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Freudenthal triple systems via root system methods

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Freudenthal triple systems via root system methods

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B.S., Massachusetts Institute of Technology, 1982

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An abstract of
A dissertation submitted to the Faculty of the Graduate School
of Emory University in partial fulfillment
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2009

Abstract

Freudenthal triple systems via root system methods
By Fred W. Helenius

For a Lie algebra $\mathfrak{g}$ of type B, D, E or F, we can apply a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and then define a quartic form and a skew-symmetric bilinear form on $\mathfrak{g}_1$, thereby constructing a Freudenthal triple system. The structure of the Freudenthal triple system is examined using root system methods available in the Lie algebra context. In the important cases $\mathfrak{g} = E_8$ and $\mathfrak{g} = D_4$, we determine the groups stabilizing the quartic form and both the quartic and bilinear forms.

Freudenthal triple systems via root system methods

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This dissertation lists only one author on its title page, but there are many more people who were essential to its creation. It is on the one hand a product of the intellectual tradition of mathematics, and many of those who contributed to the circle of ideas that it joins are listed in the bibliography. It is also a product of the community in which it was created, the Department of Mathematics and Computer Science of Emory University. Every member of the department over the past five years, whether faculty, staff or student, has contributed to the development of this document. I wish to thank:

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Chapter 1

Introduction

In this chapter, we begin by providing an overview of the contents of the subsequent chapters. Section 1.2 contains a brief history of the algebraic structures known as Freudenthal triple systems. Finally, Section 1.3 describes related work in the existing literature, noting in particular which of the results we present are parallel to those in other published papers.

1.1   Outline

Following this introductory chapter, Chapter 2 describes the background material needed for the results that will follow. This includes the definition and fundamental properties of the root system of a Lie algebra, the classification of irreducible root systems, the properties of the structure constants that define the multiplication in a Lie algebra; most of the material in the first three sections is standard, so proofs are only provided when there is no convenient reference to the literature. The later sections are concerned with a specific class of Lie algebras in which we will identify a particular subspace, denoted $\mathfrak{g}_1$, which will be equipped with the necessary operations to construct a Freudenthal triple system. We cite known results about the orbits in $\mathfrak{g}_1$ under a group action, and then collect some useful facts about the roots corresponding to the root subspaces that make up $\mathfrak{g}_1$.

The real work begins in Chapter 3, which contains results that are valid for any of the Lie algebras we consider. We first define a quartic form and a bilinear form on $\mathfrak{g}_1$, and establish their basic properties. After using the forms to define a triple product on $\mathfrak{g}_1$, we define a class of elements called "strictly regular" for which the triple product behaves in a particularly simple way. Several alternative characterizations of strictly regular elements are given and specific examples are identified; these enable us to produce a formula for the triple product in a special case. This in turn allows us to finally show that the structure defined on $\mathfrak{g}_1$ by these operations satisfies the axiomatic definition of a Freudenthal triple system. In the final section of the chapter, we show how to compute the quartic form on $\mathfrak{g}_1$ in several cases; these cases cover all possibilities for simply-laced Lie algebras.

Chapter 4 contains results which only apply to particular Lie algebras. The first three sections apply to simply-laced Lie algebras. In the first, we exhibit a decomposition of $\mathfrak{g}_1$ into four eigenspaces in a way that mirrors the construction of Freudenthal triple systems from Jordan algebras; in the second, we give simple characterizations of elements in the different orbits in $\mathfrak{g}_1$; in the the third, we study the groups that preserve either the quartic form or the bilinear form up to a scalar factor. Results from all three sections are then applied to precisely determine the groups that stabilize either the quartic form or both forms in the case that the Lie algebra is of type E_8. The last section applies similar techniques to answer the same questions for the Lie algebra D_4.

The final chapter summarizes the results and indicates some directions that future work may take.

1.2   History

The history of the study of Lie algebras and their representations is a complex and interesting subject; here we only mention the steps that led directly to the definition of a Freudenthal triple system. For more detail about the early period of this history, up to 1926, the reader may wish to refer to the comprehensive survey by Hawkins ([16]).

The classification of simple Lie groups and their Lie algebras over $\mathbb{C}$ was first attempted by Wilhelm Killing in a series of articles published in *Mathematische Annalen* from 1888 to 1890. In the last article, [19], he exhibited his classification, which consists of four sequences of Lie algebras of the so-called "classical" types and six others. Killing had made a mistake—two of his six new Lie algebras were actually isomorphic—but he had also inaugurated the study of "exceptional" Lie algebras.

Killing's work was corrected, revised and expanded by Élie Cartan in his 1894 thesis, [6], in which he established the notation now used for the five exceptional Lie algebras: E_6, E_7, E_8, F_4 and G_2. Cartan also studied the representations of these Lie algebras; our interest lies in the 56-dimensional faithful irreducible representation of E_7 that he described. This representation, now known as the *minuscule* representation of E_7 (meaning that its weights form a single orbit under the Weyl group), is the prototypical example of a Freudenthal triple system. Cartan found that there was a quartic form on this representation that was invariant under the action of the Lie group E_7, and gave an expression for the quartic form as a sum of 7784 terms ([6], p. 274). This unwieldy expression, however, was not correct.

Cartan's error was noted by Hans Freudenthal in [12], one of several papers in which he studied E_7, its minuscule representation and the invariant quartic form using a variety of techniques.

In [23], Kurt Meyberg gave an axiomatic definition (which he credited to

T.A. Springer) for an algebraic structure which he called a *Freudenthalsches Tripelsystem*, or, in English, a Freudenthal triple system. This structure generalizes the properties of the minuscule representation of E_7 and its invariant quartic form as described by Freudenthal and applies over general fields, with some restrictions on the characteristic. An equivalent definition was found independently by Robert B. Brown ([5]). In the next section we will review related work on these structures.

1.3   Prior work

The work presented in this dissertation involves two related lines of inquiry that have been considered by many authors. The first consists of the study of Freudenthal triple systems in general or specifically of the minuscule representation of E_7 and its invariant quartic. One of our principal results, which has been obtained previously in varying degrees of generality, is the determination of the group of linear transformations which stabilize the quartic form.

The second line of inquiry concerns the construction of Freudenthal triple systems from Lie algebras. Although such constructions have been given before, the one we use does not seem to have been explicitly described.

We now examine a broad selection of related work, in chronological order. The descriptions are limited to the topics in these papers that are related to our results; in most cases the papers contain much unrelated material as well.

As mentioned in the previous section, Freudenthal studied E_7 extensively. In a 1953 paper, [12], he defines the quartic form (correcting Cartan's error) and uses a remarkable series of indicial tensor calculations to determine the stabilizer. In [13], published one year later, he obtains the same result, but using a definition of the quartic form in terms of 3×3 matrices of octonions

(in effect, the exceptional Jordan algebra). In both papers he implicitly works over $\mathbb{C}$.

In unpublished notes ([25]) from 1962, Seligman also uses the exceptional Jordan algebra to determine the stabilizer of the quartic form, but takes care to obtain a result valid for fields of characteristic other than 2 or 3.

Meyberg's 1968 paper, [23], as mentioned earlier, introduces the abstract definition of a Freudenthal triple system.

In 1969, Brown ([5]) also gives axioms defining a Freudenthal system, but with the aim of applying them to determine the stabilizer of the quartic form on the 56-dimensional representation of E_7. This approach entails a large number of intricate calculations, yielding a result valid for fields of characteristic not 2 or 3. Brown also shows how some Freudenthal triple systems may be constructed from Jordan algebras.

Ferrar's 1972 paper, [11], uses the axiomatic definition to study Freudenthal triple systems, putting particular emphasis on analyzing their structure by use of so-called strictly regular elements. The properties he deduces were a useful guide in our investigation; the parallels between his results and ours are detailed at the end of this section. Like Brown, he also constructs Freudenthal triple systems from Jordan algebras.

The 1978 paper by Kantor and Skopec, [18], is the earliest we know of that constructs Freudenthal triple systems from Lie algebras. They use a grading on the Lie algebra as we do, but define the operations in a less transparent way. To obtain their results they have to weaken the definition of Freudenthal triple system by allowing the quartic form to be zero and require the characteristic to be zero. Where both our methods apply, the resulting Freudenthal triple systems appear to coincide.

In a 1979 paper, [1], Allison used the language of structurable algebras to apply a construction similar to that of Kantor and Skopec to Lie algebras in characteristic other than 2, 3 or 5. Since he does not use the terminology of

Freudenthal triple systems, it is not obvious when his construction produces them.

Cooperstein's paper from 1995, [9], presents a unique construction of the quartic form on a 56-dimensional space, beginning with two 28-dimensional spaces acted on by a group of type A_7. The introduction of an incidence structure preserved by the stabilizer is used to identify the group; here the result is valid outside of characteristic 2.

The 2001 paper by Lurie, [21], contains yet another determination of the stabilizer of the quartic form, but extends the result so that it is valid in all characteristics. To include characteristic 2 it is necessary to replace the quartic form with its linearization; an impressive array of technical machinery is also employed.

In 2003, Clerc ([8]) published a paper that includes many elements of our construction of a Freudenthal triple system from a Lie algebra: he defines a grading of the algebra into five parts, and defines the same quartic form on one of the parts. However, Clerc does not explicitly identify the resulting structure as a Freudenthal triple system. He does examine the orbits under a group action, obtaining a result similar to our Proposition 4.2.

Although Springer's involvement with Freudenthal triple systems extends over decades, the paper we cite ([26]) is an expository work from 2006. His approach to determining the stabilizer of the quartic form, which we partially adopt in our proof, depends upon first determining the smaller automorphism group resulting when additional structure is defined on the 56-dimensional Freudenthal triple system.

Krutelevich's 2007 paper, [20], considers Freudenthal triple systems defined from Jordan algebras. Like Clerc, he considers their orbit structure, obtaining a resulting very similar to ours.

As its title indicates, the present work is distinguished by the fact that the structure which we construct from a Lie algebra is examined and proved to

be a Freudenthal triple system by means of root system computations. The operations we use are defined directly in terms of the Lie algebra multiplication. As a result, the calculations we require, although not always easy, are generally much simpler than those required when working with the axiomatic definition of a Freudenthal triple system. In fact, we make no essential use of the axiomatic definition; it does not appear until section 3.3, and then only for the verification that the structure we study is in fact a Freudenthal triple system.

Our determination of the stabilizer of the quartic form on the minuscule representation of E_7 is clearly not a new result. However, we also apply our techniques to answer the question for an 8-dimensional Freudenthal triple system derived from the Lie algebra D_4; this result is apparently new.

Although the axiomatic approach used by Ferrar in [11] is quite different from the methods used here, our choice of results to prove was often guided by the content of his article. The table below indicates results here that parallel those of Ferrar as well as results in the articles by Clerc ([8]) and Krutelevich ([20]). The many papers cited above that determine the stabilizer of the quartic form, parallel to Theorem 4.6, are not shown in the table.

Lemma 3.11	[11], Cor. 2.5
Prop. 3.15	[11], Cor. 6.2
Lemma 3.16	[11], (5)
Prop. 3.21	[11], Lemma 3.1
Lemma 3.23	[11], Lemma 3.6
Prop. 4.1	[11], §4
Prop. 4.2	[8], §§8,9; [20], Def. 22
Prop. 4.4	[11], Lemma 7.3

Table 1.1: Parallel results in other papers

Chapter 2

Technical background

This chapter presents essential facts about the root systems of Lie algebras and the structure constants defining the Lie algebra multiplication as well as results concerning the orbits of in a particular subspace of Lie algebra under a group action. The material in the first three sections is standard; the main reference used is the textbook by Humphreys ([17]). Similar statements can also be found in Bourbaki ([4]). The next two sections contain more specialized results relevant to the particular situation we study; they are drawn from articles by Röhrle ([24]) and by Borel and Tits ([3]). The final section is a one-page summary of the notations and definitions in this chapter.

2.1   Root systems

Throughout this chapter, we consider a Lie algebra $\mathfrak{g}$ over a field F. Specifically, we assume that $\mathfrak{g}$ is the Lie algebra of a semisimple linear algebraic group G that is split over F; thus ([2], Theorem 13.18) $\mathfrak{g}$ has a Cartan subalgebra which we fix and denote by $\mathfrak{h}$, and $\mathfrak{g}$ decomposes into the direct sum of $\mathfrak{h}$ and the root subspaces, with each root subspace being one-dimensional. In other words, even though Humphreys works over $\mathbb{C}$ in [17], his results also apply to the Lie algebras we consider.

The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are vectors in the dual space $\mathfrak{h}^\vee$, but as their coordinates are rational ([17], §8.5), they can be seen as vectors in $\mathbb{R}^n$,

where n is the rank of $\mathfrak{g}$. Thus for any roots β, γ of $\mathfrak{g}$ we have the usual inner product (β, γ). A particular combination of inner products arises often enough to deserve a special notation; we define $\langle -, - \rangle$ as follows:

$$\langle \beta, \gamma \rangle = 2 \frac{(\beta, \gamma)}{(\gamma, \gamma)}.$$

Note that the expression $\langle \beta, \gamma \rangle$ is linear in the first variable, but not in the second. We will sometimes use this notation with arguments which are sums of roots, but not necessarily roots themselves.

We can now define the subject of this section.

A subset Ψ of $\mathbb{R}^n$ is a *root system* if

- Ψ consists of finitely many nonzero vectors that span $\mathbb{R}^n$,

- for $\beta \in \Psi$, the only other scalar multiple of β in Ψ is $-\beta$,

- for all $\beta, \gamma \in \Psi$, the value $\langle \beta, \gamma \rangle$ is an integer,

- for all $\beta, \gamma \in \Psi$, the reflection of β in the hyperplane orthogonal to γ, which is given by $\beta - \langle \beta, \gamma \rangle \gamma$, is also in Ψ.

It will be no surprise that the roots of a semisimple Lie algebra form a root system ([17], §8.5).

The roots of $\mathfrak{g}$ give rise to a *root space decomposition*, namely

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Psi} F x_\beta$$

where x_β is a nonzero representative of the one-dimensional root subspace corresponding to the root β ([17], §§8.1, 8.4).

The root space decomposition interacts in a simple way with the Lie algebra product:

Fact 2.1. *For roots $\beta, \gamma \in \Psi$ such that $\beta + \gamma \neq 0$, the product $[x_\beta, x_\gamma]$ is in the root space corresponding to $\beta + \gamma$, if that is a root; otherwise, it must be zero.*

This is Proposition 8.1 in [17]. Fulton & Harris refer to this fact as "the fundamental calculation" ([14], §11.1).

Within any root system, there are severe constraints on the angle between two roots and the ratio of their lengths. There are only four possibilities ([17], §9.4):

1. The roots are orthogonal; in this case their lengths may have any ratio.

2. The angle between the roots is $\pi/3$ or $2\pi/3$ and their lengths are equal.

3. The angle between the roots is $\pi/4$ or $3\pi/4$ and one length is $\sqrt{2}$ times the other.

4. The angle between the roots is $\pi/6$ or $5\pi/6$ and one length is $\sqrt{3}$ times the other.

By choosing a hyperplane through the origin that does not include any of the roots, it is possible to partition Ψ into two sets: we call the roots on one side of the hyperplane *positive roots* and the others *negative roots* ([17], §10.1). Since $-\beta$ is a root whenever β is, there are equal numbers of positive and negative roots.

Once such a partition of Ψ is chosen, there is a unique set of n positive roots, say α_i for $1 \leq i \leq n$ (again, n is the rank of $\mathfrak{g}$), called *simple roots* such that every positive root is a sum of simple roots; that is, every positive root β can be written as $\beta = \sum_{i=1}^{n} k_i \alpha_i$ with the coefficients k_i being nonnegative integers ([17], §10.1). We call $\sum_{i=1}^{n} k_i$ the *height* of β and denote it by $\mathrm{ht}\,\beta$.

We define a partial order on the positive roots as follows: if $\beta = \sum_{i=1}^{n} k_i \alpha_i$ and $\beta' = \sum_{i=1}^{n} k_i' \alpha_i$ are positive roots, then β' is above β if $k_i' \geq k_i$ for each i. In other words, β' is above β if β' can be obtained from β by adding simple roots.

A root system is *reducible* if it can be partitioned into two nonempty sets such that each root in one set is orthogonal to all roots in the other; otherwise

it is *irreducible*. In an irreducible root system, the roots can have at most two different lengths which we call *short* and *long* ([17], Lemma 10.4.C). If all roots are the same length, we call them long and say that Ψ is *simply laced*. Henceforth we will assume that Ψ is irreducible.

In an irreducible root system, there is a unique maximal element in the partial order defined above; it is called the *highest root*. We will denote the highest root by ρ. The highest root, ρ, is always a long root ([17], Lemma 10.4.D).

When performing computations in root systems, it is frequently important to know whether the sum (or difference) of two roots is again a root. The following facts will be used repeatedly in such situations.

Fact 2.2. *For $\beta, \gamma \in \Psi$, if $\langle \beta, \gamma \rangle < 0$ then $\beta + \gamma$ is a root or zero. Likewise, if $\langle \beta, \gamma \rangle > 0$ then $\beta - \gamma$ is a root or zero.*

This is Lemma 9.4 in [17].

