence children's understanding of conservation so as to "speed up" the learning process. In general, this line of research was not fruitful and was abandoned as a research program. Little emphasis was given to how teachers might influence learning through their instructional programs or how instructional programs were influenced by what teachers believed about mathematics or its teaching.

Most research on teaching mathematics that was conducted in the 1970s used a behavioristic framework to shape both questions and methodologies. In general, various characteristics of teaching were quantified and correlated with certain learning outcomes. Although this line of research led to a few interesting results in terms of revealing high inference variables that were significantly correlated to learning, e.g., clarity and flexibility, in general, this line of research was deemed too limited. There was the concern that the mathematics being assessed was too narrow and that the notions of clarity, flexibility, and the like defied definition in a mathematics classroom. Primarily, however, there was an emerging methodological crisis in the late 1970s that suggested researchers were not focusing on what was really significant about the teaching of mathematics. There was a brief foray into investigations about teachers' attitudes toward mathematics and teaching. This research was generally well received but it lacked a cognitive component and ultimately failed to be free of the shackles of the dominant behaviorist paradigm. Still, there was a sense that in order to understand the teaching and learning of mathematics one must contend with something other than well-defined variables and a tightly quantifiable notion of teaching behavior.

Perhaps spurred by Kuhn's (1962) classic work The Structure of Scientific Revolution, combined with the movement toward teaching experiments, borne out of Russian methodologies in mathematics education, sense-making became the epistemological orientation adopted by most researchers. Researchers exhibited a paradigm shift away from viewing teaching and learning from an ontological perspective and toward a constructivist orientation in which an individual's construction of meaning was paramount. Given the diverse forces that were influencing research in mathematics education, there evolved a sense that the setting in which the teacher operated influenced what was being taught and learned in the classroom. Setting became recognized not just as the physical arrangement of classrooms but of what was inside the teachers' head about mathematics and its teaching. In the early 1980s Stephen Brown and I conducted an NSF supported research project in which we began to study teachers' decision-making strategies. In particular, we were interested in what decisions teachers made and what influenced those decisions. Somewhere along the line, we became interested in the more general question of how teachers made sense of their professional life. To make a long story short, this led to the notion of beliefs and to how we could unpack this aspect of human knowing since it might influence life in the classroom. Our methodology involved various techniques drawn from anthropological research and led us down the path of what is now usually referred to as teachers' beliefs about mathematics and about the teaching and learning of mathematics. The two notable studies that were conducted within this framework were the stories about a beginning teacher's view of problem solving (Cooney, 1985), and about three middle grade teachers' conceptions of mathematics (Thompson, 1982;1984).

Over the past 15 years, there has been considerable research that has focused on some aspect of teachers' beliefs. I say "some aspect" because it is not always clear just what the author means by beliefs. An apparent synonym is conceptions or even attitude. Given that mathematics educators have a background in mathematics, there is a strong urge to define what we mean prior to engaging in our research involving a particular construct. In response to this inherent need, I'll try to make a distinction between belief and knowledge, with the realization that the attempt only partially addresses the need for explicit explication.

The notions of belief

Belief is usually seen as a construct that has a cognitive component but is a weaker condition than knowing. Scheffler (1965) claimed that X knows Q if and only if

i. X believes Q
ii. X has the right be to sure Q
iii. Q

Accordingly, X believing Q is a necessary condition for X knowing Q. Indeed, it would be strange to say that I know that wood floats but I don't believe it. On the other hand, it would not be strange to say that I believe wood is lighter than plastic, but I don't know it.

In order for X to have the right to be sure that Q is the case, there must be reasonable evidence to support the existence of Q, that is, criterion ii. The evidence may, of course, point us in the wrong direction but some evidence must exist. We can wrongly believe that the universe is geo-centered or that "bleeding" is an appropriate medical treatment. But to know that these propositions are true, evidence must exist—as historically we know was the case. The last criterion, the actual existence of Q, is a tricky criterion. How can we be sure that Q actually exists? Science has revealed that the universe is not geocentered and that bleeding is a dangerous medical treatment. But can we be so sure that what now appears to be the case is, in fact, the case? The famous book Flatland cautions us to not always think that the perspective we now hold with good evidence is reality. The very essence of constructivism is that we can never know reality as such but rather we construct models that have viability (Von Glasersfeld, 1991) for describing the world in which we exist. For the constructivist, we can never determine that condition iii holds. Rather, we can speak with confidence, but not certainty, that Q holds. This perspective suggests that criteria ii and iii ought to be somehow combined into one condition that might be stated as follows.

ii (revised). X has reasonable evidence to support Q

Even here, we are not out of deep water because we have the question about what constitutes evidence. What is evidence for some is not for others. Nevertheless, condition iiR avoids the trap of trying to determine what an individual knows or believes based on some perceived absolute reality.

The logic of the above explication suggests that belief is a necessary but not sufficient condition for knowing. What does it mean for me to say, "I believe life is good." or that "I believe the essense of mathematics is problem solving." or that "I believe it is cold outside. I think most of us would agree that replacing the word "believe" with "know" would result in a questionable proclamation. Further, it would be strange to act in a way that seemed counter to my stated belief. For example, if I really believed it was cold outside, it seems unlikely I would venture outside for an extended period of time wearing only shorts and a tee shirt. We might say, "He says he believes it is cold outside but I don't believe he means it." which suggests a certain insincerity or inconsistency on my part. Scheffler (1965) addressed this point when he wrote,

> A belief is a cluster of dispositions to do various things under various associated circumstances. The things done include responses and actions of many sorts and are not restricted to verbal affirmations. None of these dispositions is strictly necessary, or sufficient, for the belief in question; what is required is that a sufficient number of these clustered dispositions be present. Thus verbal dispositions, in particular, occupy no privileged position vis-à-vis belief (p. 85)

Based on this definition we encounter difficulties with our present day research on beliefs. According to Scheffler, the determination of one's beliefs requires a variety of types of evidence including not only what a person says but also what the person does. What do we conclude, for example, when a teacher steadfastly maintains that the essence of mathematics is problem solving, yet we see only procedural knowledge being emphasized in the classroom? Typically, the researcher claims that there exists an inconsistency between the teacher's belief and his/her practice. But other interpretations exist. Consider the following possibilities:

1. We do not have a viable interpretation of what the teacher means by problem-solving.
2. The teacher cannot act according to his/her belief because of practical or logistical circumstances.
3. The teacher holds the belief about problem solving subservient (or peripheral in Green's (1971) terms), to the belief that the teaching of mathematics is about certainty and procedural knowledge.

Acceptance of any one of these three alternative interpretations would not lead us to a conclusion about inconsistency but rather would require us to understand in a deeper sense how the teacher constructs meaning.

Beliefs and the relevance of ways of knowing

When the emphasis of our research shifts towards a sense-making perspective, boundary lines become blurred as we seek to understand what drives an individual—be it rationality or irrationality. At least one of the goals of the research is to develop viable schemes for describing teachers' beliefs. This is not to dismiss research that quantifies and correlates beliefs with other behavior. Indeed, this research can provide significant insights into what teachers value and the relative importance they assign to different aspects of mathematics or the teaching of mathematics. Neither is it to excuse sloppiness in the face of the challenge of digging deeply. But it is to emphasize the necessity of telling stories about teachers' lives in the classroom and what shapes those lives. Good stories are not simply descriptions but are grounded in theoretical constructs that have the power to explain. I will now offer a glimpse as to the kind of theoretical orientation I consider important and helpful in the telling of those stories.

Given the uncertainty that exists between criteria ii and iii, we are always faced with the condition that knowing is a relative condition necessarily dependent on the context in which one lives. This is not to say that such words as true and false have no meaning but rather that these terms, when applied to the human condition, must be tempered with the realization that we all see the world differently. Because of this, knowing is a relativistic construct. Accordingly, the professional development of teachers can be seen as progressing from seeing the doing of mathematics and the teaching of mathematics in dualistic terms to seeing these activities in relativistic terms. I contend that this perspective can lead us to conceptualize how teachers come to know and the relative flexibility of their knowledge. Specifically, a dualistic orientation toward mathematics leads to an emphasis on product, such as the acquisition of procedures, without accompanying meaning. Similarly, a dualistic orientation toward teaching mathematics leads to an instructional style determined by telling, certainty, and a priori conceived teaching strategies. In short, preconceived instructional styles are necessarily insensitive to the contexts in which they are used —contexts that are determined, in part, by what students know and believe about mathematics. In contrast, a relativistic view of mathematics has an emphasis on process and leads to a dynamic view of mathematics. A relativistic view of mathematics teaching would be based on the context of teaching, the most important of which is student understanding. The question becomes less of whether an activity or teaching strategy is good or bad per se and more a question of what context an activity becomes effective for student learning.

This line of reasoning has led me and many of my students to use those theoretical perspectives that attend to ways of knowing, i.e., that consider an individual's personal journey from a dualistic orientation to a relativistic one. This includes schemes developed by Perry (1970), Belenky, Clinchy, Goldberger, and Tarule (1986), and Baxter-Magolda (1992). Each of these schemes provides a basis for conceptualizing one's beliefs and how those beliefs are related to the way the individual comes to know. Cooney, Shealy, and Arvold (1998) have posited a similar scheme that is specific to mathematics in which ways of teachers' knowing are characterized. Consistent with these schemes and also used in our research are the perspectives offered by Green (1971), Schön (1983), and King and Kitchener (1994). In sum, these theoretical orientations offer a means for describing and shaping our ways of thinking about teacher development. If we define reform teaching as that teaching which attends to context, including basing instruction on students' knowledge, then teaching becomes a matter of being adaptive (Cooney, 1994) rather than a matter of using a particular sequence of instructional strategies.

The development of a reform-oriented teacher, so defined, is rooted in the ability of the individual to doubt, to reflect, and to reconstruct. Teacher education then becomes a matter of focusing on reflection and on the inculcation of doubt in order to promote attention to context. This opens new vistas for creating situations in teacher education in which teachers can develop a reflective posture toward their teaching. This requires considerable maturity with respect to the knowledge domains of mathematics, pedagogy of mathematics, and student learning (Lappan-Theule-Lubienski, 1994). A limited view of mathematics, preconceived teaching strategies, and a reductionist view of learning clearly work against the development of the reflective teacher. Consequently, our teacher education programs should model the kind of knowledge development that we expect our teachers to exhibit for their students. For otherwise, we will find ourselves mired in a significant moral dilemma as our medium and our message are inconsistent if not incoherent.

A final thought

Given the nature of our research, concern over a precise definition of belief pales in comparison to understanding the nature of teachers' professional development. It might be argued that a more precise definition could better shape our research and provide a framework in which our research would advance more rapidly. But I question the value of this perspective. It seems to me that the human condition is always beset with a strange mixture of rationality and irrationality that defy sharp lines of demarcation. There is much to be appreciated about the artistry of teaching and what contributes to that artistry. Certainly, quantification with well-defined constructs can enable us to better understand the teaching and learning of mathematics. (See, for example, Sfard, 1999). But to the extent that our research is about individual's sense-making activities, we must recognize that the individual is driven by many considerations, some of which are amenable to a form of Aristotelian logic and others that are not. I suggest that neither situation, logical or not, should dissuade us from a certain empiricism that demands consistency and the kind of analysis that reveals insights about how we come to know. I see this task as the challenge research on teachers' beliefs presents to us as we seek to construct a scientific basis for both our practice and our future research.

References

Baxter-Magolda, M. (1992). *Knowing and reasoning in college*. San Francisco: Jossey-Bass, Inc.

Belenky, M., Clinchy, B., Goldberger, N., & Tarule, J. (1986). *Womens' way of knowing: The development of self, voice, and mind*. New York: Basic Books.

Cooney, T. (1985). A beginning teacher's view of problem solving. Journal for Research in Mathematics Education. 16, 324-336.

Cooney, T. (1994). Teacher education as an exercise in adaptation. In D. Aichele & A. Coxford (Eds.) Professional development for teachers of mathematics. 1994 Yearbook of the National Council of Teachers of Mathematics. (pp. 9-22). Reston, VA: National Council of Teachers of Mathematics.

Cooney, T., Shealy, B., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education*, 29, 306-333.

Green, T. (1971). *The activities of teaching*. New York: McGraw-Hill.

King, P. & Kitchener, K. (1994). *Developing reflective judgment*. San Francisco: Jossey Bass, Inc.

Kuhn, T. (1962). *The structure of scientific revolution*. Chicago: University of Chicago Press.

Lappan, G. & Theule-Lubienski, S. (1994). *Training teachers or educating professionals? What are the issues and how are they being resolved?* In D. Robitaille, D. Wheeler, & C. Kieran (Eds.) *Selected lectures from the 7th International Congress on Mathematical Education*. (pp. 249-261). Les Presses de L'Université: Quebec.

Perry, W. (1970). *Forms of intellectual and ethical development in the college years*. New York: Holt, Rinehart, & Winston.

Scheffler, I. (1965). *Conditions of knowledge*. Chicago: Scott Foresman and Company.

Schön, D. (1983). *The reflective practitioner: How professionals think in action*. New York: Basic Books, Inc.

Sfard, A. (1999). "On the needs and obligations of the "Guide-on-the-side": rethinking the place of the teacher in the reformed mathematics classroom." Paper presented at the International Conference on Mathematics Teacher Education, Taipei, Taiwan.

Thompson, A. (1982). *Teachers' conceptions of mathematics*. Unpublished doctoral dissertation. University of Georgia, Athens.

Thompson, A. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. *Educational Studies in Mathematics*. 15, 105-127.

A virtual panel evaluating characterizations of beliefs

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Introduction

The importance of beliefs/conceptions in the study of cognitive and metacognitive phenomena, as well as teachers' behavior and attitudes in the classroom is widely recognised. The following passage taken from (Schoenfeld, 1992) expresses very significantly this importance with respect to the study of students' performances:

> "purely cognitive" behavior is extremely rare, and that what is often taken for pure cognition is actually shaped - if not distorted - by a variety of factors. [...] The thesis advanced here is that the cognitive behaviors we customarily study in experimental fashion take place within, and are shaped by, a broad social-cognitive and metacognitive matrix. That is, the tangible cognitive actions produced by our experimental subjects are often the result of consciously or unconsciously held beliefs about (a) the task at hand, (b) the social environment within which the task takes place, and (c) the individual problem solver's perception of self and his or her relation to the task and the environment.

As for teachers, many studies have shown that beliefs/conceptions are behind teachers' behavior in their classroom and act as a filter to indications of curriculum developers.

In order to make as clear as possible the theories developed about beliefs we have identified the following elements of reflection:

- terminology: some points need of a careful discussion in order to agree on terminology and description of terms. Even the words 'belief' and 'conception' have different meanings for different researchers.

- methodology: How do we detect and analyse beliefs, which in their nature are entities hidden and elusive?

- effect: How may studies on beliefs/conceptions affect the classroom practice and strategies for teacher education?

The present study concerns the first element, that is to say we will try to investigate on what authors intend when mention beliefs, conception. We will see that this implies also to take into consideration the concept of 'knowledge', which has strong historical and epistemological links with the terms in question. We are aware that our way of approaching the subject is closer to the way used by mathematicians than the one used by psychologists. The firsts develop their theories starting from well stated axioms and definitions, while the others often keep their assumptions and terms quite fuzzy. Our choice originated from the difficulty to deal with different studies in which it may happen that authors use different terms to express same things or same terms to express different things.

To give a first idea of the problem we quote some position found in the domain of mathematics education. Thompson (1992, p.130) claims that «the distinction [between beliefs and conceptions] may not be a terribly important one». Other authors share, more or less explicitly, this opinion. But there are other researchers who clearly distinguish the meanings of the two terms. This is, for example, the position emerging from the following passage of Ponte (1994):

> 'knowledge refers to a wide network of concepts, images, and intelligent abilities possessed by human beings. Beliefs and conceptions are regarded as part of knowledge. Beliefs are the incontrovertible personal "truths" held by everyone, deriving from experience or from fantasy, with a strong affective and evaluative component (Pajares, 1992). They state that something is either true or false, thus having a prepositional nature. Conceptions are cognitive constructs that may be viewed as the underlying organizing frames of concepts. They are essentially metaphorical' (p.169).

In (Sfard, 1991) conceptions are described as follows:

> 'the whole cluster of internal representations and associations evoked by the concept - the concept's counterpart in the internal, subjective "universe of human knowing" - will be referred to as "conception"' (p.3).

Conceptions may be considered the personal/private side of the term 'concept' defined by Sfard as follows:

> 'The word "concept" (sometimes replaced by "notion") will be mentioned whenever a mathematical idea is concerned in its "official" form as a theoretical construct within "the formal universe of ideal knowledge"' (ibid.).

This position resounds the famous theory about images - concept images and concept (formal) definitions - of Tall and Vinner (1981). These authors, as well as Sfard, do not mention beliefs.

Thompson (1984 and 1992), as other researchers do, uses the term 'conception' not referring to a single mathematical idea, but to the whole mathematics. For her the 'conception' of the nature of mathematics

> 'may be viewed as that teacher's conscious and subconscious beliefs, concepts, meanings, rules, mental images, and preference concerning the discipline of mathematics. Those beliefs, concepts, views, and preferences constitute the rudiments of a philosophy of mathematics, although for some teachers they may not be developed and articulated into a coherent philosophy' (Thompson 1992, p.132).

We see that the use of the terms 'belief' and 'conception' and the mutual relationship between the underlying related concepts is fuzzy; even more fuzzy is the relation between beliefs/conceptions and knowledge. Briefly we can say that when we consider knowledge we pass from the simple personal/affective dimension to the social dimension. Knowledge is something that is subject to the evaluation of a community (school, mathematicians, ...). Thompson, Nespor and Abelson distinguish knowledge from belief systems on the basis of the possibility of objective (= outside the individual) evaluation of validity. This distinction is depending on the interpretation of knowledge they give. According to Thompson (1992):

> ‘From a traditional epistemological perspective, a characteristic of knowledge is general agreement about procedures for evaluating and judging its validity; knowledge must meet criteria involving canons of evidence. Beliefs, on the other hand, are often held or justified for reasons that do not meet those criteria, and, thus, are characterized by a lack of agreement over how they are to be evaluated or judged’ (p.130).

In (Philippou & Christou, 1998) beliefs may be defined as one’s amalgamated mixture of subjective knowledge and feelings about a certain object or person. Beliefs are seen as distinct from knowledge; the latter must involve a certain degree of objectivity and validation vis-à-vis reality.

Such views does not fit what happens in school practice. As Confrey (1990, p.111) claims:

> ‘in most formal knowledge, students distinguish between believing and knowing. To them there is no contradiction in saying, “I know that such and such is considered to be true, but I do not believe it.” To a constructivist, knowledge without beliefs is contradictory’.

This coexistence of knowledge and beliefs, which may be mutually in contrast, is evidenced by the schizophrenia observed in the perception of what students learn in school: proof which are carried out, but do not convince, symbols which are manipulated, but have not meaning.

Also philosophy is interested in the subject. In the dialogue *Theaetetus* Plato showed that a useful definition of knowledge is elusive, see (Lindgren, 1999; Rodd, 1997). In this concern Rodd (1997, p.65) quotes the following passage from *Theaetetus*:

> *Socrates* And it is utterly silly, when we are looking for a definition of knowledge, to say it is right opinion with knowledge, whether of difference or of anything else whatsoever. So neither perception, Theaetetus, nor true opinion, nor reason or explanation combined with true opinion could be knowledge.

Artigue (1990) outlines links between conceptions and epistemology. From the survey this author gives of the French research in mathematics education it emerges that the word ‘belief’ does not appear in the French literature (in mathematics education), while many authors (among them Brousseau, Chevallard, Vergnaud) deal with the term ‘conception’. According to this author the notion of conception concerns two distinct requirements (p.265, our translation):

> - to emphasize the multiplicity of possible points of view on the same mathematical objects, to make different the representations and ways of treatment that one associates to it, to emphasize their (more or less good) adaptation to the solution of a given class of problems,
>
> - to help researchers in mathematics education to fight again the illusion of making transparent the didactic communication provoked by the empiricist models of learning, by allowing to distinguish knowledge that the teacher wishes to transmit and knowledge actually built by the student.

Our research

On the ground of the previous considerations we admit that the different assumptions of researchers make it impossible to reach a complete agreement on the mutual relationships in the *triad 'beliefs - conceptions - knowledge'*. Nevertheless it is our opinion that researchers have to make clear their position on this point in order to improve the communication in the scientific community and to make their studies actually understandable.

On the ground of this assumption we have worked out a questionnaire based on nine characterisations of the terms of the triad present in literature. The authors of the characterisation that we put in the questionnaire were not indicated. We invited a group of experts in the field to express their opinions on these characterisations, based on the following points:

- if they agree with the given characterisation
- possible improvements
- reasons for disagreement
- personal characterisation.

In the following we report the nine characterizations given in the questionnaire sent to the researchers. Each characterization was accompanied by the sentences

- 'Do you consider the characterization to be a proper one?
- Please, give the reasons for your decision!'.

<u>Characterization #1:</u>

"we use the word belief to reflect certain types of judgments about a set of objects"

<u>Characterization #2:</u>

"beliefs constitute the individual's subjective knowledge about self, mathematics, problem solving, and the topics dealt with in problem statements"

<u>Characterization #3:</u>

"we use the word conceptions to refer to a person's general mental structures that encompass knowledge, beliefs, understandings, preferences, and views"

<u>Characterization #5:</u>

"Belief systems often include affective feelings and evaluations, vivid memories of personal experiences, and assumptions about the existence of entities and alternative worlds, all of which are simply not open to outside evaluation or critical examination in the same sense that the components of knowledge systems are"

Characterization #5:

"Beliefs and conceptions are regarded as part of knowledge. Beliefs are the incontrovertible personal 'truths' held by everyone, deriving from experience or from fantasy, with a strong affective and evaluative component."

Characterization #6:

"we understand beliefs as one's stable subjective knowledge (which also includes his feelings) of a certain object or concern to which tenable grounds may not always be found in objective considerations"

Characterization #7:

"beliefs - to be interpreted as an individual understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior"

Characterization #8:

"A teacher's conceptions of the nature of mathematics may be viewed as that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics."

Characterization #9:

"Attitude is a stable, long-lasting, learned predisposition to respond to certain things in a certain way. The concept has a cognitive (belief) aspect, an affective (feeling) aspect, and a conative (action) aspect."

Your characterization:

Please, write your own characterization for the concept of 'belief'?

The author of the statements are respectively: Hart, Lester, McLeod, Nespor, Pajares, Pehkonen, Schoenfeld, Thompson, Törner & Grigutsch.

Results

In Table 1 we report a summary of results. The answers are organized in a five-step scale: Y ('fully agreement'), P+ ('partial agreement with a positive orientation'), P ('partial agreement'), P- ('partial agreement with a negative orientation'), N ('fully disagreement'). * means that there is answer, - means that there is not answer.

	1	2	3	4	5	6	7	8	9	10
1 Chapman	Y	Y	P	N	N	N	Y	Y	Y	-
2 Cooney	P+	P	P	P-	N	N	Y	Y	Y	-
3 Ernest	P	P	Y	P	N	N	P+	P+	P+	-
4 Grigutsch	P+	P	P	Y	Y	Y	Y	P	P+	
5 Hersh	Y	Y	N	N	N	Y	Y	Y	Y	-
6 Kloosterman	P	N	N	P	N	N	P	Y	P	*
7 Leder	N	Y	Y	Y	Y	N	Y	Y	P	-
8 Lerman	N	P+	P+	N	P+	P+	N	Y	Y	*
9 McLeod	P+	Y	P+	P	N	Y	Y	Y	Y	*
10 Philippou	Y	Y	Y	N	N	Y	Y	Y	P+	-
11 Presmeg	Y	Y	Y	N	N	Y	Y	N	N	*
12 Schoenfeld	N	P	Y	P-	N	P	P	Y	P-	-
13 Stacey	P-	P	P+	N	N	Y	P	P	Y	*
14 Tirosh	Y	N	Y	Y	N	N	N	N	P-	*
15 Törner	P+	P	P	P	N	Y	Y	Y	P+	-
16 Vinner	Y	N	Y	Y	N	N	N	N	N	*
17 Wilson	Y	Y	Y	N	N	N	Y	Y	Y	*
18 Yackel	N	P	Y	N	N	N	Y	N	N	*
Y = YES	7	7	17	4	2	6	11	12	8	
P+ = PARTLY YES	4	1	3	-	1	1	1	-	4	
P= PARTLY	2	7	4	4	-	2	3	2	2	
P- = PARTLY NO	1	-	-	2	-	-	-	-	4	
N = NO	4	3	4	8	15	9	3	4	4	

References

Artigue, M.: 1990, 'Épistémologie et didactique', *Recherches en didactique des mathématiques*, v.10, 241-286.

Lindgren, S.: 1999, 'Plato. knowledge and mathematics education', in G. Philippou (editor), *MAVI-8 Proceedings* (Nicosia, Cyprus), 73-78.

Pajares, M. F.: 1992, 'Teachers' beliefs and educational research: cleaning up a messy construct', *Review of educational research*, v.62, 307-332.

Pehkonen, E.: 1988, 'Conceptions and images of mathematics professors on teaching mathematics in school', *International journal of mathematical education in science and technology*, v.30, 389-397.

Philippou, G. N. & Christou, C.: 1998, 'The effect of a preparatory mathematics program in changing prospective teachers' attitudes towards mathematics', *Educational studies in mathematics*, v.35, 189-206.

Ponte, J. P.: 1994, 'Knowledge, beliefs, and conceptions in mathematics teaching and learning', in L. Bazzini (editor), *Proceedings of the Fifth International Conference on Systematic Cooperation between Theory and Practice in Mathematics Education*, 169-177.

Rodd, M. M.: 1997, 'Beliefs and their warrants in mathematics learning', in E. Pehkonen (editor), *Proceedings of PME 21*, v. 4, 64-71.

Schoenfeld, A. H.: 1983, 'Beyond the purely cognitive: beliefs systems, social cognitions, and metacognitions as driving forces in intellectual performance', *Cognitive Science*, v.7, n.4, 329-363.

Sfard, A.: 1991, 'On the dual nature of the mathematical objects', *Educational studies in mathematics*, v.22, 1-36.

Tall, D. & Vinner, S.: 1981, 'Concept image and concept definition with particular reference to limits and continuity', *Educational studies in mathematics*, v.12, 151-169.

Thompson Gonzalez, A.: 1984, 'Teachers' conception of mathematics and mathematics teaching on instructional practice', *Educational studies in mathematics*, v.15, 105-127.

Thompson, A.: 1992, 'Teachers' beliefs and conceptions: A synthesis of the research', in A. D. Grows (editor), *Handbook of research on mathematics learning and teaching*, Macmillan, New York, 127-146.

Conflicts between mathematics graduates' proof behaviours and their stated beliefs about proof

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University mathematics lecturers tend to assume that if they only present mathematics "as it is", then undergraduates who make an effort to understand will automatically converge toward perceptions of mathematical objects and mathematical procedures which reflect those accepted by mathematicians.

Even if the truth is much more complex, there are good reasons to defend this approach as the only available option - given the constraints on the lecturer. However, it is only effective provided the students start out in a position from which it is possible to be pulled in the desired direction. Two students may begin in a near-by positions which are outwardly indistinguishable, and yet respond differently to the same experiences: one may be pulled into the orbit of mathematics, while the other lands up in some "parallel universe". Hence, either (a) one needs a school system which reliably produces 18 year olds who are in an initial position which will make it possible for them to be pulled into the orbit of mathematics, or (b) one needs to take corrective action early on in an undergraduate course.

The analysis summarised above is not generally recognised - so no such early corrective action is taken. What is recognised is that undergraduate mathematics cohorts in the UK are becoming increasingly "bimodal" - as judged by university assessment. The left hand hump is generally assumed to consist of those who "find mathematics hard", or who do not "have what it takes". I will present some evidence to support the conclusion that many of those in the left hand hump are in fact operating in a "parallel universe", struggling to learn and to reproduce the outword trappings of mathematics, while basing their learning strategies on methods which are inimical to mathematics.

The evidence I will present comes from an optional module in the final semester of an undergraduate honours mathematics program. The module is called "Mathematics as a human endeavour" and attracts 30-40 students of all kinds. Most of those who think it looks like an easy option disappear within the first week, leaving a group of students who are at least willing to persist with a module designed to make them think. The best students in a year group are often advised to take more technical coourses; nevertheless the class regularly contains a true spread - including some of the best students. There are also some genuine strugglers (with nowhere else to go!). The purpose of the course is to encourage students to reflect (before they finally graduate!) on aspects of mathematics which are usually overlooked - or even suppressed. The course has two parts - for each of which students have to submit an extended project.

The first part is intended to convey what it means to "do" mathematics. This involves getting students to explore, to develop, to conjecture and to prove results as part of an extended study. Shorter tasks are explored in class and as homework, and their experiences are discussed in class to bring out important features before they themselves tackle an extended problem (e.g. "Which Fibonacci numbers are divisible by n?") - where there is plenty of scope for them to make partial progress and to prove significant partial results, while experiencing something of the "never-ending" character of true research. (As far as I know the Fibonacci example does not yet have a definitive answer.)

The aim is to force students to distinguish clearly between the subjective, but psychologically important, exploratory phase – which they are supposed to record in pencil on the left hand two thirds of the page - and the precise mathematical formulation of conjectures, simple definitions, remarks, lemmas, theorems, and proofs - which are recorded in ink on the right hand two thirds of the page. This provides experience on an elementary level (though with problems that are by no means easy) of what is, and what is not a "proof".

The second part of the course - which begins while they are still working on this extended study - explores three themes which allow us to examine how three very different ideas permeate mathematics across cultures, and across time. It is a study of "mathematical culture", not of history. The three current themes are

(i) Infinity, (ii) Proof, (iii) the solution of equations.

Students are then required to choose a simple theme which interests them and which reflects some aspect of "mathematics as a human endeavour", and to write a 10 page essay - showing that they have made a serious effort to come to grips with some aspect of their chosen topic.

The setting is one which deliberately encourages reflection; so it provides considerable scope for "seeing" unsuspected student perceptions and beliefs - beliefs which are perhaps always present but which are usually kept out of sight.

In tackling the short and extended problems in the first section of the course, one sees how profoundly UK students have failed to distinguish between

* subjective "exploratory" work and
* strict mathematics.

When we come to the section on "Proof", this point is reinforced. Considerable time is spent

(a) discussing why proof is important,

(b) identifying the key ingredients that give proof in mathematics ist special position (as compared with common sense and the scientific method),

(c) examining the development of proof through time, and

(d) analysing flawed "proofs" from elementary mathematics.

This gives plenty of opportunity to see how students perform at the most elementary technical level - where error has little to do with the confusion created by unfamiliar material.

In order to look more closely at the observations I have routinely made as in previous years, for the last two years the opening hour of the section on proof has been spent asking students to complete a questionnaire with 25 statements. For each statement they are asked to circle A (agree), S (have some sympathy with) or D (disagree). Students are also encouraged to add their own remarks in the space provided. (The design of the questionnaire and the analysis of students' responses has been done in collaboration with Dr Candida Moreira, who has also administered the same questions and task to her own undergraduates in Porto.)

On the questionnaire, the statements are mixed up rather than grouped, and appear in both positive and negative forms (sometimes on closely related themes). In contrast the statements below have been grouped to provide some structure.

We begin with the statements relating to the nature of mathematics and of mathematical activity:

1. A set of axioms for an area of mathematics should consist of statements which are completely clear and self-evident.
2. Mathematicians discover theorems by "step-by-step deduction", starting from a set of axioms.
3. Mathematical entities appear to exist independently of human beings, in some kind of realm which we can observe and talk about.
4. Mathematical activity is in some ways just playing with symbols and manipulating symbolic expressions according to certain rules.

The next group contains statements relating to the mechanics of proof.

5. The existence of a single counterexample is enough to show that a proposition is false.
6. The kind of reasoning used in a mathematical proof is similar to that used when discussing things carefully in everyday conversation.
7. The fact that we have proved a statement deductively does not guarantee that there might not be "counterexamples".
8. To be reliable, a proof in mathematics must stick to the language and rules of inference which are accepted by the mathematical community.
9. On the basis of experimental evidence (for example, (a) from measuring, or (b) from analysing particular instances of a problem and noticing a pattern) one can sometimes deduce general statements that apply to all relevant cases.

Next the statements on the role of proof.

10. If a statement is always true, there is no real need to prove it.

11. In mathematics one can convince oneself (without making a mistake) that a theorem is true even though one does not know a proof.
12. The main role of "proof" in mathematics is to "convince" (oneself or others) of the truth of what is proved.
13. The main reason for insisting on proofs is to show which mathematical truths depend on which.
14. It is sometimes permissible in mathematics to use a proposition which has not yet been proved in order to prove another theorem.
15. The experience of having to write deductive proofs in mathematics helps one to handle situations in everyday life which involve "reasoning".
16. Giving a proof by contradicition doesn't really explain why the result is true.
17. I expect a mathematical proof to convey some insight as to why the proposition is true.
18. To prove propositions which are self-evident is a mistake.

Finally a mixed bag - beginning with four statements which could be labelled "Myself and proof".

19. Deductive proofs do not often convince me, because they use lots of propositions which I am not sure about.
20. I often feel overwhelmed when a mathematical proof involves definitions and notation which I don't really understand.
21. Many theorems are proved in such a strange way that I have no idea how they could ever have been discovered.
22. I still have misgivings about "proof by induction".
23. I have got used to the fact that what counts as a "proof" depends on what lecture course we are in (or on who is giving the lecture course).
24. My idea of what constitutes a proof changed more during the first year than in the second or third years.
25. I don't see why so much time is spent in mathematics lectures going through proofs of theorems.

The most striking feature of this exercise is the extent to which students appeared to welcome the chance to express themselves on such topics. They make serious efforts to express themselves, and write much more extensively than one might expect UK mathematics students to do (even if what they write is sometimes incoherent).

Though the views expressed were in some cases surprising and even apparently self-contradictory, the majority expressed views which would have gladdened the heart of most mathematics lecturers. If one were to judge on the basis of what the students *wrote*, then one would conclude that they accepted the central role of proof, and recognised its importance for anyone wishing to study mathematics. At the very least one would conclude that the experience of being exposed to endless proofs in undergraduate courses is less negative than is often claimed.

A more careful scrutiny of ther responses reveals certain limitations of the views expressed (such as the fact that almost no students had grasped the finer points - such as the intended significance of statement 13). But the overall impression was surprisingly positive.

This generally positive statement of beliefs (explicit examples will be presented in the talk) took on a very different complexion, however, when one looked at students abilities to recall even the simplest proofs, or to identify the flaws in a patently false proof. When invited to *act*, their responses indicated that the *words* used on the questionnaire should be treated with extreme caution: it was as though they had learned some tribal mantra (in order to "belong") without knowing - or even desiring to know - what purpose it served in practice.

Towards the end of the section on proof students were given a sheet which invited them to prove the Euclidean result that the angles in a triangle sum to two right angles.

Only 20% or so of the class (in the final semester of a 3 year honours mathematics program) managed to produce something approximating a proof; and only half of these declared the relevant construction explicitly. A further 20% or so produced proofs which assumed results equivalent to what was to be proved. Most of the responses were presented in a way which raised serious questions about whether the students understood what was meant by a "proof".

They were then given a second sheet on which three different "proofs" of this result were presented:

(i) the classical Euclidean proof;

(ii) the English primary school approach in which the corners of a paper triangle are torn off and fitted along a ruler; and

(c) a flawed proof (in which a point P is marked inside triangle ABC and joined to all three vertices - the sum of angles in a triangle is assumed to be constant and this constant is then evaluated by adding up the angles in all three triangles).

Not one of the students gave a satisfactory criticism of the second "proof" (namely that it related to *paper* triangles rather than to mathematical triangles). 40% objected on the grounds of "lack of generality" - in that only one triangle had been checked: none of these students mentioned the problem of accuracy. A similar number objected on the grounds of inaccuracy: none of these students mentioned the lack of generality. 12% made inscrutable comments, while 5% made no comment at all.

Around 33% of students identified the assumption of constancy in the third proof - though many of them asserted that the proof assumed that the poroof assumed the result which was to be proved. A further 33% expressed some misgivings with the third proof, but they were mostly concerned with the assumption that the angles at the point P added to four right angles! Around 35% of the students declared the third proof to be totally rigorous.

These results were reflected elsewhere. On the final examination similar questions were asked which indicated that some of the simplest lessons had been learned; but the responses were still very weak (especially in identifying the flaws in erroneous "proofs"of elementary results.

The point I would wish to emphasise here is that, although declared "beliefs" are of some interest, if we are to discuss something more substantial than mere words, "beliefs" have to be assessed also through actions and competences. Two students may have stated "beliefs" which are outwardly indistinguishable, yet which in practice have quite different meanings, and in time have totally different effects.

I will present a selective analysis of student responses to the questionnaire - including some statisitics and some suggestive individual responses, and draw some tentative conclusions.

Affect, meta-affect, and mathematical belief structures[2]

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Summary notes

This will be a theoretical talk, exploring some perspectives on beliefs related to mathematics that come from an analysis of the affective domain. Special attention is given to the representational function of affect, and to the interplay between meta-affect and beliefs in sustaining each other.

1. Some perspectives on affect

A. Affect as a *system of representation,* encoding information about the external physical and social environment, mathematics, cognitive and affective configurations of the individual, cognitive and affective configurations of others.

B. Affect as an extraordinarily powerful, essentially human, evolutionary *language for communication;* interaction with the affect of others; shared affect across groups of people.