We will also use a partial converse to this:

Fact 2.3. *If β, γ are long roots and $\langle \beta, \gamma \rangle \geq 0$, then $\beta + \gamma$ is not a root.*

Proof. Let r be the length of β and γ, then

$$\langle \beta, \gamma \rangle = 2\frac{(\beta, \gamma)}{(\gamma, \gamma)} = 2\cos\theta,$$

where θ is the angle between β and γ. By hypothesis, $\cos\theta \geq 0$, so $0 \leq \theta \leq \pi/2$. The length of $\beta + \gamma$ is then $2\sin(\theta/2)r \geq \sqrt{2}r$, so $\beta + \gamma$ cannot be a root since it is longer than a long root. $\square$

The hypothesis that the roots are long is necessary as the sum of orthogonal short roots may be a root; in the root system for the Lie algebra G_2 (see Section 2.2), $\beta + \gamma$ may be a root even if $\langle \beta, \gamma \rangle > 0$.

Fact 2.4. *If the sum (or difference) of long roots is a root, it is a long root.*

Proof. Since the negative of a long root is also long, it suffices to consider sums of roots. Suppose β and γ are long roots; the angle θ between them must be one of $\pi/2$, $\pi/3$ or $2\pi/3$. If $\beta + \gamma$ is also a root, θ must be greater than $\pi/2$ by the previous fact; thus $\theta = 2\pi/3$. The length of $\beta + \gamma$ is then $2\sin(\pi/3) = 1$ times the length of β and γ, so it is long. $\square$

2.2   Classification of root systems

The classification of simple Lie algebras possessing a Cartan subalgebra, due to Killing ([19]) and Cartan ([6]), proceeds by the following reductions: such a simple Lie algebra has an irreducible root system ([17], Proposition 14.1), and an irreducible root system is determined by the configuration of a set of simple roots ([17], Proposition 11.1).

The geometry of a set of simple roots can be described by giving the relative lengths of and angle between each pair of simple roots. The angle θ between any two simple roots cannot be acute: suppose β, γ were simple roots with $\langle \beta, \gamma \rangle > 0$; then $\beta - \gamma$ would be a root (Fact 2.2), which is impossible since $\beta - \gamma$ is neither a positive root nor a negative root. Thus the possible relations between two simple roots are reduced to these cases: the roots may be orthogonal, they may be of equal length with $\theta = 2\pi/3$, the ratio of their lengths may be $\sqrt{2}$ with $\theta = 3\pi/4$, or the ratio of their lengths may be $\sqrt{3}$ with $\theta = 5\pi/6$. These relationships may be encoded succinctly by means of a *Dynkin diagram*. A Dynkin diagram is a graph with vertices representing the simple roots. Orthogonal roots are not joined by an edge, roots with $\theta = 2\pi/3$ are joined by a single edge, roots with $\theta = 3\pi/4$ are joined by a double edge, and roots with $\theta = 5\pi/6$ are joined by a triple edge. In the latter two cases, the edge is decorated with a wedge pointing toward the shorter root. If the root system is irreducible, the corresponding Dynkin diagram is connected.

For example, the Dynkin diagram shown in Figure 2.1 describes a root system with four simple roots; the two on the left being long and the other two short. This is the Dynkin diagram for the root system F_4, which corresponds to a 52-dimensional Lie algebra.
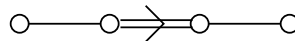
Figure 2.1: The Dynkin diagram for the root system F_4

By straightforward geometric arguments ([17], §11), all connected Dynkin diagrams are shown to belong to four infinite sequences (corresponding to the so-called *classical* Lie algebras) except for five *exceptional* cases. Conversely, every such Dynkin diagram corresponds to an isomorphism class of simple Lie algebras ([17], Theorems 12.1, 18.4).

The classical types are shown in Figure 2.2, where there are n vertices in each diagram.

Figure 2.2: Dynkin diagrams for root systems of types A, B, C, D

The exceptional cases are shown in Figure 2.3.
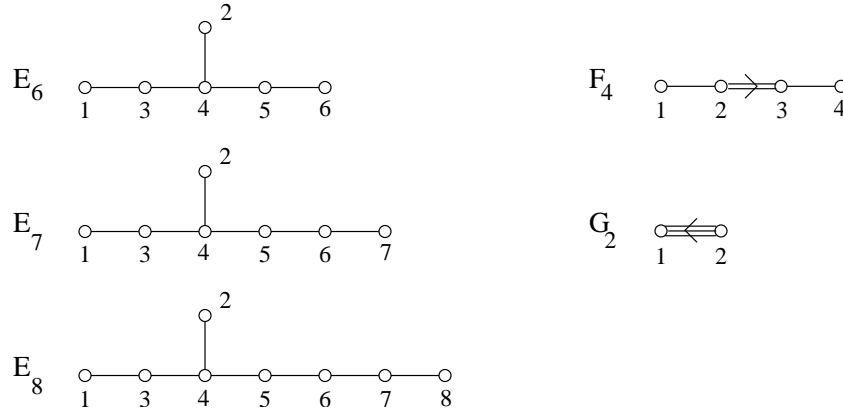
Figure 2.3: Dynkin diagrams for root systems E_6, E_7, E_8, F_4, G_2

2.3   Structure constants
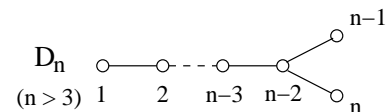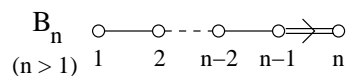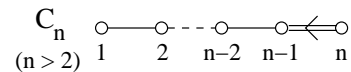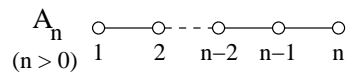
The multiplication in an algebra over a field can be defined by choosing a basis and writing each possible product of two basis elements as a linear combination of basis elements. The coefficients in the linear combinations are called *structure constants*; they depend upon the choice of basis. A good choice of basis will result in simple structure constants. For a split semisimple Lie algebra, there is basis called a *Chevalley basis* for which the structure constants become particularly simple.

For roots β and γ that are not opposite, we define the structure constant $c_{\beta,\gamma}$ so that $[x_\beta, x_\gamma] = c_{\beta,\gamma} x_{\beta+\gamma}$ if $\beta + \gamma$ is a root; if $\beta + \gamma$ is not a root, we define $c_{\beta,\gamma} = 0$. We will also find it convenient to define $x_{\beta+\gamma}$ to be zero in the case that $\beta + \gamma$ is not a root.

A Chevalley basis for $\mathfrak{g}$ consists of an element x_β in the root subspace corresponding to β for each root β and elements $h_i = [x_{\alpha_i}, x_{-\alpha_i}]$ for α_i a simple root, $1 \leq i \leq \operatorname{rk} \mathfrak{g}$, such that the multiplication of these elements satisfies the following ([17], §25.2):

- $[h_i, h_j] = 0$.

- $[h_i, x_\beta] = \langle \beta, \alpha_i \rangle x_\beta$.

- If β, γ are roots and $\beta + \gamma \neq 0$, then $c_{\beta,\gamma} = -c_{-\beta,-\gamma}$.

- $[x_\beta, x_{-\beta}]$ is a $\mathbb{Z}$-linear combination of the h_i, denoted by h_β. This element satisfies $[h_\beta, x_\gamma] = \langle \gamma, \beta \rangle x_\gamma$.

If the roots β, γ are long and $\beta + \gamma$ is a root, then $c_{\beta,\gamma} = \pm 1$ ([17], Proposition 25.2(c)).

Henceforth, we will always assume that the basis elements x_β, h_i are in a Chevalley basis.

Theorem 4.1.2 in [7] provides the following information about the structure constants:

Fact 2.5. *Let $\beta, \gamma, \delta, \epsilon$ be roots in Ψ.*

(a) *In all cases, $c_{\beta,\gamma} = -c_{\gamma,\beta}$.*

(b) *If β, γ, δ are long roots such that $\beta + \gamma + \delta = 0$, then $c_{\beta,\gamma} = c_{\gamma,\delta} = c_{\delta,\beta}$.*

(c) *If β, γ are long roots, then $c_{\beta,\gamma} = -c_{-\beta,-\gamma}$.*

(d) *If $\beta, \gamma, \delta, \epsilon$ are long roots such that $\beta + \gamma + \delta + \epsilon = 0$ and no two are opposite, then*

$$c_{\beta,\gamma}c_{\delta,\epsilon} + c_{\gamma,\delta}c_{\beta,\epsilon} + c_{\delta,\beta}c_{\gamma,\epsilon} = 0. \tag{2.6}$$

For (b), (c) and (d) we have simplified the statements given in [7] by requiring the roots to be long.

For the reader's convenience, we prove these rules here.

Proof. Since $c_{\beta,\gamma}x_{\beta+\gamma} = [x_\beta, x_\gamma] = -[x_\gamma, x_\beta] = -c_{\gamma,\beta}x_{\beta+\gamma}$, (a) follows immediately.

In (b), the sum of each pair of roots is again a root (e.g., $\beta + \gamma = -\delta$), so we have in particular that $\langle \delta, \beta \rangle = \langle \delta, \gamma \rangle = -1$. By the Jacobi identity,

$$0 = [x_\beta, [x_\gamma, x_\delta]] + [x_\gamma, [x_\delta, x_\beta]] + [x_\delta, [x_\beta, x_\gamma]]$$
$$= c_{\gamma,\delta}[x_\beta, x_{-\beta}] + c_{\delta,\beta}[x_\gamma, x_{-\gamma}] + c_{\beta,\gamma}[x_\delta, x_{-\delta}]$$
$$= c_{\gamma,\delta} h_\beta + c_{\delta,\beta} h_\gamma + c_{\beta,\gamma} h_\delta.$$

Computing the Lie bracket of this expression with x_δ yields

$$0 = c_{\gamma,\delta}[h_\beta, x_\delta] + c_{\delta,\beta}[h_\gamma, x_\delta] + c_{\beta,\gamma}[h_\delta, x_\delta]$$
$$= c_{\gamma,\delta}\langle \delta, \beta \rangle x_\delta + c_{\delta,\beta}\langle \delta, \gamma \rangle x_\delta + c_{\beta,\gamma}\langle \delta, \delta \rangle x_\delta$$
$$= (-c_{\gamma,\delta} - c_{\delta,\beta} + 2c_{\beta,\gamma})x_\delta.$$

Thus $-c_{\gamma,\delta} - c_{\delta,\beta} + 2c_{\beta,\gamma} = 0$; by permuting the indices, we also have $-c_{\delta,\beta} - c_{\beta,\gamma} + 2c_{\gamma,\delta} = 0$. By subtracting, we find $c_{\beta,\gamma} = c_{\gamma,\delta}$; by again permuting the indices, we have $c_{\beta,\gamma} = c_{\gamma,\delta} = c_{\delta,\beta}$.

Although (c) holds more generally, for us it is a consequence of having chosen to work in a Chevalley basis.

For (d), we again apply the Jacobi identity:

$$0 = [x_\beta, [x_\gamma, x_\delta]] + [x_\gamma, [x_\delta, x_\beta]] + [x_\delta, [x_\beta, x_\gamma]]$$
$$= c_{\gamma,\delta}[x_\beta, x_{\gamma+\delta}] + c_{\delta,\beta}[x_\gamma, x_{\delta+\beta}] + c_{\beta,\gamma}[x_\delta, x_{\beta+\gamma}]$$
$$= c_{\gamma,\delta} c_{\beta,\gamma+\delta} x_{-\epsilon} + c_{\delta,\beta} c_{\gamma,\delta+\beta} x_{-\epsilon} + c_{\beta,\gamma} c_{\delta,\beta+\gamma} x_{-\epsilon},$$

where we have extended our notation slightly by writing, for example, $c_{\beta,\gamma+\delta}$ even though $\gamma + \delta$ might not be a root; however, in that case the factor $c_{\gamma,\delta}$ is already zero. When $\gamma + \delta$ is a root, we have $c_{\beta,\gamma+\delta} = c_{\epsilon,\beta}$ by (b); in the other case, substituting one for the other is harmless. Thus we have

$$c_{\gamma,\delta} c_{\epsilon,\beta} + c_{\delta,\beta} c_{\epsilon,\gamma} + c_{\beta,\gamma} c_{\epsilon,\delta} = 0.$$

Applying (a) to the second factor of each term and reordering the terms gives the result. $\qquad \square$

2.4   Our situation

Our goal is to use root system techniques to define and describe an algebraic structure (namely, that of a Freudenthal triple system) on a subspace of a Lie algebra. The methods we use are inspired by Röhrle's work in [24]. In this section we describe restrictions that we require on the Lie algebra to use his results and summarize the definitions and results that we will apply to this situation.

Let F be an arbitrary field of characteristic $\neq 2, 3$, and let G be a simple, connected linear algebraic group that is split over F, and let $\mathfrak{g}$ be its Lie algebra. As mentioned in Section 2.1, $\mathfrak{g}$ then has a root space decomposition and so belongs to one of the types given by the classification in Section 2.2. The restriction on characteristic will be needed when we define the Freudenthal triple system.

Let Ψ be the root system of $\mathfrak{g}$ with respect to a fixed Cartan subalgebra $\mathfrak{h}$; thus $\Psi \subset \mathfrak{h}^\vee$. Also fix a set of simple roots in Ψ, and let ρ be the highest root in the resulting partial order on the positive roots.

The negative of the highest root, $-\rho$, can be added to the usual Dynkin diagram for each root system to form an *extended* (or *completed*) *Dynkin diagram*. Figure 2.4 shows the extended Dynkin diagrams for all the root systems of simple Lie algebras; $-\rho$ is represented by the unmarked vertex in each diagram.

Except in type A, we see that $-\rho$ (and thus ρ itself) is orthogonal to all but one of the simple roots. In the remaining cases other than type C, the unique simple root not orthogonal to ρ is long. Henceforth we will assume $\mathfrak{g}$ is not of type A or C, so there is a unique simple root α such that $\langle \alpha, \rho \rangle = -\langle \alpha, -\rho \rangle = 1$ and α is a long root. We also need to assume that the rank of $\mathfrak{g}$ is at least 4, so we are thereby assuming that $\mathfrak{g}$ is not of type G (and also is not B_3). In most of Chapter 4, we will also assume that $\mathfrak{g}$ is simply-laced and thus of type D or E.
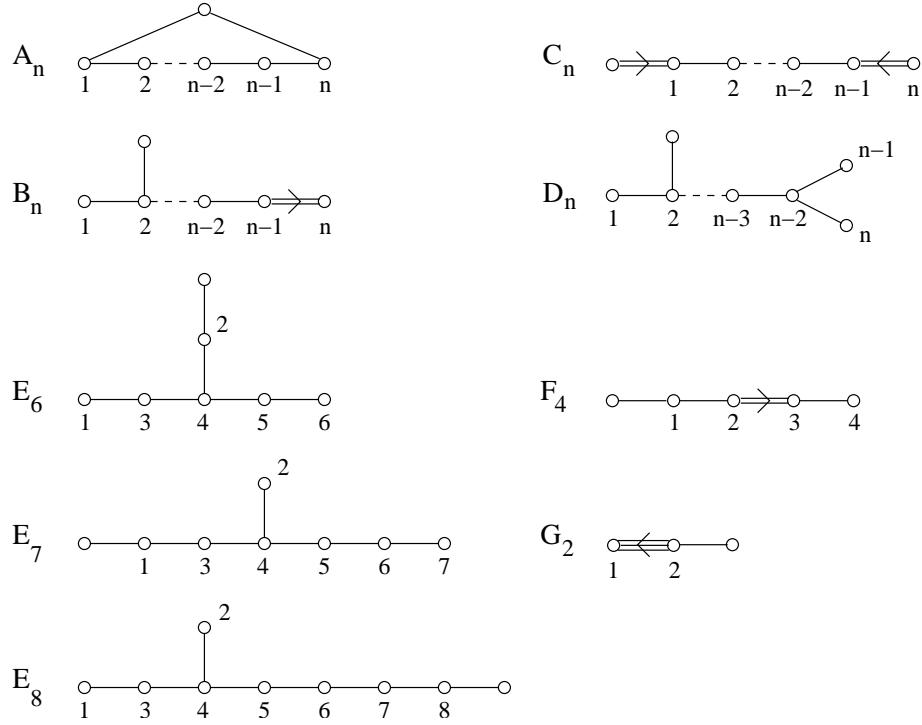
Figure 2.4: Extended Dynkin diagrams

For each $\beta \in \Psi$, the α-*height* of β is given by $\langle \beta, \rho \rangle$; equivalently, the α-height of β is the coefficient of α when β is expressed as a linear combination of the simple roots. Since $\langle \beta, \rho \rangle = 2\frac{(\beta,\rho)}{(\rho,\rho)}$ and ρ is long, it can only take on values from -2 to 2. In particular, $\langle \beta, \rho \rangle = 2$ only if $\beta = \rho$; likewise, $\langle \beta, \rho \rangle = -2$ only if $\beta = -\rho$.

The α-height induces a grading on the Lie algebra $\mathfrak{g}$: We write $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where, for each $k \neq 0$, $\mathfrak{g}_k$ is the direct sum of the root subspaces for roots of α-height k; $\mathfrak{g}_0$ is the direct sum of the root subspaces for roots of α-height 0 and of $\mathfrak{h}$. Equivalently, each $\mathfrak{g}_k$ contains all $x \in \mathfrak{g}$ for which $[h_\rho, x] = kx$. Since $\langle \beta, \rho \rangle = -2$ (resp. 2) only when $\beta = -\rho$ (resp. ρ), we see that $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ are one-dimensional, consisting of the root subspaces corresponding to $-\rho$ and ρ, respectively.

The grading on $\mathfrak{g}$ allows us to define several operations on the subspace $\mathfrak{g}_1$ in a natural way. If we take the element $x_{-\rho} \in \mathfrak{g}_{-2}$ and form the Lie bracket of it with some $x \in \mathfrak{g}_1$, the grading implies that the result $[x, x_{-\rho}]$ is in $\mathfrak{g}_{-1}$. If we again apply x, we obtain $[x, [x, x_{-\rho}]]$ in $\mathfrak{g}_0$. Continuing twice more, the value $[x, [x, [x, [x, x_{-\rho}]]]]$ is in $\mathfrak{g}_2$; that is, it is a scalar multiple of x_ρ. The coefficient of x_ρ is thus the value of a quartic form $q(x)$ defined on $\mathfrak{g}_1$:

$$[x, [x, [x, [x, x_{-\rho}]]]] = q(x)x_\rho.$$

Using the standard notation $\operatorname{ad} x$ for the map $y \mapsto [x, y]$, we can write this more concisely as

$$(\operatorname{ad} x)^4(x_{-\rho}) = q(x)x_\rho.$$

Given the quartic form $q(x)$, there is a corresponding fully-symmetric 4-linear form $q(x, y, z, w)$ defined by linearization. To specify the scalar factor, we define $q(x, x, x, x) = q(x)$ for all $x \in \mathfrak{g}_1$.

Since the Lie bracket of any two elements of $\mathfrak{g}_1$ lies in $\mathfrak{g}_2$, we can define a skew-symmetric bilinear form $\langle x, y \rangle$ on $\mathfrak{g}_1$ by $[x, y] = \langle x, y \rangle x_\rho$. (We used the same notation for a function of two roots; since this is a function of Lie algebra elements, no confusion should result.) If we fix any three elements $x, y, z \in \mathfrak{g}_1$, the expression $q(w, x, y, z)$ is thus a linear form on w. The form $\langle -, - \rangle$ will be shown to be nondegenerate (Lemma 3.1); thus there is a unique element in $\mathfrak{g}_1$ which we denote by xyz such that $q(w, x, y, z) = \langle w, xyz \rangle$ for all $w \in \mathfrak{g}_1$. We call xyz the *triple product* of x, y, z; since q is a symmetric 4-linear form, the triple product is symmetric and trilinear.

By a result of Vinberg ([28], Proposition 2), if F is algebraically closed the Levi complement of a parabolic subgroup of the linear algebraic group G acts on the unipotent radical of the parabolic subgroup with finitely many orbits. Let G_0 be the subgroup of G that corresponds to $\mathfrak{g}_0$, more precisely, the centralizer of h_ρ in G. In terms of the Lie algebra, G_0, acts on $\mathfrak{g}_1$ and actually partitions $\mathfrak{g}_1$ into finitely many orbits.

In Theorem 2.6 of [24], Röhrle gives the number of G_0-orbits in $\mathfrak{g}_1$ for each Lie algebra $\mathfrak{g}$ satisfying our common hypotheses. For the Lie algebras E_6, E_7, E_8, there are five orbits. Each orbit is represented by an element of the form $\sum_{i=1}^{k} x_{\beta_i}$ for $k = 0, \ldots, 4$ where $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a set of mutually orthogonal roots of α-height 1 ([24], Theorem 4.8). We refer to these as orbit 0 through orbit 4. We may, and frequently do, take $\beta_1 = \alpha$; indeed, the sets of four mutually orthogonal roots of α-height 1 exhibited in Section 2.5 all contain α.

For Lie algebras of type D_n, each orbit has a representative as above, but there are either two ($n > 4$) or three ($n = 4$) distinct orbits generated by sums with two terms; that is, "orbit 2" is split into two or three orbits in this case; we refer to each of them as a level 2 orbit. Similarly, for Lie algebras of type B_n, $n \geq 4$, or F_4 there are two level 2 orbits.

For all of the types, orbit 4 is also represented by $x_\alpha + x_{\rho - \alpha}$ ([24], Corollary 4.4).

The semisimple part of G_0, which we denote by $(G_0)^{\mathrm{ss}}$, also acts on $\mathfrak{g}_1$; here there are finitely many orbits in the projective space $\mathbb{P}(\mathfrak{g}_1)$. These projective orbits correspond to the nonzero orbits under the action of G_0. The action of $(G_0)^{\mathrm{ss}}$ is of interest because of the following fact.

Fact 2.7. *The quartic form, skew-symmetric bilinear form and triple product on $\mathfrak{g}_1$ are preserved by the action of $(G_0)^{ss}$.*

Proof. The elements of $(G_0)^{\mathrm{ss}}$ act on $\mathfrak{g}$ by Lie algebra homomorphisms, so their action preserves the Lie bracket. For any basis element of the Lie subalgebra of $\mathfrak{g}$ corresponding to $(G_0)^{\mathrm{ss}}$, i.e., any x_β where β is a root of α-height 0 or any h_γ where γ is a simple root other than α, we have $[x_\beta, x_\rho] = 0$ and $[h_\gamma, x_\rho] = 0$ because ρ is orthogonal to every root of α-height 0. Similarly, we also have $[x_\beta, x_{-\rho}] = 0$ and $[h_\gamma, x_{-\rho}] = 0$. Thus elements of $(G_0)^{\mathrm{ss}}$ fix x_ρ and $x_{-\rho}$. The quartic form and bilinear form we have defined on $\mathfrak{g}_1$ depend

only on the Lie bracket, x_ρ and $x_{-\rho}$, so both are preserved by the action of $(G_0)^{\mathrm{ss}}$; likewise for the triple product, which is defined in terms of the two forms. $\qquad\square$

By Théorème 3.13 in Borel & Tits ([3]), the orbits are "nested" in the following sense: the closure of any of the G_0-orbits is its union with all smaller (i.e., lower level) orbits. In particular, the largest orbit, orbit 4, is dense in $\mathfrak{g}_1$.

The statements about orbits are made under the assumption that F is algebraically closed. In general, geometric statements about orbits will at least be true over the algebraic closure of F. The algebraic consequences, such as Fact 2.7 above, remain true for any F, since they involve polynomial relations defined over F. To avoid repetition in what follows, we make this convention: *all statements about orbits are understood to refer to the orbits over the algebraic closure.*

The preceding Fact and the nesting of the orbits provide us with a valuable proof technique. To prove an algebraic relation holds on all of $\mathfrak{g}_1$, it suffices to show it for elements of the dense orbit. However, if the relation is defined in terms of the quartic and bilinear forms and the triple product and is preserved by scaling, it suffices to show it only for a single representative of the dense orbit, such as $x_\alpha + x_{\rho-\alpha}$; then the action of $(G_0)^{\mathrm{ss}}$ combined with scaling guarantee it holds for the entire orbit. Similarly, if we show some scale-independent relation of the forms holds for a representative of a given orbit, it holds for the entire orbit and the smaller ones in its closure; if the relation fails, it cannot hold anywhere in that orbit or any larger one.

2.5   Roots of α-height 1

In much of the sequel we will be concerned with roots of α-height 1 and the root subspaces Fx_β where β is such a root. Indeed, the direct sum of these

root subspaces, the space we have called $\mathfrak{g}_1$, will prove to be (Theorem 3.26) the Freudenthal triple system of the title. Here we establish a few useful facts about these roots.

We begin by considering the map $\beta \mapsto \rho - \beta$ where β is a root of α-height 1. This map preserves many properties of roots of α-height 1.

Fact 2.8. *If β is root of α-height 1, then $\rho - \beta$ is also a root, is also of α-height 1, and has the same length as β. If β and γ are orthogonal roots of α-height 1, then $\rho - \beta$ and $\rho - \gamma$ are also orthogonal.*

Proof. We have $\langle \beta, \rho \rangle = 1$, so $\rho - \beta$ is a root by Fact 2.2. The α-height of $\rho - \beta$ is $\langle \rho - \beta, \rho \rangle = \langle \rho, \rho \rangle - \langle \beta, \rho \rangle = 2 - 1 = 1$. The highest root ρ is long, so if β is long, then so is $\rho - \beta$ by Fact 2.4. If β is short, $\rho - \beta$ cannot be long, for we then have that $\rho - (\rho - \beta) = \beta$ is long.

If $\langle \beta, \gamma \rangle = 0$, then

$$
\begin{aligned}
\langle \rho - \beta, \rho - \gamma \rangle &= \frac{2}{(\rho - \gamma, \rho - \gamma)}(\rho - \beta, \rho - \gamma) \\
&= \frac{2}{(\rho - \gamma, \rho - \gamma)}((\rho, \rho) - (\gamma, \rho) - (\beta, \rho) + (\beta, \gamma)) \\
&= \frac{(\rho, \rho)}{(\rho - \gamma, \rho - \gamma)}(\langle \rho, \rho \rangle - \langle \gamma, \rho \rangle - \langle \beta, \rho \rangle) \\
&= \frac{(\rho, \rho)}{(\rho - \gamma, \rho - \gamma)}(2 - 1 - 1) \\
&= 0.
\end{aligned}
$$

$\square$

Fact 2.9. *If $\beta_1, \beta_2, \beta_3, \beta_4$ are mutually orthogonal roots of α-height 1, then $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\rho$.*

This is Corollary 1.4 in [24].

Proof. Since β_1 has α-height 1, $\rho - \beta_1$ is a root. Since β_2 is orthogonal to β_1, $\langle \rho - \beta_1, \beta_2 \rangle = \langle \rho, \beta_2 \rangle - \langle \beta_1, \beta_2 \rangle = \langle \rho, \beta_2 \rangle = 1$, so $\rho - \beta_1 - \beta_2$ is also a root. In

the same way, $\rho - \beta_1 - \beta_2 - \beta_3$ and, finally, $\rho - \beta_1 - \beta_2 - \beta_3 - \beta_4$ are roots; since the latter has α-height -2, it must be $-\rho$. $\qquad\square$

Fact 2.10. *If four roots of α-height 1 are mutually orthogonal, then they are all long roots.*

Proof. Call the roots $\beta_1, \beta_2, \beta_3, \beta_4$. By Fact 2.9, $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\rho$; since the roots are mutually orthogonal we then have

$$
\begin{aligned}
4(\rho, \rho) &= (2\rho, 2\rho) \\
&= (\beta_1 + \beta_2 + \beta_3 + \beta_4, \beta_1 + \beta_2 + \beta_3 + \beta_4) \\
&= (\beta_1, \beta_1) + (\beta_2, \beta_2) + (\beta_3, \beta_3) + (\beta_4, \beta_4).
\end{aligned}
$$

Since ρ is long, $(\beta_i, \beta_i) \leq (\rho, \rho)$ for each i, $1 \leq i \leq 4$. Thus we must have $(\beta_i, \beta_i) = (\rho, \rho)$ for each i; that is, each root is long. $\qquad\square$

Röhrle remarks that sets of four mutually orthogonal roots of α-height 1 are easily exhibited for the Lie algebras we are considering, but does not give explicit examples. For completeness, we do so here. The verification is simplified by the observation that, once we have three such roots, we get the fourth "for free".

Lemma 2.11. *If β, γ and δ are three mutually orthogonal long roots of α-height 1, then $\epsilon = 2\rho - \beta - \gamma - \delta$ is a root of α-height 1 that is orthogonal to β, γ and δ.*

Proof. By Fact 2.8, $\rho - \beta$ and $\rho - \gamma$ are orthogonal long roots of α-height 1. Since $\langle \rho - \gamma, \delta \rangle = \langle \rho, \delta \rangle - \langle \gamma, \delta \rangle = 1 - 0 = 1$, $\rho - \gamma - \delta$ is a root. We then have

$$
\begin{aligned}
\langle \rho - \gamma - \delta, \rho - \beta \rangle &= \langle \rho - \gamma, \rho - \beta \rangle - \langle \delta, \rho - \beta \rangle \\
&= 0 + \langle \beta - \rho, \delta \rangle \\
&= 0 - 1 \\
&= -1,
\end{aligned}
$$

so $\epsilon = (\rho - \gamma - \delta) + (\rho - \beta)$ is a root. We check that

$$
\begin{aligned}
\langle \epsilon, \beta \rangle &= \langle 2\rho - \beta - \gamma - \delta, \beta \rangle \\
&= 2\langle \rho, \beta \rangle - \langle \beta, \beta \rangle - \langle \gamma, \beta \rangle - \langle \delta, \beta \rangle \\
&= 2 - 2 - 0 - 0 \\
&= 0,
\end{aligned}
$$

so ϵ is orthogonal to β; by symmetry, it is orthogonal to γ and δ as well. $\square$

Types B_n and D_n are similar, so we handle them simultaneously. They have the following in common, which are all the facts we will need:

- The roots $\alpha_1, \alpha_2, \alpha_3$ are all long (since we are assuming $n \geq 4$ in the B_n case).

- The root $\alpha = \alpha_2$, so $\langle \alpha_2, \rho \rangle = 1$ and $\langle \alpha_1, \rho \rangle = \langle \alpha_3, \rho \rangle = 0$.

- The root α_2 is joined by a single edge to each of α_1 and α_3 in the Dynkin diagram, but α_1 and α_3 are not joined; that is, $\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_3, \alpha_2 \rangle = -1$ and $\langle \alpha_1, \alpha_3 \rangle = 0$.

We claim that the following are mutually orthogonal long roots of α-height 1: $\alpha = \alpha_2$, $\beta = \alpha_1 + \alpha_2 + \alpha_3$, $\gamma = \rho - \alpha_1 - \alpha_2$, $\delta = \rho - \alpha_2 - \alpha_3$. First, we see that the first three are roots: α is a simple root; $\langle \alpha_1, \alpha_2 \rangle = -1$ implies that $\alpha_1 + \alpha_2$ is a root, and $\langle \alpha_1 + \alpha_2, \alpha_3 \rangle = \langle \alpha_1, \alpha_3 \rangle + \langle \alpha_2, \alpha_3 \rangle = 0 + (-1) = -1$ implies that β is a root; $\rho - \alpha_2$ is a root by Fact 2.8 and $\langle \rho - \alpha_2, \alpha_3 \rangle = \langle \rho, \alpha_3 \rangle - \langle \alpha_2, \alpha_3 \rangle = 0 - (-1) = 1$ implies that γ is a root. Each of the three is of α-height 1, and is long by Fact 2.4. Since $\alpha + \beta + \gamma + \delta = 2\rho$, by Lemma 2.11 it now suffices to check that the first three are mutually orthogonal:

$$
\begin{aligned}
\langle \beta, \alpha \rangle &= \langle \alpha_1, \alpha_2 \rangle + \langle \alpha_2, \alpha_2 \rangle + \langle \alpha_3, \alpha_2 \rangle \\
&= -1 + 2 - 1 \\
&= 0.
\end{aligned}
$$

$$\langle \gamma, \alpha \rangle = \langle \rho, \alpha_2 \rangle - \langle \alpha_1, \alpha_2 \rangle - \langle \alpha_2, \alpha_2 \rangle$$
$$= 1 - (-1) - 2$$
$$= 0.$$

$$\langle \gamma, \beta \rangle = \langle \rho, \beta \rangle - \langle \alpha_1, \beta \rangle - \langle \alpha_2, \beta \rangle$$
$$= \langle \beta, \rho \rangle - \langle \beta, \alpha_1 \rangle - \langle \beta, \alpha \rangle$$
$$= 1 - (\langle \alpha_1, \alpha_1 \rangle + \langle \alpha_2, \alpha_1 \rangle + \langle \alpha_3, \alpha_1 \rangle) - 0$$
$$= 1 - (2 - 1 + 0)$$
$$= 0.$$

In the four remaining cases (E_6, E_7, E_8 and F_4) we simply list suitable sets of four mutually orthogonal roots of α-height 1 (Table 2.1). We use the compact notation of [4] for the roots; for example, 1220 means $\alpha_1 + 2\alpha_2 + 2\alpha_3$. In each case, α is one of the roots used; we also include the value of ρ for reference.

	E_6	E_7	E_8	F_4
ρ	122321	2234321	23465432	2342
α	010000	1000000	00000001	1000
β	112221	1224321	23465421	1242
γ	111210	1122100	11232221	1220
δ	011211	1122221	12233221	1222

Table 2.1: Sets of four mutually orthogonal roots of α-height 1

2.6   Summary

Table 2.2 is a partial list of notations and definitions from this chapter included for ease of reference.