C. Affect *intertwined with cognition;* affective configurations can stand for, evoke, enhance or subdue, and otherwise interact with cognitive configurations in highly context-dependent ways (e.g., affect and metaphor); systems of cognitive representation (verbal/syntactic, imagistic, formal notational, strategic/ heuristic)

D. *Domains* of affective representation [McLeod, DeBellis and Goldin]: (1) emotions (rapidly changing, mild to very intense feeling, local, embedded), (2) attitudes (moderately stable, balance of affect and cognition), (3) beliefs (stable, highly cognitive, may be highly structured), (4) values (deep personal "truths"), ethics, morals (stable, highly affective, may be highly structured)

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[2] Parts of this talk are based on joint work with Valerie A. DeBellis.

E. Affective *competencies and structures;* local and global affect, affective pathways and networks, with accompanying meanings; defense mechanisms; change in global affect (e.g., anger --> forgiveness); examples (mathematical intimacy, mathematical integrity)

F. *Meta-affect:* affect about affect, affect about and within cognition (which may again be about affect), monitoring of affect, and affect as monitoring

2. Affect and meta-affect

A. An example

One first thinks of fear as a negative emotional state, to be avoided or soothed. A young child may be terrified of the dark, or of being alone. An adolescent may experience fear of rejection or of failure.

But a moment's reflection reminds us that in some circumstances, many people find fear highly pleasurable. People enjoy amusement park rides, where the more terrifying the roller coaster experience, the more exhilarating and "fun" it is. Why is this? The cognition that the person is "safe" on the roller coaster permits the fear to occur in a meta-affective context of excitement and joy. The person may also feel a satisfying sense of her own bravery, of having conquered fear, and the anticipatory joy (Vorfreude) of stepping onto the solid earth again.

Suppose, however, that a cable breaks during such a ride, and the roller coaster begins to swerve uncontrollably. The affect changes entirely. The fear feels entirely different — because the meta-affect has changed. Even if the person is really in no danger, the removal of the belief that she is safe changes everything. The terrifying ride is no longer fun; it is horrible.

B. What makes the different meta-affect possible? Not just cognition, but beliefs and values

Note how the "cognitive" belief, that the ride is in fact safe, may be essential to the joyful meta-affect. Yet cognition alone does not suffice. Adults may have "panic attacks" in a variety of circumstances — a fear of crowds, of flying, or of public speaking — when they "know they are really safe" but do not "believe" it.

Values (tacit or overt) also play a role — values of life and safety, of approval by peers or authority, of personality traits such as bravery.

C. Mathematical examples

Many people experience fear of mathematics — or they may fear a particular topic in mathematics, such as fractions or algebra. One student may experience fear immediately on being given a mathematical problem to solve; another upon realizing that he doesn't know how to proceed with the problem. Some may be afraid of the test, or of the teacher, or of the computer. Even advanced graduate students may fear exposure of a (self-perceived) mathematical inadequacy. The meta-affect here is not usually joyful.

Let us consider examples of less extreme feelings. Some students may experience frustration during mathematical learning or problem solving. The sense is unpleasant, and the meta-affective context is one of anxiety or fear. The frustration signals anticipation of failure, with attendant negative emotions.

Another student, solving the same problem, may also experience frustration — but the meta-affect is joyful. The student anticipates success, or at least a learning experience. The frustration signals that the problem is nontrivial, deep, or interesting, and heightens the anticipation of joy in success. The "cognitive" belief in his or her likelihood of success, together with some other beliefs about mathematics yielding to insightful processes, may contribute to positive meta-affect.

D. Powerful affective representation inheres not so much in the affect, as in the meta-affect

E. Affect stabilizes beliefs, and beliefs establish meta-affective contexts

Stable beliefs are comfortable (which is not the same as saying they are pleasant). They reinforce defenses against pain and hurt.

3. Some terminology relating to beliefs

A. Essential differences among (1) working assumptions or conjectures, (2) weakly- or strongly-held beliefs, (3) warrants for beliefs, (4) psychological functions of beliefs, (5) individual and shared beliefs, (6) knowledge (beliefs that in some sense apart from the fact of belief, are true), and (7) individual and shared values

B. Notions of *truth* in various contexts, pertaining to each of these.

C. In some contexts—e.g., the physical world, or mathematical problems—beliefs do not affect truth. In others—e.g., an estimation of the individual's own mathematical ability—they may have a partial influence. In still others—e.g., personal values—the belief creates its own truth (self-referential).

4. Some types of mathematical beliefs

This preliminary typology is included to lend specificity to the general points. Belief *structures* often cut across several of the categories.

A. Beliefs and conceptions about the physical world, and about the correspondence of mathematics to the physical world (e.g., number, measurement)

B. Specific beliefs about mathematical facts, rules, equations, theorems, etc. (e.g., law of exponents)

C. Beliefs about how mathematical truths are established (i.e., about mathematical validity)

D. Beliefs about effective mathematical reasoning methods and strategies, heuristics

E. Beliefs about the metaphysics or philosophy of mathematics

F. Beliefs about mathematics as a social phenomenon

G. Beliefs about aesthetics, beauty, meaningfulness, or power in mathematics

H. Beliefs about individual people who do mathematics, their traits and characteristics

I. Beliefs about the learning of mathematics, the teaching of mathematics, and the psychology of doing mathematics

J. Beliefs about oneself in relation to mathematics, including one's ability, emotions, history, integrity, motivations, self-concept, stature in the eyes of others, etc.

5. Belief structures, truth, warrants for belief, and meta-affect

A. The role of beliefs in establishing and sustaining meta-affect

B. The role of affect and meta-affect in establishing and sustaining beliefs

C. Examples of warrants for beliefs in relation to mathematics: mathematically illustrative examples, kinds of evidence, counterexamples to conjectures, intuitions, diagrams, rational arguments, proofs; but also (sometimes) appeal to authority, feelings, personal history and education, values, social acceptability, and fashion

D. Discussion of truth in relation to warrants for belief, and the role of meta-affect

References

DeBellis, V. A. (1996). Interactions between affect and cognition during mathematical problem solving: A two year case study of four elementary school children. Rutgers Univ. doctoral dissertation. Ann Arbor, MI: Univ. Microfilms 96-30716.

DeBellis, V. A. (1998). Mathematical intimacy: Local affect in powerful problem solvers. In S. Berenson et al. (Eds.), *Proceedings of the 20th Annual Meeting of PME-NA*, vol. 2, 435-440. Columbus, OH: ERIC.

DeBellis, V. A. & Goldin, G. A. (1991). Interactions between cognition and affect in eight high school students' individual problem solving. In R. G. Underhill (Ed.), *Procs. of the 13th annual meeting of PME-NA.* Blacksburg, VI: Virginia Tech., vol. 1, 29-35.

DeBellis, V. A. & Goldin, G. A. (1993), Analysis of interactions between affect and cognition in elementary school children during problem solving. In J. R. Becker & B.

Pense (Eds.), *Procs. of the 15th annual meeting of PME-NA.* Pacific Grove, CA: San Jose State Univ. Ctr. for Math. and Computer Sci. Educ., vol. 2, 56-62.

DeBellis, V. A., & Goldin, G. A. (1997). The affective domain in mathematical problem solving. In E. Pehkonen (Ed.), *Procs. of the 21st annual meeting of PME:* Vol. 2 (pp. 209-216). Helsinki, Finland: University of Helsinki Dept. of Teacher Education.

DeBellis, V. A., & Goldin, G. A. (1999). Aspects of affect: Mathematical intimacy, mathematical integrity. In *Procs. of the 23rd annual meeting of PME.* Haifa, Israel.

Goldin, G. A. (1987). Cognitive representational systems for mathematical problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics.* Hillsdale, NJ: Erlbaum, 125-145.

Goldin, G. A. (1988), Affective representation and mathematical problem solving. In M. J. Behr, C. B. Lacampagne, & M. M. Wheeler (Eds.), *Procs. of the 10th annual meeting of PME-NA.* DeKalb, IL: Northern Illinois Univ. Department of Mathematics, 1-7. A revised version of this paper will be published shortly in *Mathematical Thinking and Learning* (1999, in press).

Goldin, G. A. (1998). Representations, learning, and problem solving in mathematics. *Journal of Mathematical Behavior 17, 2,* 137-165.

Leder, G. (1982). Mathematics achievement and fear of success. *Journal for Research in Mathematics Education, 13,* 124-135.

Leder, G. (1993). Reconciling affective and cognitive aspects of mathematics learning: Reality or a pious hope? In I. Hirabayashi *et al.* (Eds.), *Procs. of the 17th annual meeting of PME.* Tsukuba, Japan: Univ. of Tsukuba, vol.1, 46-65.

Leron, U. & Hazzan, O. (1997). The world according to Johnny: A coping perspective in mathematics education. *Educational Studies in Mathematics, 32,* 265-292.

Lester, F. K., Garofalo, J., & Lambdin Kroll, D. (1989). Self-confidence, interest, beliefs, and metacognition: Key influences on problem-solving behavior. In D. B. McLeod & V. M. Adams (Eds.), *op. cit..*, 75-88.

McLeod, D. B. (1988). Affective issues in mathematical problem solving: Some theoretical considerations. *Journal for Research in Mathematics Education, 19,* 134-141.

McLeod, D. B. & Adams, V. M., Eds. (1989). *Affect and mathematical problem solving: A new perspective.* NY: Springer-Verlag.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning,* 575-596. NY: Macmillan.

McLeod, D. B. (1994). Research on affect and mathematics learning in the JRME: 1970 to the present. *Journal for Research in Mathematics Education, 25,* 637-647.

Picard, R. W. (1997). Affective computing. Cambridge, MA: The MIT Press.

Rogers, T. B. (1983). Emotion, imagery, and verbal codes: a closer look at an increasingly complex interaction. In J. Yuille (Ed.), *Imagery, Memory, and Cognition: Essays in Honor of Allan Paivio,* 285-305. Hillsdale, NJ: Erlbaum.

Schoenfeld, A. (1985). *Mathematical Problem Solving.* Orlando, FL: Academic Press.

Vinner, S. (1997). From intuition to inhibition—mathematics, education, and other endangered species. In E. Pehkonen (Ed.), *Procs. of the 21st Annual Conference of PME,* vol. 1, 63-78. Lahti, Finland: University of Helsinki.

Zajonc, R. B. (1980). Feeling and thinking: Preferences need no inferences. *American Psychologist, 35,* 151-175.

Teachers’ perceptions and beliefs about factors that influence change in their pedagogy

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For more than a decade the mathematics education community has made recommendations for reform based on theoretical work coming out of cognitive science. Individual teachers, schools and entire school systems have sought models and methods to assist them as they attempt to respond to these recommendations and numerous projects have been initiated to support teacher change consistent with the recommendations (Annenberg Foundation, 1996; Grouws and Schultz, 1996). In this paper I will briefly describe the professional development model used in one such project and I will report the beliefs and perceptions of a group of teachers who, four years after the end of the project, were asked to look back and reflect on the factors that most impacted their change.

The teacher development model

Crucial to the success of any reform effort is the professional development model chosen to implement the effort (Clarke, 1997). The model used in the project was the Reflective Teaching Model (RTM) (Hart, 1996; Hart and Najee-ullah, 1997). The model is grounded in the theories of constructivism and metacognition. It also is based on beliefs in the value of shared authority and modeling.

Philosophy and assumptions

Constructivism. Within the theory of constructivism two dichotomous views exist: the radical constructivist who views learning as an individual cognitive process (von Glasersfeld, 1983), and the social constructivist who views learning as a social process (Bauersfeld, 1992; Ernest, 1992). The Reflective Teaching Model is built on the position that both perspectives are helpful in understanding knowledge development (Simon, 1995). In the RTM teachers and mathematics educators are *learners* of new thinking and pedagogy associated with mathematics education. As learners within the model, they engage in both individual and group experiences that challenge existing beliefs and that provide opportunities to construct new knowledge; to examine, share and rethink their ideas; to evaluate, argue and justify their thinking; and to reflect on their experiences.

Metacognition. The theory of metacognition (Flavell, 1979) refers in part to our ability to monitor and regulate what we are doing and thinking while we are experiencing it and our ability to reflect on experiences and to learn from them. Metacognitive activity supports the construction of new knowledge by making it public to the learner. Metacognitive knowledge (beliefs) can be challenged through this process, which lays the groundwork for real change in teaching practice. For the teacher in the RTM, developing the ability to *monitor the teaching act,* to regulate teaching behavior *while teaching*, as well as to *reflect on that lesson after it is over* are fundamental parts of the experience and critical to the process of change (Hart, Schultz, Najee-ullah & Nash, 1992).

Shared Authority. The RTM is also based on the belief that sharing authority (Cooney, 1993; Hart 1993) is a critical factor in the interaction of teachers and teacher educators. The ability of a teacher or teacher educator to relinquish intellectual control and allow others to share in the generation of mathematical or pedagogical ideas is a subtle but significant shift in roles from the traditional teacher educator as teller. It builds trust, ownership, and cohesion among those involved.

Modeling. Much human behavior is learned through observation and modeling (Bower and Hilgard, 1981; Rotter, 1992). Modeling is a particularly effective way to teach abstract behaviors such as examination of values and standards of conduct. Given the very complex nature of teaching within a constructivist framework of learning, we believe it is reasonable to assume that modeling can be a critical component in facilitating teacher change. The more complex the skill or thinking process the greater the need for an opportunity to observe performance of the skill or thinking process. More recently, Hermann (1990), found that >mental modeling' of reasoning processes enhanced the effectiveness of instruction. It follows that the mental modeling of the thinking processes engaged in planning lessons or of reflecting on the act of teaching assists teachers in developing their own mental models of the thinking involved in teaching.

The model/experience/reflect framework

All activities in the RTM follow a model/experience/reflect framework. This includes all aspects of summer or school year staff development. The facilitators first model activities, the teachers then experience these activities and they conclude with a reflection on the process.

Essential activities in the RTM

Based on the assumptions and the framework just described, teachers engage in several essential activities during a project using the RTM. They engage in an initial inservice that introduces the philosophy and language of reform and lays the groundwork for building relationships necessary to carry on long-term support. They engage in think-aloud paired problem solving sessions where they learn to listen to others solve problems and learn to communicate their own problem solving processes. They engage in monthly plan/teach/debrief cycles where pairs work together to think-aloud through the process of planning a lesson, teaching the lesson and debriefing on the lesson. Finally they engage in various formats for reflecting on their teaching: videotaping, oral reflections, reflection logs, etc. Each of these experiences is modeled first by RAM facilitators, experienced by the teachers and reflected on by all.

The study

The research described in this paper involves teachers who participated in the RTM through the Atlanta Math Project (AMP), a National Science Foundation Teacher Enhancement Project conducted from 1990-1994: starting with 13 teachers in 1990 and concluding with a database of 98 in 1994. At this conclusion of the formal funding period, the AMP *teachers* independently formed a professional group (the AMP Council) dedicated to continuing the professional development and collegial relations begun in the Atlanta Math Project. At the beginning of the 1998-1999 academic year the Council had a membership of 53 from the original 98. The remaining 45 who were not participating in the Council were identified as follows: nine of the 45 were administrators and supervisors, eight teachers had retired, four had moved and no forwarding address was found, eight had left teaching, two were deceased, and 14 were still practicing in the Atlanta area.

Rationale for the study

Other researchers have looked at factors that influence teacher change. For example, in his research using innovative curricular materials Clark (1997) identified 12 factors as having most influenced the changing role of two grade-6 U.S. teachers. Pehkonen (in press) interviewed 13 *innovative* German teachers. Through his interviews he identified 13 factors that were key in causing break-throughs in their teaching. The teachers who had participated in AMP offered a slightly different perspective on the issue of factors that influence teacher change. They all had a common experience in AMP. Fifty-three of the teachers remained connected through the Council yet 14 were not. And, all had had several years to reflect on their experiences and situate those related to AMP within a larger framework of their change process. What beliefs did those teachers hold about their own change? What do they believe were the factors that most influenced them? Do teachers who were no longer members of the council believe differently? The purpose of this research was to ask the teachers to look back on their experience and attempt to answer these questions.

Methods of inquiry and data sources

Drawing on results from the research mentioned above and the fundamentals of the RTM, a survey was developed with sixteen likert-scale items. On the survey AMP teachers were asked to rate and prioritize factors that may or may not have impacted their teaching over the last few years. Eight general factors from teacher change research (e.g., the reform movement in general, innovative curricular materials) as well as eight factors specifically related to the model (e.g., the plan/teach/debrief sequence) were listed.

After the surveys were returned and analyzed a sample of teachers was interviewed to confirm and expand the survey data. Teachers were asked to write the answers to three questions. They were then asked to explain their written comments while the interviewer took notes. They were asked if they saw themselves having a before and after with respect to the reform in mathematics education and if so, to describe how their teaching had changed. They were asked to identify factors that contributed to that change and finally they were asked to identify factors that may have hindered the change. Descriptive statistics were used to analyze the survey items. For the interview questions, qualitative methods were used to analyze responses looking for themes that confirmed the surveys or for new themes that were not identified in the survey.

Participants

The survey was mailed to the 53 members on the AMP Council database current in the Fall, 1998 and to the 14 members who were still teaching but no longer participating in the AMP Council. Follow-up phone calls were made to those who did not respond. Thirty-three surveys were returned from the AMP Council. Four were unopened with no forwarding address, resulting in 29 surveys available for analysis. Eight surveys were returned from the teachers who were no longer participating in the AMP Council. Ten of the AMP Council members were interviewed. Four of those not participating in the Council were interviewed.

Results

Survey. Surprisingly, the results were not glaringly different across the two groups on the survey. The majority of both groups indicated most of the 16 factors as being very or somewhat helpful in their change. However, some trends were apparent.

Among the Council members, three factors were identified by 90% of the respondents as being ***very*** helpful in supporting their change: colleagues in AMP, modeling in AMP and collegiality/ collaboration in AMP. Two factors were identified as only somewhat helpful or an actual hindrance by the teachers: the principal/school administration (76%) and my day-to-day working conditions (69%). Among the non-Council members the three very helpful factors mentioned above were identified 75% of the time. However, two other non-AMP factors were equally identified: the reform in general and innovative curricula. The non-Council members identified the same negative factors: the principal/school administration and my day-to-day working conditions.

Interview. When asked in the follow-up interviews to discuss the factors that they perceived as having most influenced change in their teaching, the AMP Council members reiterated the three factors found in the survey: modeling of ideas and strategies, colleagues in AMP, and collegiality in AMP. Two other themes, while not as vivid as the others, were frequently mentioned: (1) the importance of reflection in their teaching, and (2) how the plan/teach/debrief cycle structured their experience.

Not surprisingly the non-council members were less focused on their AMP experience but seemed to incorporate that experience with their over-all professional growth. The four factors mentioned most by these teachers were (1) learning from *their* students, (2) learning from other colleagues in general, (3) the NCTM standards, and (4) becoming a student themselves in inservice and college courses. They spoke positively of AMP in general but seldom referred to specific factors within the model which influenced them. One theme did emerge that was consistent with the AMP Council group: the value of collegial support in the project.

During the interviews other factors not identified on the survey were mentioned as hindrances to change. Both groups mentioned parents who were not supportive. The AMP Council members mentioned standardized testing. The non-Council group strongly identified colleagues who do not agree with the reform position and lack of planning time.

Summary and discussion

What can we learn from the beliefs of these teachers, twenty-nine who have continued for four years to remain professionally aligned through the AMP Council and eight who are no longer involved? One clear factor emerged from the data. Both groups believe that working with colleagues and developing the spirit of collegiality both within and outside of the project is critical to teacher change. Second, the Council members believe it was important for them to have the ideas and strategies of reform modeled for them; that planning together, observing each other teach and debriefing together was important; and that reflection encouraged the change process. The non-Council members point to their students as critical factors in their change process as well as the reform movement in general, innovative materials and themselves as learners both within and outside of the project.

It is interesting to look at the results from these groups in contrast to the U.S. and German groups mentioned earlier. Five of the dominant factors from the AMP groups appeared in Clarke's (1997) U.S. results. The most dominant theme of developing collegial relations (mentioned by both AMP groups) appears in his list as does curricular materials and the reform movement in general (mentioned by the non-Council group). However, he does not make the distinction between positive and negative influences in his listing. Both AMP groups mentioned two of his items, (the day-to-day working conditions and the principal and school administration) as their most negative factors.

Pehkonen (in press) also identified factors common to the teachers in the AMP groups. His teachers, who had not participated in a group project, were more consistent with the non-Council members who had been away from AMP for a while. His second most identified factor (*experiences with pupils in the school*) was identified first as a positive change factor among the non-Council teachers. Both the German and non-Council groups mentioned factors of further learning (inservice and university) as positive factors. Finally, the German teachers mentioned parents only once, however both AMP groups mentioned it as a real detriment to teacher change when parents were not supportive of the change in teaching practices or of the curriculum.

Determining what factors of teacher change projects are making a difference is not always easy: success in whose eyes--from what perspective? The teachers' beliefs about the factors that influenced their change are critical. If we consider the responses from these teachers we must build projects that builds strong collegial networks, that models the processes being advocated and that works closely with teachers in planning, teaching and reflecting on their lessons. We must also consider the support from the principal, the administration, and the parents as well as a teachers day-to-day working conditions.

References

Bauersfeld, H. (1992). Classroom cultures from a social constructivist's point of view. *Educational Studies in Mathematics* , 23(5), 467-481.

Bower, G.H. & Hilgard, E.R. (1981). *Theories of learning* (5th ed.). Englewood Cliffs, NJ: Prentice-Hall.

Clarke, D.M. (1997). The changing role of the mathematics teacher. *Journal of Research in Mathematics Education,* 28(3), 278-308.

Cooney, T. (1993). On the notion of authority applied to teacher education. In J.R. Becker & B.J.Pence, eds.) *Proceedings of the 15th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education*, vol. 1, pp. 40-46.

Ernest, P. (!992). The nature of mathematics: Towards a social constructivist account. *Science and Education*,1(1), 89-100.

Flavell, J.H. (1979). Metacognition and cognitive monitoring. *American Psychologist,* 34 (10), 906-911.

Grouws, D.A. & Schultz, K.A. (1996). Mathematics teacher education. In. J. Sikula (Ed.), *Handbook of research on teacher education* (pp. 442-458). New York: Macmillan Library Reference.

Hart, L.C. (1996). The Decatur Elementary Math Project. In E. Jabuwoski (Ed.) *Proceedings of the 18th Annual Meeting of the Psychology of Mathematics Education - North American Chapter,* University of Florida, pp. 491-496.

Hart, L.C. (1993). Shared authority: A roadblock to teacher change? In J.R. Becker & B.J.Pence, eds.) *Proceedings of the 15th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education*, vol. 2, 189 - 195.

Hart, L.C. & Najee-ullah, D.H. (1997). *The Reflective Teaching Model [CDrom*]. Atlanta, GA: Galileo, Inc.

Hart, L.C., Schultz, K., Najee-ullah, D., & Nash, L. (1992). Implementing the Professional Standards for Teaching Mathematics: The role of reflection in teaching. *Arithmetic Teacher,* 40 (1), 40-42.

Hermann, B.A. (1990). Teaching preservice teachers how to model thought processes. *Teacher education and special education,* 13 (2), 73-81.

Pehkenon, E. (In press). *Teachers' professional development: What are the key factors for mathematics teachers to change?* University of Helsinki.

Rotter, J.B. (1992). *The development and applications of social learning theory: Selected papers.* New York: Praeger.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal of Research in Mathematics Education,* 26(2), 114-145.

The guide to math & science reform [Computer software]. (1996). South Burlington, VT: The Annenberg/Corporation for Public Broadcasting-Elementary and High School Project for Mathematics and Science. (**http://www.learner.org/k12/)**

von Glasersfeld, E. (1983). Learning as a constructive activity. *Proceedings of the Fifth Annual Conference of the Psychology of Mathematics Education*, Vol. 1, (pp. 41 - 69). Montreal: U. of Montreal.

Wertsch, J. & Toma, C. (1995). Discourse and learning in the classroom: A sociocultural approach. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 3-16). Hillsdale, NJ: Lawrence Erlbaum.

Mathematical beliefs and motivation of high school students in the United States

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In the United States, as in many other countries, major initiatives for reforming mathematics instruction specify increased focus on reasoning, problem solving and conceptual understanding. Emphasis on memorizing step-by-step procedures for solving specific types of problems is diminishing. Unfortunately, this change in emphasis has the potential to reduce the motivation of students who believe that mathematics is a set of procedures to be learned and that one mainly needs to memorize to learn mathematics. In other words, if the rationale for a more conceptual approach is not fully understood, students in reform-oriented classrooms may quit trying to learn when the instructor stresses comprehension of concepts over memorization of steps to get an answer (Kloosterman, 1996). In this paper, I address the issue of what United States high school students (ages 14 to 18) think mathematics is and how important memorization is in the learning process.

The data reported in this paper come from a larger study that took place during the 1995-96 academic year. In that study, 56 United States high school student volunteers were interviewed using an extensive interview protocol (Kloosterman, 1997). Questions focused on a variety of factors with the potential to influence motivation. These included (a) background in mathematics, (b) feelings about school and mathematics, (c) effort in mathematics, (d) out-of-school influences, (e) self-confidence in mathematics, (f) natural ability in mathematics, (g) goal orientation and effort, (h) study habits in mathematics, (i) specific mathematics content, (j) assessment practices, and (k) how teachers influence motivation and work habits in mathematics classes. Students also completed the *Indiana Mathematics Belief Scales* (*IMBS* – Kloosterman & Stage, 1992) and their teachers completed a short questionnaire about their motivation and achievement in mathematics.

The students, who were enrolled in mathematics courses ranging from pre-algebra to calculus, were selected from 4 schools ranging in size from 500 to 2000 students and representing both rural and small city populations. At the time of the study, a few of the teachers at the schools were involved in a project to bring instruction in their schools in closer alignment with reform practices but most of the instruction the students were receiving would have been considered “traditional.” Interviews took two 45 to 50 minute periods to complete. Student comments were noted in writing and audio taped.

For the purposes of this paper, transcripts of student responses to two of the 56 interview questions were carefully analyzed. Written notes summarizing responses to several other questions were also considered when relevant. The two questions were selected because they were the ones that focused most directly the nature of mathematics. The first question was

Suppose an alien from outer space landed in your back yard and started asking you what math was like in Indiana. What would you tell him? What words best describe mathematics?

The second question, which focused on memorization and the extent to which mathematics is a collection of facts and procedures to be memorized, was

How important is memorization in mathematics? Are you good at memorizing? Can someone who is not good at memorizing be good in mathematics (or even "OK" in math?)

After reading through the transcripts and interview notes several times, three primary themes related to beliefs about the nature of mathematics emerged. Once the themes were specified, the data were reviewed one more time for comments that were contrary to the themes. None of the themes were true for every student, but this final review of the data confirmed that they were true for most of them.

The first two themes are related and thus I consider them together.

Theme 1: The nature of mathematics as a discipline is not an issue that United States high school students think about.

Theme 2: When students are pressed to talk about the nature of mathematics, they mention that mathematics can be used to solve a variety of problems and that it involves numbers. They often mention the procedural nature of mathematics. They sometimes mention the logical nature of mathematics but almost never mention the deduction or proof.

Because the nature of mathematics should become apparent after years of exposure to the subject, I had assumed that when questioned, students would be able to provide their views about what comprises mathematics. Theme one reflects the fact that this turned out not to be the case. A number of students talked about whether they liked mathematics, why one needed to learn mathematics, and the daily procedures in mathematics classes (grading homework, listening to an explanation, etc.).

Lana (junior, second-year algebra): I'd probably explain the benefits of math and tell the alien that everything you do basically has to do with math. The words I'd use to describe it? Probably the same words I'd used before – interesting, difficult at times but sometimes it's really easy too. I'd probably tell the alien that about what math class I'm in right now and describe it to him or it or whatever it is. And tell it the progress I'm making.
Interviewer (I): How would you describe the math that you're in right now?
Lana: Advanced, difficult, a lot of application, that's it.

Sid (junior, first-year algebra): Its fun, the teachers I have … makes it fun. It ain't dull. They [teachers] do extra stuff to show you more how to do it [and] let you learn better.

I: What kind of extra stuff do you mean when you say extra stuff.

Sid: Like projects and stuff. Like extra sometimes it's extra credit assignments to do and helps you get better.

Noah (senior, precalculus): I would say to the alien to only take what you need to take because from my point of view a lot of it after a certain point is purposeless except if you're going to go on to college and that's the only other time you're going to use a lot of the extended math.

I: Okay.

Noah: If I could elaborate a little…. I believe everyone should know how to do simple equations, word problems for example. But when you get into this cosine and sin and I don't even know how to describe all this stuff. I have a hard time seeing when it's going to be used unless you're going to teach trig or something and go on to be a mathematician. I have a hard time finding use for this, so I'd say I'd tell them that definitely take math. It's very important but to an extent.

I: Okay. You say you had a hard time seeing how things in trig for example are going to be used. Do you have to do word problems or any things like that?

Noah: Very rarely.

Fifteen of the 56 students mentioned specifically that mathematics involved steps, procedures, or formulas.

Lisa (junior, precalculus): Basic at first, just like steps and skills, and of course as you progress in math it gets more challenging. Can be difficult at times but it can always be solved. Basically I don't think there isn't ever an answer. I mean there probably is.

I: You said it was basic at first – steps and skills. Does that change?

Lisa: Yes. The steps get more complex and more involved in harder classes.

I: Is math still a set of steps? Is that the way you think of it even at the higher levels?

Lisa: Basically yeah. I think math problems have a lot of formulas. I do math in steps instead of just being able to, you know, you [I] kind of have to look at things and make sure I know what I'm doing.

Seven students spoke of mathematics as a way of thinking or a logical system. One of the students, who was in his last year of high school, spoke specifically about the structure of mathematics. However, he was like many other students in that it was difficult to know exactly what he was thinking.

Arlan (senior, calculus): Its like a structure, math is like the study of a structure.

I: Math is study of a structure?

Arlan: It's the study of an imaginary structure that you can use to analyze all kinds of problems and we use to label things. Because the structure fits to a label, we label objects by using math.

I: Okay.

Arlan: It's hard to kind of explain.

Only one student mentioned proof as an important aspect of mathematics. This student was taking geometry at the time she was interviewed and "two-column proof" was a major topic in her course.

Allie (junior, geometry): How would I describe math here in Indiana?

I: Yeah. Just the idea is you're describing it to somebody who doesn't have any conception of what math is. How do you get this thing to understand what math is?

Allie: Dealing with a lot of numbers, problem solving, proofs. I guess that's all….

I: When you say problem solving, what do you mean by that?

Allie: Like word problems.

Overall, the majority of students just assume that "math is math" and that on the surface, it is the same for everyone. Most students we interviewed found it difficult to describe the nature of mathematics, even with significant prompting. The fact that 15 of the 56 students mentioned the stepwise nature of mathematics without any prompting indicates that many students do feel mathematics is a set of rules to be mastered.

Upon further reflection, these findings seem quite reasonable. Rather than asking what math is, parents and peers ask students what they like about a class, or what they are learning in a class. In fact, a sizable number of those interviewed seemed to interpret the question about the nature of mathematics as a question about the nature of mathematics <u>class</u>. In short, because students are not regularly asked about the nature of mathematics, they do not form opinions on this issue. I will return to the implications of this finding later in the paper.

Theme 3: Students tend to feel that memorization, and the ability to memorize procedures, is an important part of being successful in mathematics. On the other hand, they also feel that students who are not good at memorizing can still learn mathematics if they work hard enough.

In the United States, there is a common perception that one either has a "math mind" or one does not (National Research Council, 1989). The 1996 National Assessment of Educational Progress (NAEP) reported that 89% of grade 4 students and 73% of grade 8 students agreed with the statement "Everyone can do well in mathematics if they try." By grade 12, however, only 50% agreed, with 29% disagreeing and 21% undecided (Mitchell, Hawkins, Jakwerth, Stancavage, & Dossey, 1999). Mathematics has traditionally been taught as a series of procedures to be applied to specific types of problems rather than as an ability to build and analyze models of complex real-life situations. As such, memorization of procedures can be important to learning mathematics. From a motivation perspective, those students who have doubts about their ability to memorize formulas and procedures have good reason to question whether they have ability to do mathematics and thus whether they should bother to try.

Like the question about the nature of mathematics, the question about memorization in mathematics caught a number of students by surprise. They understood what they were being asked and many had very strong opinions. On the other hand, many of the comments they had seemed contradictory in that they would claim that memorization was very important and then they would also claim that students who did not memorize well could still do well in mathematics. Others would say that memorization was not that important and then talk about how difficult it was to know mathematics if you did not remember the procedures. Overall, 20 of the 56 students used the terms very important or really important when first asked about the need to memorize. Another 19 said memorization was important but were not emphatic about it. Fifteen said that it was somewhat important or that it was important in some ways but not others, one said that it was not important, and one said it might be important but that she really did not know. Although there was overwhelming agreement that memorization was important in some if not all aspects of mathematics, the nature of the specific comments proved to be more interesting than the actual numbers in each category.

Following are the comments provided by Beth, a sophomore taking first-year algebra, when she was asked about the importance of memorization when trying to learn mathematics. Beth's comments are quite typical in that (a) she felt memorization was important, (b) even though she said memorization was very important, she felt that someone who did not memorize well could be good at mathematics, and (c) the interviewer had to probe to get her to say why she felt the way she did.

Beth: [Memorization is] real important. Because if you don't remember for example, your basic adding and subtracting, then how [are] you really going to do anything? And your multiplication, you'll have to go back to really, you know, if you don't remember it then, then it'll take you forever to do anything.

I: So… it sounds like here you're giving examples of basic facts…. addition, subtraction, multiplication. Is there any other place where you need memorization … or is it always just a matter of memorizing these isolated pieces of math?

Beth: Yeah, like I don't know. (Giggle.) Just, I mean, if you don't remember it then you'll just have to always have somebody tell you how to do it and explain it again and again. And it helps a lot to remember so you don't have to go over it and over it…..

I: And does that … apply to, for instance, the kind of math you're doing now in Algebra?

Beth: Yeah.

I: And what kinds of things do you memorize in there?

Beth: Like, just basically how to do it and how to like take one paragraph, or one parentheses and split it into two and like, like stuff like that.

I: Those are like steps or procedures?

Beth: Yeah.

I: Okay. Are you good at memorizing?

Beth: Well, math I am. Yeah.
I: Can someone who's not very good at memorizing be good at math?
Beth: Yeah. I mean, they'd have to have it all written down or have to have someone really explain it to 'em. But, they can still do good in math.

Discussion

The comments discussed in this paper focus on a small fraction of the beliefs that are important with respect to student motivation to learn mathematics. Issues such as beliefs about oneself as learner, usefulness of mathematics in a career, intrinsic beauty of mathematics, and short term rewards such as grades all can play a role in how hard students work in a mathematics class. It is clear that the question "what is mathematics" is difficult for many students to answer, probably because they are never asked to think about the nature of mathematics. Responses to the alien questions as well as other questions throughout the interview indicated that students felt there was a significant procedural aspect to mathematics but most also felt that some people could be successful without having to memorizing everything. The inherent contradiction between believing that memory was very important, and believing that people who cannot memorize can still do mathematics, was typical of many student comment. That is, logical inconsistencies in students' beliefs were common and seemed to indicate that students were often doing mathematics assignments without thinking about what it takes to learn or why knowing mathematics is important.

Going back to the issue of reform mathematics, I am still concerned that when students are (a) expected to complete mathematics tasks that require far more than procedural understanding, (b) work cooperatively with peers, and (c) solve problems when the teacher will not explain the steps to get the answer, they will hit a roadblock. There is plenty of evidence that with time and with adequate rationale for doing mathematics differently, students can thrive in a reform mathematics environment (e.g., Hiebert et al., 1997). Because the students we interviewed had not experienced reform mathematics instruction, it was not possible to determine their beliefs about such instruction. It appears that reform instruction goes against many of their intuitions about what mathematics should be but it also appears that they simply do what they are told. One important goal of reform instruction, at least from my perspective, is that students are actually expected to see mathematics as a discipline for dealing with complex questions and problems. Learning procedures to apply in specific situations may be important but it is only one aspect of necessary mathematical knowledge.

In sum, students need to be exposed to an issue they are not exposed to now. That issue is what it means to know and do mathematics. When students are asked to think about what the discipline of mathematics entails, the chances of them learning to think mathematically will be much greater.

References

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Kloosterman, P. (1996). Students' beliefs about knowing and learning mathematics: Implications for motivation. In M. Carr (Ed.), *Motivation in mathematics* (pp. 131-156). Cresskill, NJ: Hampton Press.

Kloosterman, P. (1997, March). *Assessing motivation in high school mathematics*. Paper presented at the annual meeting of the American Educational Research Association. Chicago.

Kloosterman, P., & Stage, F. K. (1992). Measuring beliefs about mathematical problem solving. *School Science and Mathematics*, 92, 109-115.

Mitchell, J. H., Hawkins, E. F., Jakwerth, P. M., Stancavage, F. B., & Dossey, J. A. (1999). Student work and teacher practices in mathematics. *Washington, DC: National Center for Education Statistics.*

National Research Council. (1989). Everybody counts, A report to the national on the future of mathematics education. *Washington, DC: National Academy Press.*

Measuring mathematical beliefs and their impact on the learning of mathematics: A new approach

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Introduction

It is now widely accepted that cognitive as well as affective factors - such as attitudes, beliefs, feelings, and moods - must be explored if our understanding of the nature of mathematics learning is to be enhanced. How students' beliefs and attitudes about mathematics influence their learning of this subject has attracted considerable research attention. Yet, finding ways to infer beliefs and attitudes from behaviours has continued to be a challenge to researchers. In an influential article, Schoenfeld (1992) argued: "The older measurement tools and concepts found in the affective literature are simply inadequate; they are not at a level of mechanism and most often tell us that *something* happens without offering good suggestions as to how or why" (p. 364).