F	a field of characteristic $\neq 2, 3$
G	a simple, connected linear algebraic group, split over F, not of type A or C, with rank ≥ 4
$\mathfrak{g}$	the Lie algebra of G
$[x, y]$	the Lie bracket in $\mathfrak{g}$
$\mathfrak{h}$	a fixed Cartan subalgebra of $\mathfrak{g}$
Ψ	the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$
(β, γ)	the inner product on the roots (or on $\mathfrak{h}^\vee$)
$\langle \beta, \gamma \rangle$	$2\dfrac{(\beta, \gamma)}{(\gamma, \gamma)}$
x_β, h_i	elements of a Chevalley basis for $\mathfrak{g}$
$c_{\beta, \gamma}$	the structure constant defined by $[x_\beta, x_\gamma] = c_{\beta, \gamma} x_{\beta + \gamma}$
α_i	a fixed set of simple roots of Ψ
ρ	the highest root with respect to the α_i
α	the unique simple root not orthogonal to ρ
$\mathfrak{g}_{-2}, \ldots, \mathfrak{g}_2$	parts of a grading of $\mathfrak{g}$
$q(x)$	the quartic form on $\mathfrak{g}_1$ defined by $(\operatorname{ad} x)^4(x_{-\rho}) = q(x)x_\rho$
$q(w, x, y, z)$	the linearization of $q(x)$ given by $q(x, x, x, x) = q(x)$
$\langle x, y \rangle$	the bilinear form on $\mathfrak{g}_1$ defined by $[x, y] = \langle x, y \rangle x_\rho$
xyz	the triple product on $\mathfrak{g}_1$ defined by $q(w, x, y, z) = \langle w, xyz \rangle$

Table 2.2: Summary of principal notations and definitions

Chapter 3

General Results

In this chapter, we present results that apply to all the types of Lie algebras we consider (viz., types B, D, E and F); the next chapter will contain results that apply only to some types or just to a specific Lie algebra.

In the previous chapter, we established a $\mathbb{Z}/5\mathbb{Z}$-grading on the Lie algebras in question and used it to define a quartic form and a bilinear form on the grade 1 elements. We repeat the definitions of these forms and establish their basic properties in section 3.1. After characterizing the so-called strictly regular elements (section 3.2), we will verify that $\mathfrak{g}_1$ is a Freudenthal triple system (section 3.3). Finally, we show how to explicitly compute the quartic form in the simply-laced case (section 3.4).

3.1 The bilinear and quartic forms

In this section, we define a bilinear form and a quartic form on the space $\mathfrak{g}_1$ and establish basic facts about them. For later use, we also compute the value of the quartic form and its associated 4-linear form on some special arguments.

The bilinear form on $\mathfrak{g}_1$ is defined in a natural way. Given any $x, y \in \mathfrak{g}_1$, the Lie algebra product lies in $\mathfrak{g}_2 = Fx_\rho$, so we define the bilinear form $\langle x, y \rangle$ to be the resulting coefficient of x_ρ; that is, $\langle x, y \rangle$ is given by $[x, y] = \langle x, y \rangle x_\rho$. This form is clearly skew-symmetric.

Lemma 3.1. *The bilinear form $\langle -, - \rangle$ on $\mathfrak{g}_1$ is nondegenerate.*

Proof. The elements x_β with β a root of α-height 1 form a basis for $\mathfrak{g}_1$. Consider the matrix of the form with respect to this basis; the entries are of the form $\langle x_\beta, x_\gamma \rangle$ with β, γ roots of α-height 1. Such an entry is zero unless $[x_\beta, x_\gamma]$ is a nonzero element of Fx_ρ; that is, unless $\beta + \gamma = \rho$. By Fact 2.8, $\rho - \beta$ is a root of α-height 1; hence each row and each column of the matrix contains exactly one nonzero entry. Such a matrix (sometimes called a *monomial matrix*) can be written as the product of a diagonal matrix with nonzero entries on the diagonal and a permutation matrix, hence it is invertible. Thus the form is nondegenerate. $\square$

The definition of the quartic form is also straightforward. Since $x_{-\rho}$ is an element of $\mathfrak{g}_{-2}$, for any $x \in \mathfrak{g}_1$ the value $[x, [x, [x, [x, x_{-\rho}]]]]$, or, more briefly, $(\operatorname{ad} x)^4(x_{-\rho})$, is in $\mathfrak{g}_2$. Thus we may define the quartic form $q(x)$ for $x \in \mathfrak{g}_1$ by $(\operatorname{ad} x)^4(x_{-\rho}) = q(x)x_\rho$. This in turn gives rise to a fully symmetric 4-linear form $q(x, y, z, w)$ defined by setting $q(x, x, x, x) = q(x)$ and extending by linearization.

Lemma 3.2. *Let $\beta_1, \beta_2, \beta_3, \beta_4$ be roots of α-height 1. The value of the 4-linear form $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$ is given by*

$$q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})x_\rho =$$
$$\frac{1}{4!} \sum_{\pi \in S_4} (\operatorname{ad} x_{\beta_{\pi(1)}} \circ \operatorname{ad} x_{\beta_{\pi(2)}} \circ \operatorname{ad} x_{\beta_{\pi(3)}} \circ \operatorname{ad} x_{\beta_{\pi(4)}})(x_{-\rho}),$$

where S_4 is the symmetric group on $\{1, 2, 3, 4\}$.

Proof. Let λ, μ, ν be indeterminates. By the definition of the quartic form, we have

$$q(x_{\beta_1} + \lambda x_{\beta_2} + \mu x_{\beta_3} + \nu x_{\beta_4})x_\rho = (\operatorname{ad} x_{\beta_1} + \lambda x_{\beta_2} + \mu x_{\beta_3} + \nu x_{\beta_4})^4(x_{-\rho}).$$

Replacing the quartic form on the left-hand side by the equivalent 4-linear form and expanding this expression by linearity, the resulting coefficient of $\lambda\mu\nu$ is $24q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})x_\rho$. On the right-hand side, the coefficient of $\lambda\mu\nu$ is $\sum_{\pi \in S_4}(\operatorname{ad} x_{\beta_{\pi(1)}} \circ \operatorname{ad} x_{\beta_{\pi(2)}} \circ \operatorname{ad} x_{\beta_{\pi(3)}} \circ \operatorname{ad} x_{\beta_{\pi(4)}})(x_{-\rho})$. Equating the coefficients yields the result. $\qquad\square$

Corollary 3.3. *Let $\beta_1, \beta_2, \beta_3, \beta_4$ be roots of α-height 1; then the 4-linear form $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4}) = 0$ whenever $\beta_1 + \beta_2 + \beta_3 + \beta_4 \neq 2\rho$.*

Proof. If the summand $(\operatorname{ad} x_{\beta_{\pi(1)}} \circ \operatorname{ad} x_{\beta_{\pi(2)}} \circ \operatorname{ad} x_{\beta_{\pi(3)}} \circ \operatorname{ad} x_{\beta_{\pi(4)}})(x_{-\rho})$ in the previous lemma is nonzero, it must be some multiple of x_ρ. By Fact 2.1, that means we must have $\beta_1 + \beta_2 + \beta_3 + \beta_4 + (-\rho) = \rho$. The result follows. $\quad\square$

To establish that the quartic form is nonzero, we will use some of the facts that were given in Section 2.3 about the structure constants that define the multiplication in $\mathfrak{g}$; the reader may wish to review them. These facts allow us to compute the value of the 4-linear form on some special arguments.

Lemma 3.4. *If β is a long root of α-height 1, then*

$$q(x_\beta, x_\beta, x_{\rho-\beta}, x_{\rho-\beta}) = 1. \tag{3.5}$$

Proof. By Fact 2.8, $\rho-\beta$ is also a long root of α-height 1, so the 4-linear form is defined on the specified arguments. We begin by finding $q(x_\beta + \lambda x_{\rho-\beta})$, which is given by $(\operatorname{ad} x_\beta + \lambda x_{\rho-\beta})^4(x_{-\rho}) = q(x_\beta + \lambda x_{\rho-\beta})x_\rho$. The left-hand side can be calculated by repeatedly applying $\operatorname{ad} x_\beta + \lambda x_{\rho-\beta}$. For the first step,

$$[x_\beta + \lambda x_{\rho-\beta}, x_{-\rho}] = c_{\beta,-\rho}x_{\beta-\rho} + \lambda c_{\rho-\beta,-\rho}x_{-\beta},$$

where the structure constants are not zero since $\beta - \rho$ and $-\beta$ are roots. Writing a for $c_{\beta,-\rho}$ and b for $c_{\rho-\beta,-\rho}$, we continue, keeping in mind the multiplication rules for a Chevalley basis given in Section 2.3. For the second step, we have

$$[x_\beta + \lambda x_{\rho-\beta}, ax_{\beta-\rho} + \lambda bx_{-\beta}] = \lambda ah_{\rho-\beta} + \lambda bh_\beta;$$

the other terms are zero since $2\beta - \rho$ (resp. $\rho - 2\beta$) is not a root; indeed, $\langle \beta - \rho, \beta \rangle = 1$, so Fact 2.3 applies.

The remaining two steps are as follows:

$$[x_\beta + \lambda x_{\rho-\beta}, \lambda a h_{\rho-\beta} + \lambda b h_\beta] = -2\lambda^2 a x_{\rho-\beta} - 2\lambda b x_\beta + \lambda a x_\beta + \lambda^2 b x_{\rho-\beta},$$

$$[x_\beta + \lambda x_{\rho-\beta}, -2\lambda^2 a x_{\rho-\beta} - 2\lambda b x_\beta + \lambda a x_\beta + \lambda^2 b x_{\rho-\beta}] =$$
$$3\lambda^2 c_{\beta,\rho-\beta}(b - a) x_\rho.$$

Since β, $-\rho$ and $\rho - \beta$ are long roots that sum to zero, we can apply Fact 2.5 to find $a = c_{\beta,-\rho} = c_{\rho-\beta,\beta} = -c_{\beta,\rho-\beta}$ and $b = c_{\rho-\beta,-\rho} = c_{\beta,\rho-\beta} = -a$. Since the structure constant $c_{\beta,\rho-\beta}$ is ± 1, the result is $6\lambda^2 c_{\beta,\rho-\beta}^2 = 6\lambda^2$. On the other hand, the term in λ^2 resulting from expanding $q(x_\beta + \lambda x_{\rho-\beta})$ by linearity is $6\lambda^2 q(x_\beta, x_\beta, x_{\rho-\beta}, x_{\rho-\beta})$, so we have

$$q(x_\beta, x_\beta, x_{\rho-\beta}, x_{\rho-\beta}) = 1,$$

as required. $\qquad\square$

Since there is always a long root of α-height 1 (e.g., α itself), we have established that the 4-linear form and thus also the quartic form are not identically zero. In particular, taking $\beta = \alpha$ and $\lambda = 1$, we have $q(x_\alpha + x_{\rho-\alpha}) = 6$.

In the next section we will also need to know that the 4-linear form is nonzero in another special case. We show this after the following lemma, which is a fact about structure constants that will also be used in Section 3.4.

Lemma 3.6. *Let β and γ be two orthogonal long roots of α-height 1. Each of $\beta - \rho$, $\gamma - \rho$ and $\rho - \beta - \gamma$ is a root; each of the structure constants $c_{\beta,\gamma-\rho}$, $c_{\gamma,\beta-\rho}$, $c_{\beta,-\rho}$, $c_{\gamma,-\rho}$ is ± 1 and their product is 1.*

Proof. By Fact 2.8, $\rho - \beta$ and $\rho - \gamma$ are long roots, so their negatives are as well. Since β and γ are orthogonal, $\langle \rho - \beta, \gamma \rangle = \langle \rho, \gamma \rangle = 1$, so $\rho - \beta - \gamma$ is

a root; by Fact 2.4 it is long. Since these are roots, the specified structure constants are nonzero; since all roots involved are long, they are ± 1.

We apply (2.6), replacing $\beta, \gamma, \delta, \epsilon$ with $\rho - \beta - \gamma, \beta, \gamma, -\rho$ to yield

$$c_{\rho-\beta-\gamma,\beta}c_{\gamma,-\rho} + c_{\beta,\gamma}c_{\rho-\beta-\gamma,-\rho} + c_{\gamma,\rho-\beta-\gamma}c_{\beta,-\rho} = 0.$$

As β and γ are orthogonal long roots, $\beta + \gamma$ (likewise $-\beta - \gamma$) is not a root; thus the structure constants in the middle term are zero. Thus we find

$$c_{\rho-\beta-\gamma,\beta}c_{\gamma,-\rho} = -c_{\gamma,\rho-\beta-\gamma}c_{\beta,-\rho}.$$

By Fact 2.5, we have $c_{\rho-\beta-\gamma,\beta} = c_{\beta,\gamma-\rho}$ and $c_{\gamma,\rho-\beta-\gamma} = c_{\beta-\rho,\gamma} = -c_{\gamma,\beta-\rho}$; substituting these yields

$$c_{\beta,\gamma-\rho}c_{\gamma,-\rho} = c_{\gamma,\beta-\rho}c_{\beta,-\rho}.$$

Since each side is ± 1, the product of all four structure constants is 1. $\qquad\square$

Lemma 3.7. *If β and γ are two orthogonal long roots of α-height 1, then*

$$q(x_\beta, x_\gamma, x_{\rho-\beta}, x_{\rho-\gamma}) = -\frac{1}{2}c_{\beta,-\rho}c_{\gamma,-\rho} \neq 0. \tag{3.8}$$

Proof. By Lemma 3.2, there are 24 terms to consider. We divide them into three classes.

Class 1: These are the terms in which the first two elements applied to $x_{-\rho}$ are x_β and $x_{\rho-\beta}$, in either order, or, likewise, x_γ and $x_{\rho-\gamma}$. The result in $\mathfrak{g}_0$ is thus in $\mathfrak{h}$. By Fact 2.8, since β and γ are orthogonal, so are $\rho - \beta$ and $\rho - \gamma$. As a result, half the terms in this case are zero; e.g., $[x_{\rho-\beta}, [x_\beta, x_{-\rho}]]$ is a multiple of $h_{\rho-\beta}$, and $[x_{\rho-\gamma}, h_{\rho-\beta}] = \langle \rho - \beta, \rho - \gamma \rangle x_{\rho-\gamma} = 0$. There are

eight terms in all:

$$[x_\gamma, [x_{\rho-\gamma}, [x_{\rho-\beta}, [x_\beta, x_{-\rho}]]]] = 0,$$
$$[x_{\rho-\gamma}, [x_\gamma, [x_\beta, [x_{\rho-\beta}, x_{-\rho}]]]] = 0,$$
$$[x_\beta, [x_{\rho-\beta}, [x_{\rho-\gamma}, [x_\gamma, x_{-\rho}]]]] = 0,$$
$$[x_{\rho-\beta}, [x_\beta, [x_\gamma, [x_{\rho-\gamma}, x_{-\rho}]]]] = 0,$$
$$[x_{\rho-\gamma}, [x_\gamma, [x_{\rho-\beta}, [x_\beta, x_{-\rho}]]]] = -c_{\rho-\gamma,\gamma}c_{\beta,-\rho}x_\rho,$$
$$[x_\gamma, [x_{\rho-\gamma}, [x_\beta, [x_{\rho-\beta}, x_{-\rho}]]]] = -c_{\gamma,\rho-\gamma}c_{\rho-\beta,-\rho}x_\rho,$$
$$[x_{\rho-\beta}, [x_\beta, [x_{\rho-\gamma}, [x_\gamma, x_{-\rho}]]]] = -c_{\rho-\beta,\beta}c_{\gamma,-\rho}x_\rho,$$
$$[x_\beta, [x_{\rho-\beta}, [x_\gamma, [x_{\rho-\gamma}, x_{-\rho}]]]] = -c_{\beta,\rho-\beta}c_{\rho-\gamma,-\rho}x_\rho.$$

By Fact 2.5, we have $c_{\gamma,-\rho} = c_{\rho-\gamma,\gamma} = -c_{\gamma,\rho-\gamma} = -c_{\rho-\gamma,-\rho}$, and, replacing γ with β, $c_{\beta,-\rho} = c_{\rho-\beta,\beta} = -c_{\beta,\rho-\beta} = -c_{\rho-\beta,-\rho}$. Thus each of the four nonzero terms computed above is equal to $-c_{\gamma,-\rho}c_{\beta,-\rho}x_\rho$.

Class 2: Here the terms are those in which the first two elements applied to $x_{-\rho}$ are x_β and $x_{\rho-\gamma}$, in either order, or, likewise, x_γ and $x_{\rho-\beta}$. Since $\beta - \gamma$ (resp. $\gamma - \beta$) is not a root, each of these eight terms is zero:

$$[x_{\rho-\beta}, [x_\gamma, [x_{\rho-\gamma}, [x_\beta, x_{-\rho}]]]] = 0,$$
$$[x_\gamma, [x_{\rho-\beta}, [x_{\rho-\gamma}, [x_\beta, x_{-\rho}]]]] = 0,$$
$$[x_{\rho-\beta}, [x_\gamma, [x_\beta, [x_{\rho-\gamma}, x_{-\rho}]]]] = 0,$$
$$[x_\gamma, [x_{\rho-\beta}, [x_\beta, [x_{\rho-\gamma}, x_{-\rho}]]]] = 0,$$
$$[x_{\rho-\gamma}, [x_\beta, [x_{\rho-\beta}, [x_\gamma, x_{-\rho}]]]] = 0,$$
$$[x_\beta, [x_{\rho-\gamma}, [x_{\rho-\beta}, [x_\gamma, x_{-\rho}]]]] = 0,$$
$$[x_{\rho-\gamma}, [x_\beta, [x_\gamma, [x_{\rho-\beta}, x_{-\rho}]]]] = 0,$$
$$[x_\beta, [x_{\rho-\gamma}, [x_\gamma, [x_{\rho-\beta}, x_{-\rho}]]]] = 0.$$

Class 3: The remaining terms are those in which the first two elements applied to $x_{-\rho}$ are x_β and x_γ, in either order, or, likewise, $x_{\rho-\gamma}$ and $x_{\rho-\beta}$.