In the remainder of this paper a novel and rich approach for capturing affect, including beliefs, is described. This technique, the *Experience Sampling Method* (ESM), has been discussed in some detail by Csikszentmihalyi, Rathunde, and Whalen (1993) in their study of talented teenagers, but has apparently not been used before in mathematics education research. The methodology, scope of the information gathered, and specific findings are described below3.

The experience sampling method

Through extensive monitoring of activities over an extended period of time, ESM allows insights into the motivations, attitudes, and beliefs associated with students' behaviours. Specifically, on receipt of a signal, participants are requested to chart their daily activities, and reactions to those activities, through completion of a specially designed form, the *Experience Sampling Form* or ESF. Excerpts from the *Experience Sampling Form* are shown in Figure 1. As can be seen, respondents have the opportunity to describe and comment on the activities being undertaken as well as the attitudes, beliefs, emotions, and moods elicited by those activities.

3 The project, now in its third year, is funded by the Australian Research Council . I wish to acknowledge the close cooperation of my co-chief investigator, Helen Forgasz, and the invaluable research support provided by Chris Brew

Name ________________________________
Date __________ **Time Beeped** __________ **Time filled out** _________

As you were beeped:
(You may have been engaged in an academic, household, employment related or leisure pursuit. Any of these are relevant for the purpose of this study)

Where were you? ____
Who were you with? ____
……….........….. ____

What were you thinking about? ____________________________________

Tick the column which best describes your response to each of the following questions.

	Not at all	**a little**	**moderately**	**quite a bit**	**completely**
Were you concentrating?					
Was it hard to concentrate?					
Were you living up to your own expectations?					
……..…......					

In the table below you are asked to describe your mood as you were beeped …. Tick the appropriate column which best describes your mood along (the) continuum.

	Very	**Quite**	**NEITHER**	***Quite***	***Very***	
happy						*sad*
lonely						*sociable*
distracted						*focussed*
……..…						*……..…*

If you had the choice when you were beeped:

What would you prefer to have been doing? ______________________________
…...................…..... ____

Has anything happened, or have you done anything, since you were last beeped which has affected the way you now feel? Not applicable ( ) No ( ) Yes ( )

If "yes", please elaborate: ____
____

Any other comments? ____
____

Figure 1: Excerpts from the *Experience Sampling Form*

The sample

Twenty mature age students4 participated in the ESM5. They were asked to carry an electronic beeper for six consecutive days which included a weekend. Six signals were sent between the hours of 7am and 10pm on week days and between 10am and 10pm on weekend days. Respondents were expected to complete the ESFs within 30 minutes of receipt of the signal. In addition to summary ESM data obtained for the full sample, more detailed information is given for two students, Caitlin and Boyd6. These two students were selected for intensive reporting because they were conveniently enrolled in the same institution and in the same second year mathematics subject.

Relevant data

Out of the approximately 720 ESFs distributed7 to the 20 students, 582 forms were returned. This response rate (81%) exceeded our expectations, since we had informed students that completion of at least four of the six sheets each day would be quite acceptable. We interpreted the much higher response rate as indicative of the group's strong commitment to the research project.

An excerpt from an ESF is shown in Figure 2.

TIME (BEEPED / FILLED OUT)	STUDENT	ACTIVTY	AFFECT
Wednesday, 9.20 am / 9.20 am	BOYD	On a train, returning from work. Although there were other passengers he was alone, in a sense. He was reading *The*	He was concentrating "a little", felt in control of the situation, and that he was living up to his own expectations. The activity was quite important to him. He was feeling quite passive, excited, clear, tense, and focussed. He would prefer to be

4 Students who are 21 or over on March 1 of the year in which University entry is sought.

5 The following additional data gathering instruments were used in the study (sample sizes varied): a survey questionnaire which covered: biographical and background details, enrolment issues, affective dimensions, and perceptions of the learning environment; interviews; regular e-mail or snail-mail contacts, and a "tag-a-student" period in which time was spent with students on campus.

6 pseudonyms

7 Human error led to the sending of five rather than six signals on a small number of days. One participant had returned to his home in the country for the weekend and was out of reach of our signals. The total number of ESFs that could have been completed was thus less than 720.

TIME (BEEPED / FILLED OUT)	STUDENT	ACTIVTY	AFFECT
		French mathematician by Tom Petsinis. He was enjoying it but "… concerned that Everiste Galois may not make it to his entrance exam for the Polytechnic on time".	at home, in bed: "I've left work and I'm heading back home and I feel stuffed and I want to be at home in bed."
Thursday, 10.06 am / 11.15 am	*CAITLIN*	*In a mathematics tutorial, with her tutor and classmates, listening to an introduction to the lesson*	*Caitlin is concentrating hard, is feeling quite good about herself, and feels that she is living up to her own expectations (completely) and those of others (quite a bit). The activity is very important to her, and she is very happy to be doing it. She is feeling generally content "but still sick"*
Thursday, 8.35 pm / 8.35 pm	BOYD	Alone, at home in the lounge room, working on problems regarding the Fourier Series. He was thinking back to a conversation he'd had earlier that day with Jim, a post graduate student about "motivational problems with studying". Knowing that Jim had similar problems to his	He was finding it a little hard to concentrate. He considered the activity important in its own right and very important to his overall goals. Boyd was feeling quite active, confused and focussed. He wrote "I hope I've taken some action in dealing with motivational problems in study. The most important step of course is simply doing something." He would have preferred to be playing with prime numbers. "I like playing with ideas in mathematics and setting myself little experiments to find out what happens. It's kind of like doodling for me even though these mathematical doodles have little to do with the

TIME (BEEPED / FILLED OUT)	STUDENT	ACTIVTY	AFFECT
		own was reassuring	course material. I see different tricks and techniques in mathematics as being rather similar to the different materials and mediums available to an artist. The more tricks I learn, the larger my palette becomes and the more art it is possible to create through mathematics. I see mathematics being as much an art as it is a science. Maybe this can be a motivational inspiration in itself. By studying the course material I can expand my technique."
Friday, 7.53 am / 8.00 am	*CAITLIN*	*In bed, with her dog, reading a logic textbook, "preparing to work on an assignment (due today!)"*	*She is trying to concentrate but finding it rather hard to do so. She feels that she is not living up to the expectation of others. The activity is very important to her and also in relation to her overall goals. Caitlin informs us "I have decided to take the day off work and to see a doctor, so I've conceded defeat to this bug." She wishes she was not feeling sick and adds: "I plan to work pretty much full-time next week to make up for this week. I'm stressed about being behind with work and study, but feel confident that I can make it up".*
Saturday, 4.25 pm / 4.50 pm	BOYD	Alone, at home, making a cup of tea ("an important source of pleasure to me"), thinking about prime numbers	He was feeling quite happy and active, was happy to be by himself but would have preferred to be studying Fourier Series. Boyd tells us: "Due to fatigue on Saturday afternoons my mental state seems to fall to a lowest common denominator regarding will power and I end up spending too much time on

TIME (BEEPED / FILLED OUT)	STUDENT	ACTIVTY	AFFECT
			primes”

Figure 2: Excerpts from a completed *Experience Sampling Form*

The full data set was used to determine the spread and extent of activities in which the students were engaged (see Figure 3). However, for tracing data pertaining to affective issues we initially counted as valid only those responses completed within 30 minutes of receipt of the signal, following Csikszentmihalyi et al. (1993). Of the forms returned, 364 (62% of the overall response of 582 ESF) fulfilled this requirement. Closer examination of the ESFs completed outside the 30 minute cut off period revealed that there were often good reasons why the forms were not completed at the time of the signal (e.g. examinations, laboratory classes, cinema, in transit). We decided to include as valid those ESFs which had clearly been completed as soon as possible after receipt of the signal. This increased the number of valid ESFs to 492 (84% of the overall response of 582 ESF).

The response rate generally fell over the six day period. Typically it was high for the first two days, fell away on the third day and remained steady over the remainder of the signalling period.

Snap shots of daily activities

The activities of the students when beeped were divided into eight major categories (study, paid work, relaxation, family, chores, transit, sleeping, eating). When coding was not clear cut, e.g., when the respondent was eating with other family members, the dimension emphasized elsewhere on the same ESF determined the category. A summary of the daily activities captured for Caitlin and Boyd is shown in Figure 3.

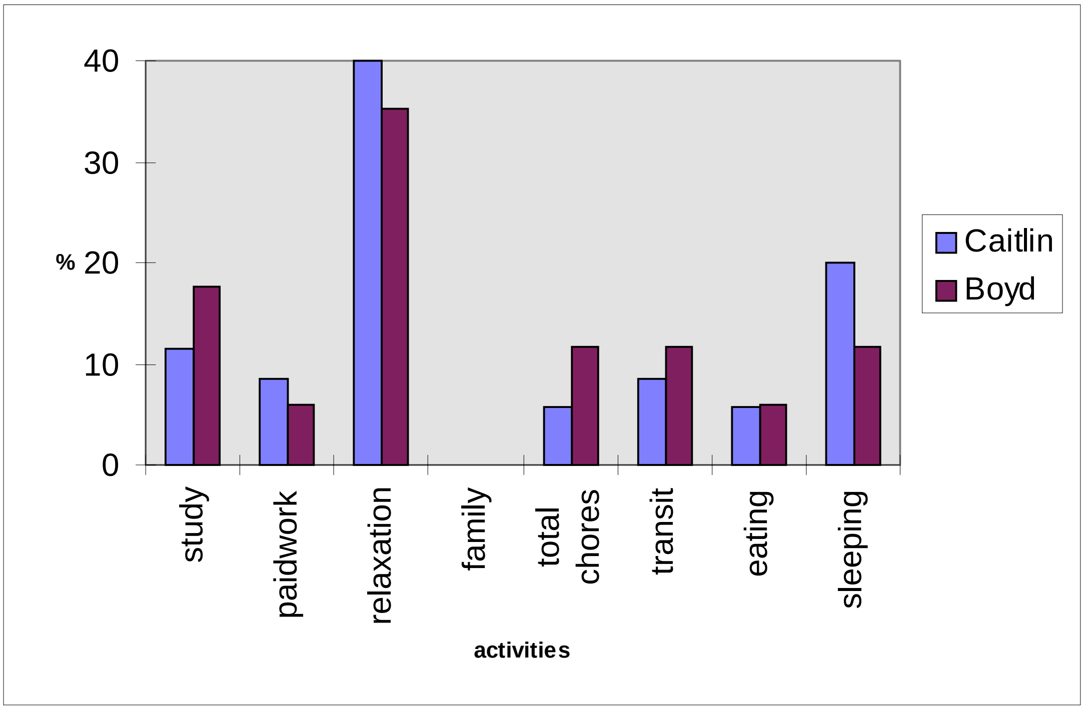

Figure 3: Overview of Caitlin and Boyd's activities when beeped

When beeped, Caitlin and Boyd could be relaxing, busy with their paid work commitments, doing routine chores in the house, in transit to or from university or work, eating, or even sleeping. (Boyd's part time job involved night work. Caitlin, we learnt, was fighting a heavy cold.) Clearly, studies occupied only a part of the day.

How Caitlin and Boyd *felt* when they were beeped as they were in class or studying is shown in Figure 4.

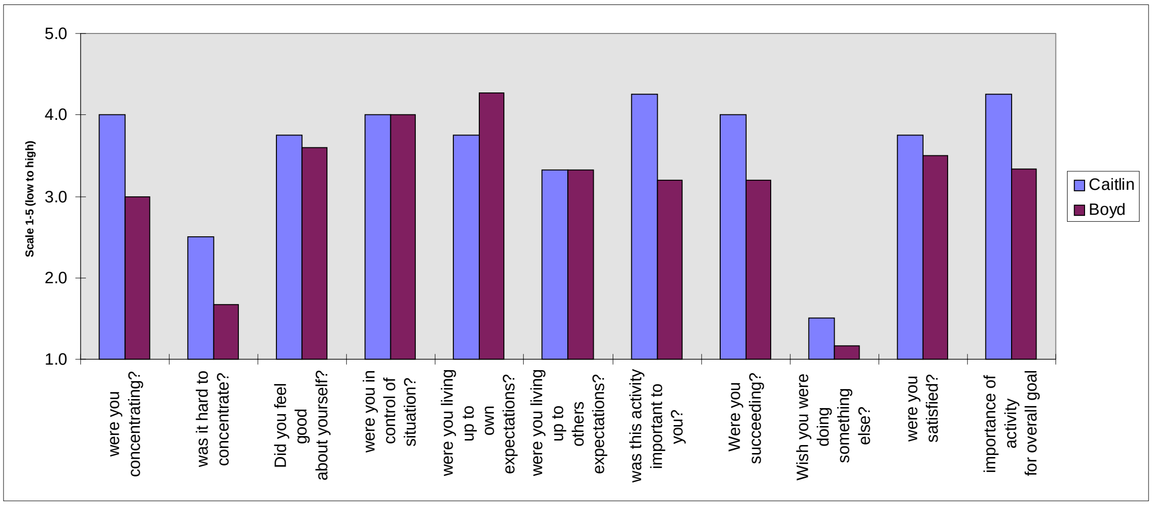

Figure 4: Results from selected ESFs for two case studies - data represent the average result for ESFs completed during study activity (n=4 for Caitlin and n=6 for Boyd).

Comments

Caitlin's and Boyd's responses to the ESF prompts reinforced the impressions gained from other data we had gathered through interviews and regular e-mail interactions. Boyd was involved in study activities almost 20% of the times he was beeped. For Caitlin this figure was just over 10%. Nevertheless Caitlin's commitment to her work and academic confidence became clear as she was monitored over the week. For example, when beeped as she was driving to the university, late for a lecture Caitlin wrote that s*he very much wished she had been doing something else and was feeling most dissatisfied with her efforts. She would prefer to be in her class and added: "I haven't started work on my assignment, so I feel like I've wasted a lot of the day"*. On another occasion she volunteered *"I'm a bit stressed about the work and study not getting done due to my illness"*. Still fighting a heavy cold she was increasingly concerned about her work: *"I have decided to take the day off work and to see a doctor, so I've conceded defeat to this bug."* She wished she was not feeling sick and added: *"I plan to work pretty much full-time next week to make up for this week. I'm stressed about being behind with work and study, but feel confident that I can make it up"*. Later in the week she told us, via an ESF entry, that she was working on a mathematics assignment as she was beeped and that she was *concentrating, feeling good about herself, …, and living up to her own expectations and those of others. The activity was challenging to her and very important. She was … feeling quite alert and happy, and starting to feel healthier*. Excerpts such as these are powerful indicators of Caitlin's beliefs about herself and as a student of mathematics.

When we compared Boyd's and Caitlin's descriptions of their mood for the times the beeper signals caught them during studying activities, Caitlin consistently revealed herself to be happier, more confident, and more focussed. Boyd's entries showed that the relatively high time spent travelling on public transport to and from university and work was often spent productively: by reading a book about mathematicians and their work and to attempt some of the examples given. For instance, when he was beeped while he was on a train, on his way home from work he wrote that "*even though there are other passengers [I am] alone, in a sense"*. He was reading *The French mathematician* by Tom Petsinis. He was enjoying it but "*… concerned that Everiste Galois may not make it to his entrance exam for the Polytechnic on time*". Study time was often spent on mathematically related matters of interest as well as on assignments set: One time he was beeped as he *was thinking about specific aspect of the question ("whether cosine was positive or negative")*. Another time he was *sorting through his applied partial differential calculus notes and listening to a tape. "They need to be organised on a regular basis as there are many loose sheets"*. Caught at home, during study time, he wrote that he *was looking for a formula which generates prime numbers. [It] is probably a very challenging activity in itself but since it has nothing to do with my course material in applied mathematics I actually find it to be more of a pleasant distraction rather than a challenge. … Most of the fun is searching for it. In a very real sense, once the treasure is discovered the game is over."*

ESF entries such as those reproduced offer unique insights into the "how and why" of many of Boyd's actions, as well as his beliefs about study, mathematical thinking and the intrinsic value of mathematics.

References

Csikszentmihalyi, M., Rathunde, K., & Whalen, S. (1993) *Talented teenagers*. Cambridge: University of Cambridge Press

Schoenfeld, A H (1992) Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D A Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning* (pp. 334-370) New York: MacMillan

Research on mathematics teachers' beliefs: A situated perspective

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Introduction

Much work has gone into the analysis of teachers' beliefs about mathematics, about mathematics education and the possible connections between them and it remains of great interest today. It has been argued (e.g. Thompson, 1984) that teachers' beliefs are critical factors determining how they teach. So-called mismatches between theories and practices have been discussed in the literature (Cooney 1985; Thompson 1982, 1984, 1992; Lerman 1986, 1990), although it is sometimes referred to as espoused and enacted theories of mathematics teaching (Ernest 1989). In her review of the research in this field, Thompson (1992, p. 138) suggested that the relationship between teachers' conceptions of mathematics and their practice is complex and argued for viewing the relationship as a dialectic one, citing the work of Cobb, Wood & Yackel (1990).

Teacher educators will be familiar with the problem that "Inservice teachers' resistance to change and preservice and beginning teachers' reversion to teaching styles similar to those their own teachers used are legendary" (Brown, Cooney & Jones, 1990, p. 649; see also Crawford & Deer, 1993; Lerman, 1997; Klein, 1997b). It appears that courses do not provoke students to confront their naive notions of teaching mathematics. At the heart of the research on teachers' beliefs is the argument that teachers' or student teachers' beliefs and conceptions need to change for their teaching to change. Beliefs are taken to be an internal mental landscape that can be charted by suitable research instruments. They are assumed to be stable across the range of teaching sites and across the researcher's data collection sites but they are assumed to be amenable to change over time as a result of interventions or activities. I have some concerns about the first two of these assumptions and, as a consequence, wish to re-interpret the notion of 'change' as more elaborate than arising from a change in beliefs.

In the literature, research into the analysis and classification of teachers' beliefs (Thompson, 1982; Lerman, 1986; Wilson & Goldenberg, 1998) forms one major focus and monitoring and accounting for changes in teachers' beliefs over time and through periods of curriculum change, advanced study or research programmes forms another focus (Pehkonen, 1995; Wilson & Goldenberg, 1998). I will first examine the problems of researching teachers' beliefs, and then look at research on changes in teachers' beliefs through pre-service, in-service, curriculum innovation and research projects.

Clearly space will not permit me to examine the full range of literature (but see Lerman, 1999). Finally I will propose that a situated perspective offers a more coherent theoretical framework which is also more fruitful for research on change. In the past I have engaged in studies of teachers' beliefs (1986, 1990, 1994). In this paper I will be offering a critique of my own work, as well as that of others, in a search for theories of teacher development which build on person-in-context as the unit of analysis rather than decontextualised beliefs.

Beliefs research

Thompson (1984, 1992) Cooney (1985), Lerman (1986), Ernest (1989) and others have researched the structures of teachers' beliefs. Ponte (1994) draws a distinction between *knowledge, beliefs and conceptions*. I have suggested (1994) that research which examines teachers' beliefs and theories in one context and attempts to examine practice, or beliefs about practice, in another context is based on a notion that the core of a subject's identity is somehow unified and decontextualised. It is as if the teacher brings his/her theories (relatively stable mental objects) to bear on practice (empirical settings). Given this separation, attitudes and beliefs about the teaching of mathematics can be examined by an instrument in one setting, interviews in a 'laboratory' or questionnaire completion on one's own for example, and their impact examined in another setting, the classroom. There is then the argument that whatever mismatches there appear to be result from the influence of one particular factor, in a school environment perhaps, that distorts or over-rides beliefs. In the literature of research in this area one finds many examples like the following:

> For some teachers, these conceptions *(about teaching in general and about students, and the social and emotional makeup of the class)* are likely to take precedence over other views and beliefs specific to the teaching of mathematics. (Thompson 1984 p. 124-125, my italics)

In most cases researchers do not distinguish between settings of data collection either, assuming that questionnaires, interviews and other methods give access to teachers' beliefs independent of that setting. I want to suggest that whilst there is a 'family resemblance' between concepts, beliefs and actions in one context and another they are qualitatively different by virtue of those contexts. I want to argue that contexts in which research on teachers' beliefs and practices are carried out should be seen as a whole, in the sense that the cognitive and emotional responses of the subject(s) through and with the methods, tools, social structuring in relation to the researcher(s), language etc. form the 'findings'; they are not separable. So too the classroom must be seen as a specific context. It is not that the opinions and beliefs of a teacher in, say, an interview context are unrelated to that teacher's practice. I have characterised it as a family relationship in that there are strong links and resemblances, but the classroom is its own setting. For example, Gattuso (1994) describes the effect of mood, personal relationships to students and other factors on teaching styles. She came to the classroom with a set of beliefs about how one should teach and indeed of how she herself taught, but a monitoring of her teaching demonstrated the differences. Again it is not to be seen as some factors interfering with a set of beliefs; those beliefs are related to the context within which they are elicited and cannot be otherwise. I am suggesting that it makes no sense to claim that, at the time she was reacting to a student to whom she felt unsympathetic in a formal manner, simply giving rather curt answers to his questions, can be described as a mismatch between espoused and enacted beliefs. Teachers' beliefs are contextualised: to the data gathering situation; to the interviewer/interviewee relationship; to the location of classroom, laboratory or other setting; to the particular group of students, etc. They are related to practice and beliefs about practice but they are not simply mappable one to the other.

In short I have argued here for a unit of analysis of person-in-context. Studies of teachers' beliefs would need to be built on this basis. Further, research methods such as interviewing would be seen as *productive* of the responses, not merely reflective of an inner decontextualised reality.

Changes in beliefs

Many studies on change in teachers' beliefs take place in a programme of intervention. I will begin, however, by examining a recent paper by Wilson & Goldenberg (1998) in which they study change in beliefs without a programme of intervention. They first present a model for mapping teachers' belief, based on Perry (1970), and then use that framework to examine one teacher over a two-year period. Their version of the Perry model uses 4 categories of beliefs: dualism, pluralism, extreme relativism and experimentalism. They claim that these are not to be seen as stages in teacher development, although their search for progress through these stages towards experimentalism, which they see as the desirable perspective for reform teaching, suggests otherwise. What is remarkable in that study is the absence of any discussion about what might lead to changes in beliefs or indeed a lack of changes. There is a brief mention of the support, mainly teaching materials, provided by university personnel "which may account for some of (the teacher's) instructional decisions" (p. 284), but in terms of processes of change, or what I would wish to call learning, there is no mention of theory. This is typical of much research on teachers' beliefs. Many readers would conjecture that a teacher being observed and interviewed over a two-year period by university researchers would construct a picture of the researchers' agenda and point of view, which might lead to changes in their practices and beliefs. In fact it would be remarkable were there to be no influence, although the authors claim "We remind readers that we did very little to intervene in Mr Burt's practice or to sway his beliefs, and that we eschewed opportunities to more actively cause change along the lines of our model." (p. 290).

An example of research which monitors changes in beliefs during a programme of intervention is a paper by Grant, Hiebert and Wearne (1998). The paper describes a project in which nine teachers watched reform oriented lessons taught by three project teachers over a number of weeks. According to the researchers, four teachers missed "the point of the instruction. They tended to focus on individual features, predominantly the use of manipulatives, as the crux of the alternative instruction" (p. 225). Three other teachers "recognised some of the features of the instruction ... However ... they did not connect these features with larger goals but treated them in narrow, overly-constrained ways" (p. 226). The final two teachers "tended to recognize and internalize the instruction" (p. 227). The research also examined the beliefs of the three project teachers, and found that they "were all quite articulate in discussing the intent of the instruction and its salient features in a way consistent with our views" (p. 230). In the face of this evidence, Grant, Hiebert and Wearne attribute the key to the teachers' beliefs filtering "what they see and what they internalize" (p. 233). Again, the question that is not addressed is how beliefs might change, what the mechanics of this process might be. One might conjecture, for example, that the culture of the community of the project team as a whole, engaged in teaching 'expert' lessons to measure the changes in beliefs of the observing teachers, might play a significant role in the project teachers' developing perceptions of the nature of teaching and learning mathematics. The researchers refer to Guskey's (1986) suggestion that teachers' beliefs may change when there is an intrinsic reward, such as seeing their students' improved success, but although that did occur here, in the form of students' improved end of year scores, it did not seem to bring about the changes hoped for by the research project in a larger number of the observing teachers.

Grant *et al* (1998) refer to Schifter's work as a successful example of changing beliefs. In a recent paper, Schifter (1998) explores two "avenues for promoting teachers' mathematical investigations. The first avenue is *exploration of disciplinary content* ... The second avenue is *examination of student thinking*" (p. 57, italics in original). 36 elementary school teachers and 6 staff members from three institutions are engaged in a four-year teacher-enhancement project. During sessions in college, teachers engage in mathematical investigations at their own level and experience models of different mathematical learning experiences, including problems being set without prior instruction, group problem solving, etc. Through reflection on the nature of their own learning experiences in mathematics some, at least, of the teachers recognise possibilities for transforming their own classrooms. At the same time the journals that they keep include records of their students' mathematical activities, and analysis of their thinking. Schifter also refers to assigned readings as an element that contributes to teachers' changing practices. Schifter demands yet more, though, the development by teachers of "a new ear, one that is attuned to the mathematical ideas of one's own students" (p. 79). Narratives of incidents in their classes, as well as narratives from the staff members of the project team, offer opportunities for the teachers to discuss their students' learning with other teachers in the project. The final element is that of staff visits to their classrooms, in which the staff help teachers to work on aspects of their lessons which the teachers identified at the beginning of the year.

The reports of teachers' development are impressive, as they are in others of Schifter's writing (e.g. Schifter, 1995). The range of experiences in which teachers engage, during the project, clearly has a significant effect on their perceptions of their own teaching. From the accounts in these papers it seems clear that many of the teachers have learnt a great deal about mathematics and about listening to and analysing their students' mathematical activity, and their practice has been transformed.

How might we account for the situations in which teachers' beliefs and/or practices have changed and others where they have not? It seems clear that there is an interaction between what any teacher brings to the practices, be they pre-service, in-service, research sites or whatever, and the practices as constituted by the whole 'community'. However, measuring changes in beliefs is deeply problematic, given that any data collection is contextualised and elicits responses that are part of that context. If we take account of the person-in-context we would need to examine who is seen as the *master* (Lave & Wenger, 1991) or perhaps what are the models of mastery in the practice.

A situated perspective

I want to propose, therefore, that it is not the internal map of beliefs and awarenesses which changes but the identities of teachers, as models of mastery within communities are provided within the activities. These situations can be seen as *productive* of new elements of identities, or perhaps new identities, for the teachers. Some situations are particularly fruitful in teacher development, such as setting up on-going communities of teachers or teachers engaged in research together (Jaworski, 1998). Where they are not successful (e.g. Frykholm, 1999) we may talk of students' identities as teachers not having developed. Similarly, so many preservice teacher education courses engage with students' identities as *students on courses*, and the same with the teachers in other situations to which Cooney and his colleagues refer, and do not impinge on their identities as *teachers in classrooms*. The expression of knowledge, beliefs or conceptions, however they are distinguished and by whatever research method they are elicited, are an expression of the person-in-context. In that identities change through these situations, we may better refer to person-in-context-in-person (Lerman, forthcoming).

References

Brown, C., Cooney, T. A. & Jones, D. (1990) Mathematics Teacher Education. In W. R. Houston (Ed.) *Handbook of Research on Teacher Education* (639-656) New York: Macmillan.

Cobb, P., Wood, T. & Yackel, E. (1990) Classrooms as learning environments for teachers and researcher. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.) *Constructivist Views on the Teaching and Learning of Mathematics* Journal for Research in Mathematics Education Monograph No. 4, (pp. 125-146). Reston, VA: National Council of Teachers of Mathematics.

Cooney, T. (1985) A Beginning Teacher's View of Problem Solving. *Journal for Research in Mathematics Education,* 16, 324-336.

Crawford, K. & Deer, E. (1993) Do we practise what we preach?: putting policy into practise in teacher education. *South Pacific Journal of Teacher Education* 21(2), 111-121.

Ernest, P. (1989) The Impact of Beliefs on the Teaching of Mathematics. In P. Ernest (Ed.) *Mathematics Teaching: The State of the Art* (pp. 249-254). Lewes: Falmer.

Frykholm, J. A. (1999) The Impact of Reform: Challenges for Mathematics Teacher Preparation *Journal of Mathematics Teacher Education* 2(1), 79-105.

Gattuso, L. (1994) What Happens When Robots Have Feelings? In S. Lerman (Ed.), *Cultural Perspectives on the Mathematics Classroom.* (pp. 99-114), Dordrecht: Kluwer.

Grant, T. J., Hiebert, J. & Wearne, D. (1998) Observing and Teaching Reform-Minded Lessons: What Do Teacher See? *Journal of Mathematics Teacher Education*, 1(2), 217-236.

Guskey, T. R. (1986) Staff Development and the Process of Teacher Change. *Educational Researcher* 15(5), 5-12.

Jaworski, B. (1998) Mathematics Teacher Research: Process, Practice and the Development of Teaching. *Journal of Mathematics Teacher Education*, 1(1), 3-31.

Klein, M. (1997b) Constructivist Practice in Preservice Teacher Education in Mathematics: Aboriginal and Torres Strait Islander Voices Heard Yet Silenced. *Equity & Excellence in Education,* 30(1), 65-71.

Lave, J. & Wenger, E. (1991) *Situated Learning: Legitimate Peripheral Participation.* New York: Cambridge University Press.

Lerman, S. (1986) *Alternative Views of the Nature of Mathematics and their Possible Influence on the Teaching of Mathematics.* Unpublished doctoral dissertation, University of London.

Lerman, S. (1990) Alternative Perspectives of the Nature of Mathematics and their Influence on the Teaching of Mathematics. *British Educational Research Journal,* 16(1), 53-61.

Lerman, S. (1997) The Psychology of Mathematics Teacher Learning: In Search of Theory. In E. Pehkonen (Ed.) *Proceedings of the Twenty-First Meeting of the International Group for the Psychology of Mathematics Education* Vol. 3, pp. 200-207), Lahti, Finland.

Lerman, S. (1998) Cultural Perspectives on Mathematics and Mathematics Teaching and Learning. In F. Seeger, J. Voigt & U. Waschescio (Eds) *The Culture of the Mathematics Classroom: Analyses and Changes* (p. 290-307) New York: Cambridge University Press.

Lerman, S. (1999) A Review of Research in Mathematics Teacher Education. In F-L Lin (Ed.) *Proceedings of International Conference on Mathematics Teacher Education*, National Taiwan Normal University, pp. 110-133.

Lerman, S. (forthcoming) The Social Turn in Mathematics Education Research. In J. Boaler (Ed.) *Social Analyses of Mathematics Teaching and Learning*

Pehkonen, E. (1995) What are the key factors for mathematics teachers to change? In D. Carraher & L. Meira (Eds.) *Proceedings of 19th Meeting of International Group for the Psychology of Mathematics Education*, Recife, Brazil, Vol. 2, pp. 178-185.

Perry, W. G. (1970) *Forms of Intellectual and Ethical Development in the College Years.* New York: Holt, Rinehart & Winston.

da Ponte, J-P (1994) Mathematics teachers' professional knowledge. In J-P da Ponte and J-F Matos (Eds.) *Proceedings of 18th Meeting of International Group for the Pschology of Mathematics Education*, Lisbon, Portugal, Vol. 1, pp. 195-210.

Schifter, D. (1998) Learning Mathematics for Teaching: From a Teachers' Seminar to the Classroom. *Journal of Mathematics Teacher Education*, 1(1), 55-87.

Schifter, D. (Ed.) (1995) *What's Happening in Math Class? Reconstructing Professional Identities* Volume 2, New York: Teacher's College Press.

Thompson, A. G. (1982) *Teachers' Conceptions of Mathematics: Three Case Studies.* Unpublished doctoral dissertation, University of Georgia, Athens, USA.

Thompson, A. G. (1984) The Relationship of Teachers' Conceptions of Mathematics and Mathematics Teaching to Instructional Practice. *Educational Studies in Mathematics,* 15(2), 105-127.

Thompson, A. G. (1992) Teachers' Beliefs and Conceptions: A Synthesis of the Research. In D. A. Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning*. (pp. 127-146). New York: Macmillan.

Wilson, M. & Goldenberg, M. P. (1998) Some Conceptions are Difficult to Change: One Middle School Teacher's Struggle. *Journal of Mathematics Teacher Education*, 1(3), 269-293.

Elementary teacher students' beliefs and learning to teach mathematics

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Learning how to teach mathematics as a situated learning.

For some time now it has been assumed that learning is not a passive activity. The constructivist postulate states that the learner constructs his/her new knowledge by taking as a reference his/her prior knowledge. On the other hand, from the viewpoint of situated learning it is also defended that the context and the nature of the activities that the individual carries out form part of what is learnt. Learning seen in this way is linked to the characteristics of the learner's forms of participation in learning environments.

Learning how to teach mathematics may be seen as situated learning. Thus, we may consider both of the aforementioned aspects -the active construction of knowledge and the influence of the context and the form of participation- in the characterisation of the process of learning how to teach. In this case, the content of the learning are beliefs, mathematics knowledge, mathematical specific pedagogical content knowledge and pedagogical skills. From this point of view, the relationship between the context and cognition is emphasised and it is assumed that the form of participating for the elementary teacher students in the learning tasks will constitute features of the knowledge and the beliefs generated (Llinares, 1998).

Elementary teacher students' belief system as a mediating system in learning to teach mathematics.

Understanding the process of learning to teach mathematics involves studying the overlapping roles of beliefs, knowledge and fatures of learning environments (Borko et al, 1992; Nesbitt & Bright, 1999). The aim of this paper is to point out how, in the process for learning to teach, the elementary student teachers' belief system and knowledge system are entangled with aspects of the school mathematics culture (for example, the meanings associated with common practices in the classroom). This link between beliefs, knowledge and school culture is a characteristic of this learning process (Llinares, 1994). The focus will be the role played by the awareness of the elementary teacher student about his/her beliefs and knowledge about learning to teach.

Understanding these relationships better may provide us with information about two aspects: (i) for the design of learning environments for learning how to teach; and (ii) on the differences in the training programmes. These relationships shall be exemplified using three examples from research concentrating on characterising the process of learning to teach as situated learning. The examples concentrate on the relationships generated between mathematical understanding, specific mathematical pedagogical content knowledge, pedagogical reasoning, beliefs and the features of school culture. Furthermore, the examples will illustrate the mediation played by the system of beliefs about what it means to do mathematics, the role of representation modes in mathematics learning, the teacher's role in teaching and in the developing of pedagogical reasoning and instructional skills.

In the situations that we are going to describe, when a task is presented to elementary teacher students in a specific social context, it attempts to make them generate an activity (doing mathematics, analysing a learning difficulty or planning and managing the instruction). The relationship between the task and the activity (whether cognitive, affective or social) determines the learning environment and the features of elementary teacher students forms of participating will constitute features of the knowledge and the beliefs generated.

Doing mathematics: Understanding and beliefs about mathematics.

The following example from a mathematics summer course for elementary education majors was documented by M. Civil (1996) and it illustrates the web between the understanding mathematics and a set of beliefs about what constitutes doing mathematics in the classroom, how validity is determined, and what it means to understand something in mathematics. Civil describes how participation in a course on mathematics, which was intended to portray the current recommendations for mathematics teaching, illustres the relationships among the beliefs about mathematics and what is meant by doing mathematics and learning how to teach. In this course, the beliefs about what it means to do mathematics hold by teacher students clearly showed their prior experience with mathematics:

> *"I never really thought much about maths problems. Growing up, I was shown a formula, plugged in numbers, and got praised if I calculated the correct answer. I never really looked at the answer and said 'But, why is this the answer?'. I just accepted it as right and went on to the next problem (Betsy's journal) (Civil, 1996; 146)*

These prior beliefs functioned as mediating factors when it came to approach how to do mathematics in the classroom in a different way. One of the teacher student in M. Civil's study said:

> *Donna. I get out very frustrated from this class; like I do not get anything*
> *Marta: Would you feel better if I gave you the formula, plug it in, and get an answer?*

Donna: Yes
Marta: Why?
Donna: Because I would feel I have achieved something, at least I've got an answer.

In this experiment, the beliefs about what it means to do mathematics functioned as a determining factor when doing mathematics in a different way. In a certain sense, a teacher student's awareness about his/her beliefs and knowledge is one aspect for mediation in developing this new practice of doing mathematics. One characteristic in this situation was the relationship established between the "activity of actually performing mathematics" and the activity for "teaching mathematics". Doing mathematics in a different way is not enough for that the teacher students teach mathematics differently. This relationship should also be seen through the role of mediation played by beliefs. As one of the teacher students in M. Civil's study pointed out:

"I want to teach both skills and problem solving techniques, but I'm not sure how to go about it. I don't want to have to rely on a textbook to teach maths, but at the same time I'm scared of approaching maths from any other way (Donna's final essay) (Civil, 1996; 151)

Nevertheless, although Civil reports that during this course some teacher students began to change their way of looking at mathematics and what it meant to do mathematics, it was also not that there does not seem to be a direct link a linear one) between starting to do mathematics in a different way and teaching mathematics in a way that might provide this view of mathematics. Civil's research shows that for the teacher students "doing mathematics" and "teaching mathematics" are two separate activities that are backed by systems of beliefs and knowledge that are not the same. In this sense, the relationship between doing mathematics and teaching mathematics in the process for learning how to teach mathematics is not very clear. In Civil's research (1996) some student teachers began to value the approach developed in the course that they attended, insofar as they felt more comfortable about doing mathematics in another way, but they still had their doubts about whether they, as teachers, could teach mathematics that way. The teacher students therefore showed their insecurity. Providing formulae and procedures and stating the way to use them and apply them was seen to be safer than developing a more open approach which might allow the students to consider alternative approaches.