Since $\beta + \gamma - \rho$ is a root by Lemma 3.6, the result in $\mathfrak{g}_0$ is nonzero and not in $\mathfrak{h}$, so we compute each term by simply accumulating the structure constants. Here are the eight terms with the results simplified by use of the properties of the structure constants:

$$\begin{aligned}
[x_{\rho-\gamma}, [x_{\rho-\beta}, [x_\gamma, [x_\beta, x_{-\rho}]]]] &= c_{\rho-\gamma,\gamma}c_{\rho-\beta,\beta+\gamma-\rho}c_{\gamma,\beta-\rho}c_{\beta,-\rho}x_\rho \\
&= c_{\gamma,-\rho}c_{-\gamma,\rho-\beta}c_{\gamma,\beta-\rho}c_{\beta,-\rho}x_\rho \\
&= -c_{\gamma,-\rho}c_{\beta,-\rho}x_\rho,
\end{aligned}$$

$$\begin{aligned}
[x_\gamma, [x_\beta, [x_{\rho-\gamma}, [x_{\rho-\beta}, x_{-\rho}]]]] &= c_{\gamma,\rho-\gamma}c_{\beta,\rho-\beta-\gamma}c_{\rho-\gamma,-\beta}c_{\rho-\beta,-\rho}x_\rho \\
&= c_{-\rho,\gamma}c_{\gamma-\rho,\beta}c_{\rho-\gamma,-\beta}c_{-\rho,\beta}x_\rho \\
&= -c_{\gamma,-\rho}c_{\beta,-\rho}x_\rho,
\end{aligned}$$

$$\begin{aligned}
[x_{\rho-\beta}, [x_{\rho-\gamma}, [x_\gamma, [x_\beta, x_{-\rho}]]]] &= c_{\rho-\beta,\beta}c_{\rho-\gamma,\beta+\gamma-\rho}c_{\gamma,\beta-\rho}c_{\beta,-\rho}x_\rho \\
&= c_{\beta,-\rho}c_{-\beta,\rho-\gamma}c_{\gamma,\beta-\rho}c_{\beta,-\rho}x_\rho \\
&= -c_{\beta,\gamma-\rho}c_{\gamma,\beta-\rho},
\end{aligned}$$

$$\begin{aligned}
[x_\beta, [x_\gamma, [x_{\rho-\gamma}, [x_{\rho-\beta}, x_{-\rho}]]]] &= c_{\beta,\rho-\beta}c_{\gamma,\rho-\beta-\gamma}c_{\rho-\gamma,-\beta}c_{\rho-\beta,-\rho}x_\rho \\
&= c_{\beta,\rho-\beta}c_{\beta-\rho,\gamma}c_{\rho-\gamma,-\beta}c_{\beta,\rho-\beta}x_\rho \\
&= -c_{\beta,\gamma-\rho}c_{\gamma,\beta-\rho},
\end{aligned}$$

$$\begin{aligned}
[x_{\rho-\beta}, [x_{\rho-\gamma}, [x_\beta, [x_\gamma, x_{-\rho}]]]] &= c_{\rho-\beta,\beta}c_{\rho-\gamma,\gamma+\beta-\rho}c_{\beta,\gamma-\rho}c_{\gamma,-\rho}x_\rho \\
&= c_{\beta,-\rho}c_{-\beta,\rho-\gamma}c_{\beta,\gamma-\rho}c_{\gamma,-\rho}x_\rho \\
&= -c_{\beta,-\rho}c_{\gamma,-\rho}x_\rho,
\end{aligned}$$

$$[x_\beta, [x_\gamma, [x_{\rho-\beta}, [x_{\rho-\gamma}, x_{-\rho}]]]] = c_{\beta,\rho-\beta} c_{\gamma,\rho-\gamma-\beta} c_{\rho-\beta,-\gamma} c_{\rho-\gamma,-\rho} x_\rho$$

$$= c_{-\rho,\beta} c_{\beta-\rho,\gamma} c_{\rho-\beta,-\gamma} c_{-\rho,\gamma} x_\rho$$

$$= -c_{\beta,-\rho} c_{\gamma,-\rho} x_\rho,$$

$$[x_{\rho-\gamma}, [x_{\rho-\beta}, [x_\beta, [x_\gamma, x_{-\rho}]]]] = c_{\rho-\gamma,\gamma} c_{\rho-\beta,\gamma+\beta-\rho} c_{\beta,\gamma-\rho} c_{\gamma,-\rho} x_\rho$$

$$= c_{\gamma,-\rho} c_{-\gamma,\rho-\beta} c_{\beta,\gamma-\rho} c_{\gamma,-\rho} x_\rho$$

$$= -c_{\gamma,\beta-\rho} c_{\beta,\gamma-\rho},$$

$$[x_\gamma, [x_\beta, [x_{\rho-\beta}, [x_{\rho-\gamma}, x_{-\rho}]]]] = c_{\gamma,\rho-\gamma} c_{\beta,\rho-\gamma-\beta} c_{\rho-\beta,-\gamma} c_{\rho-\gamma,-\rho} x_\rho$$

$$= c_{\gamma,\rho-\gamma} c_{\gamma-\rho,\beta} c_{\rho-\beta,-\gamma} c_{\gamma,\rho-\gamma} x_\rho$$

$$= -c_{\gamma,\beta-\rho} c_{\beta,\gamma-\rho}.$$

These eight terms thus consist of four terms equal to $-c_{\beta,-\rho} c_{\gamma,-\rho} x_\rho$ and four equal to $-c_{\beta,\gamma-\rho} c_{\gamma,\beta-\rho} x_\rho$.

Combining all the terms, we have

$$q(x_\beta, x_\gamma, x_{\rho-\beta}, x_{\rho-\gamma}) = -\frac{1}{3} c_{\beta,-\rho} c_{\gamma,-\rho} - \frac{1}{6} c_{\beta,\gamma-\rho} c_{\gamma,\beta-\rho};$$

but it follows from Lemma 3.6 that the two products of structure constants are equal. Thus we have

$$q(x_\beta, x_\gamma, x_{\rho-\beta}, x_{\rho-\gamma}) = -\frac{1}{2} c_{\beta,-\rho} c_{\gamma,-\rho}.$$

In particular, it is not zero. ☐

3.2   Strictly regular elements

We recall the definition of the triple product on $\mathfrak{g}_1$. For any fixed $x, y, z \in \mathfrak{g}_1$, the expression $q(w, x, y, z)$ with $w \in \mathfrak{g}_1$ is a linear function of w. Since the skew-symmetric bilinear form $\langle -, - \rangle$ is nondegenerate (Lemma 3.1), we may

define the *triple product* of x, y, z to be the unique element xyz of $\mathfrak{g}_1$ such that $q(w, x, y, z) = \langle w, xyz \rangle$ for all $w \in \mathfrak{g}_1$.

Following Ferrar ([11], §3), we call a nonzero element $x \in \mathfrak{g}_1$ *strictly regular* if $xxy \in Fx$ for all $y \in \mathfrak{g}_1$. In this section we will give several equivalent characterizations of strictly regular elements.

Lemma 3.9. *The basis element x_α is strictly regular.*

Proof. Let β, γ be roots of α-height 1. By Corollary 3.3, if $\langle x_\gamma, x_\alpha x_\alpha x_\beta \rangle = q(x_\gamma, x_\alpha, x_\alpha, x_\beta)$ is nonzero, then $2\alpha + \beta + \gamma = 2\rho$. Since the simple root α has height 1, this implies $\mathrm{ht}(\beta + \gamma) = 2 \, \mathrm{ht} \, \rho - 2$. As ρ is the unique highest root, β and γ have smaller heights than ρ, so this can only occur if both have height $\mathrm{ht} \, \rho - 1$. Since the only simple root not orthogonal to ρ is α, the only root of that height is $\rho - \alpha$, and $\langle x_\gamma, x_\alpha x_\alpha x_\beta \rangle$ is therefore zero unless $\beta = \gamma = \rho - \alpha$. The orthogonal complement of any $x_\alpha x_\alpha y$ thus includes the space generated by all the x_γ, $\gamma \neq \rho - \alpha$. Since this is the orthogonal complement of x_α, we have $x_\alpha x_\alpha y \in Fx_\alpha$. $\square$

Corollary 3.10. *For any long root β of α-height 1, x_β is strictly regular.*

Proof. Since the property of being strictly regular depends only on the triple product, it is preserved by the action of $(G_0)^{\mathrm{ss}}$ by Fact 2.7. It is also preserved by scaling, so it is preserved by the action of G_0. By Lemma 2.1 in [24], all the elements x_β with β a long root of α-height 1 are in the same G_0-orbit, so they are are all strictly regular since x_α is. $\square$

Lemma 3.11. *Let $x \in \mathfrak{g}_1$ be such that $xxy = 0$ for all $y \in \mathfrak{g}_1$; then $x = 0$.*

Proof. The set of all x such that $xx\mathfrak{g}_1 = \{0\}$ is invariant under the action of G_0 on $\mathfrak{g}_1$, so it is a union of G_0-orbits; it is also closed (in the Zariski topology). Thus it suffices to show that $xx\mathfrak{g}_1 \neq \{0\}$ for a representative x of the smallest nonzero orbit (i.e., orbit 1); this follows if there are $y, z \in \mathfrak{g}_1$

such that $q(x, x, y, z) \neq 0$. A representative of the smallest nonzero orbit is $x = x_\alpha$; we let $y = z = x_{\rho-\alpha}$. By (3.5), we have $q(x, x, y, z) = 1$. $\qquad \square$

Lemma 3.12. *Let $\alpha, \beta, \gamma, \delta$ be mutually orthogonal roots of α-height 1; then $q(x_\alpha, x_\beta, x_\gamma, x_\delta) \neq 0$.*

We will give an explicit expression for this value in Proposition 3.35.

Proof. Since $x_\alpha + x_\beta + x_\gamma + x_\delta$ is a representative of the dense orbit and q is not identically zero, $q(x_\alpha + x_\beta + x_\gamma + x_\delta) \neq 0$. Expanding the corresponding 4-linear form, we obtain five kinds of terms, corresponding to the five partitions of 4:

- Those with four equal arguments, e.g., $q(x_\beta, x_\beta, x_\beta, x_\beta)$. Since 2β is not a root, we cannot have $4\beta = 2\rho$, so this expression is zero by Lemma 3.2.

- Those with exactly three equal arguments, e.g., $q(x_\beta, x_\beta, x_\beta, x_\gamma)$. Since $\alpha, \beta, \gamma, \delta$ are mutually orthogonal, they are long by Fact 2.10. Thus x_β is strictly regular (Corollary 3.10), so this expression is $\langle x_\gamma, x_\beta x_\beta x_\beta \rangle = \lambda \langle x_\gamma, x_\beta \rangle$ for some $\lambda \in F$; but $\langle x_\gamma, x_\beta \rangle = 0$ because $\gamma + \beta$ is not a root. Thus these terms are also zero.

- Those with two pairs of equal arguments, e.g., $q(x_\beta, x_\beta, x_\gamma, x_\gamma)$. The sum of the orthogonal long roots β and γ is not a root; in particular it is not ρ. Thus $2\beta + 2\gamma \neq 2\rho$, so this expression is zero.

- Those with exactly two equal arguments, e.g., $q(x_\beta, x_\beta, x_\gamma, x_\delta)$. By Fact 2.9, $\alpha + \beta + \gamma + \delta = 2\rho$; thus $2\beta + \gamma + \delta \neq 2\rho$, so those terms are zero.

- Those with four unequal arguments, e.g., $q(x_\alpha, x_\beta, x_\gamma, x_\delta)$, which by elimination must be nonzero. $\qquad \square$

Proposition 3.13. *The strictly regular elements of $\mathfrak{g}_1$ are those contained in the smallest nonzero orbit.*

Proof. The set of strictly regular elements is a union of orbits; its union with 0 is a closed set. Since x_α is a representative of the smallest nonzero orbit and is strictly regular by Lemma 3.9, all elements of the smallest nonzero orbit are also strictly regular. It thus suffices to show that representatives of level 2 orbits are not strictly regular. Let $\alpha, \beta, \gamma, \delta$ be four mutually orthogonal roots of α-height 1. We take $x_\alpha + x_\beta$ as a representative of a level 2 orbit.

We compute

$$
\begin{aligned}
\langle x_\delta, (x_\alpha + x_\beta)(x_\alpha + x_\beta)x_\gamma \rangle &= q(x_\alpha + x_\beta, x_\alpha + x_\beta, x_\gamma, x_\delta) \\
&= q(x_\alpha, x_\alpha, x_\gamma, x_\delta) + 2q(x_\alpha, x_\beta, x_\gamma, x_\delta) \\
&\quad + q(x_\beta, x_\beta, x_\gamma, x_\delta) \\
&= 2q(x_\alpha, x_\beta, x_\gamma, x_\delta),
\end{aligned}
$$

the other terms being zero since $\alpha + \alpha + \gamma + \delta$ and $\beta + \beta + \gamma + \delta$ cannot equal 2ρ since $\alpha + \beta + \gamma + \delta = 2\rho$ by Fact 2.9. By Lemma 3.12, the result is nonzero, so in particular the triple product $(x_\alpha + x_\beta)(x_\alpha + x_\beta)x_\gamma$ is not orthogonal to x_δ. However, $\langle x_\alpha + x_\beta, x_\delta \rangle = \langle x_\alpha, x_\delta \rangle + \langle x_\beta, x_\delta \rangle = 0$ since neither $\alpha + \delta$ nor $\beta + \delta$ is a root. Hence the triple product $(x_\alpha + x_\beta)(x_\alpha + x_\beta)x_\gamma$ is not a scalar multiple of $x_\alpha + x_\beta$; thus $x_\alpha + x_\beta$ is not strictly regular. $\square$

Lemma 3.14. *The strictly regular elements span $\mathfrak{g}_1$.*

Proof. By Proposition 3.13, orbit 1 consists of strictly regular elements. The span of orbit 1 is invariant under the action of G_0; thus it is a union of orbits. Both x_α and $x_{\rho-\alpha}$ are in orbit 1, so $x_\alpha + x_{\rho-\alpha}$ is in their span, but is also a representative of the dense orbit. Thus all of the dense orbit is in the span of orbit 1. Since the dense orbit is not contained in a proper subspace, the span of orbit 1 is all of $\mathfrak{g}_1$. $\square$

An element $x \in \mathfrak{g}_1$ is *rank one* if $xx\mathfrak{g}_1$ is a one-dimensional vector space over F.

Proposition 3.15. *An element $x \in \mathfrak{g}_1$ is strictly regular if and only if it is rank one.*

Proof. Suppose x is strictly regular. By definition, $xx\mathfrak{g}_1$ is contained in the one-dimensional space Fx. In the case $x = x_\alpha$, we know $x_\alpha x_\alpha \mathfrak{g}_1$ is not zero because $\langle x_{\rho-\alpha}, x_\alpha x_\alpha x_{\rho-\alpha} \rangle = q(x_{\rho-\alpha}, x_\alpha, x_\alpha, x_{\rho-\alpha})$, which is 1 by (3.5). The condition that $xx\mathfrak{g}_1$ is not zero is invariant under the action of G_0, so it holds for all of orbit 1.

As in the proof of the previous proposition, let $\alpha, \beta, \gamma, \delta$ be four mutually orthogonal roots of α-height 1, and choose $x = x_\alpha + x_\beta$ as a representative of a level 2 orbit. Since the set of rank one elements is a closed union of orbits, it will suffice to show that x is not rank one. We have

$$\langle x_{\rho-\beta}, xxx_{\rho-\alpha} \rangle = q(x_{\rho-\beta}, x_\alpha, x_\alpha, x_{\rho-\alpha}) + $$
$$q(x_{\rho-\beta}, x_\beta, x_\beta, x_{\rho-\alpha}) + 2q(x_{\rho-\beta}, x_\alpha, x_\beta, x_{\rho-\alpha})$$
$$= 2q(x_{\rho-\beta}, x_\alpha, x_\beta, x_{\rho-\alpha})$$
$$\neq 0,$$

by Corollary 3.3 and (3.8). However,

$$\langle x_{\rho-\beta}, xxx_\gamma \rangle = q(x_{\rho-\beta}, x_\alpha, x_\alpha, x_\gamma) + q(x_{\rho-\beta}, x_\beta, x_\beta, x_\gamma) + 2q(x_{\rho-\beta}, x_\alpha, x_\beta, x_\gamma)$$
$$= 0,$$

where we know the first term is zero because it is $\langle x_{\rho-\beta}, x_\alpha x_\alpha x_\gamma \rangle$ and the triple product is a scalar multiple of x_α; the other two terms are zero by Corollary 3.3. On the other hand, we know that xxx_γ is nonzero since

$$\langle x_\delta, xxx_\gamma \rangle = q(x_\delta, x_\alpha, x_\alpha, x_\gamma) + q(x_\delta, x_\beta, x_\beta, x_\gamma) + 2q(x_\delta, x_\alpha, x_\beta, x_\gamma)$$
$$= 2q(x_\alpha, x_\beta, x_\gamma, x_\delta),$$

where once again the other terms are zero by Corollary 3.3; the remaining term is not zero by Lemma 3.12. Thus $xxx_{\rho-\alpha}$ is not orthogonal to $x_{\rho-\beta}$ but xxx_{γ} is; hence they do not lie in the same one-dimensional subspace. Therefore x is not rank one. □

The following result allows us to compute the triple product and the 4-linear form if two of the arguments are the same strictly regular element. It will be useful in several proofs and computations.