Understanding the pupils' difficulties: mathematical knowledge, pedagogical content knowledge and beliefs about the role of the teacher and about learning and teaching.

The following example from a mathematics methods course was documented by Llinares (1998). One of the activities within the course was an analysis of cases that described the difficulties that Primary School pupils had with mathematics. The purpose of this type of task concentrated on increasing the teacher students' understanding about the pupils' ways of thinking. One of the cases presented consisted of two cartoon strips showing the difficulties of two pupils from the 5^{th} grade with the concept of fractions and the modes of representation used. The first cartoon showed one pupil's response to the task of "represent 5/4 of a rectangle". The pupil drew a rectangle and divided it into five parts as if he were making divisions in a circle (from a point in the centre trying to represent circular sections), shading them in to show that it had 5 quarters. The task in the second cartoon strip asked: "How many counters were two thirds of 6 counters?" And it showed the attempt to solve it made by one pupil who drew a circle and then divided it into three parts placing one counter on each part and giving two counters as the answer.

The discussion generated between one group of student teachers showed the links between understanding the concept of fractions and the different modes for representation with their beliefs about learning and their role as teachers. The dialogue illustrated the discrepancies between understanding fractions as an operator and the use of the counters as a mode of representation. During these interactions what is highlighted is that the "struggle" of the student teachers to make sense of the pupils' answers put into play not just their understanding of school mathematics, but also their beliefs about learning, the teacher's role and teaching. The statements made by Carmen, a teacher student, in her attempts to understand the pupils difficulties illustrate the issues of fairness of the task:

> *Carmen: But ... theyve never done it before!*
> ...
> *Carmen: What I don't understand is why they (teachers) give these exercises to the pupils, if the kids don't know how to solve them.*

The low level of understanding of the mathematical situation and the role played by the modes of representation in learning situations favoured the justification of certain beliefs about teaching and the teacher's role. Furthermore, when they attempt to understand the pupils' difficulties, it seems that the teacher student's low level of understanding of different interpretations of the concept of fractions strengthened certain beliefs about learning ("but ,.. They [the pupils] have never done it before") or about teaching and the role of the teacher ("why they [teachers] give these exercises to the pupil if the kids don't know how to solve them"). The subsequent discussion when linked to the analysis of this case allowed the student teachers to clarify their understanding of the concept of fractions but we do not know whether or not they managed to alter their beliefs.

The hypothetical nature of the dialectical relationship between beliefs and understanding is complex and possibly may not be understood solely from the cognitive perspective. The contexts of learning and the different ways of participating in them may provide us with another perspective about the relationship between understanding and beliefs in the process of learning to teach mathematics.

Pedagogical reasoning: Beliefs and the role of habitual practices in school mathematics

The example described here was taken from Sanchez & Llinares (1996). The latter described a teacher student's attempt (Carlota) to manage a problem-solving situation with a group of Primary school pupils during her student teaching (practice period). One of the difficulties that Carlota had to face arose from the meaning that the pupils gave to the notion of problem-exercise and the role that the teacher had to play in these situations. The lack of any custom amongst the pupils in carrying out sessions for group problem-solving turned into a non-stop session of asking the student teacher for instructions and asking her to tell them whether what they were doing was right or wrong. The following interaction is representative of what happened in this experiment in relation to this aspect:

Sara:(talking to Carlota) "I do it this way, look..."
Carlota: "Well, you divide 43 by 4 for nothing."
Sole "Is this alright?"
Dolores: "So, have we got it right?"

On the other hand, Carlota's pupils in this experiment saw solving the problem like identifying the appropriate operation and doing it, and then they sought from Carlota confirmation that what they had done was in fact correct or not. Carlota's reflection about what happened showed that she seemed to be convinced that if she did not provide this type of confirmation the pupils would not carry on trying to solve the problem. She said that this was so: "*Because I am the teacher*". Carlota assumed that her role as a teacher consisted of telling the pupils whether what they were doing was right or wrong. In this sense, Carlota made her belief compatible with the belief that it justifies and supports the usual practices in a traditional class. To a certain extent, her awareness about this belief was used as support for developing the practice thus conditioning her critical reflection. Carlota's story shows us how teacher students may develop a set of instructional principles taking from their course on methods which reflect the constructivist philosophy taught in the course. Nevertheless, the teaching performed during the teacher students' student teaching (though in a context of micro-teaching such as this one) and the subsequent reflection highlighted the influence of the usual practices in the mathematics class and the compatibility (strengthening) of Carlota's beliefs with the culture that supports these practices.

Understanding and Beliefs in learning how to teach mathematics.

The three examples described above show different aspects of the process of learning to teach in which it highlights the problems in the relationship between understanding, student teachers' beliefs and the school culture. The three contexts from which the different examples were taken - mathematics course, course concentrating on the analysis of the pupils' ways of thinking and student teaching- were concentrated on three important aspects on the process for learning how to teach mathematics: mathematical knowledge, specific mathematical pedagogical content knowledge, and pedagogical reasoning. On the basis of the examples described above, we underline the role played by the systems of beliefs as mediators in the process of learning aspects of each one of these components of the knowledge needed to teach. Furthermore, we have attempted to show how the student teachers' awareness about their own beliefs and knowledge actually play different roles in learning how to teach. From this point of view, this awareness may be considered to be key to self-regulation in this learning.

References

Borko, H.; Eisenhart, M.; Brown, C. Underhill, R.; Jones, D. & Agrad, P. (1992) *Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily?.* Journal for Research in Mathematics Education, 23(3), 194-222.

Civil, M. (1996) *Thinking about mathematics and its teaching: An experience with preservice teachers.* In Gimenez, Llinares & Sanchez (Eds.) Becoming a Primary Teacher. Issues from Mathematics Education. Indugraphic: authors. Badajoz: Spain; pp. 135-156

Llinares, S. (1994) *The Development of Prospective Elementary Teachers Pedagogical knowledge and reasoning. The school mathematical culture as reference.* In N. Malara & L. Rico (Eds.) First Italian-Spanish Research Symposium in Mathematics Education. Dipartimento di matemática, Università di Modena, Italy; pp.165-172.

Llinares, S. (1996-a) *Improving knowledge, professional growth and monitoring the development of mathematics teachers: a neccesary integrating of theoretical frameworks.* In Addenda -PME 20; Universitat de València, Spain; 23-31.

Llinares, S. (1996-b) *Contexts and learning to teach mathematics: The case of prospective elementary teachers.* In Gimenez, Llinares & Sanchez (Eds.) Becoming a Primary Teacher. Issues from Mathematics Education. Indugraphic: authors. Badajoz: Spain; pp.13-31.

Llinares, S. (1998) *Preservice elementary teachers and learning to teach mathematics. Relationships among context, task and cognitive activity.* In N. Ellerton (Ed.) Mathematics Teacher Development: International Perspectives. Meridian Press, West Perth, Wetern , Australia.

Nesbitt, N & Bright, G.W. (1999) *Elementary Preservice Teachers Changing Beliefs and Instructional Use of Children s mathematical thinking.* Journal for Research in Mathematics Education, 30(1), 89-110

Sánchez, V. & Llinares, S. (1996) *Habitual school practices and problem solving situations: the case of Carlota.* In Gimenez, Llinares & Sanchez (Eds.*) Becoming a Primary Teacher. Issues from Mathematics Education.* Indugraphic: Authors. Badajoz: Spain; pp. 179-202

Learning with and about mathematics curriculum: The role of teachers' conceptions

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Ongoing reform efforts in the United States are based on views of mathematics, learning, and teaching that depart significantly from school mathematics traditions. Reforms aim to revise the conventional view of mathematics learning as the mastery of a fixed set of facts and procedures to more centrally locate the processes of investigation, sense-making, and communication in classroom activities. Cobb, Wood, Yackel, and McNeal (1992) make the powerful distinction between *inquiry mathematics* and *school mathematics*. In contrast with traditional classroom activities that emphasize repetition, practice, and routinized means to some focused endpoint, inquiry mathematics instruction emphasizes student engagement in problem-solving and theory-building about important mathematical situations and concepts.

Bringing about such dramatic changes in mathematics instruction can be a daunting task for teachers. Perhaps the greatest obstacles for teachers is a lack of personal familiarity with mathematical problem-solving and sense-making – processes that most have never experienced themselves, as students or teachers. Even when teachers' efforts to change are supported by curriculum materials designed to aid in the enactment of reform visions, teachers struggle to bring about significant changes in classroom practice. When teachers' conceptions and practices are deeply tied to traditional mathematics pedagogy, innovative curricula can be very difficult to implement (Cohen, 1990; Grant, Peterson, & Shojgreen-Downer, 1996; Wilson & Lloyd, in press). Instructional reform is unlikely to take hold unless we can identify viable ways to encourage and enable teachers to make significant shifts in their conceptions.

The purpose of this paper is to draw attention to the role of teachers' conceptions in their experiences with reform-oriented curriculum materials. Of particular interest is what and how teachers learn from their experiences with reform-oriented curricula. There is a great diversity of possible experiences that teachers may have with innovative K-12 mathematics curricula. For instance, during inservice workshops or preservice courses at the university, teachers may work collaboratively "as students" on the mathematical lessons outlined in the materials. Doing so can offer teachers critical opportunities as learners because they can personally experience unfamiliar mathematics in novel ways. Another rich context for educative experiences with curricula is teachers' own classrooms. As teachers implement curriculum materials in their classrooms (or as student teachers), they may develop new mathematical and pedagogical skills on the basis of their design of lessons, interactions with students, use of technology, and so on.

My inquiry about the nature of teachers' learning from experiences with innovative curriculum materials relies upon analysis of teachers' mathematical and pedagogical conceptions. Entwined within these conceptions are teachers' conceptions about mathematics curriculum. A better understanding of teachers' conceptions of mathematics curriculum is vital to the success of current reform efforts. Although textbooks have long held prominent roles in guiding practice in American classrooms (Tyson-Bernstein & Woodward, 1991), we know surprisingly little about how teachers' conceptions of curriculum materials relate to their conceptions of mathematics, teaching, and learning, and how they develop during teacher education and school-based experiences.

Teacher learning with reform-oriented mathematics curriculum materials

To support teachers in reforming mathematics classroom activity, numerous sets of reform-oriented curriculum materials have been developed (e.g., Investigations of Number, Data, and Space; Connected Mathematics Project; Core-Plus Mathematics Project; Interactive Mathematics Program; Mathematics in Context; etc.). Although the curriculum materials of these programs incorporate specific aspects of reform recommendations in diverse ways (emphasizing different themes or activities), the materials share certain qualities that distinguish them from traditional mathematics textbooks.

First, reform curricula explicitly incorporate reform ideas about mathematics and pedagogy by emphasizing inquiry mathematics: student explorations of real-world mathematical situations and discussions of problem-centered activities. Furthermore, the materials are formatted to support these mathematical and pedagogical differences. American texts are typically divided into chapters outlining self-contained daily lessons for the teacher to present (composed primarily of definitions and examples of the lesson's content) followed by practice exercises for the student. In contrast, most reform-oriented curriculum materials are published in unit booklets (offering greater flexibility in ordering) that pose large-scale problems and situations, centered on particular mathematical themes and content areas, for students to investigate and debate.

A second substantive difference is that reform-oriented materials generally offer more extensive information for teachers than do traditional texts. In addition to providing problem solutions, the teachers' guides for most of these new materials offer details about different representations of content, historical information about mathematical and pedagogical ideas, examples of what students might believe or understand about particular activities and content, potentially fruitful questions for eliciting discussion, and so on. The inclusion of these details has been motivated in part by the failure of the "teacher-proof" curriculum materials of the 1950s and 1960s to facilitate substantial educational change. After all, it is *teachers* who determine how the innovations envisioned by reformers and curriculum designers become implemented in mathematics classrooms (Cooney, 1988; Freudenthal, 1983).

Contexts for Teacher Learning with Reform-Oriented Curricula

There are many potential ways that teachers can learn from engagement with reform-oriented curricula. This section identifies two different contexts for learning with curriculum materials. My presentation at Oberwolfach will include specific examples to illustrate details of each of these sites for teacher learning with curriculum.

<u>Teacher learning with curriculum in the mathematics classroom</u>. When implementing innovative curriculum materials in K-12 classrooms, teachers are afforded frequent and extensive learning opportunities. Because the act of teaching (regardless of the context - reform or otherwise) is a learning process, instruction necessarily impacts teachers' conceptions. As Ball (1994) describes, teachers continually construct new knowledge from classroom experiences:

> Teachers must figure things out as they teach. They are constantly faced with the data of their own experience. They must develop knowledge of particular children, of the material they are teaching, and of ways to engage students in the content. (p. 9)

The potential for learning from classroom experiences increases as teachers attempt to enact reform visions. Existing conceptions and practices may be directly challenged by the process of interpreting and implementing reform recommendations and curricula. Implementation of innovative curriculum material offers one potentially powerfully site for teachers to learn about themselves, their students, mathematics, and the teaching and learning of mathematics (Ball & Cohen, 1996; Lloyd, 1996; Russell et al., 1995). The experience of implementing novel curricula, however, is not readily available to many teachers.

<u>Teacher learning with curriculum in university and in-service settings</u>. A useful preservice and inservice activity involves inviting teachers to carefully work through the mathematical activities presented in innovative K-12 curriculum materials. Engagement with curriculum *as learners* allows teachers to think about challenging mathematics and the nature of mathematical activity, reflect on the process of learning mathematics to develop empathy for future students, and contemplate teaching mathematics to create new personal visions of classroom practice (Lloyd & Frykholm, in press). Such experiences are critical:

> Teachers themselves need experiences in doing mathematics – in exploring, guessing, testing, estimating, arguing, and proving ... they should learn mathematics in a manner that encourages active engagement with mathematical ideas. (MSEB and NRC, 1989, p. 65)

As teachers revisit mathematical content from new perspectives, they can begin to translate the knowledge developed as learners into pedagogical content knowledge – knowledge of mathematics *for teaching* (Shulman, 1987). Teachers can also begin to revise their views of the types of activities that give rise to rich mathematical understanding, and their views of what constitutes evidence of student understanding. Davenport and Sassi (1995) report that the veteran teachers in their study were profoundly affected by reading detailed classroom narratives in curriculum materials and other print resources. Such detailed descriptions of lessons, including important images of students engaging in meaningful mathematics, helped the teachers to develop visions for their own classrooms. It is likely that preservice and beginning teachers would also benefit in multiple ways by reading and reflecting on the variety of information provided in innovative curriculum materials.

Conceptions of mathematics, learning, teaching, and mathematics curriculum

What role do conceptions play in teachers' learning in the two curriculum contexts identified in the previous section (in classrooms and university/inservice settings)? The model in Figure 1 offers a proposed representation of some of the relationships and constructs involved with this complex process. A central assumption of this model is that teachers' conceptions and their activities in learning contexts are inter-related. For example, classroom implementation of curriculum both influences and is influenced by a teacher's conceptions.

To this point, I have not specified what types of conceptions I am most interested in. This model highlights teachers' conceptions of mathematics, student learning, and mathematics pedagogy. In addition, the model suggests the important role of teachers' conceptions about mathematics curriculum. This branch of teachers' conceptions is intimately related to conceptions of mathematics, learning, and teaching. Teacher learning *about* curriculum is one goal of learning *with* innovative curriculum materials. For instance, reflection on the design and uses of curriculum may help teachers to develop rich conceptions of the role of curriculum in mathematics teaching.

The following four sections elaborate the roles in teacher learning of the four areas of conceptions that comprise the centerpiece of the model in Figure 1. The first step in this elaboration is to clarify my use of the term *conceptions.* Conceptions encompass an individual's knowledge and beliefs. This inclusive terminology avoids the difficult and often contentious distinction between knowledge and beliefs. Although philosophers commonly invoke a sharp distinction by associating disputability with beliefs and truth with knowledge, researchers vary greatly on how they conceptualize and differentiate between knowledge and beliefs (Alexander &

Dochy, 1995; Pajares, 1992). Some researchers, including Dewey (1910), describe beliefs as components of knowledge. For example, Alexander, Schallert, and Hare (1991) propose that "knowledge encompasses all a person knows or believes to be true, whether or not it is verified as true in some sort of objective or external way" (p. 371). My thinking follows this line by describing an individual's knowledge and beliefs – the mental structures that shape how he or she conceives of his or her world – as conceptions.

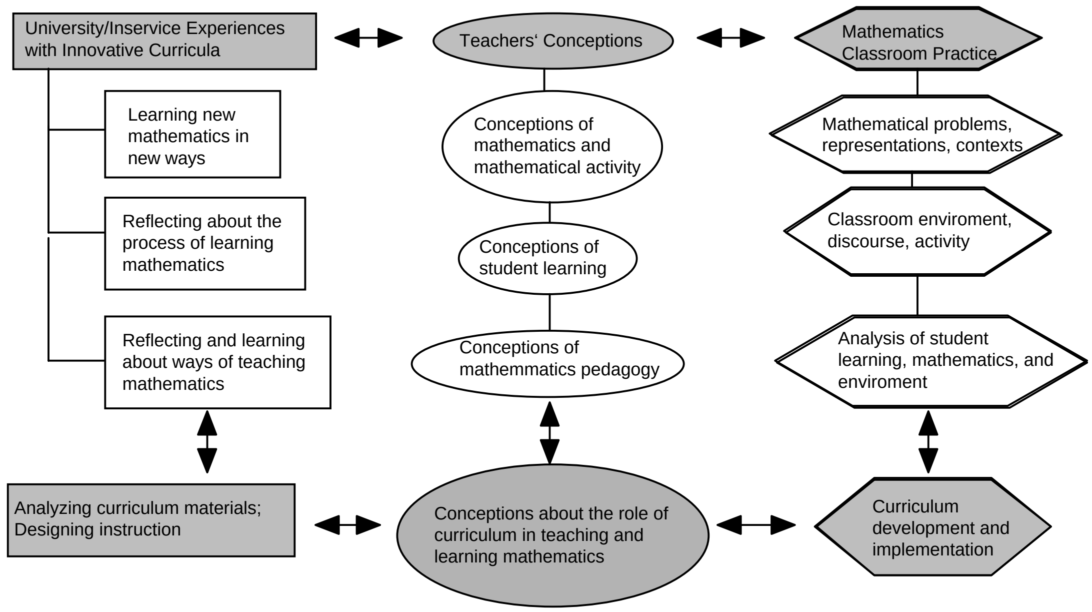

Figure 1. Model of Teacher Learning With and About Mathematics Curriculum

Conceptions of Mathematics and Mathematical Activity

Teachers' conceptions of mathematical subject matter are critical influences on their classroom instruction (Brophy, 1991; Fennema & Franke, 1992; Thompson, 1992) and on their implementations of innovative curriculum materials (Lloyd, 1996; Lloyd & Wilson, 1998). One aim of engaging teachers with innovative curriculum materials is to allow them to learn or re-learn mathematical subject matter currently recommended for school mathematics. Whether it occurs in the classroom while teaching with novel curriculum materials or in university settings while working on curricular activities as students, for most teachers, this learning involves revisiting mathematical ideas that they learned as children to extend their knowledge to include more conceptual or relational understandings. Teachers' learning may also involve exploring entirely new mathematics. For example, many teachers have never learned about probability, statistics, and discrete mathematics – areas now viewed as "big ideas" in the K-12 curriculum. Because reform-oriented curriculum materials have been designed to include these topics, and emphasize conceptual connections, they are an excellent source of mathematical activities that can give these teachers first-hand experiences with the types of mathematics they are expected to teach.

Teachers' conceptions of *mathematical activity* also need to be challenged and developed. Whereas many teachers learned mathematics by memorizing rules, they must now learn to view rich mathematical understanding as the capacity to use mathematics to reason, to communicate, and to pose and solve meaningful problems (Hiebert et al., 1996; NCTM, 1989). Doing so involves learning that a mathematical idea or solution should be judged to be appropriate or correct because it is meaningful and works, not just because the teacher, textbook, or some other outside authority says it is so (Cooney, 1994; Wilson & Goldenberg, 1998; Wilson & Lloyd, in press). Reform-oriented curriculum materials portray mathematics as a vibrant and useful subject to be explored and understood. When teachers learn personally from their work with these materials, or share in their students' engagement with these materials, they are better prepared to make, and more personally invested in making, important changes in their views of appropriate mathematical activity for the classroom.

Conceptions of Learning Mathematics

Mathematics is learned through an active, social process of construction (Cobb, 1995; Davis & Maher, 1990; MSEB & NRC, 1989; von Glasersfeld, 1984). Helping teachers to make sense of constructivist learning theories is a major challenge for those involved with the professional development of teachers. A teacher's conceptions about how students engage in mathematical activity and learn are critical factors in their ability to design and carry out inquiry-based instruction. Researchers on the Cognitively Guided Instruction (CGI) project have promoted the theory that teacher development involves a fundamental change in the content and organization of teachers' knowledge about children's mathematical thought (Fennema et al., 1996). Teachers need to have a sense of how student understanding develops so that they can anticipate what sorts of mathematical activities will help specific students' learning (Even & Tirosh, 1995). For most teachers, development of this sense will involve a shift in how they conceptualize the mathematical learning process.

As a field we know very little about how teachers can learn to center their instructional plans on student development. How do teachers come to view student learning as both the goal and guide of their mathematics classroom practices? A worthy area for research is the potential for teachers' own experiences doing mathematics in reform-oriented ways to support their development of instructional practices that honor and build on students' understandings. Teachers' reflection on experiences learning mathematics with reform-oriented curriculum materials can allow teachers to extract important theories about the nature of the mathematical learning process. Doing so may help them to recognize the significance of the learning that can occur during inquiry and student-centered activities. Further, when they work with teachers' editions (which include descriptions of possible student responses or work) and use innovative curricula to plan instructional activities, teachers may learn about the processes through which young children develop understandings during particular classroom activities.

Conceptions of Mathematics Teaching

The responsibilities of mathematics teachers are extensive. The NCTM (1991) *Professional Standards* delineate four categories of mathematics teachers' work: setting goals and selecting or creating mathematical tasks to help students achieve these goals; stimulating and managing mathematical discourse; creating a classroom environment to support teaching and learning mathematics; and analyzing student learning, the mathematical tasks, and the environment in order to make ongoing instructional actions. These responsibilities suggest the complexities and challenges of what reform-minded teachers aim to achieve in their classrooms. Understanding learners and subject matter as interactive is one of the most important conceptions of teaching. Ball (1993) describes this conception as a "bifocal perspective – perceiving the mathematics through the mind of the learner while perceiving the mind of the learner through the mathematics" (p. 159). For teachers to appreciate and strive for this relationship is a major challenge of reform-oriented teacher development programs. How can teachers come to view learners and mathematics interactively? How does such a relationship develop and play out in the classroom?

Experiences learning and teaching with innovative curriculum materials may compel teachers to recognize that the nature of mathematics communicated in the classroom is intimately linked to the way it is shared with students. If teachers wish to communicate vibrant and useful images of mathematics, they must incorporate a range of pedagogical strategies that engage students in genuine problem-solving and problem-posing. Preservice teachers, whose process of learning to teach in reformed ways is compounded by the pressures of teaching for the first time, may greatly benefit from explicit attention to the development of models of practice during teacher education experiences. Reform-oriented curriculum materials offer useful images of what reformed mathematics teaching can look like (Davenport & Sassi, 1995).

Conceptions of Mathematics Curriculum

The notion that teachers possess conceptions of curriculum is certainly not new. For instance, Shulman (1987) identifies "curriculum knowledge, with particular grasp of the materials and programs that serve as 'tools of the trade' for teachers" (p. 8). However, in the present climate of reform in mathematics education, teachers' conceptions of curriculum have seldom been discussed. Because textbooks and curriculum materials are often teachers' sole contact with reform visions (Ball & Cohen, 1996), this neglect is particularly alarming. Given the prominence of textbooks in teachers' classroom decision-making (Bush, 1986; Tyson-Bernstein & Woodward, 1991), we would be wise to attend more fully to teachers' conceptions of mathematics curriculum and their role in the teacher change process. Reform recommendations and associated curriculum materials cannot and do not bring about change alone – educational change is a complex human endeavor involving teachers *and* texts (Cooney, 1988; Freudenthal, 1983).

Teachers' conceptions of mathematics curriculum include more than a familiarity with the currently available materials for designing mathematics instruction. Conceptions of curriculum encompass understandings of the role of curricular materials in the teaching and learning process, the philosophies of teaching and learning that underlie diverse curriculum materials, knowledge of the appropriateness of particular materials for certain classes and individuals, and the practical and intellectual understandings necessary for making adjustments to curricular approaches. The notion that a textbook outlines *one* of many possible mathematical and pedagogical approaches is central to teachers viewing curriculum as adaptable. Teachers are often dissatisfied with features of textbooks and curriculum materials but tend not to change or adapt those features (e.g., Lloyd, in press). Teachers' treatment of curriculum as fixed suggests that teachers may struggle to conceive of curriculum as a flexible guide that permits and encourages alterations with respect to the changing needs and demands of particular students. As Prawat (1992) explains, a static view of curriculum is one impediment to significant teacher change:

> Instead of viewing students and curriculum interactively . . . teachers tend to regard them as similar factors that somehow must be reconciled. . . . Teachers focus on the packaging and delivery of content, instead of on more substantive issues of knowledge selection and construction. (p. 389)

If teachers are to view students and curriculum dynamically, they need to learn to make classroom-based developments within the curriculum implementation process. After all, curriculum developers may wish to create certain learning experiences for students, but they cannot fully anticipate how particular students will interact with the mathematical activities. Teachers require support not only in coming to recognize the need to adapt curriculum, but also in learning *how* to adapt it.

Experiences with innovative curriculum materials can directly challenge teachers' conceptions of curriculum. The distinctions between reform-oriented and traditional curricula provide immediate opportunities for teachers to explore, and possibly experience, multiple approaches to mathematical subject matter and mathematics pedagogy. Teachers' recognition of the multiplicity of curricular approaches is critical to their movement toward adopting more innovative instructional practices. As teachers identify and weigh the value of specific characteristics of curriculum, they may be pressed to recognize the need to make contextual, classroom-based decisions about instructional design. When teachers' conceptions of curriculum include an inquiry perspective toward *their own* development of pedagogy, we may see a corresponding increase in teachers' ability and inclination to honor and capitalize on students' processes of mathematical sense-making in the classroom. In other words, just as the quality of students' learning hinges upon their ability to make sense of mathematical problems and situations, teachers' development hinges upon educative opportunities to engage in sense-making and problem-solving about mathematics curriculum.

We should take more seriously the powerful role that curriculum materials can play in the learning of teachers throughout their careers. Most teachers rely upon one or two primary textbooks to guide their classroom instruction. If teachers can learn to use their textbooks for their own personal development, then they will be better prepared to learn from and deal productively with the types of materials that will continue to emerge in school settings in the future. Teachers need guidance in learning to make reasoned pedagogical decisions about how to judiciously incorporate the recommendations of curriculum materials into their own instruction. Such learning must extend beyond making choices among particular practices or activities to the broader development of sensible and useable theories of teaching and learning. Explicit emphasis in professional development activities on the role of curriculum materials in students' learning will support teachers in more effectively using textbooks and other resource materials to teach themselves and their students in the future.

References

Alexander, P. A., & Dochy, F. J. (1995). Conceptions of knowledge and beliefs: A comparison across varying cultural and educational communities. *American Educational Research Journal, 32*(2), 413-442.

Alexander, P. A., Schallert, D. L., & Hare, V. C. (1991). Coming to terms: How researchers in learning and literacy talk about knowledge. *Review of Educational Research, 61,* 315-343.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93*, 373-397.

Ball, D. L. (1994, November). *Developing mathematics reform: What don't we know about teacher learning – but would make good hypotheses*? Paper prepared for conference on Teacher Enhancement in Mathematics K-6, National Science Foundation, Washington D. C.

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is - or might be - the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher*, *25*(9), 6-8, 14.

Brophy, J. (Ed.). (1991). *Advances in research on teaching: Teachers' knowledge of subject matter as it relates to their teaching practice* (Vol. 2). Greenwich, CT: JAI.

Bush, W. S. (1986). Preservice teachers' sources of decisions in teaching secondary mathematics. *Journal for Research in Mathematics Education, 17*(1), 21-30.

Cobb, P. (1995). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, *23* (7), 13-20.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal*, *29*, 573-604.

Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis*, *12*(3), 327-345.

Cooney, T. J. (1988). The issue of reform: What have we learned from yesteryear. *Mathematics Teacher, 81*, 352-363.

Cooney, T. J. (1994). Research and teacher education. *Journal for Research in Mathematics Education*, *25*, 608-636.

Davenport, L. R., & Sassi, A. (1995). Transforming mathematics teaching in grades K-8: How narrative structures in resource materials help support teacher change. In B. S. Nelson (Ed.), *Inquiry and the development of teaching* (pp. 37-46). Newton, MA: Education Development Center.

Davis, R. B., & Maher, C. A. (1990). The nature of mathematics: What do we do when we do mathematics? In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist* *views on the teaching and learning of mathematics*. Journal for Research in Mathematics Education, Monograph Number 4, 65-78.

Dewey, J. (1910). *How we think*. Boston: D. C. Heath.

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics*, *29*, 1-20.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*(4), 403-434.

Freudenthal, H. (1983). Major problems of mathematics education. In M. Zweng, T. Green, J. Kilpatrick, H. Pollak, and M. Suydam (Eds.), *Proceedings of the Fourth International Congress on Mathematical Education* (pp. 1-7). Boston: Birkhauser.

Grant, S. G., Peterson, P. L., & Shojgreen-Downer, A. (1996). Learning to teach mathematics in the context of systemic reform. *American Educational Research Journal*, *33*(2), 509-541.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, *25*(4), 12-21.

Lloyd, G. M. (1996). Change in teaching about functions: Content conceptions and curriculum reform. In E. Jakubowski, D. Watkins, H. Biske (Eds.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for Psychology of Mathematics Education* (vol. 2, pp. 200-206). Columbus, OH: ERIC CSMEE.

Lloyd, G. M. (in press). Two teachers' conceptions of a reform curriculum: Implications for mathematics teacher development. *Journal of Mathematics Teacher Education*.

Lloyd, G. M., & Frykholm, J. A. (in press). Middle school mathematics curricula at the university? Using innovative materials to challenge the conceptions of preservice elementary teachers. *Education*.

Lloyd, G. M., & Wilson, M. (1998). Supporting innovation: The impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, *29*, 248-274.

Mathematical Sciences Education Board and National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education*. Washington DC: National Academy Press.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Parajes, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307-332.

Prawat, R. (1992). Teachers' beliefs about teaching and learning: A constructivist perspective. *American Journal of Education*, *100*(3), 354-395.

Russell, S. J., Schifter, D., Bastable, V., Yaffee, L., Lester, J. B., & Cohen, S. (1995). Learning mathematics while teaching. In B. S. Nelson (Ed.), *Inquiry and the development of teaching: Issues in the transformation of mathematics teaching* (pp. 9-16). Newton, MA: Center for the Development of Teaching.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1-22.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Tyson-Bernstein, H., & Woodward, A. (1991). Nineteenth century policies for twenty-first century practice: The textbook reform dilemma. In P. Altbach, G. Kelly, H. Petrie, & L. Weis (Eds.), *Textbooks in American society* (pp. 91-104). Albany: SUNY Press.

von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality* (pp. 17-40). New York: Norton.

Wilson, M., & Goldenberg, M. P. (1998). Some conceptions are difficult to change: One middle school mathematics teacher's struggle. *Journal of Mathematics Teacher Education, 1*, 269-293.

Wilson, M. R., & Lloyd, G. M. (in press). The challenge to share mathematical authority with students: High school teachers reforming classroom roles. *Journal of Curriculum and Supervision*.

Mathematical beliefs and curriculum reform

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During the last decade, efforts to change and improve the mathematics curriculum in the United States have been heavily influenced by the Curriculum and Evaluation Standards of the National Council of Teachers of Mathematics (NCTM, 1989). The NCTM Standards project, including recommendations for teaching (NCTM, 1991) and assessment (NCTM, 1995), has been a stimulus to other curriculum areas as well. The resulting effort to reform the curriculum in all K-12 subject matter areas has had a broad impact on educational policymakers in the USA (McLeod, Stake, Schappelle, Mellissinos, & Gierl, 1996; Ravitch, 1995).

NCTM has now embarked on a new project, Standards 2000 (NCTM, 1998), to revise and update their earlier recommendations. During the development of the curriculum standards (NCTM, 1989), as well as in the Standards 2000 project (NCTM, 1998), various groups have been asked to advise NCTM on its curriculum reform efforts. In this paper the views of various groups are examined, including especially the views of parents and mathematicians, in order to identify prominent beliefs of each group about mathematics and to consider whether significant changes have occurred over recent decades. The purpose of this paper is to investigate how beliefs about mathematics can influence the recommendations on curriculum that come from the various groups that are concerned about mathematics education.

Educational governance in the USA

A wide variety of groups are involved in influencing mathematics education in the USA. Every local community has its own school board elected by the public, and these local political leaders are responsible for overall leadership of the school's curriculum. In spite of the fragmented nature of school leadership, one may still discern a great deal of similarity in the school mathematics curriculum as one moves from school to school and from district to district. NCTM leaders were very aware of these patterns in curriculum and the kinds of parental beliefs that they reflected at the beginning of the current reform effort. These leaders were aware of the challenge that they faced in trying to change beliefs about mathematics and the mathematics curriculum that were deeply embedded in the culture. As NCTM president Shirley Hill observed in her presidential address (Hill, 1980), the public and its representatives on school boards tended to think of mathematics only in terms of basic computational skills, and it was NCTM's obligation as a professional organization to try to change that traditional view of mathematics.

Beliefs expressed by parents

NCTM has had substantial documentation on what parents expected from school mathematics for many years. In one part of the Priorities in School Mathematics project (NCTM, 1981), data from a survey of parents documented their mostly traditional views of mathematics. Although other groups in the survey were responding in a variety of different ways to issues like how technology might change the curriculum, parents maintained their beliefs that mathematics was primarily about computation and that calculators should not be used in the classroom.

These views were confirmed in the process of review that led up to the publication of the Curriculum and Evaluation Standards (NCTM, 1989). For example, in 1988 the Mathematical Sciences Education Board (MSEB) conducted a review of the draft of the 1989 Standards using focus groups such as parents, mathematicians and statisticians, school administrators, school board members, and representatives of science and industry (MSEB, 1988). Again, the parents were among the most traditional in their approach to mathematics. They expressed their concern about any change in emphasis on traditional computational skills, particularly any use of calculators in the elementary grades (MSEB, 1988; McLeod et al., 1996).

These traditional views of the mathematics curriculum continued to dominate parental concerns in the 1990s. Mathematically Correct, a national organization devoted to the "concerns raised by parents and scientists" about the school mathematics curriculum, states its views in great detail on its web site (http://ourworld.compuserve.com/homepages/mathman/).

Material from the web site provides a traditional view of mathematics that is stated explicitly as the "truth" as opposed to the views expressed in the NCTM Standards documents. The web site focuses extensively on the need to "restore basic skills to math education." In one of its segments, contributed by William G. Quirk, the web site explains that learning mathematics involves building a knowledge base that "consists of math facts tightly linked to math skills." "Math facts" are said to consist of undefined terms, definitions, axioms, and theorems, and examples of these items are given in an illustrative proof of the theorem that 2 + 2 = 4, noting that the authors "are assuming the traditional 'formalist' philosophy of mathematics."

In commenting on NCTM's curriculum recommendation, the web site of Mathematically Correct observed that NCTM "sees math through the subjective prism of progressive social science," honoring children's answers as valuable and praiseworthy even if not mathematically correct. In contrast, the web site observes that mathematics is not subjective and that "precision and exactness" are the hallmarks of mathematics. According to these critics, arithmetic is not "natural or intuitive"; you need to use the "immense power of human memory" to be successful in mathematics.

The comments on the Mathematically Correct web site are a small indication of how reform in mathematics education has become a part of the broader "culture wars" in the USA. These disagreements, which seem to center around the implications of postmodernism for the broader society, have become influential in the "reading wars" and the "math wars" in California (see, for example, Sowder, 1998).

The traditional beliefs about mathematics are also reflected in the responses of some teachers to the NCTM Standards (NCTM, 1989). When a consultant to an elementary school worked with teachers over the course of a year to focus on mathematical understanding rather than just memorization of computational algorithms, some teachers concluded that the consultant was telling them not to teach computational skills at all (McLeod et al., 1996). Such a conclusion was a surprise to the consultant and a powerful example of how difficult it is to communicate when people have different beliefs, and how difficult it is to change beliefs about the mathematics curriculum.

Beliefs expressed by mathematicians

Mathematicians differ in their beliefs about mathematics (Toerner, 1998). Mathematicians in the USA also have a long history of disagreeing about what should be included in school mathematics. As Kilpatrick (1997) pointed out, the reforms attempted in the "new math" era of the 1960s were valued by many mathematicians for the emphasis on unifying concepts like sets and functions. Others thought that the new curricula were too formal and put too much emphasis on pure mathematics. Morris Kline was among the most outspoken critics, arguing that the emphasis should be on mathematics as one of the liberal arts with connections to science and history. Unfortunately, it seemed that the part of Kline's critique that reached the public was his catchy book title "Why Johnny Can't Add." Kilpatrick (1997) noted that the title was not Kline's idea, but his publisher's. The title did seem to capture the interest of the public and to solidify the public's rejection of "new math."

During the 1970s the role of mathematicians in curriculum issues was much reduced. The changes in the field of mathematics, with the rise of computer science and the increased emphasis on applied mathematics, seemed to be the focus of mathematicians. The lack of federal funding for curriculum development was another major factor that led many mathematicians to pursue other interests. Also, the 1970s were a period of substantial growth and developments of leadership in mathematics education, particularly at NCTM (McLeod et al., 1996). As a result the attempts to resist the narrowing of the mathematics curriculum that occurred as part of the "back to basics" shift of the 1970s were led mainly by NCTM and other groups with a sustained interest in precollege mathematics education.

During the 1988 review of the draft of the Curriculum and Evaluation Standards (NCTM, 1989), there were focus groups for both mathematicians and statisticians. The report of their recommendations indicated substantial support for the general direction of the NCTM Standards project, but both groups raised questions. For example, some mathematicians expressed concern about any reduction in emphasis on computation, and the statisticians were concerned that the standards on statistics and probability were overly ambitions (MSEB, 1988). Nevertheless, all the major organizations in the mathematical sciences were listed in the 1989 Standards as "Endorsers" who supported the vision of school mathematics contained in the document. This support from organizations like the American Mathematical Society (AMS), American Statistical Association (ASA), and the Mathematical Association of America (MAA) was an important part of the positive reception that the NCTM Standards initially received.

In the Standards 2000 project, NCTM has made special efforts to include mathematicians in planning and reviewing the document. For example, NCTM engaged Association Review Groups (ARG) from the major organizations in the mathematical sciences to provide a formal review of the draft (NCTM, 1998). The reviews by the representatives of AMS, ASA, MAA, and other associations (available through the NCTM Standards 2000 web site at http://www.nctm.org/standards2000/) provide significant documentation of the discussion and recommendations by ARG participants.

In contrast to 1988, when the Standards project was not well known as a significant force in mathematics education, the 1998 draft of the Standards 2000 project has received considerable attention from mathematical organizations. Even AMS, an organization of research mathematicians that in former years showed little interest in mathematics education, invested considerable effort in its response to the Standards 2000 project. As with the other organizations, a wide range of views were expressed about mathematics. Although it is not possible to summarize the comments briefly, it is possible to identify some of the comments that suggest particular beliefs about mathematics.

Comments from the AMS web site often supported the NCTM in its efforts to reform the curriculum. For example, the notion that "mathematics should make sense to every child" was supported, as was the notion that computational skill should be "PART (but not ALL) of the school mathematics experience." But many of the mathematicians continued to ask for more emphasis on fluency with basic facts and the mastering of fundamentals. These comments were often associated with a concern about the use of calculators, especially in the early grades, and the suggestion that perhaps the decision on calculator use should be left to the individual teacher.

In their discussion of proof, the AMS ARG indicated proof "is the heart of the subject," and indicated support for the idea of "local proof" as suggested in the 1989 Standards. They also recommended that reasoning "should play a major role in high school geometry, but perhaps not [receive] a full axiomatic treatment."

The MAA ARG recommended more clarity about the balance between pure and applied mathematics, and seemed to support the NCTM emphasis on problem solving, suggesting more emphasis on problem formulation. They also argued for an emphasis on the learning of skills, sometimes along with understanding and sometimes with the suggestion that rote learning may be appropriate. There seemed to be a great deal of disagreement among members of the MAA ARG about certain issues. Some ARG members seemed to reluctantly agree that NCTM had made a significant effort only to point out that the effort was nevertheless lacking in some important respect. For example, a critic noted that the Standards 2000 project had paid some attention to the theoretical nature of mathematics, but put too much stress on utilitarian aspects of mathematics without noting how the applications often came from seemingly unrelated ideas in abstract mathematics. Another critic thought the treatment of statistical topics was a strength, but then wondered what material in the current curriculum would be replaced if that amount of statistics was included.

The ASA ARG was perhaps the most supportive of the Standards 2000 project, possibly because the discipline of statistics has itself undergone substantial change in recent years. The ASA regularly emphasized understanding and reasoning rather than procedural aspects of statistics, and suggested that NCTM was doing a good job in dealing with algorithmic thinking. They also suggested replacing “traditional axiom-proof mathematics with more applied mathematics.”

Since the traditional curriculum has placed little emphasis on statistical ideas, it is perhaps natural that statisticians seem more pleased with the NCTM Standards in general than the AMS and MAA ARG members. But the kinds of changes in the curriculum that NCTM recommends — more problem solving, communication, applications, use of technology—are characteristic of the kinds of changes that statistics has undergone in recent years.

One might also speculate that statistics has a different point of view since it has never been in a position of ownership with regard to school mathematics. Some mathematicians clearly resist the way that NCTM has taken the leadership on reform issues. In responding to the NCTM initiative to reform school algebra, a prominent critic stated that NCTM had no role in such an effort; in his view, only the best algebraists were qualified to participate in a reconceptualization of school algebra.

As Steen (1996) pointed out, there are some people who think that their group owns school mathematics. He criticizes NCTM and some parent groups for being among those who sometimes act as if they are the only owners of mathematics. It appears that some mathematicians also need to expand their ideas about who owns mathematics. Steen argued that all of us have a stake in school mathematics, and many voices deserve to be heard. Perhaps through careful listening to others we can identify where our beliefs come into conflict and have a better chance to resolve some of our disagreements about the school mathematics curriculum.

References

Hill, S. A. (1980). An agenda for action: President's address, 58th annual meeting. *Mathematics Teacher, 73,* 473-480.

Kilpatrick, J. (1997). Confronting reform. *American Mathematical Monthly, 104,* 955-962.

Mathematical Sciences Education Board. (1988). *Review by selected constituencies outside the mathematics teaching community of the Curriculum and Evaluation Standards for School Mathematics.* Washington, DC: National Research Council.

McLeod, D. B., Stake, R. E., Schappelle, B., Mellissinos, M., & Gierl, M. J. (1996). Setting the Standards: NCTM's role in the reform of mathematics education. In S. A.

Raizen & E. D. Britton (Eds.), *Bold ventures: Volume 3: Case studies of U.S. innovations in science and mathematics education* (pp. 13-132). Dordrecht: Kluwer.

National Council of Teachers of Mathematics. (1981). *Priorities in school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1998). *Principles and standards for school mathematics.* Reston, VA: Author.

Ravitch, D. (1995). National standards in American education: A citizen's guide. Washington, DC: Brookings Institution.

Sowder, J. (1998, March). "What Are 'Math Wars' in California All About? Reasons and Perspectives." Paper presented as the Annual Phi Beta Kappa Lecture, San Diego State University, San Diego.

Steen, L. A. (1996). Who owns school mathematics? *The California Mathematics Council Communicator, 20* (3), 25-29.

Toerner, G. (1998). Mathematical beliefs and their impact on mathematics teaching and learning. In J. Leinonen (Ed.), *Proceedings of the Research Association of Teaching Mathematics and Science* (pp. 221-246). Roveniemi, Finland: University of Lapland, 1998.

Images des mathématiques chez des futurs maîtres

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RÉSUMÉ.

Dans le présent article, nous exposons les résultats d'une recherche sur la perception qu'ont des mathématiques des étudiantes et des étudiants se préparant à enseigner au primaire ou au secondaire ($N = 54 + 42$). Les données ont été recueillies à l'aide d'un questionnaire comprenant deux questions ouvertes ainsi qu'une série d'énoncés à évaluer selon l'échelle de Likert. Nous comparons les réponses obtenues par ces deux moyens. Nous comparons également les résultats concernant les personnes se destinant à l'enseignement au primaire et ceux qui touchent les personnes se dirigeant vers l'enseignement au secondaire.

ABSTRACT.

This article presents the results of a study about the views of mathematics held by 54 primary and 42 secondary prospective teachers. Data were collected through a questionnaire that included two open questions as well as a series of items to be assessed on a Likert scale. We compare the results obtained by these two means. We also compare the results concerning the primary student teachers with those concerning the secondary student teachers.

Balancing between cognition and affect: Students' mathematics-related beliefs and their emotions during problem solving

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Introduction

Recent theories on cognition and learning (e.g., Greeno, Collins, & Resnick, 1996; Salomon & Perkins, 1998) point to the social-historical embeddedness and the constructive nature of thinking and problem solving. To their opinion each form of knowing and thinking is constituted by the meanings and rules that function in the specific communities it is situated in (e.g., the scientific community, the classgroup,…). Acquiring knowledge or learning, consequently, consists of getting acquainted with the concepts and rules that characterize the activities in the respective contexts. As such, learning becomes fundamentally a social activity. From such a perspective learning is primarily defined as a form of engagement that implies the active use of certain cognitive and metacognitive strategies, but can not be reduced to it. After all, more and more researchers (e.g., Bereiter & Scardamalia, 1993) are convinced that only referring to cognitive and metacognitive factors doesn't capture the heart of learning. Several studies (e.g., Connell & Wellborn, 1990; Schiefele, & Csikszentmihalyi, 1995) point to the key-role conative and affective factors play as constituting elements of the learning process, next to and in close interaction with (meta)cognitive factors.

Recent developments in the field of research on mathematical problem solving, tend to illustrate this change in perspective. Studies on students' beliefs about mathematics (e.g., Schoenfeld, 1985) or motivational beliefs (e.g., Pintrich & Schrauben, 1992; Seegers & Boekaerts, 1993), as well as research on the influence of emotions (Cobb, Yackel, & Wood, 1989; McLeod, Metzger, & Craviotto, 1989) and other affective factors as "students' perceived confidence' (Vermeer, 1997) are directed at unraveling the role of conative and affective factors in mathematical problem solving. On a conceptual level researchers try to capture the interrelated influence of (meta)cognitive, conative and affective factors on mathematical learning and problem solving in the notion 'mathematical disposition'. As an equivalent of mathematical competence it refers to the integrated mastery of cognitive, conative and affective aptitudes that results in a sensitivity for occasions when it is appropriate to use mathematical skills and knowledge and an inclination to do so (De Corte, Verschaffel,

& Op ‘t Eynde, in press). It is not surprising then that a mathematical disposition becomes *the goal* of mathematics education (see NCTM, 1989)

This increased concern for motivational and affective influences on cognition undoubtedly stimulates research on these variables. This is, however, more the case for studies on motivation and volition (e.g., Corno, 1993; Niemivirta, 1996; Pintrich & Garcia, 1994) then for research on affective variables. Traditionally, specific affective variables as, for example, feelings, emotions and moods have been less extensively studied, then motivational variables (see Boekaerts, 1994; Pekrun, 1990). Only since the last decade, scholars (e.g., Boekaerts, 1992, 1995; Mc. Leod, 1992; Snow, Corno, & Jackson III, 1996) have started to study and to conceptualize the different affective variables in relation to each other, on the one hand, and to motivation and cognition, on the other hand. McLeod (1988,1989, 1992) did pioneering research on the role of affect in mathematics education. He argued that "if research is going to help us understand the role of affect in mathematics learning and teaching, studies of affect must be integrated with studies of cognition" (1992, p. 588)

In studying the role of mathematics-related beliefs in mathematical problem solving we situate our research within this integrated perspective. Student's mathematics-related beliefs are situated at the intersection of the cognitive and the motivational, or better affective, domain. On the one hand, students' beliefs determine how one chooses to approach a problem and which techniques and strategies will be used (Schoenfeld, 1985a). On the other hand, it is argued that they provide an important part of the context within which emotional responses to mathematics develop (McLeod, 1992). Rarely scholars have addressed in their research this relation between students' mathematics-related beliefs and emotions experienced during problem solving in the classroom. We need to develop a better understanding of how mathematics-related beliefs determine the emotions they experience *and* the influence they have on students' problem solving behavior. This analysis of the relations between students' mathematics-related beliefs, their emotions, and their problem-solving behavior has become the focus of our research.

Investigating the role of emotions in problem solving

The emotional reactions of students have not been major factors in research on affect in mathematics education. This lack of attention to emotion is probably due to the fact that research on affective issues has generally looked for factors that are stable and can be measured by questionnaire. (McLeod, 1988). Nevertheless, there are some studies that investigate the role of emotions during mathematical learning and problem solving (e.g. McLeod, Metzger and Craviotto, 1989; Prawatt & Anderson, 1994, Seegers & Boekaerts, 1993; Vermeer, 1997). Prawatt & Anderson (1994), for example, found that students' affect in class was primarily negative and that certain affects systematically related to certain antecedents, for example confusion and lack of understanding. Knowing some immediate antecedents of emotions in class is important, but if we really want to understand the role of emotions in mathematics lessons, we should also learn more about student aptitudes that explain those specific emotional experiences, i.c. students' mathematics-related beliefs. Moreover, we should also investigate the immediate consequences of specific affective experiences in class. How do emotional experiences influence the learning process of students? What is their influence on conative and (meta)cognitive processes?

Using a qualitative approach we studied students' emotional, motivational and (meta)cognitive processes in relation to their mathematics-related beliefs when solving a mathematical problem in class. Taken into account the complexity of the phenomenon under study and the specific nature of our variables, we opted for a "multiple case study' (Merriam, 1998, Yin, 1994) using questionnaires, documents, observations and interviews to gather data.

The study was situated in the second year of junior high school (age 14) and took place in three different classes in three different schools.

Students' mathematics-related beliefs

Students of these classes were presented a self-developed questionnaire on mathematics-related beliefs. Starting from existing questionnaires (e.g., Pintrich et al, 1993; Schoenfeld, 1985b; Yackel, 1984) who usually measure only one kind of beliefs (e.g., or beliefs about math, or motivational beliefs), we developed a more integrated instrument that asked students about their beliefs on mathematics education, on their self-related beliefs in relation to math, and on their beliefs about the social context in their specific class.

1. **Beliefs about mathematics education.**
   - Beliefs about mathematics
   - Beliefs about mathematics learning and problem solving
   - Beliefs about mathematics teaching
2. **Beliefs about the self in relation to mathematics.**
   - Goal orientation beliefs
   - Task value beliefs
   - Control beliefs
   - Self-efficacy beliefs
3. **Beliefs about the social context, i.e. the class context**

Beliefs about mathematics education include students' view on the nature of mathematics, as well as, on their general ideas about what characterizes mathematical learning and teaching. Students' beliefs about the self in relation to mathematics refer to what in the motivational research literature is labeled as motivational beliefs (e.g., Pintrich & Schrauben, 1992).

The third category, beliefs about the social context of mathematics education, refers to students' views and perceptions of the classroom norms, including the social and the socio-mathematical norms, that direct teachers' and students' behavior *in their specific classroom* (Cobb & Yackel, 1998). It includes their perceptions about the role of the teacher as well as their own and their fellow students' role in their mathematics classroom, but also students' beliefs about aspects of the class culture that are specific to mathematical activity. The latter refers, for example, to students' view on what counts as a different solution or as an acceptable explanation *in their class.*

Until now there is little research that addresses these beliefs about the specific social context of the class, and how they relate to the more general beliefs about mathematics education and the self. These general beliefs are abstracted from one's experiences and from the classroom culture in which one is embedded (Schoenfeld, 1992); but, this culture is such a complex phenomenon of rules and interactions that there is clearly no linear relation between 'a class context' and more general beliefs about mathematics and the self. Therefore, in order to fully understand the influence of mathematics-related beliefs on students' learning and problem solving, it is necessary to focus in not only on their general beliefs about mathematics education and the self, but also on their beliefs about the mathematics class context in which they have to perform.

Tracking the problem-solving process

Two months after presenting the questionnaires to the classes we made a selection of six students, one high and one low achiever out of each class as evaluated by the teacher. They were asked to solve a complex realistic mathematical problem in class and had to fill in the first part of the "On-line Motivation Questionnaire (OMQ)" (Boekaerts, 1987) after they had skimmed it and before they actually started to work. Every student was asked to think aloud during the whole problem solving process that was also videotaped. Immediately after finishing, the student accompanied the researcher to a room adjoining the classroom where a 'Video Based Stimulated Recall Interview' took place (Prawatt & Anderson, 1994).

This interview procedure consisted of 3 phases. In the first phase the student and the researcher watched the videotape and the student was asked to recall what he did, thought and felt while he was solving the problem, especially during those episodes that he was not thinking aloud. After the student had described in his own words his complete problem solving story (phase 1), the interviewer asked questions for clarification (phase 2: what and how questions); on what he saw on the screen; on what the student told him; on what the student wrote down on the OMQ, and what he wrote on the answer form. Finally (phase 3), the researcher tried to unravel the subjective rationale for the students' problem solving behavior. He looked for the interpretations the student gave to certain situations. The 'Why questions' that were asked, made underlying beliefs more visible and as such clarified the relation between beliefs, emotions and problem solving behavior.

Data analysis

The analysis of the data involved a cyclic procedure. For each student, we first studied the thinking aloud and problem solving protocol, his answers on the OMQ, the interview transcript and the videotape of the problem solving process. We used these different data sources to describe chronologically as truthful and as detailed as possible the different experiences and (mental) activities that characterized the problem solving process. This resulted in six rich narratives of the way student's handled and experienced the problem that, secondly, were content analyzed in a more systematic way to develop a coding system that characterizes the essence of each subject's responses. Starting from the different phases Schoenfeld (1985) recognizes in problem solving, we divided the narratives in episodes that were further fragmented using categories as 'calculating', 'thinking what to do next', 'frustration', 'loosing concentration', etc.. Although attention was paid to cognitive, metacognitive, motivational and affective processes, the focus was on the last one, resulting in a fine-grained analysis of the emotional dimension of problem solving.

In the third phase of the analysis the focus moves from describing to explaining. The data, specifically student's task-specific perceptions (OMQ) in the orientation phase, are re-analyzed to unravel and explicate relations between these perceptions and what actually happened during problem solving. The interview transcripts are also further investigated to look for relations between students' mathematics-related beliefs and their problem-solving behavior. Students' results on the belief questionnaire are also taken into account.

Finally, after this vertical analysis of each students' problem solving process, a more horizontal approach was taken to look for recurrent patterns and/or fundamental differences that might deepen our understanding of what happens during problem solving and more specifically the role of emotions in this process.

Preliminary results

Students' experience different emotions during mathematical problem solving. We found students to be

- annoyed - frustrated - angry
- worried - anxious
- relieved
- happy
- nervous
- not happy

Moreover, some of these were frequently observed in a systematic order that was characterized by an intensification of the emotion experienced.
Lukas was first annoyed when he did not immediately know how to solve the problem. When after 20 seconds he still couldn't solve it, he became frustrated and later on angry on himself.

Most of the negative emotions were experienced the moment students were not able anymore to solve the problem fluently. This cognitive blockage has apparently an emotional correlate (see also Mandler, 1989). The experience of the emotion always triggered students to redirect their behavior using alternative cognitive strategies or heuristics to find a way out of the problem. However, there are big differences in the effectiveness and efficiency of the cognitive strategies used.

Only when students experienced a strong negative emotion, for example when they became really angry, they thought of using an emotional regulation strategy. However, none of them actually used one.

The kind of emotions students experience during problem solving and their intensity appear to be determined by students' beliefs, and more specifically their beliefs about the self. The task specific perceptions in the orientation phase are closely related to these beliefs, although they sometimes differ, due to the specific context.
Steve (the low achiever) who is absolutely convinced that he is no good in mathematics, scores high on perception of self-efficacy for this problem, because he knows that he is good in word problems.
Although the influence of beliefs on students' problem solving is mediated through these task specific perceptions (see Seegers & Boekaerts, 1993), it turns out that these more general beliefs also have a direct influence through the experienced emotions during problem solving, specifically when they differ from the task specific perceptions. Steve who is as convinced as Lukas that he will be able to solve this specific task, starts doubting his capacities at the first cognitive blockage. "If I'm already not able to solve this, than I surely will not be able to solve the problem" Lukas confronted with the same problem never started questioning his capacities. Apparently the difference in their beliefs on their general mathematics competence made the difference. Those who belief in themselves at this level act self-confident throughout the problem solving process. Others, who score low on the confidence in their mathematics capacities, start questioning their capacities the minute they encounter a problem.

Students' mathematics-related beliefs, especially their beliefs about the self, have been shown to influence the problem-solving process indirectly through the task-specific perceptions during the orientation phase (see, Seegers & Boekaerts, 1993). However, our preliminary results indicate that these beliefs also determine students' problem solving in a more direct way. Students' emotions during problem solving and their impact on their further behavior seems to be at least partly a function of students' general beliefs about the self, independently of their earlier task -specific perceptions.

References

Bereiter, C., & Scardamalia, M. (1993). *Surpassing Ourselves: An inquiry into the nature and implications of expertise*. Chicago: Open Court.

Boekaerts, M. (1995). Self-Regulated Learning: Bridging the gap between metacognitive and Metamotivation theories. *Educational Psychologist*, 30(4), 195-200.

Boekaerts, M. (1994). Action control: How relevant is it for classroom learning? In J. Kuhl, & J. Beckmann (Eds.), *Volition and personality: Action versus state orientation* (pp. 427-435). Seattle: Hogrefe & Huber Publishers.

Boekaerts, M. (1992). The adaptable learning process: Initiating and maintainig behavioural change. *Applied Psychology: An International Review*, 41(4), 377-397.

Boekaerts, M. (1987). Situation specific judgements of a learning task versus overall measures of motivational orientation. In E. De Corte, R. Lodewijks, R. Parmentier, & P. Span (Eds), *Learning and Instruction* . Oxford: Pergamon Press.

Cobb, P., & Yackel, E. (1998). A constructivist perspective on the culture of the mathematics classroom. In F. Seeger, J. Voigt, & U. Waschescio (Eds.), *The culture of the mathematics classroom* (pp. 158-190).

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while engaged in mathematical problem solving. In D. B. McLeod, & V. M. Adams (Eds.), *Affect and Mathematical Problem Solving: A new perspective* . New York: Springer-Verlag.

Connell, J. P., & Wellborn, J. G. (1990). Competence, Autonomy, and relatedness: A motivational analysis of self-system processes. In M. Gunnar , & L.A. Sroufe (Eds.), *Minnesota Symposium on Child Psychology* Vol. 23). Hillsdale, N.J.: Erlbaum.

Corno, L. (1993). The Best-Laid Plans: Modern Conceptions of Volition and Educational Research. *Educational Researcher*, 22(2), 14-22.

De Corte, E., Verschaffel, L., & Op 't Eynde, P. (in press). Self-regulation: A characteristic and a goal of mathematics learning. In M. Boekaerts, P. R. Pintrich, & M. Zeidner (Eds.), *Handbook of self-regulation* . San Diego: Academic Press.

Greeno, J. G., Collins, A. M., & Resnick, L. B. (1996). Cognition and learning. In D. C. Berliner, & R. C. Calfee (Eds.), *Handbook of Educational Psychology* (pp. 15-46). New York: Simon & Schuster Macmillan.

Mandler, G. (1989). Affect and Learning: Causes and Consequences of Emotional Interactions. In D. B. McLeod, & V. M. Adams (Eds.), *Affect and Mathematical Problem Solving: A new perspective* (pp. 3-19). New York: Springer-Verlag.

McLeod, D. B. (1992). Research on affect in mathematics education: a reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics* (pp. 575-596). New York: Macmillan.

McLeod, D. B. (1989). The role of affect in mathematical problem solving. In D. B. McLeod, & V. M. Adams (Eds.), *Affect and Mathematical Problem Solving: A new perspective* (pp. 20-36). New York: Springer-Verlag.

McLeod, D. B. (1988). Research on learning and instruction in mathematics: The role of affect. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics. Papers from the First Wisconsin Symposium for Research on Teaching and Learning Mathematics* (pp. 60-89). Wisconsin: Wisconsin Center for Education Research, University of Wisconsin

McLeod, D. B., Metzger, W., & Craviotto, C. (1989). Comparing experts' and novices' affective reactions to mathematical problem solving: An exploratory study. In G. Vergnaud (Ed.), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education* (Vol. 2pp. 296-303). Paris: Laboratoire de Psychologie du Developpement et de l'Education de l'Enfant.

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass Publishers.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Niemivirta, M. (1996). *Intentional and adaptive learning modes: The self at stake*. Paper presented at the second European Conference on Education, September 1996, Sevilla, Spain.

Pekrun, R. (1990). Emotion and motivation in educational psychology. In P. J. D. Drenth, J. A. Sergeant, & R. J. Takens (Eds.), *European perspectives in psychology: Theoretical, psychometrics, personality, development, educational, cognitive, gerontological* (Vol. 1pp. 265-295). Chisester, UK: John Whiley & Sons.

Pintrich, P. R., & Garcia, T. (1994). Self-regulated learning in college-students: Knowledge, strategies, and motivation. In P. R. Pintrich, D. R. Brown, & C. E. Weinstein (Eds.), *Student motivation, cognition, and learning Essays in honor of Wilbert J. McKeachie* (pp. 113-133). Hillsdale, NJ: Lawrence Erlbaum Associates.

Pintrich, P. R., Smith, D. A. F., Garcia, T., & McKeachie, W. J. (1993). Reliability and predictive validity of the Motivated Strategies for Learning Questionnaire (MSLQ). *Educational and Psychological Measurement*, 53, 801-813.

Pintrich, P. R., & Schrauben, B. (1992). Students' motivational beliefs and their cognitive engagement in academic tasks. In D. Schunk, & J. Meece (Eds.), *Students' perceptions in the classroom: Causes and consequences* (pp. 149-183). Hillsdale, NJ: Lawrence Erlbaum Associates.

Prawat, R. S., & Anderson, A. L. H. (1994). The affective experiences of children during mathematics. *Journal of Mathematical Behavior*, 13, 201-222.

Salomon, G., & Perkins, D. N. (1998). Individual and social aspects of learning. In P. D. Pearson, & A. Iran-Nejad (Eds.), *Review of Research on Education* Vol. 23 (pp. 1-25). Washington, DC: AERA.

Schiefele, U., & Csikszentmihalyi, M. (1995). Motivation and ability as factors in mathematics experience and achievement. *Journal for Research in Mathematics Education*, 26(2), 163-181.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics . In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Schoenfeld, A. H. (1985a). *Mathematical problem solving*. Orlando,Florida: Academic Press.

Schoenfeld, A. H. (1985b). Students' beliefs about mathematics and their effects on mathematical performance: A questionnaire analysis. Paper presented at the Annual Meeting of the American Educational Research Association, March 31 - April 4, 1985, Chicago, Illinois.

Seegers, G., & Boekaerts, M. (1993). Task motivation and mathematics in actual task situations. *Learning and Instruction*, 3, 133-150.

Snow, R. E., Corno, L., & Jackson III, D. (1996). Individual differences in affective and conative functions. In D. C. Berliner, & R. C. Calfee (Eds.), *Handbook of Educational Psychology* (pp. 243-310). New York: Simon & Schuster Macmillan.

Vermeer, H. J. (1997). *Sixth-grade students' mathematical problem-solving behavior: Motivational variables and gender differences*. Leiden: UFB, Leiden University.

Yackel, E. (1984). Mathematical belief systems survey. Purdue University West Lafayette, In.

Yin, R. K. (1994). *Case study research: Design and methods*. Thousand Oaks: Sage Publications.

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Beliefs as obstacles for implementing an educational change in problem solving

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Summary

From an international need for change, a framework of teacher change is discussed, emphasizing the complexity of change. The basic concepts in teachers' and pupils' beliefs are described. The theoretical statements on beliefs as obstacles for an educational change are illustrated within the research program in Helsinki (1989-98) which aims for improvement in mathematics teaching through problem solving in the Finnish lower secondary school.

A THEORATICAL BACKGROUND

The purpose of school education in each country is, more or less, to develop individuals who are independent, self-initiative, critical thinking, motivated and many-sided skilled, and who will manage in societal settings they will counter later on in their life. The key question is that conventional school instruction does not seem to be optimal for this goal. There is a need for an educational change which idea is accepted all over the world.

Complexity of the change

During the last two decades, there have been several trials to improve and to develop mathematics instruction. Careful development programs have been created, interesting materials for problem solving and small group working have been developed, and appropriate models for assessment have been produced, but these actions did not seem to have a major influence the way of teaching in school mathematics (Fennema & Nelson 1997). Although the research community in mathematics education know rather well what kind of change in school instruction we want to have, most of practicing teachers are still in favor of conventional teaching. Cooney and Shealy (1997, 106) put it words, as follows:

> *"For many teachers, changing their teaching of mathematics is problematic and fraught with difficulties … We expect some teachers will change and others will not, despite our best efforts, and that some will change regardless of whether there are external efforts at all."*

Behind this unwillingness to change, there might be their experiences during studies in teacher training, as well as the teaching habits they have developed and considered as effective. Cobb and his group describe such a teacher as an example in their research report (Cobb & al. 1990): She was voluntarily taking part in the research project which aimed to change mathematics teaching in elementary school, but she always fell back to her own teaching practice as soon as the researcher was no more in her class. The break-through in her change happened until when she saw the results of her teaching, when she realized that her best pupils who can do marvelously the complicated algorithmic tasks had very faint understanding on the algorithms they were doing.

Although we know that conventional teaching methods produce superficial learning, for some of teachers these are the best functioning teaching methods. A plenty of trials to break this belief and to improve school teaching in mathematics are implemented all over the world, and many of them are reported in scientific journal or in the PME proceedings (see e.g. the overview of Pehkonen 1994 or the book Fennema & Nelson 1997). Many of the research reports give an optimistic view on the possibilities for teacher change, but when looking nearer in the research results, there are usually some teachers who have changed and some others who have not. Thus, there seems to exist a factor (or factors) which hinders some teachers from changing.

The framework of beliefs

When trying to bring new approaches in mathematics teaching in school, it seems that teachers' own view of mathematics, i.e. their conception on good mathematics teaching, conducts very strongly their decisions on instruction. If new teaching method is not in concordance with a teacher's view of teaching, the reform will not be successful, although the teacher will be trained. Therefore, the teacher's own beliefs and conceptions on teaching are in a key position (Lerman 1993).

Here we understand that an individual's *beliefs* are formed on the base of his subjective experiences and are rather stable ways of thinking which are usually, more or less, emotion-laden (for more details on the definition of beliefs see e.g. Pehkonen & Törner 1996). *Conceptions* are understood as an individual's conscious beliefs for which he is usually able to give reasons. *Views* belong to conceptions, but they are more spontaneous and have more affective coloring. Beliefs do not occur totally separately, but in clusters which together form a person’s belief system. When discussing on a person’s mathematical belief system, we often use the concept *mathematical world view* which originates from Schoenfeld (1985), or shortly *mathematical view*.

Usually, the use of new approach requires a change in a teacher’s role. The teacher is no more only a transmitter of knowledge, but a guide and a facilitator for learning as well as a planer of learning situations (e.g. Ernest 1991). In order this could be possible, the teacher’s beliefs on teaching and learning often should be changed. If his beliefs on implementation of teaching and on pupils’ learning possibilities will stay conventional, he can feel pupils’ active working in new learning environments problematic, since then several different actions might happen simultaneously in the classroom. Especially a less experienced teacher may feel the unorder as a threat which he is not able to deal with. (Blumenfeld & al. 1991)

RESEARCH PROGRAM "USE OF OPEN TASKS IN MATHEMATICS"

The research projects realized by the author during last ten years in Helsinki are so closely connected with each others that one can speak about a research program. The ultimate goal of the program is to develop mathematics teaching in the lower secondary school in Finland through open tasks. These are used in the form of problem fields; some examples one can find i.a. in Pehkonen (1995b, 1997). Here we will describe only such research projects which have had the funding from the Academy of Finland. In these research projects one can see the above described phenomena ”teachers unableness to change”, although the teachers in question were willing to.

The project "Open Tasks in Mathematics"

The first three-year project ”Open Tasks in Mathematics” was carried out during the years 1989-92 in Helsinki (Finland); it is described in detail in Pehkonen & Zimmermann (1990), and briefly in Pehkonen (1995a). The project was implemented in grades 7-9 (i.e. 13-15 year-old pupils) in lower secondary school, and concentrated on the use of problem fields (a certain type of open problems) in addition to the conventional mathematics teaching. The main experiment began in the fall of 1989 in Helsinki with ten grade 7 classes, and continued with those classes through the whole lower secondary school (up to grade 9), i.e. to spring 1992. Half of the classes formed an experimental group, and the other half a control group.

<u>The description of the project</u>. The purpose of the research project was to develop and foster methods for teaching problem solving (in the sense of open problems) in Finnish lower secondary schools. We tried to stay within the frame of the ”normal” teaching, i.e. in the frame of the valid curriculum, and to take into account the teaching style of the co-operating teachers when using problem fields. The objectives of the research project can be categorized into five fields of emphasis:

(1) To clarify possibilities and methods for the use of problem fields in teaching.

(2) To foster pupils’ attitudes against mathematics and mathematics teaching.

(3) To develop the flexibility of pupils’ thinking.

(4) To examine how pupils’ problem solving ability develops when normal teaching methods are used.

(5) To develop teachers’ conceptions of mathematics and mathematics teaching.

Both in the beginning and at the end of the experimental phase, teachers’ and pupils’ conceptions of mathematics teaching have been gathered using questionnaires and interviews. In the main experiment, the experimental group and the control group differed in the point that from the mathematics lessons of the experimental group about 20 % (i.e. once a month about 2-3 lessons) was reserved for dealing with problem fields. There was a questionnaire for each problem field in which the pupils’ conceptions of using that problem field were ascertained. The teachers’ conceptions of using problem fields were obtained with short interviews after each term. Pupils solved in their class work some open problems, and the problems were same for both.

On the results. Here we will discuss briefly two main groups of the results: Firstly, results in respect of pupils (for more details see e.g. Pehkonen 1995a), and secondly, some results concerning teachers (for more details see e.g. Pehkonen 1993).

Summarizing the results in respect of pupils in the research project, one may state that the pupils experienced the problem fields used as an interesting form of learning mathematics. They liked very much most of them, and were motivated and activated also during other parts of mathematics lessons. Some of them did not consider the solving of problem fields as mathematics, but some kind of amusement. Their mathematical views did not change statistically significantly during the three years of the experiment. But the non-significant changes in the questionnaire data, in the classroom observations, and in their teacher's evaluations indicate that there existed a change, and the change was in most cases positive.

Based on the research findings concerning teachers, some questions arose: On which reasons do a teacher actually form his assessment of the selection of an open-ended problem (or more generally of mathematical teaching material)? It seems that about a third of the teachers based their assessment on the convenience to use the material. Which kind of objectives should we pose for those conducting the change in teaching? In order to reach change with the aid of teaching material, one may choose between, at least, two strategies:

(1) One emphasizes the pupil-centerness and the mathematical content of the tasks. This leads to the problem of an effective teacher in-service training.

(2) One is satisfied with the offering of easy-to-use materials to teachers. This leads to the problem of producing material.

Summarizing, the results suggest that the open approach, when used in addition to the conventional teaching methods, seems to be a promising one. Furthermore, the results of the project showed that such an approach will function in classroom, although resistance for change could be observed in the case of some teachers and pupils which resistance might be due to their beliefs on mathematics teaching. The pupils clearly preferred this kind of mathematics teaching where one important factor was the freedom let to pupils to decide their own learning rate. In the case of teachers, there were some teachers who almost automatically used the ideas in the right way. But there were also teachers who used ”the whole group teaching”-strategy, although other methods were suggested, and failed in their trials.

The project "Development of Pupils' Mathematical Beliefs"

The second project was running in the years 1996-98 and formed a continuation to the earlier one. This focussed on one group of the difficulties (pupils' beliefs) which we encountered when trying to implement the problem solving approach in a classroom in the first project.

The project "Development of Pupils' Mathematical Beliefs” was designed to consider how pupils' mathematical beliefs and belief systems develop within different classroom contexts in lower secondary school for the period of three years, i.e. in the grades 7-9, or with 13-15 year-old pupils (Hannula & al. 1996). The project included two full-time researchers (Mr. Markku Hannula and Ms. Marja-Liisa Malmivuori / Ms.

Kirsti Hoskonen) for three years. The focus of our research was how pupils' beliefs direct their learning and behavior, and what kind of development can be observed in pupils' beliefs. In the implementation of the project, we used mainly intensive case studies, although many kinds of both quantitative and qualitative research methods were used.

The project concentrated on pupils from (mainly) two schools from Helsinki area. In one school (in one class), mathematical beliefs were actively tried to affect and promote by the teacher (Hannula), who at the same time acted both as a mathematics teacher and as a researcher (the action research part). A special emphases was given also to the kind of effects which were viewed to contribute especially to girls' experiences and beliefs in learning mathematics, i.e. to their self-confidence in mathematics. In another school, mathematics teaching of two experienced innovative teachers was observed from outside (the holistic part). For the data gathering we used all kind of methods (questionnaires, interviews, observation), and thus developed a holistic view of mathematics teaching in that school, i.e. in those classes. As a third point, one researcher (Hoskonen) concentrated on investigation of the structure of one pupil's mathematical beliefs. With the aid of deep interviews, she collected information from one female pupil using Kelly's repertory grid technique (Kelly 1955).

There are some preliminary results from the project: Based on the classroom experiences and the research results, Hannula was able to construct a model for the change of pupils' beliefs (Hannula 1998a, 1998b). In the case study of the entire school, we could follow the development of pupils' interests and beliefs on problem solving, which might be consequences of the problem-rich learning environments offered by the teachers (Järvenpää & al. 1999). In the case of one pupil, Hoskonen was able to sketch the pupil's mathematical world view (Hoskonen 1998, 1999).

An additional feature of this project was the inquiry of mathematics professors' beliefs (Pehkonen 1999b). The aim was to find out what kind of beliefs professors at mathematics departments (in different universities) have on school teachers' mathematics view. The central results can be summarized as follows: Professors estimated that the teachers' basic knowledge was poor and old-fashioned, requiring improvement. Furthermore, they emphasized the meaning of a teacher's personality and his own view of mathematics for his teaching. Also they considered the teachers' working environment as being very poor, and stressed that it should be improved.