Lemma 3.16. *For x strictly regular and any $y, z \in \mathfrak{g}_1$,*

$$xxy = \langle y, x \rangle x, \tag{3.17}$$

$$q(x, x, y, z) = \langle y, x \rangle \langle z, x \rangle. \tag{3.18}$$

Proof. Since x is strictly regular, for any $y \in \mathfrak{g}_1$ we have $xxy \in Fx$. If $\langle y, x \rangle = 0$, then for any $z \in \mathfrak{g}_1$ we have $\langle z, xxy \rangle = q(z, x, x, y) = \langle y, xxz \rangle = 0$, thus $xxy = 0$. Define $f : \mathfrak{g}_1 \to F$ by $xxy = f(y)x$; then f is a linear form and $f(y)$ is zero whenever $\langle y, x \rangle$ is zero. Thus $f(-)$ is a scalar multiple of $\langle -, x \rangle$.

By Proposition 3.13, x is in orbit 1; by Lemma 3.9, so is x_{α}. Hence there is some element $g \in (G_0)^{\text{ss}}$ such that $g \cdot x = cx_{\alpha}$ for some $c \in F^{\times}$; by Fact 2.7, the action of g stabilizes the bilinear and 4-linear forms. Let $x' = g^{-1} \cdot x_{\rho-\alpha}$; since the bilinear form is preserved, we have $\langle x', x \rangle = \langle x_{\rho-\alpha}, cx_{\alpha} \rangle = \pm c$. We can now compute $q(x, x, x', x')$ in two ways. On the one hand, since the 4-linear form is preserved, we have

$$\begin{aligned}
q(x, x, x', x') &= q(cx_{\alpha}, cx_{\alpha}, x_{\rho-\alpha}, x_{\rho-\alpha}) \\
&= c^2 q(x_{\alpha}, x_{\alpha}, x_{\rho-\alpha}, x_{\rho-\alpha}) \\
&= c^2 \quad \text{(by (3.5))} \\
&= \langle x', x \rangle^2.
\end{aligned}$$

On the other hand, it is $\langle x', xxx' \rangle = \langle x', f(x')x \rangle = f(x')\langle x', x \rangle$. Thus $f(x') = \langle x', x \rangle$, and therefore $f(y) = \langle y, x \rangle$ for any $y \in \mathfrak{g}_1$.

By the definition of f, we now have $xxy = \langle y, x \rangle x$ for all $y \in \mathfrak{g}_1$. Further, for any $z \in \mathfrak{g}_1$ we have $q(x, x, y, z) = \langle z, xxy \rangle = \langle y, x \rangle\langle z, x \rangle$. $\qquad\square$

Lemma 3.19. *Let β, γ be roots of α-height 1. The triple product $x_\beta x_\beta x_\gamma$ is zero unless $\beta + \gamma = \rho$.*

Proof. Since x_β is strictly regular (Corollary 3.10), (3.17) gives $x_\beta x_\beta x_\gamma = \langle x_\gamma, x_\beta \rangle x_\beta$. As $\langle x_\gamma, x_\beta \rangle$ is zero unless $\beta + \gamma = \rho$, the result follows. $\qquad\square$

Lemma 3.20. *If $\beta_1, \beta_2, \beta_3$ are roots of α-height 1 such that $x_{\beta_1} x_{\beta_2} x_{\beta_3} \neq 0$, then $\beta_4 = 2\rho - \beta_1 - \beta_2 - \beta_3$ is a root and $x_{\beta_1} x_{\beta_2} x_{\beta_3} \in Fx_{\rho - \beta_4}$.*

Proof. By Corollary 3.3, if $\langle x_\eta, x_{\beta_1} x_{\beta_2} x_{\beta_3} \rangle = q(x_\eta, x_{\beta_1}, x_{\beta_2}, x_{\beta_3}) \neq 0$ for some root η of α-height 1, then we have $\eta + \beta_1 + \beta_2 + \beta_3 = 2\rho$. Since, by hypothesis, the triple product is nonzero, there must be some such η and it must be $\beta_4 = 2\rho - \beta_1 - \beta_2 - \beta_3$. For any basis element x_η of $\mathfrak{g}_1$ other than x_{β_4}, we have $q(x_\eta, x_{\beta_1}, x_{\beta_2}, x_{\beta_3}) = 0$; therefore the triple product is orthogonal to all basis elements other than x_{β_4}. Thus it is a scalar multiple of $x_{\rho - \beta_4}$. $\qquad\square$

Proposition 3.21. *An element $x \in \mathfrak{g}_1$ is strictly regular or zero if and only if $xxx = 0$ and $x = xxy$ for some $y \in \mathfrak{g}_1$.*

Proof. First, assume that x is either strictly regular or zero. If $x = 0$ then $xxx = 0$ and $xxy = x$ for any $y \in \mathfrak{g}_1$. On the other hand, if x is strictly regular, then (3.17) gives $xxx = \langle x, x \rangle x = 0$. Furthermore, by Lemma 3.11, xxy is not identically zero, but it is in Fx since x is strictly regular. Thus xxy takes on all values of Fx, including x itself.

The set of elements x for which $xxx = 0$ is a closed union of G_0-orbits; for the present we will show that it does not include orbit 3. Let $\beta_1, \beta_2, \beta_3, \beta_4$ be mutually orthogonal roots of α-height 1, and take $x = x_{\beta_1} + x_{\beta_2} + x_{\beta_3}$ as

a representative of orbit 3. We have $xxx = 6x_{\beta_1}x_{\beta_2}x_{\beta_3}$ since the other terms vanish by Lemma 3.19. Thus we have $\langle x_{\beta_4}, xxx \rangle = 6q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$, which is not zero by Lemma 3.12. Hence $xxx \neq 0$.

Thus if $xxx = 0$ then x is in the closure of the level 2 orbits; so we need only show that if x is in a level 2 orbit then there is no $y \in \mathfrak{g}_1$ such that $x = xxy$. We take $x = x_\beta + x_\gamma$ as a representative of a level 2 orbit, where β, γ are orthogonal roots of α-height 1. For any root η of α-height 1, we have

$$xxx_\eta = x_\beta x_\beta x_\eta + x_\gamma x_\gamma x_\eta + 2x_\beta x_\gamma x_\eta.$$

Since x_β is strictly regular, the first term lies in Fx_β and is zero unless $\eta = \rho - \beta$ (Lemma 3.19); likewise, the second is in Fx_γ and is zero unless $\eta = \rho - \gamma$. If the final term is a nonzero element in the span of x_β and x_γ, then η must be $\rho - \gamma$ or $\rho - \beta$ by Lemma 3.20. Thus $x_{\rho-\beta}$ and $x_{\rho-\gamma}$ are the only basis elements of $\mathfrak{g}_1$ that can yield a triple product with nonzero coordinates for x_β or x_γ. Hence if there is some y such that $xxy = x_\beta + x_\gamma$, then there is such a y in $Fx_{\rho-\beta} \oplus Fx_{\rho-\gamma}$.

Write $y = ax_{\rho-\beta} + bx_{\rho-\gamma}$; then we have

$$xxy = ax_\beta x_\beta x_{\rho-\beta} + bx_\gamma x_\gamma x_{\rho-\gamma} + 2ax_\beta x_\gamma x_{\rho-\beta} + 2bx_\beta x_\gamma x_{\rho-\gamma}.$$

Define a_{ij} for $1 \leq i, j \leq 2$ by the following relations:

$$x_\beta x_\beta x_{\rho-\beta} = a_{11}x_\beta,$$
$$2x_\beta x_\gamma x_{\rho-\gamma} = a_{12}x_\beta,$$
$$2x_\beta x_\gamma x_{\rho-\beta} = a_{21}x_\gamma,$$
$$x_\gamma x_\gamma x_{\rho-\gamma} = a_{22}x_\gamma,$$

so we may write

$$xxy = \begin{bmatrix} x_\beta & x_\gamma \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{3.22}$$

We compute

$$a_{11}\langle x_{\rho-\beta}, x_\beta\rangle = \langle x_{\rho-\beta}, x_\beta x_\beta x_{\rho-\beta}\rangle$$
$$= q(x_{\rho-\beta}, x_\beta, x_\beta, x_{\rho-\beta})$$
$$= 1$$

by (3.5). Since $\langle x_{\rho-\beta}, x_\beta\rangle = c_{\rho-\beta,\beta} = \pm 1$, we have $a_{11} = c_{\rho-\beta,\beta} = c_{\beta,-\rho}$ by Fact 2.5(b). Similarly,

$$a_{22}\langle x_{\rho-\gamma}, x_\gamma\rangle = \langle x_{\rho-\gamma}, x_\gamma x_\gamma x_{\rho-\gamma}\rangle$$
$$= q(x_{\rho-\gamma}, x_\gamma, x_\gamma, x_{\rho-\gamma})$$
$$= 1;$$

thus $a_{22} = c_{\rho-\gamma,\gamma} = c_{\gamma,-\rho}$.

For a_{21}, we have

$$a_{21}\langle x_{\rho-\gamma}, x_\gamma\rangle = \langle x_{\rho-\gamma}, 2x_\beta x_\gamma x_{\rho-\beta}\rangle$$
$$= 2q(x_{\rho-\gamma}, x_\beta, x_\gamma, x_{\rho-\beta})$$
$$= -c_{\beta,-\rho}c_{\gamma,-\rho}$$
$$= -c_{\beta,-\rho}c_{\rho-\gamma,\gamma}$$

by (3.8). Thus $a_{21}c_{\rho-\gamma,\gamma} = -c_{\beta,-\rho}c_{\rho-\gamma,\gamma}$, so $a_{21} = -c_{\beta,-\rho}$. Similarly, for a_{12},

$$a_{12}\langle x_{\rho-\beta}, x_\beta\rangle = \langle x_{\rho-\beta}, 2x_\beta x_\gamma x_{\rho-\gamma}\rangle$$
$$= 2q(x_{\rho-\beta}, x_\beta, x_\gamma, x_{\rho-\gamma})$$
$$= -c_{\gamma,-\rho}c_{\beta,-\rho}$$
$$= -c_{\gamma,-\rho}c_{\rho-\beta,\beta}.$$

Thus $a_{12}c_{\rho-\beta,\beta} = -c_{\gamma,-\rho}c_{\rho-\beta,\beta}$, so $a_{12} = -c_{\gamma,-\rho}$.

Thus the 2×2 matrix in (3.22) is

$$\begin{bmatrix} c_{\beta,-\rho} & -c_{\gamma,-\rho} \\ -c_{\beta,-\rho} & c_{\gamma,-\rho} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} c_{\beta,-\rho} & -c_{\gamma,-\rho} \end{bmatrix},$$

a rank-1 matrix. Therefore xxy is a scalar multiple of $x_\beta - x_\gamma$. This cannot be $x_\beta + x_\gamma$, as required. $\qquad\square$

Lemma 3.23. *Each element in the dense orbit of $\mathfrak{g}_1$ can be expressed as the sum of two strictly regular elements in one and only one way.*

Proof. Since the action of $(G_0)^{\mathrm{ss}}$ and scaling by elements of $F^\times$ both preserve strictly regular elements, it suffices to prove this for any representative of the dense orbit. We choose $x = x_\alpha + x_{\rho-\alpha}$ as the representative, which immediately establishes the existence of an expression as the sum of two strictly regular elements.

Suppose $x = u + v$ with u, v strictly regular. The triple product xxx is thus

$$
\begin{aligned}
(u+v)(u+v)(u+v) &= uuu + 3uuv + 3uvv + vvv \\
&= \langle u, u \rangle u + 3 \langle v, u \rangle u + 3 \langle u, v \rangle v + \langle v, v \rangle v \\
&= 3 \langle v, u \rangle (u - v);
\end{aligned}
$$

in particular, this is true if $u = x_\alpha$ and $v = x_{\rho-\alpha}$, so we have shown that

$$
3 \langle v, u \rangle (u - v) = 3 \langle x_{\rho-\alpha}, x_\alpha \rangle (x_\alpha - x_{\rho-\alpha}). \tag{3.24}
$$

The quartic form $q(x) = \langle x, xxx \rangle$ is thus

$$
\begin{aligned}
\langle u + v, 3 \langle v, u \rangle (u - v) \rangle &= 3 \langle v, u \rangle (-\langle u, v \rangle + \langle v, u \rangle) \\
&= 6 \langle v, u \rangle^2;
\end{aligned}
$$

again, this must be the same as $6 \langle x_{\rho-\alpha}, x_\alpha \rangle^2$. Thus $\langle v, u \rangle = \pm \langle x_{\rho-\alpha}, x_\alpha \rangle$, so (3.24) yields $u - v = \pm (x_\alpha - x_{\rho-\alpha})$. Combined with $u + v = x_\alpha + x_{\rho-\alpha}$, one choice of sign yields $u = x_\alpha$, $v = x_{\rho-\alpha}$, and the other $u = x_{\rho-\alpha}$, $v = x_\alpha$, so the choice of u and v is determined up to order. $\qquad\square$

3.3 Freudenthal triple systems

In this section we verify that $\mathfrak{g}_1$ equipped with the quartic and bilinear forms defined above is in fact a Freudenthal triple system, a term which we now formally define. A *Freudenthal triple system* is a finite-dimensional vector space V over a field F (with characteristic not 2 or 3) such that

- There is a nonzero quartic form q defined on V. A corresponding 4-linear form, also called q, is given by linearization, with $q(x, x, x, x) = q(x)$ for all $x \in V$.

- There is a nondegenerate skew-symmetric bilinear form $\langle -, - \rangle$ defined on V. Thus for given $x, y, z \in V$ we may define the triple product xyz to be the unique vector in V such that $q(w, x, y, z) = \langle w, xyz \rangle$ for all $w \in V$.

- The triple product satisfies the following identity:

$$2(xxx)xy = \langle y, x \rangle xxx + \langle y, xxx \rangle x. \tag{3.25}$$

Definitions of Freudenthal triple system in the literature vary. For example, in [11] the coefficient 2 on the left-hand side of (3.25) is omitted; in [26] the 2 becomes a 6 and the triple product is defined so that $8q(w, x, y, z) = \langle xyz, w \rangle$. However, these variations are inessential; it is easy to convert one definition to another by rescaling the quartic and bilinear forms as needed.

Theorem 3.26. *The vector space $\mathfrak{g}_1$ equipped with the quartic form q and the bilinear form $\langle -, - \rangle$ is a Freudenthal triple system.*

Proof. We established in Section 3.1 that $\langle -, - \rangle$ is skew-symmetric and non-degenerate and that q is nonzero. Hence it remains only to show that the triple product identity (3.25) is satisfied.

We first set $x = x_\alpha + x_{\rho-\alpha}$. As in the proof of Lemma 3.23, we use (3.17) to compute

$$\begin{aligned}
xxx &= (x_\alpha + x_{\rho-\alpha})(x_\alpha + x_{\rho-\alpha})(x_\alpha + x_{\rho-\alpha}) \\
&= x_\alpha x_\alpha x_\alpha + 3x_\alpha x_\alpha x_{\rho-\alpha} + 3x_\alpha x_{\rho-\alpha} x_{\rho-\alpha} + x_{\rho-\alpha} x_{\rho-\alpha} x_{\rho-\alpha} \\
&= 3\langle x_{\rho-\alpha}, x_\alpha \rangle x_\alpha + 3\langle x_\alpha, x_{\rho-\alpha} \rangle x_{\rho-\alpha} \\
&= 3\langle x_{\rho-\alpha}, x_\alpha \rangle (x_\alpha - x_{\rho-\alpha}).
\end{aligned}$$

Thus the left-hand side of (3.25) is

$$\begin{aligned}
2(xxx)xy &= 6\langle x_{\rho-\alpha}, x_\alpha \rangle (x_\alpha - x_{\rho-\alpha})(x_\alpha + x_{\rho-\alpha})y \\
&= 6\langle x_{\rho-\alpha}, x_\alpha \rangle (x_\alpha x_\alpha y - x_{\rho-\alpha} x_{\rho-\alpha} y) \\
&= 6\langle x_{\rho-\alpha}, x_\alpha \rangle (\langle y, x_\alpha \rangle x_\alpha - \langle y, x_{\rho-\alpha} \rangle x_{\rho-\alpha}).
\end{aligned}$$

The right-hand side is

$$\begin{aligned}
\langle y, x \rangle xxx + \langle y, xxx \rangle x &= 3\langle x_{\rho-\alpha}, x_\alpha \rangle (\langle y, x_\alpha \rangle + \langle y, x_{\rho-\alpha} \rangle)(x_\alpha - x_{\rho-\alpha}) \\
&\quad + 3\langle x_{\rho-\alpha}, x_\alpha \rangle (\langle y, x_\alpha \rangle - \langle y, x_{\rho-\alpha} \rangle)(x_\alpha + x_{\rho-\alpha}) \\
&= 6\langle x_{\rho-\alpha}, x_\alpha \rangle (\langle y, x_\alpha \rangle x_\alpha - \langle y, x_{\rho-\alpha} \rangle x_{\rho-\alpha});
\end{aligned}$$

thus (3.25) holds for $x = x_\alpha + x_{\rho-\alpha}$ and any $y \in \mathfrak{g}_1$.