The project "Teachers' Conceptions on Open Tasks"

Another hindering aspect of the implementation of the problem solving approach - teachers' pedagogical content knowledge - was taken here into the focus. The purpose of the research project "Teachers' Conceptions on Open Tasks" which was implemented during 1998 was to clarify what are teachers' possibilities to apply the principles of open teaching in their daily instruction, especially in the form of open tasks. I.e. to find out teachers' pedagogical content knowledge in the case of open tasks.

The data was gathered with several methods: A questionnaire (statistical data) as well as interviews and observations (qualitative data) were used. But the main method for data gathering was a postal survey. The information obtained with the questionnaire was checked and completed with interview and observation data. The subjects of the postal survey were teachers in the lower secondary school in Finland. From all the Finnish lower secondary schools, it was selected at random every sixth (N = 135), and from each school in question a class. The number of returned filled-in questionnaires was about one half of the original sample (N = 74). The data was analyzed according to the phenomenological principles.

As an answer to the first research question of the project: ”What kind of conceptions do teachers possess on open tasks and on their role in instruction?”, we can state the following: The results obtained showed that about a half of the teachers responding were not able to formulate a proper characterization for open tasks. On one hand we have good reasons to believe that most of those teachers who have not responded belong to this group, and therefore, approximately only one quarter of the Finnish lower secondary school teachers know the term ”open task”. More on the preliminary results can be found in Pehkonen (1999a).

In connection to the project, we studies as a case two schools of a mall city (Valkeakoski) in Southern Finland, exploring teachers’ and pupils’ conceptions on open problems as well as pupils’ ability to solve such. As a result we found out that the pupils have capability to solve such problems, but they needed some hints in order not to stick in their first solution (Pehkonen & Vaulamo 1999).

CONCLUDING COMMENTS

If we will accept the result documented in a large summary by Pajares (1992) that ”*Change of beliefs in adults is rather rare*”, we must figure something out. Research must find out more effective ways of teacher change than those we know. Until today the only successful method seems to be a proper contradictory situation which puts the teacher in question to reevaluate the teaching methods he has been using (cf. Cobb & al. 1990). But this kind of producing contradictory situations for each teacher separately is a too expensive method as teacher in-service education.

In the following I have collected a group of questions which are central for teacher change. The answers to these and similar questions form the key to the change process:

- What are obstacles of change for teachers? What kind of beliefs will hinder teachers from changing? Which beliefs are favorable for change? What is the role of age in change process? Is one of the hindering factors, perhaps, the level of the security a teacher feels? Could it be true that all teachers are not able to change because of their personality characteristics? What is the connection of an individual’s flexibility in thinking (or creativity) to his ability to change?
- When research has been able to gestalt the mechanism of change, how can we continue? How to remove these obstacles of teacher change? What kind of in-service training is most effective? How steady is a teacher change gained?

References

Blumenfeld, P.C., Soloway, E., Marx, R.W., Krajcik, J.S., Guzdial, M. & Palincsar, A. 1991. Motivating project-based learning: Substaining the doing, supporting the learning. Educational Psychologist 26 (3/4), 369-398.

Cobb, P., Wood, T. & Yackel, E. 1990. Classrooms a Learning Environment for Teachers and Researchers. In: Constructivist Views on the Teaching and Learning of Mathematics (eds. Davis & al.), 125–146. JRME Monograph Number 4. Reston (VA): NCTM.

Cooney, T.J. & Shealy, B.E. 1997. On understanding the structure of teachers' beliefs and their relationship to change. In: Mathematics teachers in transition (eds. E. Fennema & B.S. Nelson), 87-109. Mahwah (NJ): Lawrence Erlbaum.

Ernest, P. 1991. The philosophy of mathematics education. London: Falmer Press.

Fennema, E. & Nelson, B.S. (eds.) 1997. Mathematics teachers in transition. Mahwah (NJ): Lawrence Erlbaum.

Hannula, M. 1998a. The case of Rita: "Maybe I started to like math more". In: Proceedings of the PME-22 conference (ed. A. Olivier & K. Newstead), Vol. 3, 33-40. University of Stellenbosch (South-Africa).

Hannula, M. 1998b. PAIRS - a Systemic View on Beliefs. In: Proceedings of the MAVI-7 workshop October 2–5, 1998 (ed. M. Hannula), 39–45. University of Helsinki. Department of Teacher Education. Research Report 198.

Hannula, M., Malmivuori, M.-L. & Pehkonen, E. 1996. Development of pupils´ mathematical beliefs: A description of a research project. In: Current State of Research on Mathematical Beliefs III. Proceedings of the MAVI-3 workshop (ed. E. Pehkonen), 39–47. University of Helsinki. Department of Teacher Education. Research Report 170.

Hoskonen, K. 1998. A student as a mathematics learner. In: Proceedings of the MAVI-7 workshop October 2–5, 1998 (ed. M. Hannula), 51–56. University of Helsinki. Department of Teacher Education. Research Report 198.

Hoskonen, K. 1999. How to study mathematics? In: MAVI-8 Proceedings. Eight European Workshop. research on Mathematical Beliefs (eds. G. Philippou), 67–72. Nicosia: University of Cyprus.

Järvenpää, M., Pehkonen, E., Pulkka, H. & Vaulamo, J. 1999. Ongelmakeskeisen matematiikan opetuksen toteutuksesta peruskoulun yläasteella [On the implementation of problem-centered mathematics instruction in lower secondary school]. To appear in: University of Helsinki. Department of Teacher Education. Research Report.

Kelly, G.A. 1955. The psychology of personal constructs. Volume 1: A theory of personality. New York: Norton.

Kupari, P. 1999. From practising computational skills to problem solving. Mathematics teachers' mathematical beliefs and the construction of their teaching. University of Jyväskylä. Institute for Educational Research. Research Reports 7. [in Finnish]

Lerman, S. 1993. The role of teachers in children’s learning of mathematics. In: Significant influences on children’s learning of mathematics (A. Bishop, K. Hart, S. Lerman & T. Nunes), 61-85. Document series 47. Paris: Unesco.

Pajares, F.M. 1992. Teachers’ beliefs and educational research: Cleaning up a messy construct. Review of Educational Research 62 (3), 307-332.

Pehkonen, E. 1993. On Teachers’ Criteria to Assess Mathematical Activities. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Volume I, 220–227. University of Tsukuba, Tsukuba.

Pehkonen, E. 1994. On Teachers’ Beliefs and Changing Mathematics Teaching. Journal für Mathematik-Didaktik 15 (3/4), 177-209.

Pehkonen, E. 1995a. On pupils’ reactions to the use of open-ended problems in mathematics. Nordic Studies in Mathematics Education 3 (4), 43–57.

Pehkonen, E. 1995b. Introduction: Use of Open-ended Problems. International Reviews on Mathematical Education 27 (2), 55-57.

Pehkonen, E. 1997. Use of problem fields as a method for educational change. In: Use of open-ended problems in mathematics classroom (ed. E. Pehkonen), 73–84. University of Helsinki. Department of Teacher Education. Research Report 176.

Pehkonen, E. 1999a. In-service teachers' conceptions on open tasks. In: MAVI-8 Proceedings. Eight European Workshop. research on Mathematical Beliefs (eds. G. Philippou), 87–95. Nicosia: University of Cyprus.

Pehkonen, E. 1999b. Conceptions and images of mathematics professors on teaching mathematics in school. International Journal of Mathematical Education in Science and Technology 30 (3), 389-397.

Pehkonen, E. & Törner, G. 1996. Mathematical Beliefs and Different Aspects of Their Meaning. International Reviews on Mathematical Education [= ZDM] 28 (4), 101–108.

Pehkonen, E. & Vaulamo, J. 1999. Pupils in lower secondary school solving open-ended problems in mathematics. In: Proceedings of the PME-23 conference (ed. O. Zaslavsky), Vol. 4, 33–40. Haifa: Israel Institute of Technology.

Pehkonen, E. & Zimmermann, B. 1990. Probleemakentät matematiikan opetuksessa ja niiden yhteys opetuksen ja oppilaiden motivaation kehittämiseen. Osa 1: Teoreettinen tausta ja tutkimusasetelma. [Problem Fields in Mathematics Teaching and Their Connection to the Development of Instruction and Pupils' Motivation. Part 1: Theoretical Background and Research Design] Helsingin yliopiston opettajankoulutuslaitos. Tutkimuksia 86.

Schoenfeld, A.H. 1985. Mathematical problem solving. Orlando (FL): Academic Press.

Efficacy beliefs with respect to mathematics teaching

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We examine the efficacy beliefs of primary school teachers with respect to teaching mathematics, analyzing data from 157 subjects through a questionnaire and 18 tape-recorded interviews. The analysis shows that a) young teachers feel quite capable to facilitate students' learning, b) efficacy feelings tend to develop negatively during the first years of service and improve after a few years of teaching experience, and c) teachers from different preservice background vary in terms of their efficacy feelings.

Theoretical background and aims

Introduction. Teachers' sense of efficacy has recently become a focus for researchers in teacher education. Numerous studies supported the conclusion that confidence in one's ability to undertake a certain action is the best predictor of his/her behavior to accomplish the task (Bandura, 1986, 1997; Guskey & Passaro, 1994; Hoy & Woolfolk, 1993; Pajares, 1996a).

The concept "teacher efficacy" grew out of two psychological strands. The first was based on Rotter's expectancy theory for internal versus external control of reinforcement and the second was based on Bandura's social cognitive theory. Self-efficacy is generally viewed as a future oriented belief; it is a motivational factor influencing the amount of effort and the persistence one will show in the face of obstacles. Bandura (1997) defined perceived self-efficacy as "beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (p. 3). In the same sense, teaching efficacy can be defined as teachers' beliefs in their capabilities to organize and orchestrate effective teaching-learning environments.

Some researchers use self-efficacy and self-esteem as synonymous concepts, while others describe self-esteem as a generalized form of self-efficacy (Pajares, 1996a). Self-efficacy differs from self-concept in the degree of specificity. Self-concept judgements are more global and less context specific than self-efficacy judgements; they are not task specific. Likewise, efficacy beliefs might be multifaceted and contextual, but the focus is on the specific ability to accomplish a criterial task. General measures of self-efficacy are weak predictors of academic performance (Pajares, 1996a), while specific measures were found to be linked to a whole set of related variables, such as enthusiasm, professional commitment, instructional experimentation etc. (Tschannen-Moran, Hoy & Hoy, 1998).

Efficacy Measurement. Two dimensions of the construct were identified in the literature, the "general teaching efficacy" (GTE) that refers to the possibility of teachers in general to influence students learning, and the "personal teaching efficacy" (PTE) that measures the individual's conviction in his/her own power to control students' motivation and achievement. The first scales developed (see Gibson and Dempo, 1984) were found to be imbalance in terms of items in the PTE and GTE dimensions and the internal-external interpretation of learning (Guskey & Passaro, 1994). Most of the PTE items were positively phrased and most of the GTE items were negatively phrased, though the internal-external dichotomy could be crossed with PTE and GTE beliefs to produce four types of items: *Personal-Internal (P-I), General-Internal (T-I), Personal-External (P-E)*, and *General-External (G-E)* (see Table 1).

Self-efficacy with respect to mathematics. So far mathematics-related efficacy beliefs is a little researched area. We have been able to locate published research on students' efficacy beliefs in mathematics learning mathematics, but not on mathematics teaching efficacy. Hacket and Betz (1989) found that self-efficacy beliefs of psychology students is moderately correlated to mathematics performance and positively correlated to attitudes and to masculine sex-role. Randhawa, Beamer, & Lundberg (1993) tested a structural model involving attitude, self-efficacy, and mathematics achievement, and found that generalized mathematics attitudes had a strong direct effect on self-efficacy and that attitudes had as strong a direct affect on performance as did self-efficacy beliefs. In a path-analysis model that controlled for the effects of anxiety, cognitive ability, mathematics grades, self-efficacy for self-regulatory learning and sex, Pajares (1996b) found that self-efficacy made an independent contribution to the problem solving performance of students.

In Cyprus primary teacher education has recently been upgraded; a university degree became mandatory as from 1995. The teacher population is therefore through a transient period and a crucial question concerns the effectiveness of the new preservice education program. The aim of this study was twofold: a) to analyze the structure of mathematics teaching efficacy beliefs of primary teachers and b) to search for differences between the various groups of the teacher population. This comparative process could serve as a mean for the evaluation of the mathematics program implemented at the University of Cyprus.

Methodology

The instrument: We used a five-point Likert-type scale with 28 items. Personal teacher efficacy (PTE) was measured through five dimensions: the internal and the external interpretation of learning control, the mathematics teaching anxiety, the mathematics teaching enjoyment, the school climate, and the efficacy beliefs of the preservice mathematics program. Four indicators, all of the external interpretation of learning control, measured the general teaching efficacy dimension (GTE). The highest efficacy level was coded 5 and the lowest as 1, meaning that 3 signified the neutral level.

The subjects: The primary teacher population of Cyprus comprises of graduates from the Pedagogical Academy (PA), graduates from the University of Cyprus (UC), and graduates from Greek Universities (GU). The questionnaire was mailed to selected schools and 157 were returned (about 65% of the total). Of them 106 (58.6%) were PA teachers, including 15 (9.7%) degree holders (PAG), 21 (13.4%) GU, and 28 (17.8%) UC graduates. This distribution is representative of the teacher population under 15 years of service.

Interviews: About two months after we received the questionnaires, we interviewed 18 of the participants. Ten of them were UC graduates, six were PA graduates, and two were GU graduates. Interviews were semi-structured, focusing on letting teachers talk about their mathematics teaching experiences and their evaluations of the pre-service program they attended. The data collection process was completed by the end of the academic year 1996-1997, just before the first UC graduates completed their first year of employment.

Results

Table 1 shows indicative items from each dimension of the scale and the percentages of positive endorsement by UC graduates, PA (including PAG), and GU graduates, respectively. On first sight it appears that there is a considerable variability of item endorsement among the subject groups. In general, teachers expressed a high level of self-confidence in teaching mathematics, while they did not feel as capable to control pupil's learning; the mean score on P-I items was higher than the mean on the P-E items. The lowest efficacy levels were found on the "preservice mathematics program" and on the "school climate" dimension.

Table 1

Indicative items with the percentage of endorsement by CU, PA, and GU teachers

	Personal Teaching Efficacy (PTE)
Internal Int. (P-I)	11. When a child becomes better in mathematics, I believe that it was due to the variety of different ways I found to help him/her (82%, 76%, 67%)*
External Int. (P-E)	2. Some children face so many difficulties in mathematics, that I feel unable to help them (68%, 63%, 54%)
Teaching Anxiety	21. Sometimes I feel anxious that a student might ask me a question that I do not know how to answer or cannot explain (86%, 89%, 90%)
Teaching Enjoyment	27. If I were to choose one subject in a colleague's class, I would have opted for mathematics (57%, 40%, 29%)
School	18. I do not feel comfortable when the school principal or the inspector

Climate	observe my mathematics class (64%, 28%, 62%)	
Preservice Program	9. My preservice mathematics program, offered my the necessary basics to become an efficient mathematics teacher (50%, 25%, 28%)	
	General Teaching Efficacy (GTE)	
General Ext. (G-E)	4. The influence of the students' environment is so decisive that the teacher of mathematics cannot do much (97%, 76%, 57%)	

*The figures in brackets refer to CU, PA and GU graduates, respectively.

More than 80% of the subjects felt quite efficient "to help pupils make progress, even in difficult topics", "to help pupils think mathematically", "to consult experienced colleagues", "to answer pupils' questions", "to correct pupils' assignments", and were unwilling "to give away mathematics". On the other side, the majority of the subjects did not endorse the items indicating efficacy "to cover the subject matter", of "the preservice program", "to help the weak students", on the "possibility of weak pupils to get help", and to "discipline students who is not used to from home".

The ANOVA showed significant differences among the four sample groups (UC, PA, UPA, and GU graduates) on the GTE dimension, on the item reflecting evaluations of their preservice mathematics program, and on the total scale by years of service. Specifically, UC graduates, more than the rest of participant groups, hold the belief that students are teachable (internal interpretation, $\underline{F} = 3.150$, d.f. = 3, $\underline{p} = .027$); they showed a higher level of efficacy beliefs on all four items of the GTE dimension of the scale. In addition, UC graduates expressed significantly higher efficacy beliefs regarding their preservice mathematics program ($\underline{F} = 8.992$, d.f. =3, $\underline{p} = .000$); On the same item, the PA graduates were found to hold the most negative evaluations ($\overline{\underline{X}}_{UC} = 3.29$, $\overline{\underline{X}}_{PA}=2.25$, $\overline{\underline{X}}_{PAG}=2.27$, $\overline{\underline{X}}_{GU} = 2.90$).

The ANOVA on the total scale gave an $\underline{F} = 3.257$, with d.f. = 3, p= .042, indicating significant variation of the efficacy beliefs among the three age groups. The mean values indicated that teachers' beliefs tend to get worst during the first period of their professional life and are subsequently improved through experience ($\overline{\underline{X}}_1 = 3.59 \Downarrow \overline{\underline{X}}_2 = 3.37 \Uparrow \overline{\underline{X}}_3 = 3.65$). This affirms and extends earlier findings (Hoy & Woolfolk, 1993).

Analysis of the clinical interviews

We present excerpts from the interviews on three dimensions: perceived efficacy to influence non-motivated pupils (NM), about the preservice program (PSP), and the school climate (SCL). Efficacy judgments are classified as positive, neutral, or negative, and the participants were listed in one of these categories.

Efficacy to help even non-motivated pupils

Q1-NM: How confident do you feel to help the non-motivated pupils?

Positive view: I am sure that I can help all students to make progress. For the 2-3 non-motivated or slow learners that are normally found in every class, one has to make special arrangements i.e., to simplify activities, allow for more time, cooperate with parents etc.

Neutral view: I can help slow learners, but "it is not possible for all students to reach the same level". In every class "there are a few special cases for which it is very hard to do anything". "They need special attention and I have no time".

Q2-NM: In the case that a child makes progress, to whom should this be credited?

Positive view: A student's learning is affected by many factors; the final outcome is due to a combination of joint efforts. In my view, however, the teacher is the first to be credited.

Neutral view: I think that a student's progress is equally due to the teacher, the parents and the student himself.

Negative view: "Teachers' influence on students' learning is limited; they are at school for only 4-5 hours a day". b) "The crucial factor is the child, quite a lot depend on him/her". c) "Everything depends on the child", d) "In some cases the teacher cannot do much".

Efficacy of the preservice mathematics program

Q1- PSP: How do you judge the preservice program you passed through?

Positive view: I believe it offered me all the necessary background to teach mathematics effectively. Sometimes, when I was a student, I wondered whether several of the issues and ideas discussed were practically applicable, now I am convinced they were useful.

Neutral view: Most useful was the Methods course. History of mathematics helped me to appreciate the developmental nature of mathematical ideas, but I think that the tutorials could have been more profitable.

Negative view: Indeed, we only did one course on teaching methods, "which was just an introduction, rather irrelevant to teaching".

Efficacy to manage the school climate

Q2- SCL: How do you feel when the principal or the inspector attends your class?

Positive view: "Well, it's natural not to feel as easy as when you are on your own; it may cause me some tension, but not really anxiety. I have nothing to hide, I want them to get the real picture of the class, to bring possible problems on the surface". "It is a matter of self-confidence".

Neutral view: "There is a certain degree of anxiety. After all, you are under assessment, they are examining your results".

Negative view: "I feel anxious and uneasy...I think it is in my character".

The overall percentages of teachers with positive endorsement were 63%, 32%, and 5% for the UC graduates and 37%, 39%, and 24% for PA graduates, respectively. The sample size of GU subjects does not allow for general conclusions, though the distribution was more or less normal between positive, neutral and the negative endorsements. CU subjects were relatively more positive about the preservice program, they felt capable to help even unmotivated pupils, and believed that pupils in general are teachable.

Conclusions

In this study, we examined the mathematics teaching efficacy beliefs of young teachers and searched for differences between groups according to education and length of service. The results indicate that young teachers feel quite efficacious in teaching mathematics, though there are components in which the level of efficacy is not satisfactory. Through the analyses of the questionnaires and the interviews, the preservice program and the school climate were found to be the most crucial dimensions. The overall mean score was above the neutral, and the interviews indicated rather positive efficacy beliefs, though not completely consonant to their reactions in the questionnaires.

The differences among age groups indicate that the newly appointed teachers have a rather optimistic view, which becomes more realistic when they come across the complexities and practical difficulties of teaching mathematics, and improves gradually by experience. Improvement of efficacy believes through experience was also found by Hoy and Woolfolk (1993), while the former stepping down during the initial employment is, to the best of our knowledge, a new result. It seems natural to take some time before the young teachers develop the necessary cognitive, behavioral, and self-regulatory tools for creating and executing appropriate courses of action, especially in mathematics.

The higher, relative to other groups, level of mathematics teaching efficacy beliefs of UC graduates, might well be ascribed to the preservice program. The fact that UC graduates expressed higher efficacy sense to control pupils' learning, that children are teachable, and that the teacher is primarily responsible for the student progress, might reflect confidence in their pedagogical education, supporting their direct evaluation of the preservice program. The whole picture seems to affirm earlier results (Philippou & Christou, 1998) indicating that the preservice program was effective to developing attitudes toward mathematics. We shall continue the follow up of our graduates, to get a better insight in their evaluations.

References

Bandura, A. (1986**).** *Social foundations of thought and actions: A social cognitive theory.* Englwood Cliffs, NJ: Prentice-Hall.

Bandura, A. (1997). *Self-efficacy: The exercise of control:* New York: Freeman**.**

Guskey, T. R., & Passaro, P. D. (1994). Teacher efficacy: A study of construct dimensions. *American Educational Research Journal*, 31, (3), 627-643.

Gibson, S., & Dempo, M. H. (1984). Teacher efficacy: A construct validation. Journal of Educational Psychology, **76**, 569-582.

Hacket, G., & Betrz, N. E. (1989). An exploration of the mathematics self-efficacy /mathematics performance correspondence. JRME, 20, (3), 261-273.

Hoy, W. K, & Woolfolk, A. E. (1993). Teachers' sense of efficacy and the organizational health of schools. *The elementary school journal*, 93, (4), 355-372.

Pajares, F. (1996a). Self-Efficacy Beliefs in Academic Settings. *Review of Educational Research*, 66, (4) 543-578.

Pajares, F. (1996b). Self-Efficacy Beliefs and mathematical problem solving of gifted \students. *Contemporary Educational Psychology*, 21, 325-344.

Philippou, G. N. & Christou, C. (1998). The Effects of a Preparatory Mathematics Program in Changing Prospective Teachers` Attitudes toward Mathematics. *Educational Studies in Mathematics*, 35, 189-206.

Randhawa, B. S., Beamer, J. E., & Lundberg, I. (1993). Role of mathematics self-efficacy in the structural model of mathematics achievement**.** *Journal of Educational Psychology*, 85, (1), 41-48.

Tschannen-Moran, M., Hoy, A. W., & Hoy, W. K. (1998). Teacher Efficacy: Its Meaning and Measure. *Review of Educational Research*, 68, 202-248.

Intuitive beliefs, formal definitions and undefined operations: The cases of division by zero

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> *...I clearly remember the day I stopped loving maths. I was in the fourth grade and we were doing division. The task was to write and solve division drills. I still remember writing 4 ÷ 0=0. My teacher crossed out my drill, saying: 'This one is wrong". When I asked: "But why? It is 0", she responded: "This one has no answer. It's a rule. You have to remember it."... I still feel the anger.... At that moment, I started hating maths. I realized, for the first time, that maths is about memorizing rules that don't make sense.*
>
> From an interview with Lilian, a 10^{th} grade student

Intuitive beliefs have the characteristics of intuitive thinking: Self evidence, intrinsic certainty, perseverance, globality and coerciveness (Fischbein, 1987). In this lecture I present my initial attempts to study one intuitive belief about mathematical operations: the numeric-answer belief.

The mathematical experiences of many children during their first years of schooling is primarily comprised of performing manipulations and arriving at numerical solutions. Such extensive experience with mathematical operations inevitably leads to a development of an intuitive belief that every mathematical operation must result in a numeric answer (i.e., a "numeric-answer belief"). Division by zero is usually the first undefined mathematical operation that students encounter during their school studies. Clearly, the mere existence of an undefined mathematical term violates the intuitive, numeric-answer belief. Adherence to this belief might result in assigning numerical values to expressions involving division by zero.

A tendency to claim that division by zero results in a number was indeed found among elementary and middle school students, prospective teachers and teachers (Ball, 1990; Blake & Verhille, 1985; Grouws & Reys 1975; Reys, 1974; Tsamir, 1996; Wheeler & Feghali, 1983). Surprisingly, I found no study that examined secondary school students' responses to tasks involving division by zero. Such inquiry could extend our understanding of the relationship between intuitive beliefs about mathematical operations and students' actual, incompatible practices, as students in secondary schools are often expected to apply their knowledge about division by zero in various situations. In my lecture I'll describe a study that explores secondary school students' ways of thinking about division by zero.

Method

One hundred and fifty-three students from Grades 9, 10, and 11 in a secondary urban school in Israel participated in this study. About half of the students in each grade level studied Advanced Mathematics Level (AML) while the other half studied Low Mathematics Level (LML). A written questionnaire including 20 multiplication and division expressions was administered to the participants during a mathematics class session of about 60 minutes. Subjects were asked to read each expression, to give a numeric solution, if possible, or to explain why it is impossible to provide a numeric solution. The questionnaire included 11 expressions involving division by zero which were mixed with other expressions to reduce the chance of receiving automatic, "undefined" responses. Typically, subjects elaborated on their responses to the questionnaires and thus provided substantial information about their reasoning. Still, in some cases, follow-up interviews in which participants were encouraged to further explain their responses were needed. In these cases, the interviewee's responses were added to the original questionnaire, providing a fuller picture of his or her related reasoning.

Results

Division of a non-zero number by zero

The majority of the participants (74%) responded that division of a non-zero number by zero is undefined. The most common incorrect responses were either zero or the dividend. Another incorrect response, "∞", was given by 12% of the AML 11th graders. The way they wrote their solutions (e.g., $12 \div 0 = \infty$), and the fact that they provided no justifications to their responses suggested that they regarded ∞ as a specific, numeric answer.

AML students used the following, three justifications to explain their "undefined" responses:

1. Relying on the definition of division as the inverse of multiplication. Most AML participants explained that any definition of expressions of the type $a \div 0$ for $a \neq 0$ would violate the definition of division as the inverse of multiplication. Some noted that such a definition would violate either the definition of division as the inverse of multiplication or the theorem $c \cdot 0 = 0$ for every *c*. This mathematically-based justification is often used by high-school teachers to explain why division of a non-zero number by zero is undefined.

2. Applying a notion of limit. The second, mathematically-based justification of the undefined responses consisted of applying a notion of the limit of $a \div x$ as x tends to zero through positive and through negative numbers. A typical reaction of this type was $12 \div 6 = 2$, $12 \div 3 = 4$, $12 \div 1 = 12$, $12 \div \frac{1}{2} = 24$, $12 \div \frac{1}{4} = 48$, $12 \div \frac{1}{16} = 192$ as I get closer and closer to zero, the numbers increase. Now, I'll do the same, but this time I'll approach zero from the left side of the number line. I have: $12 \div (-12) = (-1)$, $12 \div (-6)$

$= (-2)$, $12 \div (-3) = (-4)$, $12 \div (-1) = (-12)$, $12 \div (-\frac{1}{2}) = (-24)$, $12 \div (-\frac{1}{4}) = (-48)$, $12 \div (-\frac{1}{16}) = (-192)$. The numbers decrease. So, there is a jump at the point zero and it is impossible to find a number for $12 \div 0$". This justification was provided by 16% of the AML students in Grade 11.

3. Using the compromised, "∞-undefined" notion. Another explanation, suggested by several AML students (13%, 10% and 16% in grades 9, 10 and 11 respectively) was that "division by zero is undefined because it is infinity, and infinity is undefined". Interviews with students who used this justification revealed that for them division by zero results in the number ∞. This number ∞ was undefined either because its exact location on the number line was unknown, or because its value was not fixed. It should be noted that this "∞-undefined" response does not contradict the numeric-answer belief.

Students doing LML used one, common justification to explain their "undefined" responses:

1. Illustrating that division of a non-zero number by zero is, in practice, impossible. Participants who used this justification related to division as "sharing" and to zero as "nothing". A typical response was "$12 \div 0$ means sharing 12 cookies in equal parts among no kids. It is impossible to share 12 cookies among no kids, therefore $12 \div 0$ is meaningless, undefined".

Division of zero by zero

About 60% of all the participants correctly argued that $0 \div 0$ is undefined. The only incorrect response to these tasks, zero, was given by all other participants.

Two justifications were given by AML students to explain their "undefined" responses:

1. Viewing $0 \div 0$ as a specific instance of division by zero. Most AML students explained that division by zero is undefined for any number, including zero. In an interview with one of the participants who provided such a justification, I asked if $0 \div 0$ could also be regarded as a specific instance of $a \div 0$ for $a \neq 0$. This participant replied that "in principle, it is possible, but I know that $0 \div 0$ is a specific instance of $a \div 0$. I do not know why".

2. Relying on the single-value requirement. Few AML students specified the that an operation should fulfill the single-value requirement (10%, 9%, and 12% in grades 9, 10 and 11 respectively). A typical explanation of this kind was" "$0 \div 0$ is undefined because if $0 \div 0 = x$ then $x \cdot 0 = 0$. But this is true for every number".

The LML students provided two types of justifications to their "undefined' responses:

1. Illustrating that division of zero by zero is, in practice, impossible. Most LML students argued that practically it is impossible (meaningless) to divide zero by zero.

2. Using the compromised, "0-undefined" notion . The second justification was based on a perception of zero as an undefined number. A typical response of this type was "0 ÷ 0 is zero, but dividing by zero is problematic, therefore the answer is undefined". This response, much like the "∞-undefined" response, could coexist with the numeric-answer belief .

Conclusions and educational implications

1. Intuitive beliefs and formal definitions. Research has consistently reported the tendency of many elementary students, prospective teachers and teachers to assign numerical values to division-by-zero expressions. This study shows that a non-negligible number of secondary school students (LML and even AML) argued that division by zero results in a number. The justifications of some students to their correct, "undefined" responses revealed how they "reconciled" the apparent contradiction between their intuitive, numeric-answer belief and the non-definition of division by zero. This compromise consisted of claiming that division by zero is undefined because the result is an undefined number (either zero or infinity).

Intuitive beliefs about mathematical operations are known to largely affect students' responses to related, mathematical tasks. The intuitive, numeric-answer belief could affect students' responses not only to tasks involving division by zero (e.g., finding excluded values of rational equations) and to tasks involving other, undefined mathematical terms but also in other, seemingly different situations (e.g., simplifying algebraic expressions). Given this state of affairs, a teacher could naturally ask: "How can I help my students overcome the coercive effect of the numeric-answer belief?" The teacher's task is indeed complicated. Fischbein (1987) suggested that a major aim of mathematics education is to raise students' awareness of the role of intuitions in their thinking processes and to develop their ability to analyze and control them. In the case of division by zero, it is important to explicitly relate, in class, to the intuitive belief that every mathematical operation must result in a numerical answer, to discuss its possible sources and to demonstrate its impacts on our reasoning processes. The teacher could refer to the observed differences in students' performances in the two cases involving division by zero (a ÷ 0 for a ÷ 0, and 0 ÷ 0), drawing on the profound mathematical and psychological differences between these two cases. Other common intuitive beliefs about mathematical operations could be addressed as well (e.g., addition and multiplication makes bigger, division makes smaller), leading to a more comprehensive discussion on the differences between intuitive beliefs about and formal definitions of mathematical operations.

2. Intuitive beliefs and practical models. The sharp split in the nature of justifications provided by AML and LML students deserves consideration. These differences are mostly evident in the differential use of practically-based justifications: While most LML students used such justifications, none of the AML students did. During their years of schooling, both AML and LML secondary school students calculate, many division expressions involving fractions and negative divisors that clashed with the notion that division is meaningful only if it could be interpreted as sharing. Thus, it seems that these two groups developed different sociomathematical

norms related to what counts as an acceptable mathematical justification (Yackel & Cobb, 1996).

The reliance of LML students on the sharing model could be used as an argument against applying practical models to show the impossibility of division by zero. Clearly, it is possible to reason for (and against) the use of practical models in elementary schools with students whose acquaintance with operations is limited to non-negative, whole numbers. However, it seems that no one would argue against the need to re-evaluate the applicability of such models, if they are indeed to be used when the operation of division is extended to fractions and to negative numbers. It is essential that teachers encourage their students to reflect on the types of explanations they use and be aware of their realms of application. Such examination could then lead to an exploration of some major issues related to intuitive beliefs, formal definitions, mathematically-based and intuitively-based justifications and mathematical operations (i.e., How do mathematicians make decisions about definitions of operations? What are the main properties of mathematical definitions? What are the main properties of mathematical operations? What are the reasons behind choosing a certain definition? Is there a general policy according to which mathematical operations are doomed to be either defined or undefined?). Such discussions could assist teachers in their attempts to lessen the undesirable myth that mathematics is about memorizing unreasonable rules (see Lilian's assertion in the introduction to this summary) and to promote a view of mathematics as a human-made, reasonable discipline.

References

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education.* 21(2), 132-144.

Blake, R. & Verhille, C. (1985). The story of 0. *For the Learning of Mathematics,* 5(3), 35-47.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, Holland: Reidel.

Grouws, D. A., & Reys, R. E. (1975). Division involving zero: An experimental study and its implications. *Arithmetic Teacher*, 22, 74-80.

Reys, R. E. (1974). Division by zero: An area of needed research. *Arithmetic Teacher*, 21, 153-157.

Tsamir, P. (1996). *Teaching prospective teachers about division by zero*. Unpublished manuscript, Tel-Aviv University. Tel-Aviv.

Wheeler, M., M., & Feghali, I. (1983). Much ado about nothing: Preservice elementary school teachers' concept of zero. *Journal for Research in Mathematics Education*, 14(3), 147-155.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in mathematics Education*, 27(4), 458-477.

Domain-specific beliefs and calculus -
Some theoretical remarks and phenomenological observations

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On the basis of six triple-sectioned experience reports on calculus lessons in school and at university the problem is researched as to which related domain-specific beliefs are articulated by post-undergraduate teacher students. Beside content identification of their beliefs, the question arises as to the quasi-logical relationship of these global beliefs on mathematics stand.

Borrowing a metaphor from mathematics one can view learning as a highly multidimensional, time-variant, dynamic process with a multitude of reciprocally coupled variables, and therefore this construct can only be modeled to a marginal degree. All modeling attempts here must eventually make reductions. Representations of interactions and visualizations of dynamic processes here are only successful on a level of lower dimensions. Thus the question arises as to the choice of suitable content-relevant cuts.

The starting point.

In the literature one finds a great number of papers concerned with beliefs on mathematics, beliefs on the learning and teaching of mathematics (see survey article of Thompson 1992). Beliefs are hereby often considered to be burdened with unacceptable reductions of content and deficient views of concepts, thus inhibiting mathematics activities. An exclusively negative assessment of beliefs possibly inhibits their productive utilization, whereby one cannot fail to see that beliefs possess what one can name inertia force (Pehkonen & Törner 1996). It nonetheless appears of advantage to expand our understanding of beliefs in this sense.

Beliefs can represent closely related attitudes, which are possibly biased and incomplete but nonetheless entail an inherent logic marking an intermediate level which is not (i.e. no longer) the level on which an expert mathematician or an expert of learning processes stand. Sometimes they make absolute judgments or generalize a factual position on the premises of incomplete knowledge of the subject matter. In particular when viewed on this background beliefs can be positively considered as comprehensive indicators.

Beliefs are specified and then researched according to the various groups of their subjects. By means of various survey methods one accordingly receives different results on the problem field of beliefs on mathematics for pupils, teachers and professors, whereby one generally approaches this subject in the sense of a **Top-Down-Approach**. Under such a global approach beliefs are generally understood as ideologies or **philosophies** of mathematics and its didactics[8]; sometimes one uses the terminus ‚*view of mathematics*'. McLeod speaks of *beliefs of mathematics as a discipline*; we prefer the terminus **global beliefs**. It is generally accepted that under this methodological approach the total reality of the multifaceted subjective experiential fields of mathematics cannot be described, last but not least because the structure of the questionable beliefs according to Green (1971) is quasi-logical.

The concern with detail aspects – in the sense of a Bottom-Up-Analysis – stands in contrast to this approch. In this relation specific terms (e.g. naive concepts of sets[9]), facts and methods (division and multiplication[10]) are critically researched in terms of their functional use in view of the learner. The focus is not seldom placed on misconceptions, or better: preceptions. In analogy to the term subject-matter-knowledge used by Even (1990, 1993) we use the term **subject-matter-beliefs,** which refers to the amount and organization of knowledge and beliefs per se in the mind of the subject (see also Lloyd / Wilson (1998).

Only a few research papers address an intermediate level, thereby refering to a whole domain of mathematics and its specific fields, e.g. calculus (comp. Amit, M. & Vinner, S. 1990, Tall, D. 1990), or the field of geometry, or stochastic mathematics. It is learning mathematics at university or in course systems in school which takes place in particular within these frameworks (subjective experience fields); mathematics is possibly perceived pars pro toto[11]. It is also on this level that beliefs are to be located, which we will describe as **domain-specific-beliefs**. They appear as conceptions or fundamental ideas and their deformations. The dividing line between these beliefs and concept images (Vinner 1983) or fundamental ideas (Hofe 1995) is unclear and flowing as it can be construed either as subject-matter-orientated or as domain-specific.