Since the action of $(G_0)^{\text{ss}}$ on $\mathfrak{g}_1$ stabilizes the bilinear form and the triple product, and since (3.25) is preserved if x is adjusted by a scalar factor, it holds for the entire orbit of x, which is the dense orbit. Since the identity is a polynomial condition it also holds on the closure of that orbit, which is all of $\mathfrak{g}_1$. $\qquad\square$

3.4   Computation of the 4-linear form

In this section we show how to evaluate the expression $q(x_\beta, x_\gamma, x_\delta, x_\epsilon)$ whenever $\beta, \gamma, \delta, \epsilon$ are *long* roots of α-height 1. Among the Lie algebras we are

considering, the roots are always long in types D and E, so, by linearity, this will suffice to compute q for any values in $\mathfrak{g}_1$ in these cases.

By Corollary 3.3, $q(x_\beta, x_\gamma, x_\delta, x_\epsilon)$ is zero unless the roots satisfy $\beta + \gamma + \delta + \epsilon = 2\rho$. As the following lemma shows, this is a very restrictive condition; indeed, the subsequent proposition will show that there are only three ways long roots can add up to 2ρ.

Lemma 3.27. *Suppose $\beta_1, \beta_2, \beta_3, \beta_4$ are long roots of α-height 1 and that their sum is 2ρ. It follows that*

$$\langle \beta_1, \beta_2 \rangle + \langle \beta_1, \beta_3 \rangle + \langle \beta_1, \beta_4 \rangle = 0 \tag{3.28}$$

and

$$\langle \beta_1, \beta_2 \rangle = \langle \beta_3, \beta_4 \rangle. \tag{3.29}$$

Proof. Whenever β and γ are roots of the same length, we have $\langle \beta, \gamma \rangle = 2\frac{(\beta,\gamma)}{(\gamma,\gamma)} = 2\frac{(\beta,\gamma)}{(\beta,\beta)} = \langle \gamma, \beta \rangle$; hence we may reverse the arguments of $\langle -, - \rangle$ when both are long roots. Thus to show (3.28) we compute

$$
\begin{aligned}
\langle \beta_1, \beta_2 \rangle + \langle \beta_1, \beta_3 \rangle + \langle \beta_1, \beta_4 \rangle &= \langle \beta_2, \beta_1 \rangle + \langle \beta_3, \beta_1 \rangle + \langle \beta_4, \beta_1 \rangle \\
&= \langle \beta_2 + \beta_3 + \beta_4, \beta_1 \rangle \\
&= \langle 2\rho - \beta_1, \beta_1 \rangle \\
&= 2\langle \rho, \beta_1 \rangle - \langle \beta_1, \beta_1 \rangle \\
&= 2\langle \beta_1, \rho \rangle - 2 \\
&= 0.
\end{aligned}
$$

To show (3.29), we expand the equal expressions $(\beta_1 + \beta_2, \beta_1 + \beta_2)$ and $(2\rho - \beta_3 - \beta_4, 2\rho - \beta_3 - \beta_4)$. Taking the long roots to have unit length, we have on the one hand

$$
\begin{aligned}
(\beta_1 + \beta_2, \beta_1 + \beta_2) &= (\beta_1, \beta_1) + 2(\beta_1, \beta_2) + (\beta_2, \beta_2) \\
&= 2 + 2(\beta_1, \beta_2).
\end{aligned}
$$

Keeping in mind that, for example, $2(\rho, \beta_3) = \langle \rho, \beta_3 \rangle = 1$, we have on the other hand

$$(2\rho - \beta_3 - \beta_4, 2\rho - \beta_3 - \beta_4) = 6 - 4(\rho, \beta_3) - 4(\rho, \beta_4) + 2(\beta_3, \beta_4)$$
$$= 2 + 2(\beta_3, \beta_4).$$

Thus $2(\beta_1, \beta_2) = 2(\beta_3, \beta_4)$; that is, $\langle \beta_1, \beta_2 \rangle = \langle \beta_3, \beta_4 \rangle$. □

Proposition 3.30. *If the sum of four long roots of α-height 1 is 2ρ, then one of the following three cases must hold:*

(a) *The four roots consist of two equal pairs; that is, they are of the form $\beta, \beta, \rho - \beta, \rho - \beta$ for some β.*

(b) *The four roots consist of distinct pairs that sum to ρ; that is, they are of the form $\beta, \rho - \beta, \gamma, \rho - \gamma$ for distinct β, γ. Moreover, we may take β and γ to be orthogonal.*

(c) *The four roots are mutually orthogonal.*

Proof. Let $\beta_1, \beta_2, \beta_3, \beta_4$ be four such roots. No two can be opposite since all have α-height 1. If any two are equal, say $\beta_1 = \beta_2$, then by (3.29) we have $2 = \langle \beta_1, \beta_2 \rangle = \langle \beta_3, \beta_4 \rangle$, so $\beta_3 = \beta_4$ as well. This is case (a).

Suppose some root, say β_1, is not orthogonal to all of the others. By (3.28) we have $\langle \beta_1, \beta_2 \rangle + \langle \beta_1, \beta_3 \rangle + \langle \beta_1, \beta_4 \rangle = 0$; since each term is -1, 0 or 1 and not all are zero, we must have one of each. Without loss of generality, assume $\langle \beta_1, \beta_2 \rangle = -1$ and $\langle \beta_1, \beta_3 \rangle = 0$; then $\beta_1 + \beta_2$ is a root. Since it has α-height 2, it must be ρ. By (3.29), we also have $\langle \beta_3, \beta_4 \rangle = -1$, thus also $\beta_3 + \beta_4 = \rho$. Thus we are in case (b). As indicated, we have β_1 and β_3 orthogonal.

The only remaining possibility is that the four roots are mutually orthogonal, which is case (c). □

We now proceed to give the value of $q(\beta_1, \beta_2, \beta_3, \beta_4)$ in each of the three cases. We remind the reader that we will be making extensive use of the facts about structure constants previously mentioned in Section 2.3.

The first case was already handled in Lemma 3.4, where we showed that $q(x_\beta, x_\beta, x_{\rho-\beta}, x_{\rho-\beta}) = 1$ for any long root β of α-height 1. The second case was computed in Lemma 3.7; there we found $q(x_\beta, x_\gamma, x_{\rho-\beta}, x_{\rho-\gamma}) = -\frac{1}{2} c_{\beta,-\rho} c_{\gamma,-\rho}$ where β and γ are orthogonal long roots of α-height 1. The remaining case is covered by the following lemma.

Lemma 3.31. *If $\beta_1, \beta_2, \beta_3, \beta_4$ are mutually orthogonal roots of α-height 1, then*

$$q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4}) = c_{\beta_1,\beta_4-\rho} c_{\beta_2,\beta_1-\rho} c_{\beta_3,\beta_4-\rho} c_{\beta_4,\beta_1-\rho} \neq 0. \tag{3.32}$$

Proof. By Fact 2.9, the sum of four mutually orthogonal roots of α-height 1 is 2ρ, and by Fact 2.10 they are all long roots. We will apply (2.6) with $\beta = \beta_1$, $\gamma = \beta_2$, $\delta = \beta_3 - \rho$ and $\epsilon = \beta_4 - \rho$. Observe that $\beta + \gamma + \delta + \epsilon = 0$, as required, all four roots are long (Fact 2.4), and no two of $\beta, \gamma, \delta, \epsilon$ are opposite; for example, $\beta + \delta = 0$ implies $\beta_1 + \beta_3 = \rho$, but β_1 and β_3 are orthogonal. With these values, (2.6) becomes

$$c_{\beta_1,\beta_2} c_{\beta_3-\rho,\beta_4-\rho} + c_{\beta_2,\beta_3-\rho} c_{\beta_1,\beta_4-\rho} + c_{\beta_3-\rho,\beta_1} c_{\beta_2,\beta_4-\rho} = 0.$$

The structure constants in the first term are zero since $\beta_1 + \beta_2$ is not a root and thus $(\beta_3 - \rho) + (\beta_4 - \rho) = -\beta_1 - \beta_2$ is also not a root. The remaining structure constants are nonzero, since for distinct i, j we have $\langle \beta_i - \rho, \beta_j \rangle = \langle \beta_i, \beta_j \rangle - \langle \rho, \beta_j \rangle = -1$; thus $\beta_i + \beta_j - \rho$ is a root.

We now have

$$c_{\beta_2,\beta_3-\rho} c_{\beta_1,\beta_4-\rho} = -c_{\beta_3-\rho,\beta_1} c_{\beta_2,\beta_4-\rho},$$

or, more symmetrically,

$$c_{\beta_2,\beta_3-\rho} c_{\beta_1,\beta_4-\rho} = c_{\beta_1,\beta_3-\rho} c_{\beta_2,\beta_4-\rho}.$$

Using a_{ij} as an abbreviation for $c_{\beta_i, \beta_j - \rho}$, we can rewrite this as

$$a_{23} a_{14} = a_{13} a_{24}. \tag{3.33}$$

Since the numbering of the indices is arbitrary, we think of this as saying that, in a product of the form $a_{ij} a_{kl}$ that uses four different indices, we may interchange the first subscripts of the two factors; that is, $a_{ij} a_{kl} = a_{kj} a_{il}$ when i, j, k, l are distinct.

Since all the a_{ij} are ± 1, we can freely move them across the equals sign; in particular, we also have

$$a_{13} a_{23} = a_{14} a_{24}; \tag{3.34}$$

in other words, in a product of the form $a_{ij} a_{kj}$ involving three different indices, the repeated index may be replaced by the unused one.

The term in the sum for $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$ given by Lemma 3.2 arising from the term $(\operatorname{ad} x_{\beta_4} \circ \operatorname{ad} x_{\beta_3} \circ \operatorname{ad} x_{\beta_2} \circ \operatorname{ad} x_{\beta_1})(x_{-\rho})$ is

$$
\begin{aligned}
c_{-\rho, \beta_1} c_{\beta_1 - \rho, \beta_2} c_{\rho - \beta_3 - \beta_4, \beta_3} c_{\rho - \beta_4, \beta_4} &= c_{\beta_1, -\rho} c_{\beta_2, \beta_1 - \rho} c_{\beta_3, \beta_4 - \rho} c_{\beta_4, -\rho} \\
&= c_{\beta_1, \beta_4 - \rho} c_{\beta_2, \beta_1 - \rho} c_{\beta_3, \beta_4 - \rho} c_{\beta_4, \beta_1 - \rho} \\
&= a_{14} a_{21} a_{34} a_{41},
\end{aligned}
$$

where we have used Lemma 3.6 in the form $c_{\beta_1, -\rho} c_{\beta_4, -\rho} = c_{\beta_1, \beta_4 - \rho} c_{\beta_4, \beta_1 - \rho}$ for the second equality. Every other term in the sum is obtained by permuting the indices; we will show that the value is unchanged in each case. Since the two permutations given by $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1$ and by $1 \mapsto 2 \mapsto 1$ generate the symmetric group, it suffices to show that $a_{21} a_{32} a_{41} a_{12}$ and $a_{24} a_{12} a_{34} a_{42}$ are the same as the product above.

We first apply the principle of (3.34) in the form $a_{14} a_{34} = a_{12} a_{32}$ to find that

$$
\begin{aligned}
a_{14} a_{21} a_{34} a_{41} &= a_{12} a_{21} a_{32} a_{41} \\
&= a_{21} a_{32} a_{41} a_{12},
\end{aligned}
$$

so the first required equality holds. Proceeding from the last expression, we alternately apply (3.34) and (3.33) as follows:

$$a_{21}a_{32}a_{41}a_{12} = a_{23}a_{32}a_{43}a_{12}, \quad (\text{since } a_{21}a_{41} = a_{23}a_{43})$$
$$= a_{23}a_{32}a_{13}a_{42}, \quad (\text{since } a_{43}a_{12} = a_{13}a_{42})$$
$$= a_{24}a_{32}a_{14}a_{42}, \quad (\text{since } a_{23}a_{13} = a_{24}a_{14})$$
$$= a_{24}a_{12}a_{34}a_{42}, \quad (\text{since } a_{32}a_{14} = a_{12}a_{34})$$

which is the required product.

Thus all 24 summands are equal, so we have

$$q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4}) = a_{14}a_{21}a_{34}a_{41},$$

which, by substituting for the a_{ij}, becomes the desired equation. $\square$

To summarize, we have the following result.

Proposition 3.35. *If $\beta_1, \beta_2, \beta_3, \beta_4$ are long roots of α-height 1, then the value of $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$ is one of the following:*

- *0, if $\beta_1 + \beta_2 + \beta_3 + \beta_4 \neq 2\rho$;*

- *1, if $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\rho$ and there are two pairs of equal roots;*

- *$-\frac{1}{2}c_{\beta,-\rho}c_{\gamma,-\rho}$ if the roots are, in some order, $\beta, \gamma, \rho - \beta, \rho - \gamma$ with $\langle \beta, \gamma \rangle = 0$ for some β, γ; or*

- *$c_{\beta_1,\beta_4-\rho}c_{\beta_2,\beta_1-\rho}c_{\beta_3,\beta_4-\rho}c_{\beta_4,\beta_1-\rho}$ if the four roots are mutually orthogonal.*

In particular, $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$ is nonzero whenever $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\rho$.

Chapter 4

Special Results

This chapter presents results that are valid for specific Lie algebras or types of Lie algebras. In Section 4.1 we show that, for types D and E, the Freudenthal triple system has an eigenspace decomposition into four subspaces. For the same types, we characterize the G_0-orbits of the Freudenthal triple system in Section 4.2. The remaining sections examine groups whose actions preserve the Freudenthal triple system operations (either exactly or up to scalar multiples); in the case $\mathfrak{g} = E_8$, we find that E_7 is the group that stabilizes both the forms on the prototypical Freudenthal triple system, the 56-dimensional minuscule representation of E_7. In the case $\mathfrak{g} = D_4$, we obtain similar results, extended to allow for the symmetry of the Dynkin diagram of D_4.

4.1 Eigenspace decomposition of $\mathfrak{g}_1$

In this section we assume that $\mathfrak{g}$ is a Lie algebra of type D or E. We show that there is an element h in the Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{g}_1$ is the direct sum of the four eigenspaces under $\mathrm{ad}\,h$ corresponding to the eigenvalues $-3, -1, 1, 3$, and that the eigenspaces corresponding to the eigenvalues -3 and 3 are one-dimensional (cf. [11], §4).

Proposition 4.1. *Let $\mathfrak{g}$ be a Lie algebra of type D or E. For any root β of α-height 1 we have*

$$\langle \rho - 2\alpha, \beta \rangle = \begin{cases} -3 & \text{if } \beta = \alpha, \\ 3 & \text{if } \beta = \rho - \alpha, \\ \pm 1 & \text{otherwise.} \end{cases}$$

Moreover, the cases $\langle \rho - 2\alpha, \beta \rangle = -1$ and $\langle \rho - 2\alpha, \beta \rangle = 1$ occur equally often.

Proof. Let β be a root of α-height 1, necessarily a long root since $\mathfrak{g}$ is simply laced. For each such root, $\rho - \beta$ is another long root of α-height 1, and we have $\langle \alpha, \beta \rangle + \langle \alpha, \rho - \beta \rangle = \langle \beta, \alpha \rangle + \langle \rho - \beta, \alpha \rangle = \langle \rho, \alpha \rangle = 1$. Since $\langle \alpha, \beta \rangle = 2$ only if $\beta = \alpha$, it follows that $\langle \alpha, \beta \rangle = -1$ only if $\beta = \rho - \alpha$. Thus for the remaining pairs of roots β, $\rho - \beta$ we have $\langle \alpha, \beta \rangle = 0$ or 1 and correspondingly $\langle \alpha, \rho - \beta \rangle = 1$ or 0.

As $\langle \rho, \beta \rangle = 1$, we have $\langle \rho - 2\alpha, \beta \rangle = 1 - 2\langle \alpha, \beta \rangle$. Thus $\langle \rho - 2\alpha, \alpha \rangle = 1 - 2\langle \alpha, \alpha \rangle = -3$ and $\langle \rho - 2\alpha, \rho - \alpha \rangle = 1 - 2\langle \alpha, \rho - \alpha \rangle = 3$, with the remaining cases split equally between $\langle \rho - 2\alpha, \beta \rangle = 1$ and $\langle \rho - 2\alpha, \beta \rangle = -1$. $\qquad \square$

The above proposition can be generalized by using $\rho - 2\alpha'$ with α' any root of α-height 1 in place of $\rho - 2\alpha$; the proof goes through unchanged. However, we do not make use of this added generality.