The advantage of such a subject-matter related framework for a research report into beliefs lies in its close proximity to both the total framework of mathematics and the detailed issues of specific subject matter. As this subject-matter related framework is in most cases also the subjective experiential framework in which individual fields of mathematics are experienced, one may achieve insights into "how it is believed", a

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[8] Ernest, P. 1991. The philosophy of mathematics education. Hampshire (UK): The Falmer Press.

[9] Linchevski, L. & Vinner, S. 1988. The naive concept of sets in elementary teachers. In A. Borbas, A. (Ed.), Proceedings of the12 th International Conference of the International Group for the Psychology of Mathematics Education (PME). Volume 2 (pp. 471 - 478). Veszprém (Hungary): Hungarian National Centre for Educational Technology.

[10] Tirosh, D. & Graeber, A.O. 1989. Preservice elementary teachers' explicit beliefs about multiplication and division. Educational Studies in Mathematics 20 (1), 79 - 96.

[11] The basic criticism on such a mediative approach will be left out here.

question which Green (1971, p. 47) considers to be of greater relevance than the question of "what is believed".

At a first glance one could be of the opinion that these differentiations of the term belief discriminate only **extensionally** within our approach. At a closer view, however, one recognizes that differences in their occupation also exist **intensionally** as differing nuances in view of the belief approaches become apparent. Through these differentiations one can make precise statements on the multifaceted term belief, and a stronger terminological integration is supported.

Some remarks on the theoretical framework of beliefs.

It is not our intention here to present an exhaustive discussion on suitable definitions of beliefs. The description of Schoenfeld (1998) may serve here as a working title, namely that we are dealing with mental constructs representing the codification of people's experiences and understandings. In the present literature up to today there is still no consensus on a single belief definition. Maybe it is this openness, i.e. vagueness of the term belief which makes its use successful and flexible.

Two levels are to be precisely defined for any definition on beliefs, firstly in view of the term ‘knowledge’, and secondly in view of the term ‘emotions’.

The difference between beliefs and objective knowledge is (...) unclear (Lester et al. 1989, p. 77). One feature of beliefs is that they can be held with varying degrees of conviction. When Scheffler (1965) argued that a claim to knowledge must satisfy a truth condition - whereas beliefs are independent of their validity - then at a first glance we are standing within mathematics on safe ground. Mathematical statements relative in school can be answered in a binary fashion and be sorted as true or false. Can one at all then talk of beliefs in the light of such objective subject matter structure here?

The validation of mathematical statements is the one thing; for the subject in question however many statements, also in mathematics, have a fuzzy character concerning precision of evidence and plausibility due to their complexity. They are true, can be proven or have been proven, are conceivable, appear possible, are considered surprising, can or may right etc. etc. Can one then, in the narrow definition of the term, speak of solid knowledge? The ideal-typical distinction between beliefs and knowledge systems with Abelson (1979) (see also Calderhead 1976) also wholly applies to knowledge of mathematical systems (comp. his terms used there) - so far as we can conceive this knowledge as subjectively and constructively realized. In other words: also in an abstract and deductively structured knowledge system it is difficult for a survey to identify information attained from subjects as objective or secured knowledge.

To illustrate this with a image: objective knowledge can be stored in „cooled bytes“ of a processor. It cannot be ignored however, that this is not possible for knowledge represented by persons. This is the noteworthy point for Ableson (p. 358): Belief systems rely heavily on evaluative and affective components. It is a well-known fact that complex issues of mathematical knowledge represented by an individual entail emotional loadings: the existence of these loadings can often be verified by the presence of a substantial amount of episodic material. There are a number of reasons which seem to speak for the view that these loadings are reduced by increasing the operationalization of knowledge. Secured knowledge for us (confirmed once again in this present report) is that emotional loadings and cognitively interpreted information bits almost never allow themselves to be completely decoupled.

Lastly, a further feature of belief systems cannot be ignored, namely their broad network structure - a modeling approach which can already be found at Green. Beliefs have a **patchwork character**, they have the structure of an evaluated graph, whereby their edges and the vertices can be interpreted in a number of ways. Pehkonen/Törner (1996) construed in relation to belief systems the image of a "bundle of spaghettis". The question arises in view of this metaphor however, what these individual spaghettis in this bundle which cannot be completely entangled, really are? Are they not possibly strings through which individual contents (subject matters) can be described, whereby the image of a subject-related belief system is then created? Alternatively conceivable would also be to interpret the spaghetti strings as personal lines of experience and as time axes of learners of mathematics.

Research questions.

This leads us to a number of research questions, which can only be marginally discussed here due to the limited scope of this paper. (1) Which domain-specific beliefs can be found in calculus? To which category in the sense of Green[12] (core beliefs versus psychological peripheral beliefs, primary versus derivative beliefs) can these beliefs be assigned? (2) Which dependency or implication structure exists between global beliefs, subject-matter-beliefs and domain-specific beliefs? (3) Is any information offered as to the origin and development of these beliefs?

Methodological considerations.

We refer to our evaluations of the essays written by post-undergraduate mathematics teacher students. Within the framework of a seminar on calculus the students (n = 10) were asked to write essays (1 to 2 pages) on the topics: (A) Calculus and I: how I (have) experienced calculus at school and university. (B) How I would have liked to have learned calculus. (C) How I would like to teach calculus.

These essay topics were subsequently handed out at intervals of three weeks. The students knew nothing of the actual case study approach of the author. Altogether six students handed in contributions on all three topics. Their essays are used here for this case study.

In the first viewing of the essays the given topics can be categorized as being similar. The research approach realized here is classified as triangulation (Cohen;

[12] One has the impression that this question within a mathematical context is not systematically researched anywhere, see e.g. Cooney, Shealy & Arvold (1998). How does one decide whether a belief deserves the predicate 'central'.

Manion 1994). This procedure can be justified through the following basic assumptions: Learning and teaching are dual processes that can be individually considered as linked together. Possibly experienced deficits are categorized - when viewed positively - as points of emphasis of one‘s own responsibly conducted lessons. Positive experiences lead to reinforcement of one‘s own actions towards others. In this respect a temporal invariant consistency in the evaluation of one's own teaching and learning processes is presumed, whereby one must bear in mind that repeated mentioning of its aspects can lead to its confirmation.

Due to the limitation of this paper an exhausting presentation of the various topics by the students cannot be expected here. No content-related expectations were placed on the students, so that freely written essays were ensured. The location and topic change, induced by the respective question formulation, enables new reflection impulses and recapitulates new aspects of a topic from the viewpoint of the other students. As the three essays are intended to illuminate different time concepts (A – past, B – present, C – future) it is to be expected that through the essays it will also be possible to determine time-invariant lines.

Parallel to this we want to, where possible, categorize the mentioned beliefs in the sense of Green with focus on the individual authors. The predication "primary" appears to us e.g. to be justified when statements are rigorously postulated in a very fundamental, irrefutable and unquestionable manner and often expressed undifferentiatedly even as a slogan (whereby mathematics often serves as the basis of justification) or simply as folk notions on learning.

The psychological centrality of a belief can be measured according to its frequency of occurrence of accent. A prominent position within any of the given essays also bears witness to the centrality of a thought.

Clusters of beliefs are verified by us when immediate text-immanent associations are realized, when statements in the text are close neighbors or possibly are even presented causally linked to each other.

Some results.

The perception of calculus in the essays concerns more or less the fields addressed by more than two thirds of the subjects: aims of mathematics education with respect to calculus; world view on mathematics and in particular calculus; calculus and learning of mathematics; calculus and the aspect of formalism in mathematics; the systemic aspect in mathematics; calculus and calculation; calculus and application versus reality; proofs and calculus; emotional dimension of mathematics learning and calculus; subjective specifica; coupling of school and university; What, then, has university mathematics conveyed?

In the following we present some of these beliefs together without confirming them through the quotations from which they are derived. Next to the explicitly expressed views one can list calculus-specific elements **not or hardly mentioned by any students**, whereby this possibly bears witness to the marginality in the self-assessment of the students. Here one can briefly mention: discussions around the central concept of calculus, namely the differentiation concept, completeness of the real number system, the concept of infinitesimals, infinity, change, growth - termini technici behind which the essentially "big ideas"[13] of calculus stand.

Functions are the "ferment" of calculus. This fundamental message is in all cases the basic belief inherent in university mathematics. Functions are omnipresent in all mathematical contexts, sometimes under other labels. Functions allow mathematical contexts to be excellently structured. The everyday school reality in the minds of the students is however characterized by colorless hues:

(1) Calculus is reduced in school down to calculating (not necessarily meaningfully) with functions.

As Baroody / Ginsburg (1987) pointed out, an overreliance on mechanical knowledge is often correlated to a limited use of conceptual knowledge and thus to some short-comings in views of mathematics. Further, one criticizes school education insofar that calculus is introduced as a completely new field of mathematics without referring to previous experience and skills acquired by working with functions. Even though university mathematics focuses on the level of differential calculation, the students state in their essays that they miss a specific contribution explaining to them the meaningfulness of this field of mathematics.

The sequence ‘differential calculus - integral calculus’ is a popular theme for didactics, whereby this question is mostly only of academic relevance. The students refer to this question in their essays:

(2) *Differential calculus is craft - integral calculus is art.*

First of all one must pay a compliment to the student who presented this slogan. Integral calculus and differential calculus have of course differing characteristics, whereby in school differential calculus is given a lot more attention than its counterpart. The subjective experiences of the four students with both mathematical fields are insofar leveled down to stereotype descriptions: here the endless number of curve discussions, and there the traditional determination of surface area. A more detailed qualitative description of the individual specifica of the two is missing, namely that beyond this limited experience lie undiscovered "worlds" - it is in this direction that the quoted belief lies.

The next beliefs of the students relate explicitly or implicitly to a corresponding view of mathematics uncritically leveled down to calculus. There are clues here that the belief can be classed as of primary importance for mathematics. That these beliefs are attributed validity for calculus is evidence for their dominance, whereby reasons related to specific fields of mathematics should not to be overlooked. On the other hand it is common knowledge that beliefs basically possess a regulating function, an adaptation

[13] see OECD, 1999. Measuring Student Knowledge and Skills. A New Framework for Assessment. Paris: OECD Programme for International Student Assessment.

function and an orientation function, which is useful when making a new field of mathematics accessible.

The next belief possesses a similar cluster pattern:

(3) *Logic is a central guideline for mathematics and in particular for calculus.*

This statement uncritically reflects in its first part the formal and systemic facets of mathematics; the generally postulated consequences for each field of mathematics and a random mathematical school subject have to be considered in a reserved manner. For the student who expressed this belief it is to him both central and primary; it appears in all of his three essays. In contrast with four of the six students the wordlogic is not present. Furthermore, this belief seems to cluster with the view that logical penetration of mathematical contexts can most suitably be achieved by dealing with the mathematical fields 'formal logic' resp. 'foundations of logic'.

(4) Exactness as a character property of mathematics can in particular be demonstrated in calculus.

Principally, a systemic aspect of mathematics stands behind this statement, which is quite a remarkable statement. Clustering is apparent. Extreme precision is difficult, at times laborious. Again only two of six students presented statements to this belief.

(5) Calculus has the special task of preparing pupils for the forthcoming university course.

This one student wrote. Calculus in school mathematics is thus attributed a prominent formal character in mathematics education. In a weakened form this tendency of tone may serve for some justifications of calculus lessons in the 70s. In the meantime this position has proven itself to be untenable and has been rejected for a number of reasons.

(6) Mathematical elegance and abstractness - a liking of mathematicians - mean a loss of illustrativeness and understandability.

This statement only reflects indirectly beliefs of students; it reflects more the observed beliefs of 'professional' mathematicians. This sufficiently well-known and often articulated assessment namely that mathematics is to be presented as abstract and as elegant as possible is experienced by the students as a burdening circumstance (first negative experiences). Students experience this in particular as a circumstance affecting them detrimentally, which leads to the development of this belief.

It is obvious that the normal consumer of mathematics and the professional mathematician experience mathematics with its subject matter and its characteristics in different frameworks. Mathematics can be economically represented as a system but does not allow itself to be efficiently absorbed as such; mathematical elegance and abstractness, for example, possess insofar differing significance for differing groups.

Finally, individual quotations present a view of the specific images of mathematics learners.

(7) The recognition of application links facilitates learning.

The roots of calculus are closely linked to application; it is in particular calculus in which the big ideas 'change', 'development' and 'dependence' can be presented in a mathematically concrete mode and be modeled. Deficits in university courses are obviously present here, as this thought is not given due appraisal by the students. It is interesting that these students do not demand proof of this for themselves. They are more interested in the potential of improved results in the learning process. In view of Green`s categorization the application relevance is to be classed as a primary belief for calculus. Effects for the learning process would only then be noted as a positive side effect. However, the assessment from the viewpoint of the students is contrary to this.

A further belief standing isolated for itself and addressing the curriculum fits into this context:

(8) Mathematical systematics facilitates mathematics learning.

A common experience lies in the core of this articulated belief: systematics makes searching, recognition, orientation and coping with the subject matter easier. Mathematics is highly structured on a formal level. This presumes one has had experience with the subject matter. That mathematical systematics does not have to conform with psychological guidelines on learning is a further sobering realization from the euphoric phase of New Maths in the 70s. We like to quote Tall (1990):

> ‘At that time, I believed, common with other professional mathematicians, that the best way to help students is to present the materials in a logical and coherent manner. [...] But exploratory investigations into students‘ conceptions revealed fundamental inadequacies.‘ (p. 49)

> ‘The sequence of topics in the mathematics curriculum is built upon the implicit assumption that simple ideas must be introduced before more complicated ones – after all, this principle is so obvious, it is self-evident. But is it? [...] It is my belief that we do students a disservice by organizing the curriculum so that they are presented only with simple ideas first and given too great an exposure to an environment which contains regularities that do not hold in general.’ (p. 56)

Conclusion.

It cannot be overlooked that the present statements of the six students are of primarily phenomenological nature, whereby individual patterns nonetheless appear to be exemplary. Beyond this the essays refer to numerous contexts outside the belief context, which we shall not discuss here.

The beliefs extracted and discussed here are of both direct and indirect nature. On the one hand they are beliefs - with the exception of (6) - which can be accounted without doubt to the students; we can speak here of professed beliefs[14]. On the other hand they are also of indirect nature, e.g. belief (6), when the effects of teaching/learning processes allow professors with great probability to conclude the original attitudes and behavioral patterns of the teachers/professors; here we can speak

[14] see Schoenfeld (1998), p. 22

with Schoenfeld (98) of attributed beliefs. We thereby do not question here that the verbalized and professed beliefs possibly have their origin in the learning process and thus can be under the influence of third persons.

Only the beliefs (1) and (2) are originally related to calculus as a mathematical field. They are concerned with the role of integral calculus in comparison to differential calculus. In belief (1) the central relevance of functions for analysis is illuminated, which in school lessons is reduced to exhausting, ritualized function discussions.

It is apparent that students in view of the beliefs (3) - (6) reflect calculus as a whole and its role in mathematics globally. Detail contents play more of a subordinated role. Also basic realizations in the field-specific central infinitesimal concepts or big ideas are hardly addressed. This may be grounded in the methodology of our approach and the freely chosen theme, but it is quite probable that this global view reflects the center of perception of the student with his/her statements. Consequently, a high correlation between the global beliefs and the domain-specific beliefs has to be assumed here.

Exactness, elegance, precision, logic etc. are qualitative features of all mathematics activities. These beliefs are postulated for calculus for superordinated reasons and point to a rather dogmatic understanding of mathematics in a rigorous manner in the sense of an "either" or "or": exactness - yes or no; logical presentation - yes or no. In a balanced understanding of mathematics however, these features can never be understood in a binary fashion but are understood in a broad, i.e. scalar fashion. Again beliefs play the roles of indicators immediately exposing deficits, for which there are plausible reasons:

Mathematics has constituted itself in the mind of the students over a number of years. New experiences and information are primarily integrated into established ideas and beliefs and do not necessarily expand consciousness automatically. One has the impression that on completion of the German Class 10 (equivalent the 5th form in England) the image of mathematics is roughly completed and that new contents are hardly integrated into the mind in a consciousness-complementing manner.

Literature

Abelson, R. 1979. Differences between belief systems and knowledge systems. *Cognitive Science* **3**, 355 - 366.

Amit, M. & Vinner, S. 1990. Some misconceptions in calculus. Anecdotes or the tip of an iceberg? In G. Booker, P. Cobb & T.N. de Mendicuti (Eds.), Proceedings of the 14 th Conference of the International Group For the Psychology of Mathematics Education (PME) with the North American Chapter 12 th PME-NA Conference. Volume 1 (pp. 3 - 10). México.

Baroody, A.J. & Ginsburg, H.P. 1987. The Relationship between initial meaningful and mechanical knowledge of arithmetic. In Hiebert, J. (Ed.), Conceptual and procedural knowledge: The case of mathematics. Hove, Great Britain: Lawrence Erlbaum

Associates Publishers (pp. 75 - 112). Hove, Great Britain: Lawrence Erlbaum Associates Publishers.

Calderhead, J. 1996. *Teachers: Beliefs and Knowledge*. In Berliner, D.C. & Calfee, R. (Eds.), Handbook of Educational Psychology. pp. 709 - 725. New York: Macmillan Library Reference USA.

Cohen, L.& Manion, L. 1994. *Research Methods in Education.* London and New York: Routledge.

Confrey, J. 1990. *A review of the research on student conceptions in mathematics, science and programming*. In C.B. Cazden (Ed.), Review of research in education (Vol. 16, pp. 3 - 55). Washington (DC): American Educational Research Association.

Cooney, T.J., Shealy, B.E. & Arvold, B. 1998. Conceptualizing belief structures of preservice secondary mathematics teachers. Journal for the Research in Mathematics Education 29, 306 - 333.

Even, R. 1990. Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics* **21**, 521 - 544.

Even, R. 1993. Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. Journal for the Research of Mathematics Education 24, 94 - 116.

Green, T.F. 1971. *The Activities of Teaching*. Tokyo: McGraw-Hill Kogakusha.

Hofe, R. vom. 1995. *Grundvorstellungen mathematischer Inhalte.* Heidelberg: Spektrum Akademischer Verlag.

Lester, F.K., Garofalo, J. & Kroll, D.L. 1989. Self-confidence, interest, beliefs, and metacognition: Key influences on problem-solving behavior. In D.B. McLeod & V.M. Adams (Eds.), *Affect and Mathematical Problem Solving. A new perspective* (pp. 75 - 88). New York: Springer-Verlag.

Lloyd, G. M. & Wilson, M.R. 1998. Supporting Innovation: The Impact of a Teacher's Conceptions of Functions on his Implementation of a Reform Curriculum. *Journal for Research in Mathematics Education* **29** (3), 248 - 274.

Nespor, J. 1987. The role of beliefs in the practice of teaching. *Journal of Curriculum Studies* **19** (4), 317 – 328.

Pajares, M.F. 1992. Teachers’ beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research* **62** (3), 307 – 332.

Pehkonen, E. & Törner, G. 1996. Mathematical beliefs and different aspects of their meaning. *International Reviews on Mathematical Education (ZDM)* **28** (4), 101 - 108.

Schoenfeld, A. 1998: Toward a theory of teaching-in-context (draft version: see http://www-gse.berkeley.edu/faculty/aschoenfeld/TeachInContext/tic.html)

Tall, D. 1990. Inconsistencies in the learning of calculus and analysis. Focus on Learning Problems in Mathematics 12 (3/4), 49 - 63.

Thompson, A.G. 1992. Teachers' beliefs and conceptions: A synthesis of the research. In D.A. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp.127 - 146). New York: Macmillan Publishing.

Vinner, S. 1983. Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology* **14**, 293 – 305.

Pupils' beliefs about the role of real-world knowledge in mathematical modelling of school arithmetic word problems

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Ascertaining the problem: pupils' suspension of sense-making in a traditional school arithmetic word problem solving setting

For several years, it has been argued that by the end of elementary school many pupils have constructed a set of beliefs and assumptions about doing mathematical application problems, wherein this activity is reduced to the selection and execution of one or a combination of the four arithmetic operations with the numbers given in the problem, without any serious consideration of possible constraints of the realities of the problem context that may jeopardize the appropriateness of their standard models and solutions. Although these claims about the existence of these beliefs and assumptions among pupils and their impact on pupils' actual problem-solving behavior in the mathematics classroom have been made by several authors, the evidence supporting these claims was rather scarce.

Some years ago, the extent to which lower secondary and upper elementary school pupils ignore plausibly relevant aspects of reality during their modelling and solving of word problems, has been simultaneously studied in Northern-Ireland by Greer (1993) and in Flanders by Verschaffel, De Corte and Lasure (1994). In both studies pupils were administered a paper-and-pencil test consisting of ten matched pairs of items, as part of a typical mathematics lesson. Each pair of items consisted of

- a straightforward or standard item (S-item) that can be solved unambiguously by applying the most obvious arithmetic operation(s) with the given numbers (e.g., "Steve has bought 5 planks of 2 meters each. How many planks of 1 meter can he saw out of these planks?" or "A boat sails at a regular speed of 45 kilometres per hour. How long does it take this boat to sail 180 kilometres at this regular speed?"), and

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- a parallel problematic item (P-item) for which the appropriate mathematical model is less obvious and indisputable, at least if one seriously takes into account the realities of the context evoked by the problem statement (e.g., "Steve has bought 4 planks of 2.5 meters each. How many planks of 1 meter can he saw out of these planks?" or "John's best time to run 100 metres in 17 seconds. How long will it take him to run 1 kilometre?").

Pupils' reactions to the P-problems were either categorized as "realistic" or "non-realistic" based on the activation and use of real-world knowledge and realistic considerations about the problem context. When a pupil gave an answer to the problem that took into account the context *or* when he produced a non-realistic answer that was accompanied by a realistic comment, his(her) overall reaction to that particular problem was scored as a "realistic reaction" (RR). Reactions without any manifest trace of the activation and use of the critical real-world knowledge, were scored as "non-realistic" (NR). Take, for instance, the planks item mentioned above. A RR score was not only given to a pupil who produced the realistic answer "4 x 2 = 8; he can saw 8 planks of 1 meter", but also to a pupil who responded with "10 planks" but who added the comment that "Steve would have a hard time putting together the remaining pieces of 0.5 meter". The code NR ("non-realistic reaction") was given to pupils who answered a problem in a non-realistic manner and did not give any further realistic comment (e.g., answering the "planks" problem as follows: "4 x 2.5 = 10; he can saw 10 planks of 1 meter"). Similarly, for the runner item a RR code was given to a pupil who gave an answer that took into account that John would not be able to run 1 kilometre at his record speed or whose answer "10 x 17 seconds = 170 seconds" was accompanied by some comment, whereas merely responding with "10 x 17 = 170 seconds" was scored as a NR. In both studies pupils demonstrated a very strong overall tendency to exclude real-world knowledge and realistic considerations when confronted with the problematic versions of the problem pairs. For instance, in Verschaffel et al.'s (1994) study, only 17 % of all reactions to the P-items could be considered as realistic (RR).

The findings of Greer (1993) and Verschaffel et al. (1994) have been replicated in several countries, namely Belgium, Germany, Japan, Northern Ireland, Switzerland, and Venezuela, mostly as part of more extensive investigations of the effects of certain variations in the presentation of the problems or in the experimental setting (see further). As in the studies of Greer (1993) and Verschaffel et al. (1994), the vast majority of pupils demonstrated little or no tendency to include real-world knowledge into their solution of most of the P-problems.

Interestingly, in some of these studies a small group of pupils who had solved the paper-and-pencil test with the P-items was afterwards selected for participation in an individual interview aimed at unravelling the reasons for the apparent neglect of real-world knowledge and context-based considerations. These interviews provided (anecdotal) evidence that at least some pupils' tendency to respond P-items in a non-realistic manner was accompanied by the explicit belief that there is a gap between the artificial world of school arithmetic word problems, on the one hand, and the real world outside school, on the other. For instance, in Caldwell's (1995, p. 39) study, one 13-years-old pupil reacted as follows to the interviewer's question as to why she did not make use of realistic considerations when solving the P-items in the context of the written test: "I know all these things, but I would never think to include them in a math problem. Math isn't about things like that. It's about getting sums right and you don't need to know outside things to get sums right".

Effects of alerting pupils about problematic word problems

Although the outcomes of the investigations of Greer (1993) and Verschaffel et al. (1994) and of the replication studies provide empirical evidence for pupils' strong tendency to exclude real-world knowledge and realistic considerations when doing word problems administered in the restricted context of school arithmetic, it remains unclear if this tendency was really due to a deep-rooted and resistant belief about the nature of (solving) school arithmetic word problems among these pupils, or if it was merely an artefact of a "tricky" experimental setting. Indeed, it could be argued that a significant number of pupils who produced NRs for P-items might have made realistic considerations during the solution process of these P-items, but might finally have decided to neglect these realistic considerations in their final answers of these items simply anticipating that such unusual answers would not be appreciated by the experimenter. To investigate the plausibility of this alternative interpretation, several follow-up studies were set up involving minimal interventions that took the form of providing pupils an explicit hint that some of the problems needed careful consideration and/or of giving them direct and explicit help to consider alternative responses taking into account realistic considerations. For instance, in studies by Yoshida, Verschaffel and De Corte (1997) and Reusser and Stebler (1997) some pupils received a written or an oral introductory warning that some of the problems would be difficult or even impossible to solve because of certain complexities or unclarities in the problem statement, and pupils were explicitly invited to mark these problems on their test sheet and to explain why they were unsolvable. However, these variations in the experimental setting intended to make pupils more alert, to sensitize them to the consideration of aspects of reality, and to legitimize alternative forms of answer produced, at best, only weak effects. Apparently, these interventions were not powerful enough to break pupils' beliefs about word problems that underlied their tendency to do word problems in a superficial and artificial way without taking into account the constraints of the problem context.

Increasing the authenticity of the experimental setting

Another set of follow-up studies investigated the impact of another kind of alteration, namely the authenticity.of the experimental setting. In these studies, one or more categories of P-items were presented in a more authentic, performance-based setting, e.g. in the context of a group discussion and/or in the presence of concrete materials and performance-based goals. For instance, DeFranco and Curcio (1997) compared upper elementary school pupils' reactions to the well-known buses item ("328 senior citizens are going to a trip. A bus can sit 40 people. How many buses are needed so that all citizens can go on a trip?") presented either as part of a traditional school arithmetic test or as embedded in a more authentic setting of making a phone call using a teletrainer to order minivans to take school children to a party, and found a drastic increase in the number of RRs (i.e., responses involving a meaningful, context-based interpretation of the outcome of the division with a remainder) from the restricted test setting to the more authentic performance setting.

The effects of traditional mathematics education on pupils' beliefs about word problems

Taken as a whole, the results from the above-mentioned studies suggest that it is not a cognitive deficit as such that causes pupils' abstention from sense-making when doing arithmetic word problems in a typical school setting, but rather that the pupils are acting in accordance with the "rules of the game" of the interactive ritual in which they are involved. Several authors have carried out analyses of the "hidden" rules and assumptions that need to be known and used by pupils to make the "game of word problems" function efficiently (De Corte & Verschaffel, 1985; Gerofsky, 1996; Kilpatrick, 1985; Lave, 1992; Reusser & Stebler, 1997; Schoenfeld, 1991). Among the assumptions thare are listed in these analyses are the following:

- Assume that every problem presented by the teacher or in a textbook is solvable and makes sense.
- Assume that there is only one correct answer to every word problem, and that this has to be a precise and a numerical one.
- Assume that this single, precise and numerical answer can and must be obtained by performing one or more mathematical operations or formulas with the numbers in the problem, and almost certainly with all of them.
- Assume that the task can be achieved using the mathematics one has access to as a student -- in fact, in most cases, by applying the mathematical concepts, formulas, algorithms, etc. recently encountered in mathematics lessons.
- Assume that the final solution, and even the intermediate results, involve “clean” numbers (i.e., whole numbers).
- Assume that the word problem itself contains all the information needed to find the correct mathematical interpretation and solution of the problem, and that no information extraneous to the problem may be sought.
- Assume that persons, objects, places, plots, etc. are different in a school word problem than in a real-world situation, and don’t worry (too much) if your knowledge or intuitions about the everyday world are violated in the situation described in the problem situation.

As long as pupils are engaged in the socio-cultural context of school mathematics, these “rules of the game” seem to govern their thinking, but when they are put in a context wherein these rules or premises are no longer valid (e.g., as in the study of DeFranco and Curcio (1997) mentioned above) most pupils seem to get rid of their inclination to model and solve these P-items in a stereotyped, meaningless way without paying attention at the realistic constraints which make the appropriateness of their routine solutions problematic.

However, it should be acknowledged that this set of beliefs and expectations about word problems is basically a hypothetical construct, for wich researchers have not come up yet with much *direct* and *compelling* empirical support (except some anecdotal explanations of pupils for their NRs to P-items wherein they articulated some of the above-mentioned rules of the game of word problems, as in the above-mentioned example from Caldwell's (1995) study).

Anticipating on more convincing evidence for the existence and the impact of these beliefs among pupils, we can ask the question: By what teaching/learning processes are these beliefs and assumptions of pupils about the game of word problems -- and especially about the unrealistic nature of these problems -- internalized by children? As with most other beliefs, the development of pupils' beliefs about word problem solving as an activity with artificial rules and without any specific relation to out-of-school reality is assumed to occur implicitly, gradually, and tacitly through being immersed in the culture of the mathematics classroom (De Corte & Verschaffel, 1985; Gerofsky, 1996; Kilpatrick, 1985; Lave, 1992; Reusser & Stebler, 1997; Schoenfeld, 1991). More specifically, this enculturation seems to be mainly due to the following two aspects of the current instructional practice and culture in which children learn to solve mathematical application problems:

- the impoverished and stereotyped diet of standard word problems occurring in mathematics lessons and tests, which can almost always be modelled and solved through the straightforward use of one or more arithmetic operations with the given numbers;

- the way in which these problems are considered and used in the current classroom practice and culture, and more specifically the lack of systematic attention to the modelling perspective by the teacher.

While there is only scarce research evidence supporting the claim that these two characteristics of current mathematics classroom practice and culture are *directly* responsible for the observed lack of sense-making among pupils, various analyses of textbooks (see e.g., Stern, 1992) and assessment materials (see e.g., Cooper, 1994), on the one hand, and of (future) teachers' beliefs and teaching behavior with respect to word problem solving (Verschaffel, De Corte & Borghart, 1997), on the other hand, provide at least *indirect* support for it.

Beyond ascertaining studies: Applying the modelling perspective

Nothwithstanding these gaps in our understanding of the nature of pupils' beliefs about school arithmetic word problems, of the relations of these different beliefs with each other and with other belief systems, and of the instructional factors that lie at the basis of their origin and development, some researchers have already started to move from ascertaining studies to intervention studies -- that is to say from studies that primarily describe the state-of-affairs as it exists to studies that seek to intervene in, and thereby improve, the state-of-affairs. For instance, Verschaffel and De Corte (1997) set up a small-scale teaching experiment in which they tried to change pupils' beliefs about the role of real-world knowledge in mathematical modelling and problem solving, and to develop in these pupils a disposition towards (more) realistic mathematical modelling. This was attempted by immersing pupils into a different classroom culture in which word problems are explicitly conceived and used as exercises in realistic mathematical modelling. Three classes from the same school participated in the experiment, comprising one experimental class and two control classes of upper elementary school children. While the pupils from the experimental class participated in a program on realistic modelling consisting of five teaching/learning units of about 2 1/2 to 3 hours each, the pupils from the two control classes followed the regular mathematics curriculum. The major characteristics of the program may be described as follows.

- The impoverished and stereotyped diet of standard word problems offered in traditional mathematics classrooms was replaced by a set of more realistic non-routine problem situations that were specifically designed to stimulate pupils to pay attention to the complexities involved in realistic mathematical modelling, and to distinguishing between realistic and stereotyped solutions of mathematical applications. Each teaching/learning unit focused on one prototypical problem of mathematical modelling (e.g., interpreting the outcome of a division problem involving a remainder, modelling the union or separation of sets with joint elements, etc.).

- Second, a varied set of highly interactive instructional techniques, i.e. small-group collaborative work followed by whole-class discussions.

- Third, the establishment of a new classroom culture by explicitly negotiating new social norms about the role of the teacher and the students in the classroom, and new socio-mathematical norms about what counts as a good mathematical word problem, a good solution procedure, and a good response (see Cobb, Yackel & Wood, 1992; Schoenfeld, 1991).

The comparison of the results of the pupils from the experimental class, on the one hand, and the two control classes, on the other hand, on a pre-test and a post-test consisting of both learning items and transfer items, as well as on a retention test, warranted a positive conclusion about the feasibility of altering pupils' beliefs about the role of real-world knowledge in mathematical modelling and problem solving in upper elementary school children, and of developing in them a disposition toward (more) realistic mathematical modelling of school word problems. Recently, a replication of this teaching experiment has been realized with German pupils by Renkl (1999), with similarly promising results.

Conclusion

Studies from Greer (1993) and Verschaffel et al. (1994), followed by several replications in different parts of the world, have documented that most 10-14-years-old pupils answer non-straightforward word problems as if they are unproblematic. As a next step, a number of studies were carried out to test the hypothesis that pupils' responses to these problematic items could be explained in terms of an experimental artefact, and that simply alerting them to the possibility that some items might have some problematic features needing interpretation, would be effective. The results of these studies provided strong evidence against this hypothesis,indicating that the pattern of responses - and the beliefs underlying them - are deeply rooted and resistant to change. By contrast, the effects of a more fundamental modification of the experimental setting, namely by presenting the tasks in more authentic settings that simulated -- at least to some degree -- the goals, social circumstances and realistic constraints that would influence performance on corresponding tasks carried out in real life, markedly increased the number of responses deemed to be showing awareness of realistic considerations. Although compelling direct empirical evidence is still scarce, it seems that pupils' tendency toward non-realistic mathematical modelling of word problems in a school setting and their beliefs underlying this tendency, are caused and shaped by several characteristics of current instructional practice, such as textbooks and assessment materials and -- last but not least -- the teacher. Finally, some studies already went beyond simply documenting the state-of-affairs and demonstrated that it is possible to break pupils' tendency to conceive and to do word problems in a stereotyped and non-realistic way, by establishing a very different set of social and socio-mathematical norms than in traditional classrooms.

References

Caldwell, L. (1995). *Contextual considerations in the solution of children’s multiplication and division word problems.* (Master’s thesis). Belfast, Northern-Ireland: University of Belfast.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, 23(1), 2-33.

Cooper, B. (1994). Authentic testing in mathematics? The boundary between everyday and mathematical knowledge in National Curriculum testing in English schools. *Assessment in Education,* 1(2), 143-166.

De Corte, E., & Verschaffel, L. (1985). Beginning first graders' initial representation of arithmetic word problems. *Journal of Mathematical Behavior*, 4, 3-21.

DeFranco, T.C., & Curcio, F.R. (1997). A division problem with a remainder embedded accross two contexts. Children’s solutions in restrictive versus real-world settings. *Focus on Learning Problems in Mathematics,* 19(2), 58-72.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. *For the Learning of Mathematics*, 16(2), 36-45.

Greer, B. (1993). The modelling perspective on wor(l)d problems. *Journal of Mathematical Behavior*, 12, 239-250.

Kilpatrick, J. (1985). A retrospective account of the past twenty-five years of research on teaching mathematical problem-solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem-solving: Multiple research perspectives* (pp. 1-16). Hillsdale, NJ: Lawrence Lawrence Erlbaum Associates.

Lave, J. (1992). Word problems: A microcosm of theories of learning. P. Light & G. Butterworth (Eds), *Context and cognition: Ways of learning and knowing* (pp. 74-92). New York: Harvester Wheatsheaf.

Renkl, A. (1999, August). *The gap between school and everyday knowledge in mathematics*. Paper presented at the 8th European Conference for Research on Learning and Instruction, Goteborg, Sweden.

Reusser, K., & Stebler, R. (1997). Every word problem has a solution: The suspension of reality and sense-making in the culture of school mathematics. *Learning and Instruction*, 7, 309-328.

Schoenfeld, A.H. (1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J.F. Voss, D.N. Perkins & J.W. Segal (Eds.), *Informal reasoning and education* (pp. 311-343). Hillsdale, NJ: Lawrence Erlbaum Associates.

Stern, E. (1992). Warum werden Kapitänsaufgaben 'gelöst'? Das verstehen von textaufgaben aus psychologischer Sicht. *Der Mathematikunterricht*, 38(5), 7-29.

Verschaffel, L., & De Corte, E. (1997b). Teaching realistic mathematical modelling and problem solving in the elementary school. A teaching experiment with fifth graders. *Journal for Research in Mathematics Education*, 28, 577-601.

Verschaffel, L., De Corte, E., & Borghart, I. (1997). Pre-service teachers'conceptions and beliefs about the role of real-world knowledge in mathematical modelling of school word problems. *Learning and Instruction,* 4*,* 339-359.

Verschaffel, L., De Corte, E., & Lasure, S. (1994). Realistic considerations in mathematical modelling of school arithmetic word problems. *Learning and Instruction*, 4, 273-294.

Yoshida, H., Verschaffel, L., & De Corte, E. (1997). Realistic considerations in solving problematic word problems: Do Japanese and Belgian children have the same difficulties? *Learning and Instruction*, 7, 329-338.

Beliefs we live by and quite often are even not aware of - their possible impact on teaching and learning mathematics

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The title of this presentation is a paraphrase of a well known title of a well known book: Metaphors We Live By (Lakoff and Johnson, 1980). The main claim of this book is that our language and our thought are full of metaphors which have great impact on the way we talk and the way we think and yet, quite often, because these metaphors are so deeply rooted in us, we are unaware of them. Hence, we are unaware of some factors which have great influence on our intellectual behavior. This is true also about beliefs and it is also true that very often we are not aware of our beliefs. Becoming aware to our beliefs is important if we want to reflect on our behavior and perhaps to change it. Before specifically discussing some beliefs I would like to elaborate on the notion of beliefs. It is a linguistic discussion and my way of conducting a linguistic discussion is to start with a lexical definition. Such a definition is supposed to tell people about the common use of a given words in different contexts. Later on, additional meanings or even technical meanings can be associated with the word in case we want to use it in a technical context. Generally, a lexical definition is very different from a mathematical definition since it does not establish the meaning of a new notion by means of previously defined notions or primary notions. Generally, the lexical definition ties one notion to other notions which have similar meaning or which are related to it. According to the Webster's Ninth New Collegiate Dictionary a belief is:

1. A state or habit of mind in which trust or confidence is placed in some person or a thing. 2. Something believed; specific: a tenet or body of tenets held by a group. 3. Conviction of the truth of some statements or the reality of some being or phenomenon esp. when based on examination of evidence.

Syn BELIEF, FAITH, CREDENCE, CREDIT *mean assent to the truth something offered for acceptance.* BELIEF *and* FAITH *are often used interchangeably but belief may or may not imply certitude in the believer whereas faith always does, even when there is no evidence or proof.*

Being sent by the dictionary from one notion to another I have found altogether that BELIEF is related to at least the following words: TRUST, CONFIDENCE, TENET, CONVICTION, FAITH, CREDENCE, CREDIT, DOCTRINE, OPINION, VIEW, PERSUASION.

Among many things which are said about BELIEF and other words related to it is worth quoting the following: *a belief is stronger than an impression and less strong than positive knowledge ... Conviction applies to a firmly and seriously held belief.*

To the above notions I would like to add some other notions which are not considered as first order relatives of BELIEF but in my discussion I will relate them to beliefs (and surely, they are second order relatives of BELIEF even by the Merriam - Webster dictionary). The notions I would like to add are: DOGMA, MODEL (and when I say model I mean a model for action or behavior), RITUAL and RHETORIC.

When I started my research in mathematics education I tried to focus on one aspect of it which was the cognitive aspect. It is a convenient way for doing research but it is frustrating in terms of educational goals. Mathematics Education is a partial domain of education and questions are supposed to be asked about the goals of education. The goals of mathematics education should be related to the general goals of education and mathematical behavior of students and teachers should be considered as a special case of general human behavior. Therefore, now, if I want to investigate some aspects of mathematical behavior I am trying to find out the general aspects of human behavior from which the mathematical aspects are derived. Thus, before dealing with mathematical beliefs and their impact on teaching and learning of mathematics I would like to reflect about beliefs in general and then to relate this reflection to beliefs which have impacts on mathematical behavior.

It seems to me that the most suitable domain for examining the general aspects of BELIEF is religion. Religion is not only a belief in God. It is a collection of many beliefs about desired behaviors. This includes instructions how to act with other human beings and instructions how to act with God. The last are the religious rituals. Some typical aspects of religious rituals are:

1. You believe that accomplishing a ritual will please God, will make you accepted by God, will open for you certain gates (the gates of heaven).

2. You do not necessarily understand why you have to follow a ritual and sometimes you are even told that you are not supposed to understand, you are not supposed to ask, you are supposed to do it and that is all. (Many Jews who do not know Hebrew and many Christians, belonging to the Catholic church who do not know Latin, say prayers the words of which they do not understand. However, they are sure that their prayers have important impact on their life and will give them credit and some good points in crucial moments.)

3. You believe that certain rituals give validity (or legal status) to a relationship between human beings or between human beings and God. (Think for instance of marriage ceremonies, divorce ceremonies, baptizing ceremonies, circumcision, funerals, religious memorial services and more).

If we accept that religious rituals satisfy some basic psychological needs in human beings we should expect that rituals will be formed and will be accomplished also in secular domains of human behavior. And, indeed, there are many secular rituals. If we ignore the context and the content of rituals they can be considered as sequences of words or actions. Thus, when looking for rituals in mathematical behavior we are supposed to look for sequences of words (also mathematical symbols in our case) and actions (mathematical actions in our case) which fulfil the following:

1*. You believe that accomplishing a given sequence of words, mathematical symbols, or mathematical action will please somebody (the system, the teachers, the parents, etc.).

2*. You do not have to understand why you and other people have to follow this sequence of words or actions.

3*. You believe that following certain sequence of words or actions will give validity (or legal status) to certain mathematical entities (it is a fuzzy notion but the domain of mathematical education is quite fuzzy for many people).

In the domain of mathematical behavior there are many procedures. (The common complaint in the mathematical education community is that mathematics teaching emphasizes too much the procedures so that many students consider mathematics as a collection of (meaningless) procedures.) Is it possible that some students are going through mathematical procedures the same way people go through rituals (either religious or secular)? Just think of some central activities in mathematical education as simplifying algebraic terms, solving equations, differentiating, solving word problems - going through them may look to many people as going through rituals. If rituals are so common in human behavior we may assume (when working within certain psychological paradigms) that there exists in us a certain psychological schema associated with rituals. I would like to call it the ritual schema. My claim is that in many students the ritual schema is activated when they do mathematics. The characterization of the ritual schema is given by 1* - 3* above.

Let us look at the following three illustrations. The first one is taken from Schor & Alston (1999). A group of elementary teachers was asked to *create meaningful situations or stories that would make sense* for some numerical expressions (p.172). One of the expressions was : $-3-(-4)$.

Here are two stories (p.173):

(a) *Sandy got squares for positive and negative numbers. –1 = a square in red color. 1 = a square in blue color. – (–1) = a square in blue color. She took 3 red squares, and then subtracted 4 in blue. How many squares in what color did she have?*

(b) *Sharifa had $3 negative (out of pocket) and she gave Maria negative one times minus $4. How much did they have together?*

If you look carefully at the examples you may admit that mathematical education at the elementary level formed a ritual which can be called the *creating a story* ritual. In this ritual students, as well as elementary teachers, are supposed to form sequences of words which can be related to given numerical expressions. There are some key words and some rules by means of which the "story" is created. It is not so important whether the "story" makes or does not make sense. As long as it is not rejected by somebody the game can continue. If it is rejected then there is a problem - I have made a move which does not suit the ritual. Anyway, I do not exactly understand the rules of this ritual.

It is reported in the above study that even some of the teachers in the group claimed they did not understand the stories and asked how these stories could be considered as real situations. Of course, we cannot know what caused the teachers to form these two meaningless stories. I suggest the *ritual schema* as a possibility.

The second illustration to the *ritual schema* is taken from a questionnaire I made long time ago (Vinner, 1983). It was created in order to investigate high school students beliefs about the notion of proof but since then I have used it with mathematics teachers in various occasions. It is the following:

In an Algebra class the teacher proved that every whole number of the form $n^3 - n$ is divisible by 6. The proof was: $n^3 - n = n(n^2 - 1)$.
Using the formula $a^2 - b^2 = (a + b)(a - b)$ we can write: $n^2 - 1 = n^2 - 1^2 = (n + 1)(n - 1)$.
Thus $n^3 - 1 = n(n + 1)(n - 1) = (n - 1)n(n + 1)$.
But $(n - 1)n(n + 1)$ is a product of three consecutive whole numbers. Therefore, one of them should be divisible by 2 and one of them (not necessarily a different one) should be divisible by 3.

A day after that, the following exercise was given to the class as a homework assignment:

Prove that $59^3 - 59$ is divisible by 6.

Here are three answers which were given by three students:

1. I computed $59^3 - 59$ and found out that it is equal to 205,320. I divided it by 6 and I got 34, 220 (the remainder was 0). Therefore, the number is divisible by 6.

2. One can write $59^3 - 59 = 59(59^2 - 1)$. But $59^2 - 1 = 59^2 - 1^2 = (59 + 1)(59 - 1)$ according to a well known formula. Therefore, $59^3 - 59 = 59(59 + 1)(59 - 1) = (59 - 1)59(59 + 1)$. We got a number which is a product of three consecutive numbers. One of them is divisible by 2 (it is 58) and one of them is divisible by 3 (it is 60). Therefore, the product is divisible by 2*3, namely, by 6.

3. Yesterday, we proved that every whole number of the form $n^3 - n$ is divisible by 6. $59^3 - 59$ has this form. Therefore it is divisible by 6.

Which answer out of the three do you prefer and why?

In the student sample that I investigated (10th and 11th graders, N = 365), 35% preferred answer 2 (14% preferred answer 1, 43% preferred answer 3 and 8% did not have any preference). If you think of it from the above perspective (the ritual schema) then at the context of the above questionnaire, going through the general proof in order to establish the particular case is pointless. However, if you look at a mathematical proof as a ritual then answer 2 is preferable. As a matter of fact, if we look at the form of traditional proofs in mathematics they do have some ritual elements (the form, the vocabulary, and the ultimate mantra, Q.E.D., at the end). Of course, I do not expect anyone to tell me explicitly that they prefer answer 2 because it reproduces the entire proof ritual. I suggest this as an explanation. The nature of this suggestion is speculative. However, when we investigate people's beliefs we observe behavior and we speculate about the beliefs which might produce it.

A more surprising distribution than the above I get when I distribute the above questionnaire to mathematics teachers. In my last sample (1999) I had 27 teachers. Eleven of them preferred answer 2, five preferred answer 1, nine preferred answer 3 and two had no preference. The arguments explaining why answer 2 is the best were like the following:

a) It was the teacher's intention. b) This answer indicates that the student really understands how to prove. c) The student reconstructed the procedure. d) The student did not substitute a number in a formula in a mechanical way. He related to the components of the task.

The answers rejecting answer 2 were similar to the following:

a) The student reproduced the entire procedure which shows that he did not exactly understand what proof is for. b) The student imitated the general proof. He learned it by heart but did not show understanding.

Examining the above arguments you realize that in some cases the very same argument can be used both for acceptance and rejection. The way I see it, the ritual schema is quite dominant in the explanations. The reason for rejecting answer 2 by some teachers was that it seemed to them as a meaningless ritual.

The third and last illustration to the *ritual schema* is anecdotal but well known to many mathematics teachers with whom I discussed it. After solving quadratic equations by means of factorization the following exercise was given to the students:

Solve $x^2 - 5x + 6 = 1$

Some of them wrote:
$(x - 2)(x - 3) = 1$

$x - 2 = 1$	$x - 3 = 1$
$x = 3$	$x = 4$

They were imitating the procedure for solving: $x^2 - 5x + 6 = 0$ as if it were a meaningless ritual.

The common tendency in the mathematical education community is to move from meaningless procedures (rituals) to meaningful actions. This tendency itself is based on the belief that meaningful learning is one of the major goals of education. This can be discussed and we may find out that this belief is not necessarily shared by everybody. At least not all our students believe in it. All mathematics educators whom I know believe in it but, of course, I do not know all mathematics educators. Do all our students in schools or in teacher education programs believe in it. If, in the line of thought I mentioned earlier, I try to look at this tendency as a special case of a general tendency in human psychology then it can be related to *man's search for meaning (*Frankl, 1978). At this point the focus of the discussion about beliefs is turned to us, the mathematics educators. We should not restrict ourselves to mathematical beliefs and their impact on our teaching and on the learning of our students. There are other beliefs which have impact on the way we teach mathematics or mathematics education. At this point I would like to return to some of the BELIEF synonyms which I mentioned earlier and to touch them briefly:

1. DOGMA. What are the beliefs of the mathematics education community which became the dogmas of the community and nobody dares to raise doubts about them? Here are some candidates: a) Mathematics is extremely important for many professions. b) Mathematics is necessary for every educated person. c) Mathematics is the language of science. d) Learning mathematics develops analytical thinking.

2. MODELS. What are the models we have adopted for our teaching? (Lecturing, in spite of all.) What is our model of a good lesson? (covering the syllabus; sticking to the lesson plan; not having too much student discussions; having questions and answers under control.) When do we get the impression that our students are really learning? (they look at us with admiration; they take notes.) What are our requirements from homework assignment? (solving extremely difficult exercises; reading a lot and summarizing, working individually.) What are our models for conferences (presentations; proceedings as outcomes; working groups; social events).

3. RHETORIC. What rhetoric do we use when we talk to our students? (mathematics is important to your future; you will fail if you do not study mathematics.) What rhetoric do we use in mathematics education conventions (technology is going to change completely mathematics teaching and learning; We should make mathematics curriculum relevant to the students; we should look at social aspects of learning mathematics.)

4. FAITH. Do we really have faith in our students' ability to study mathematics? Do we believe that mathematics is for all (as claimed in the Curriculum and Evaluation Standards for School Mathematics, 1989)? Do we really believe that mathematics education is crucial for the future of our society?

RITUALS. Are we engaged with rituals in our professional life? Can we identify ourselves in situations where we believe that saying certain key-words or going through certain actions will open for us some gates or will make us accepted by somebody (society, the human system to which we belong whether this is a family, a school, the university, a club, a political party or something else)? Here are some useful words which may open for us some gates (sometimes it is not even clear to what place we enter when these gates are opened): *The end of the second millennium, internet, virtual schools, constructivsm, educational change, professional growth, interactive, brain storms, literacy, numeracy, she* (instead of he).

Some acts which are supposed to give validity to our professional activity: *committee meetings, conferences, proceedings, papers.*

Mentioning the above does not imply that part of them or all of them are meaningless in my opinion. Sometimes, they are absolutely meaningful, especially when the context justifies it and when people know the meaning of the words they use and the purpose of their actions. However, how often are we involved in committee meetings which take place mainly for the record, and how often do we have conferences mainly in order to show scientific activity, and how often do we write papers just in order to add their titles to our publication list? What I have said now is strongly related to *rhetoric* as well as to *rituals*. Quite often, rhetoric is absolutely meaningful for the people who use it and for the people who listen to it. On the other hand, also quite often, rhetoric becomes a meaningless ritual. People say what the say even without understanding the meaning of the words, nevertheless, they know it is important for the record.

No doubt, I have deviated from the official title of the conference: *mathematical beliefs and their impact on teaching and learning of mathematics.* It is because I believe that the beliefs which are active in mathematical behavior are not necessarily been formed by the mathematical context. They may well be formed in other contexts and are applied (or become active) in mathematical contexts. Beliefs are crucial for our thought and behavior. They give us certitude. In his famous book about intuition, Fischbein (1987), speaks about the desperate need of human beings for certitude. Intuitions are also beliefs we have about certain things. Without beliefs there is no certitude and without certitude we can hardly function. Because of the need for certitude religion is so successful. Sometimes the success is well established at the paganish level. Science as well as science education are supposed to teach us critical thinking. We have to examine our beliefs. Religion and education have one thing in common. Both are supposed to recommend value systems. No doubt religion does it. I am not sure about education. It mainly emphasizes knowledge. I would be much happier if we could find some beliefs about mathematics education which can be related to human values, not only to the domain of mathematical knowledge.

References:

Curriculum and Evaluation Standards for School Mathematics (1989), NCTM, Reston, Virginia.

Fischbein, E. (1987), Intuition in Science and Mathematics. Reidel Publishing Company

Frankl,V.E. (1978), The Unheard Cry for Meaning, Simon and Schuster.

Lakoff, G. and Johnson, M. (1980), Metaphors We Live By. The University of Chicago Press.

Schor, R.Y. & Alston, A.S. (1999), Keep Change Change. The Proceedings of the 23rd conference of PME, Israel, Vol. 4, 169 - 176.

Vinner, S. (1983), The Notion of Proof - Some Aspects of Students' Views at the Senior Level. The Proceedings of the 7th conference of PME, Israel, 289 - 294.

Mathematical and pedagogical conceptions of secondary teachers

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The conceptions secondary teachers hold about mathematics and the teaching and learning of mathematics are important contextual factors influencing their practices. There have been many empirical studies describing possible relationships between mathematics teachers' conceptions and their practices (e.g., Cooney, 1985, Thompson, 1984) as well as theoretical discussions about possible relationships (e.g., Ernest, 1991). Studying secondary mathematics teachers' conceptions has become even more prevalent recently in the United States given the interest in describing the experiences of teachers trying to align their practices with recommendations promoting student inquiry, cooperation, and exploration (e.g., National Council of Teachers of Mathematics, 1989).

One prominent idea in these discussions is the role of authority in mathematics teaching and learning. *Mathematical Authority* (Wilson and Lloyd, in press) involves teachers' conceptions of mathematics. It is important for teachers to focus on mathematical content and how to make that content more accessible to students. An important component of teacher development often includes the realization of the power of moving away from an emphasis on procedures toward a greater emphasis on relational or conceptual understanding (Skemp, 1987; Hiebert and Lefevre, 1986). *Pedagogical Authority* deals with teachers' conceptions of the teaching and learning of mathematics. To acknowledge and honor diverse ways of knowing and to allow students to build their own mathematical understandings through cooperative exploration also require substantial changes for many teachers. The difficulty of this kind of change may be related to teachers' underlying pedagogical orientations: many mathematics teachers see themselves as the ultimate arbiters of mathematical correctness and find it extremely difficult to share responsibility with their students.

Authority in Mathematics Teacher Development

Mathematical Authority

Mathematical Authority is related to the writings of Paul Ernest (1991) and Alba Thompson (1992) who suggested that teachers' orientations to the subject of mathematics are intimately connected to how they teach. Mathematical Authority deals with the manner in which individuals understand and come to understand mathematical ideas. An important component of this category is what Skemp (1987) referred to as relational understanding and Hiebert and Lefevre (1986) described as conceptual knowledge. The positive impact on students of secondary teachers possessing and sharing rich, flexible, and connected understandings of mathematical concepts is illustrated, for example, in the work of Sowder, Phillip, Armstrong, and Schappelle (1998). It is essential for teachers to appreciate that they cannot simply explain all important connections and concepts to students, they must be willing at times to *share* control or responsibility of the mathematical ideas. This sharing allows students to accept responsibility for their own learning and gain ownership of the ideas considered. Such sharing is also an important part of the Mathematical Authority category.

Several reports illustrate the importance (and difficulty) of sharing Mathematical Authority with students (e.g., Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Wilson & Goldenberg, 1998; Wilson & Lloyd, in press). These examples show the importance of both deep understanding of important mathematical ideas *and* the desire (and ability) to share responsibility with students.

Pedagogical Authority

In addition to teachers possessing deep mathematical understandings and sharing these understandings with students in responsible ways, the development of teachers' voices of authority in deciding *how* to share responsibility with students is critical. Teachers' experiences are influenced by the nature of their views about Pedagogical Authority, that is, the strength of their *own* voices and conceptions for determining classroom activities and content (Wilson & Lloyd, in press). For teachers to effectively apply Pedagogical Authority it is important for them to view the classroom as a contextual domain in which "correct" or "best" depends upon circumstances. Through critical reflection on practice, using their own Pedagogical Authority, teachers determine for themselves the value of particular ideas and innovations.

There is a fine but extremely important distinction between Mathematical Authority and Pedagogical Authority. Teachers, like their students, must gain ownership of new ideas in their own ways. It is easy for reformers to simply tell teachers the best or most appropriate ways to do things, and expect teacher compliance. The reflective judgement model developed by Kitchener and King (1994) describes how adolescents and adults develop in relation to their orientations to authority. This model is central to the ideas proposed here because it emphasizes the importance of reflection and the need for the source of correctness to be internal to the individual.

The paper will describe several examples illustrating secondary teachers' attempts to develop Pedagogical Authority. Teachers' ability to share responsibility with their students is related to their own acceptance of various beliefs and behaviors. For example, Mr. Burt's (Wilson & Goldenberg, 1998) incorporation of an *experimentalist* (Dewey, 1958) teaching approach, in which he based decisions about what do on circumstances, illustrates this idea. His own critical reflection enabled him to make sense of and at times use reform ideas. However, it was not just Mr. Burt's compliance with reform ideas that illustrated his development and use of Pedagogical Authority. This idea is also illustrated by Mr. Burt's reflection about his own practice and consequent decision to use more traditional methods because they made sense to his circumstances.

Secondary teachers discussing their assessment practices provide other examples of the importance of reflection. In a recent study, it was common among the teachers interviewed to elaborate about using open-ended, innovative assessment items during class discussions but not in formal evaluations. Reflection about this inconsistency—teachers wanted their students to explore rich problem situations but did not feel an obligation to make them responsible for such connections in formal assessment activities—might enable teachers to adopt alternative assessment strategies. The paper will discuss other examples and implications for the Pedagogical Authority category.

Concluding Comments

As Goldsmith and Schifter (1997) indicated, the reconceptualization of teaching and learning roles often requires an extended period of time for teachers. Perhaps this extended time requirement is related to teachers' need to explore and develop their own strategies. It is important that the source of correctness be internal for teachers. Reflection on practice seems to be a key factor in the development by teachers of Pedagogical Authority. The two categories illustrated in this paper are intimately connected. However, the process of recognizing both Mathematical and Pedagogical Authority will help teacher educators better assist mathematics teachers who are reforming their practice, as well as researchers who are attempting to characterize teachers' experiences. There is an acknowledgement of the need for rich content knowledge by teachers, but also a recognition of the importance of teachers developing their own ways of adapting and using successful teaching strategies.

References

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*, 194-222.

Cooney, T. J. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education, 16*, 324-336.

Dewey, J. (1958). *Art as experience*. New York: Capricorn Books

Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer.

Goldsmith, L. T., & Schifter, D. (1997). Understanding teachers in transition: Characteristics of a model for the development of mathematics teaching. In E. Fennema and B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 19-54). Magwah, NJ: Lawrence Erlbaum

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural Knowledge in mathematics: An introductory analysis. In J. Hiebert, (ed.), *Conceptual and procedural knowledge: The case of mathematic*s. Hillsdale, HJ: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

King, P. M., & Kitchener, K. S. (1994). *Developing reflective judgment*. San Francisco, CA: Jossey-Bass.

Skemp, R. R. (1987). *The psychology of learning mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Sowder, J., Phillip, R., Armstrong, B., & Schappelle, B. (1998). Middle-grade teachers' mathematical knowledge and its relationship to instruction. Albany, NY: SUY Press.

Thompson, A. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*, 105-107.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Wilson, M. & Goldenberg, M. P. (1998). Some conceptions are difficult to change: One middle school mathematics teacher's struggle. *Journal of Mathematics Teacher Education* , *1*: 270-293.

Wilson, M., & Lloyd, G. M. (in press). The challenge to share mathematical authority with students: High school teachers' experiences reforming classroom roles and activities through curriculum implementation. *Journal of Curriculum and Supervision.*

Beliefs and norms in the mathematics classroom

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For more than a decade we and our colleagues[16] have collaborated to study students' mathematical learning in the context of the classroom. In the process of doing so, we have developed an interpretive framework (see Figure 1) for analyzing classrooms that coordinates both individual (psychological) and collective (sociological) perspectives. In this work we were strongly influenced by Bauersfeld, Krummheuer, and Voigt's (Bauersfeld, 1988; Bauersfeld, Krummheuer, & Voigt, 1988) long standing work in advancing symbolic interactionism as a theoretical framework for investigating mathematics teaching and learning. The central thesis of this paper is that by coordinating sociological and psychological perspectives it is possible to develop ways to explain how changes in beliefs might be initiated and fostered in mathematics classrooms. The purpose of this paper is to develop this thesis. In particular, in the paper we discuss the interpretive framework giving specific attention to students' beliefs and corresponding classroom norms. An example from a university level differential equations class is used to clarify and illustrate constructs within the framework. The example demonstrates both the normative aspects of the classroom and the corresponding student beliefs. In each of the classrooms we have studied over the past years, from elementary school mathematics to university level differential equations, students' mathematical beliefs changed dramatically over the course of the teaching experiment. In the paper we demonstrate how the theoretical constructs of the interpretive framework can be used to explain this change.

Background

Following on the seminal work of Erlwanger (1973), a number of mathematics educators have argued for the need to consider students' beliefs about mathematics when attempting to make sense of their mathematical behavior. For example, Cobb (1985) demonstrated that the mathematical activity of the young children who participated in an extended teaching experiment could not be accounted for solely in terms of their mathematical conceptions. However, by complementing a conceptual

16 The colleagues with whom we have worked over the past decade or more include Paul Cobb, Koeno Gravemeijer, Terry Wood, Grayson Wheatley and Diana Underwood. The interpretive framework that forms the basis for this paper was developed by Paul Cobb and Erna Yackel.

analysis with an analysis of the children's beliefs it was possible to explain the radically different behavior of children to whom similar concepts were attributed. At the same time, Schoenfeld's work with university level students led to similar conclusions (Schoenfeld, 1983). In terms of the framework shown in Figure 1, we would say that Cobb and others were claiming that it is insufficient to attend to the lower right cell of the framework alone when conducting a psychological analysis.

One of the consequences of this early work on beliefs was attention to context, where context was taken to mean the general implicit framework that guided an individual's activity. In this usage, context is closely related to one's worldview or paradigm. Thus, an individual's (psychological) context refers to the internal cognitive state of the individual and is to be differentiated from the setting or environment in which the person is acting. The external conditions form the setting. The distinction between this usage of context and the term setting indicates an attempt to reflect the fact that the same mathematical problem presented in two different formats, for example an addition problem written horizontally or vertically, might be interpreted by a student as two completely different tasks and might result in different solution procedures and even different answers, without the student experiencing a conflict (Cobb, 1986). Thus, in saying that we and our colleagues studied students' mathematical learning in the context of the classroom, we are implying our intention of taking into account the students' interpretations of the situation and their goals and motivations. As we have shown elsewhere, it is the situation as it is interactively constituted as a social event rather than the social setting per se that is critical in influencing the nature of children's mathematical activity (Yackel, 1995). Further, we are implying our intention of taking account of social interactions.

As early as 1986, Cobb conjectured that mathematics instruction, as a socialization process, influences students' beliefs. This conjecture, which was based on working with children in one-on-one settings, was confirmed in a classroom teaching experiment we conducted in 1986-87. As we have reported elsewhere (Cobb, Yackel, & Wood, 1989), in this classroom, at the beginning of the school year, students' beliefs were compatible with a "school mathematics tradition" but as the year progressed their beliefs became compatible with an "inquiry mathematics tradition."[17] Initially, the teacher's expectations that the children should [attempt to construct their own solutions to problems and] verbalize how they actually interpreted and attempted to solve the instructional activities ran counter to their prior experiences of mathematics instruction in school (Wood, Cobb, & Yackel, 1988). The teacher, therefore, had to exert her authority in order to help the children reconceptualize their beliefs about both their own roles as students and her role as the teacher during mathematics instruction. She and the children initially negotiated obligations and expectations at the beginning of the school

[17] In using the labels "school mathematics tradition" and "inquiry tradition" we are following Richards (1991) who characterizes the "school mathematics tradition" as one in which students are treated as passive recipients of information and the "inquiry mathematics tradition" as one designed to teach students the language of mathematical literacy. Richards likens the discourse in the school mathematics tradition to "a type of 'number talk' that is driven by computation." (p. 16) By contrast, discourse in the inquiry tradition involves discussions in which individuals interact to attempt to explain and justify their mathematical activity to one another (cf. Thompson, Philipp, Thompson, & Boyd's discussion of calculational versus conceptual orientation, 1994).

year which made possible the subsequent smooth functioning of the classroom. Once established, this mutually constructed network of obligations and expectations constrained classroom social interactions in the course of which the children constructed mathematical meanings (Blumer, 1969). The patterns of discourse served not to transmit knowledge (Mehan, 1979; Voigt, 1985) but to provide opportunities for children to articulate and reflect on their own and others' mathematical activities. (Cobb et al, 1989, p. 126)

As we will explain below, in order to investigate how it was that students' beliefs were influenced by the socialization process, we sought to analyze the social (participation) structure of the classroom. The sociological perspective we followed was that of symbolic interactionism because of its compatibility with psychological constructivism (Voigt, 1992; Yackel & Cobb, 1996).[18] In the same way that attention to students' beliefs is not a logical necessity but proves pragmatically useful because it helps to account for aspects of students' mathematical activity that otherwise are not explainable, taking a sociological perspective is not a logical necessity. However, taking such a perspective proves pragmatically useful because doing so provides means to analyze and ultimately explicate aspects of the teaching and learning of mathematics in the classroom setting that otherwise defy explanation.

The interpretive framework

First, I wish to emphasize that the interpretive framework is not the result of an a priori theoretical analysis but rather grew out of extensive classroom-based research. It evolved from our attempts to make sense of students' learning in the classroom across several year-long classroom teaching experiments in elementary school mathematics classes. Our initial efforts included considerable attention to unraveling the complexity of the classroom by focusing on the classroom social norms and later on the sociomathematical norms (Cobb, Yackel, & Wood, 1989; Yackel, Cobb, & Wood, 1991; Yackel & Cobb, 1996). These constructs are sociological in that they refer to the classroom community as a collective group rather than to the individual members of the community. Nevertheless, in attempting to analyze norms, we took the position that there is a reflexive relationship between individual activity and the activity of the collective. Therefore, our analyses of norms necessarily involved taking account of the corresponding individual components. As we have noted elsewhere (Cobb, Yackel, & Wood, 1993) we take beliefs to be the psychological correlates of norms. Thus discussions of norms and discussions of beliefs are intimately intertwined. This interrelationship between beliefs and norms is critical because it provides a means for talking about changes in beliefs. Changes in beliefs occur concomitantly with the constitution of norms.

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[18] Voigt (1992) argues that of the various theoretical approaches to social interaction, the symbolic interactionist approach is particularly useful when studying children's learning in inquiry mathematics classrooms because it emphasizes the individual's sense-making processes as well as the social processes. Rather than attempting to deduce an individual's learning from social and cultural processes or vice versa, symbolic interactionism sees individuals as developing their personal understandings as they participate in negotiating classroom norms, including those that are specific to mathematics.

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about one's role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Specifically mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

Figure 1. Interpretive Framework for Analyzing Individual and Collective Activity in Classrooms.

To clarify further, in saying that norms and beliefs are reflexively related we imply that they evolve together as a dynamic system. Methodologically, both general social norms and sociomathematical norms are inferred by identifying regularities in patterns of social interaction. Thus social norms are identified from the perspective of the observer and indicate an aspect of the social reality of the classroom. However, what becomes normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants. A student's beliefs are the understandings that he or she uses in appraising a situation. For example, a student's inferred beliefs about his or her own role in the classroom, others' roles, and the general nature of mathematical activity can be thought of as a summarization of the obligations and expectations attributed to the student across a variety of situations, that is, they are the individual's understandings of normative expectancies. Social norms can be thought of as taken-as-shared beliefs that constitute a basis for communication and make possible the smooth flow of classroom interactions (Cobb et al 1993).

Example

To illustrate the reflexive relationship between students' beliefs about their own role, others' roles and the general nature of mathematical activity in school and social norms, we consider a university level differential equations class in which the instructor sought to develop an inquiry approach to instruction.[19] This was a novel experience for the students since all of them had experienced only the school mathematics tradition in their prior grade K-12 and university mathematics instruction. Thus, at the beginning of the semester their classroom mathematical beliefs were based on expectations that the students' role in class is to follow instructions and to solve problems in the way that the instructor and/or the textbook demonstrate. Similarly, the instructor's role is to explain

[19] Chris Rasmussen was the course instructor. Erna Yackel attended every class session. They, together with mathematics educator Karen King, formed the project team for the classroom teaching experiment.

and demonstrate procedures for the students to follow. Such beliefs are in contrast with the expectations and obligations that underpin inquiry instruction, namely that each student is expected to develop a personally-meaningful solution, to explain and justify his or her thinking, to listen to and attempt to make sense of the thinking of others, and to raise questions and challenges when he or she does not understand or agree. The differing expectations of the students and the instructor led to situations of explicit negotiation. For example, on the second day of class the instructor began with a brief statement of the expectations he had for the students' mathematical activity. Then he orchestrated a whole class discussion of approximately twenty minutes in which he and the students discussed the rationale behind a differential equation they had used in the prior class session to indicate the rate of change of the recovered population in an infectious disease problem.

The critical aspect of this segment for our purposes is the explicit attention the instructor gave throughout to the negotiation of social norms compatible with the expectations listed above. Throughout the entire episode, in which the instructor spoke 26 times, there were only four occasions when he provided explanations related to the mathematical content. On all other occasions his remarks were (explicitly or implicitly) directed toward the expectations. For example he said things such as:

> Okay, can you explain to us then why it was 1/14 times I?
>
> What do the rest of the people think about that?
>
> Is that similar to what you were thinking?
>
> Anyone want to add to that explanation? Expand on it a little bit?
>
> So let's put that question out. So your question is why didn't we just say dR/dt is 1/14 or 1/15 or something like that. Is that what I heard you say?

In making these remarks, the instructor was attempting to influence the interpretations the students made of how they were to engage in the discussion. From this perspective, it might seem that the teacher is the only one in the classroom who contributes to the renegotiation of social norms. However, norms for social interaction are interactively constituted as individuals participate in the interaction (Yackel & Cobb, 1996). In this case, as the episode evolved students contributed their part to the negotiation of the social norms by increasingly acting in accordance with the expectations. As the discussion progressed, students not only responded to the instructor's questions, they initiated comments of their own that showed that they were beginning to change their understandings of the classroom participation structure. For example, a few minutes into the discussion one student said about another student's remark, "I didn't quite understand what he said" and a few seconds later explained what he did understand and then said, "What I don't understand, what I was asking about [is] ...."

The effectiveness of the renegotiation of social norms is indicated by considering classroom interactions that became typical later in the semester. For example, on one occasion after two students in the class explained how they determined that a particular phase portrait would not have two saddles next to each other, Dave spontaneously added to the discussion with this remark:

> The way I thought about it at first, to make me think that all the points weren't saddles, is that if the next one was a saddle—see how [Bill] has got the one line coming in towards [referring to the phase portrait that Bill had drawn on the blackboard]. Well, if the next one was like that, then you would have to have another point in between those two equilibrium points, like separating, like a source or something. So that's how I started thinking about it. So then $3\pi/2$ might be a source or maybe a saddle point with opposite directions.

This spontaneous remark of Dave's indicates that he has taken seriously the obligations of developing a personally-meaningful solution, of listening to and attempting to make sense of the thinking of others, and of offering explanations and justifications of his mathematical thinking. In the process of acting in accordance with these expectations he is simultaneously contributing to their ongoing constitution. In this way, the normative patterns of interaction serve to sustain the expectations and obligations on which they are based and thus to sustain individual participants' beliefs about their role and about what constitutes mathematical activity in this classroom.

In this example we have limited our discussion to social norms and beliefs about general classroom activity. Similar examples from the classroom are available for sociomathematical norms and specifically mathematical beliefs. Space limitations preclude including such examples in this paper. Through this example, we have attempted to demonstrate how social norms and individual beliefs evolve together as a dynamic system. In doing so, we give primacy neither to the social nor the psychological. Rather, we maintain that each provides a backdrop against which to consider the other. In this paper, our purpose has been to clarify that as classroom norms are renegotiated, there is a concomitant evolution of individual beliefs. In this sense, giving explicit attention to classroom norms is one means of effecting the beliefs of students in the mathematics classroom.

References

Bauersfeld, H. (1988). Interaction. construction, and knowledge: Alternative perspectives for mathematics education. In T. Cooney & D. Grouws (Eds.), *Effective mathematics teaching* (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics/Erlbaum.

Bauersfeld, H., Krummheuer, G. & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner & A. Vermandel (Eds.), *Foundations and methodology of the discipline of mathematics education* (pp. 174-188). Antwerp: Proceedings of the Theory of Mathematics Education Conference.

Blumer, H. (1969). *Symbolic interactionism.* Englewood Cliffs, NJ: Prentice-Hall.

Cobb, P. (1985). Two children's anticipations, beliefs, and motivations. *Educational Studies in Mathematics*, *16,* 111-125.

Cobb, P. (1986). Contexts, goals, beliefs, and learning mathematics. *For the Learning of Mathematics, 6*(2), 2-9.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 117-148). New York: Springer-Verlag.

Cobb, P., Yackel, E., & Wood, T. (1993). Theoretical orientation. In T. Wood, P. Cobb, E. Yackel, & D. Dillon (Eds.) *Rethinking elementary school mathematics: Insights and issues* (Journal for Research in Mathematics Education Monograph No. 6, pp. 21-32). Reston, VA: National Council of Teachers of Mathematics.

Erlwanger, S. H. (1973). Benny's concept of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1*, 7-26.

Mehan, H. (1979). *Learning lessons: Social organization in the classroom.* Cambridge, MA; Harvard University Press.

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.). *Constructivism in mathematics education* (pp. 13-52). Dordrecht, The Netherlands: Kluwer.

Schoenfeld, A. H. (1983, April). *Theoretical and pragmatic issues in the design of mathematical "problem-solving".* Paper presented at the Annual Meeting of the American Educational Research Association, Montreal.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. (1994). Calculational and conceptual orientations in teaching mathematics. In D. Achiele & A. F. Coxford (Eds.). *Professional development of teachers of mathematics* (pp. 79-92). Reston, VA: National Council of Teachers of Mathematics.

Voigt, J. (1992, August). *Negotiation of mathematical meaning in classroom practices: Social interaction and learning mathematics.* Paper presented at the Seventh International Congress on Mathematical Education, Quebec City.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactique des Mathématiques, 6,* 69-118.

Wood, T., Cobb, P., & Yackel, E. (1988, April). *The influence of change in teacher's beliefs about mathematics instruction on reading instruction.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans.

Yackel, E. (1995). Children's talk in inquiry mathematics classrooms. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 131-162). Hillsdale, NJ: Lawrence Erlbaum Associates.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27,* 458-477.

Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education, 22,* 390-408.

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