At this point, we know that the promised element of $\mathfrak{h}$ exists because the Chevalley basis gives an isomorphism between $\mathfrak{h}$ and the coroot lattice with scalars extended to F. Explicitly, we recall from Section 2.3 that, for any root β, the element $h_\beta \in \mathfrak{h}$ is defined to be $[x_\beta, x_{-\beta}]$ and has the property that $[h_\beta, x_\gamma] = \langle \gamma, \beta \rangle x_\gamma$ for any root γ. Setting $h = h_{\rho - \alpha} - h_\alpha \in \mathfrak{h}$, we then have $[h, x_\beta] = (\langle \beta, \rho - \alpha \rangle - \langle \beta, \alpha \rangle) x_\beta = \langle \rho - 2\alpha, \beta \rangle x_\beta$, yielding the eigenvalue decomposition described above.

4.2   Characterization of the orbits

Proposition 4.2. *In the cases where there are five G_0-orbits in $\mathfrak{g}_1$, namely for $\mathfrak{g}$ of type E_6, E_7 or E_8, the orbits are characterized as follows:*

- *x is in orbit 0 iff $x = 0$,*

- *x is in the closure of orbit 1 iff $xxy \in Fx$ for all $y \in \mathfrak{g}_1$,*

- *x is in the closure of orbit 2 iff $xxx = 0$,*

- *x is in the closure of orbit 3 iff $q(x) = 0$, and*

- *x is in orbit 4 iff $q(x) \neq 0$.*

Proof. The statement for orbit 0 is clear; the statement for orbit 1 is Proposition 3.13.

The conditions for orbits 2 and 3 are invariant under the action of G_0 and define closed sets, so it suffices to consider representatives of the orbits. Let $\beta_1, \beta_2, \beta_3, \beta_4$ be four mutually orthogonal roots of α-height 1.

Choose $x = x_{\beta_1} + x_{\beta_2}$ as a representative of orbit 2. The triple product xxx contains the terms $x_{\beta_1}x_{\beta_1}x_{\beta_1}$, $x_{\beta_2}x_{\beta_2}x_{\beta_2}$, $x_{\beta_1}x_{\beta_1}x_{\beta_2}$ and $x_{\beta_1}x_{\beta_2}x_{\beta_2}$. All are zero by Lemma 3.19; thus $xxx = 0$ for any x in orbit 2.

Conversely, for $x = x_{\beta_1} + x_{\beta_2} + x_{\beta_3}$ in orbit 3, we have $xxx = 6x_{\beta_1}x_{\beta_2}x_{\beta_3}$ since the other terms vanish by Lemma 3.19. Thus we have $\langle x_{\beta_4}, xxx \rangle = 6q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$, which is not zero by Lemma 3.12. Hence $xxx \neq 0$ for any x in orbit 3.

For $x = x_{\beta_1} + x_{\beta_2} + x_{\beta_3}$ in orbit 3, all the terms arising when $q(x, x, x, x)$ is expanded are zero: some x_{β_i} must be repeated, so we have terms of the form $q(x_{\beta_i}, x_{\beta_i}, x_{\beta_j}, x_{\beta_k})$ with i, j, k not necessarily distinct; such a term equals $\langle x_{\beta_j}, x_{\beta_i}x_{\beta_i}x_{\beta_k} \rangle$, which is 0 by Lemma 3.19. Thus $q(x) = 0$ for any x in orbit 3.

Finally, recall that the fourth orbit is represented by $x = x_\alpha + x_{\rho-\alpha}$ ([24], Corollary 4.4). By the remark following Lemma 3.4, we have $q(x) = 6$; hence $q(x) \neq 0$ for any x in orbit 4. $\qquad\square$

A similar result applies for Lie algebras of type D_n, except that the elements $x \in \mathfrak{g}_1$ satisfying $xxx = 0$ are those that belong to any of the level 2 orbits or their closures. As these orbits are each represented by elements of the form $x_{\beta_1} + x_{\beta_2}$, but for different choices of $\beta_1, \beta_2, \beta_3, \beta_4$, the proof goes through unchanged.

Krutelevich ([20], Definition 22) defines the *rank* of an element of Freudenthal triple system constructed from a cubic Jordan algebra using characterizations which are nearly the same as those given in the preceding proposition. His definition of rank 1 differs from the characterization of orbit 1; it is equivalent (apart from a different convention on scalars) to (3.17).

4.3   Related groups

As in Ferrar's article ([11], §7), we define two subgroups of the group of linear automorphisms of $\mathfrak{g}_1$. The first, Q, preserves the quartic form on $\mathfrak{g}_1$ up to a nonzero scalar factor, that is,

$$Q = \{\eta \in \mathrm{GL}(\mathfrak{g}_1) : \forall x \in \mathfrak{g}_1, q(\eta(x)) = rq(x) \text{ for some } r \in F^\times\}.$$

We call r the *ratio of η in Q*.

Similarly, the elements of B are those that preserve the bilinear form up to a nonzero scalar:

$$B = \{\eta \in \mathrm{GL}(\mathfrak{g}_1) : \forall x, y \in \mathfrak{g}_1, \langle \eta(x), \eta(y) \rangle = r\langle x, y \rangle \text{ for some } r \in F^\times\}.$$

In this case, we call r the *ratio of η in B*.

Lemma 4.3. *The set of strictly regular elements is invariant under any $\eta \in \mathrm{GL}(\mathfrak{g}_1)$ that preserves the quartic form.*

The following argument is adapted from Ferrar ([11], Cor. 7.2).

Proof. Suppose $x \in \mathfrak{g}_1$ is rank one; that is, the elements of the form xxy with $y \in \mathfrak{g}_1$ constitute a one-dimensional subspace. Then $q(x, x, y, z) = \langle z, xxy \rangle$ is zero for all $y \in \mathfrak{g}_1$ and all z in a codimension-1 subspace. Conversely, if $x \neq 0$ and $q(x, x, y, z) = \langle z, xxy \rangle$ is zero for all $y \in \mathfrak{g}_1$ and all z in a codimension-1 subspace, then the elements xxy lie in a 1-dimensional space. Since the elements xxy are not all zero (Lemma 3.11), x is rank one. Thus this condition on the 4-linear form characterizes the rank one elements among the nonzero elements of $\mathfrak{g}_1$.

Since any map η in $\mathrm{GL}(\mathfrak{g}_1)$ is nonsingular, it preserves the dimension of subspaces. If η preserves the quartic form (and hence the 4-linear form), then the condition on the 4-linear form is true of $\eta(x)$ if it is for x. Thus η maps rank one elements to rank one elements; by Proposition 3.15, this is the same as saying it maps strictly regular elements to strictly regular elements. $\square$

Proposition 4.4. Q *is a subgroup of* B.

Proof. Let η be an element of Q. To show that η preserves $\langle x, y \rangle$ up to a scalar factor, it suffices to show it for all x in a spanning set, such as the strictly regular elements (Lemma 3.14), and all $y \in \mathfrak{g}_1$.

By (3.18), for x strictly regular and any $y \in \mathfrak{g}_1$ we have $q(x, x, y, y) = \langle x, y \rangle^2$. By Lemma 4.3, $\eta(x)$ is also strictly regular, so

$$\langle \eta(x), \eta(y) \rangle^2 = q(\eta(x), \eta(x), \eta(y), \eta(y)) = r \cdot q(x, x, y, y) = r\langle x, y \rangle^2,$$

where r is the ratio of η in Q. Thus r is a square, say $r = s^2$; we then have $\langle \eta(x), \eta(y) \rangle = \pm s \langle x, y \rangle$. The choice of sign does not depend on y, since for any $y_1, y_2 \in \mathfrak{g}_1$ we have $\pm s \langle x, y_1 + y_2 \rangle = \langle \eta(x), \eta(y_1 + y_2) \rangle = \pm s \langle x, y_1 \rangle \pm s \langle x, y_2 \rangle$, so the signs must be the same whenever the bilinear forms are nonzero. Let

us say that x is associated with s if $\langle \eta(x), \eta(y) \rangle = s\langle x, y \rangle$ for all $y \in \mathfrak{g}_1$, or that x is associated with $-s$ otherwise.

The set of strictly regular elements associated to s (resp., to $-s$) is a relatively closed subset of the set of all strictly regular elements, and the set of strictly regular elements is the disjoint union of these two sets. However, since the set of strictly regular elements is an orbit under the action of the connected set G_0 (Proposition 3.13), it is connected. Thus all strictly regular elements are associated to the same square root of r, so η is in B. □

Corollary 4.5. *Any element $\eta \in \mathrm{GL}(\mathfrak{g}_1)$ that stabilizes the quartic form also preserves orthogonality.*

Proof. If η stabilizes the quartic form, it is in Q (with ratio 1); thus it is in B (with ratio ± 1). Therefore, for any $x, y \in \mathfrak{g}_1$, we have $\langle x, y \rangle = 0$ if and only if $\langle \eta(x), \eta(y) \rangle = 0$. □

4.4   The stabilizer of the quartic form: $G = E_8$

Suppose that G is of type E_8 and $\mathfrak{g}$ is thus the Lie algebra E_8, which has dimension 248 ([4], §VI.4.10). In this case the simple root α is, in the labeling of [4], α_8. The root subspaces within $\mathfrak{g}_0$ are then generated by the x_β where β is a root of α-height 0; that is, a root of the Lie algebra of type E_7 produced by removing $\alpha = \alpha_8$ from the Dynkin diagram of E_8. There are 126 such roots ([4], §VI.4.11); combined with the 8-dimensional Cartan subalgebra of E_8, we have $\dim \mathfrak{g}_0 = 134$. Thus G_0 is the subgroup E_7 plus a one-dimensional torus, so $(G_0)^{\mathrm{ss}}$ is E_7.

Since $\dim \mathfrak{g}_{-2} = \dim \mathfrak{g}_2 = 1$, we have $\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_1 = 56$. We see that the action of $(G_0)^{\mathrm{ss}}$ on $\mathfrak{g}_1$ is irreducible since the dense orbit cannot be contained in any proper subspace, so $\mathfrak{g}_1$ is the well-known minuscule representation of E_7.

As noted in Section 1.2, it has been known since Cartan, in the case where $F = \mathbb{C}$, that there is a quartic form on the minuscule representation, V, of E_7 that is invariant under E_7; indeed, the subgroup of $\mathrm{GL}(V)$ stabilizing this quartic form and a skew-symmetric bilinear form is exactly E_7. In this section we use our techniques to establish the subgroup stabilizing the quartic form and the subgroup stabilizing both forms in our more general context.

Theorem 4.6. *For $G = E_8$, the subgroup of $\mathrm{GL}(\mathfrak{g}_1)$ stabilizing the quartic form, $\mathrm{Stab}(q)$, is generated by E_7 and μ_4, where μ_4 is the group of the fourth roots of unity.*

Proof. First, $E_7 = (G_0)^{\mathrm{ss}}$ stabilizes the quartic form by Fact 2.7.

Next, for $k \in \mu_4$, we have $q(k \cdot x) = k^4 q(x) = q(x)$ for any $x \in \mathfrak{g}_1$, so μ_4 also stabilizes the quartic form. Thus $\mathrm{Stab}(q)$ contains the group generated by E_7 and μ_4.

To show the reverse inclusion, suppose $g \in \mathrm{Stab}(q)$. Let $v = x_\alpha + x_{\rho-\alpha}$. Since v is in the dense orbit, we have by Proposition 4.2 that $q(v) \neq 0$ and also, since $q(g \cdot v) = q(v) \neq 0$, that $g \cdot v$ is in the dense orbit. Thus there exists some $z \in E_7$ such that $zg \cdot v = kv$ for some $k \in F^\times$. Let $g' = zg$; then g' is also in $\mathrm{Stab}(q)$, so $q(v) = q(g' \cdot v) = k^4 q(v)$. Thus $k \in \mu_4$. Let $g'' = k^{-1}g'$, then $g'' \cdot v = v$, so g'' both stabilizes q and fixes v.

Lemma 4.12 below, which is the key to the proof, shows that any element that stabilizes q and fixes v is in the group generated by E_7 and μ_4; thus g'' is in that group and so is g. Therefore $\mathrm{Stab}(q) \subseteq \langle E_7, \mu_4 \rangle$. $\square$

Before completing the proof, we use the preceding theorem to determine the group that stabilizes both q and the bilinear form $\langle -, - \rangle$.

Corollary 4.7. *For $G = E_8$, the subgroup of $\mathrm{GL}(\mathfrak{g}_1)$ stabilizing both the quartic form and the skew-symmetric bilinear form, $\mathrm{Stab}(q, \langle -, - \rangle)$, is E_7.*

Proof. The previous proposition and the fact that E_7 stabilizes both forms yield the following containments:

$$E_7 \subseteq \mathrm{Stab}(q, \langle -, - \rangle) \subseteq \mathrm{Stab}(q) = \langle E_7, \mu_4 \rangle.$$

Let L_0 be the root lattice of E_7 and L_1 its weight lattice. Then L_1/L_0 is a group with two elements (see, for example, [17], §13.1 or [27], p. 45). From [27], p. 45, the center of E_7 is isomorphic to $\mathrm{Hom}(L_1/L_0, F^\times)$, so the center of E_7 consists of the elements 1 and -1. Thus the group $\langle E_7, \mu_4 \rangle$ has two components: E_7 and iE_7, where i is a primitive fourth root of unity. However, i is not in $\mathrm{Stab}(q, \langle -, - \rangle)$ since $\langle ix, iy \rangle = -\langle x, y \rangle$ for any $x, y \in \mathfrak{g}_1$. Therefore $\mathrm{Stab}(q, \langle -, - \rangle) = E_7$. $\qquad\qquad\square$

In the remainder of this section we complete the proof of the two preceding propositions by using a result of Springer ([26]) on so-called E_6-structures. An E_6-*structure* consists of two 27-dimensional F-vector spaces, A and B, with a nondegenerate pairing $\langle -, - \rangle$ along with two irreducible cubic forms, $f_1 \colon A \to F$ and $f_2 \colon B \to F$. After defining corresponding trilinear forms by $f_1(a, a, a) = f_1(a)$ and $f_2(b, b, b) = f_2(b)$, we can also define symmetric bilinear products maps $A \times A \to B$ and $B \times B \to A$, each denoted by juxtaposition, that are given implicitly by

$$3f_1(a, a_1, a_2) = \langle a, a_1 a_2 \rangle, \qquad 3f_2(b, b_1, b_2) = \langle b_1 b_2, b \rangle;$$

Finally, these maps must satisfy the following conditions:

$$(aa)(aa) = f_1(a)a, \qquad (bb)(bb) = f_2(b)b \qquad\qquad (4.8)$$

for all $a \in A$, $b \in B$.

We will take A and B to be the eigenspaces in $\mathfrak{g}_1$ described in Proposition 4.1 corresponding to the eigenvalues $+1$ and -1. Thus A is generated by elements x_β where β has α-height 1 and $\langle \alpha, \beta \rangle = 0$, whereas B is generated by elements x_γ where γ has α-height 1 and $\langle \alpha, \gamma \rangle = 1$. When $\mathfrak{g} = E_8$,

$\mathfrak{g}_1$ has dimension 56, so A and B are 27-dimensional, as required. However, none of our results before Lemma 4.12 make use of the dimension, so they apply equally well if $\mathfrak{g}$ is any Lie algebra of type D or E; in particular, we will apply Lemma 4.11 with $\mathfrak{g} = D_4$ in the next section. For the pairing on A and B, we use the skew-symmetric bilinear form $\langle -, - \rangle$ previously defined on all of $\mathfrak{g}_1$.

We define the cubic forms f_1 and f_2 as follows:

$$f_1(a) = \frac{1}{6}q(x_\alpha, a, a, a), \qquad f_2(b) = \frac{1}{6}q(x_{\rho-\alpha}, b, b, b).$$

To verify that we have an E_6-structure, we begin by computing the bilinear products on A and B. For convenience, we will assume without loss of generality that the structure constant $c_{\rho-\alpha,\alpha}$ is 1.

Lemma 4.9. *For basis elements $x_\beta, x_\gamma \in A$, if β and γ are orthogonal the bilinear product is given by $x_\beta x_\gamma = k_{\beta,\gamma} x_{\rho-\delta}$, where $\delta = 2\rho - \alpha - \beta - \gamma$ and $k_{\beta,\gamma} = \frac{1}{2}c_{\delta,\rho-\delta}q(x_\alpha, x_\beta, x_\gamma, x_\delta)$. Similarly, for basis elements $x_\beta, x_\gamma \in B$ with β and γ orthogonal, the bilinear product is given by $x_\beta x_\gamma = k'_{\beta,\gamma} x_{\rho-\delta'}$, where $\delta' = \rho + \alpha - \beta - \gamma$ and $k'_{\beta,\gamma} = \frac{1}{2}c_{\delta',\rho-\delta'}q(x_{\rho-\alpha}, x_\beta, x_\gamma, x_{\delta'})$. In both cases, if β and γ are not orthogonal, the bilinear product is zero.*

Proof. We first check that the specified δ and δ' are actually roots. In the first case, β and γ are orthogonal to each other and also to α, so we have $\langle \rho - \alpha, \beta \rangle = \langle \rho, \beta \rangle - \langle \alpha, \beta \rangle = 1$, so $\rho - \alpha - \beta$ is a root, and $\langle \rho - \alpha - \beta, \rho - \gamma \rangle = \langle \rho - \beta, \rho - \gamma \rangle - \langle \alpha, \rho - \gamma \rangle = -1$, so $\delta = 2\rho - \alpha - \beta - \gamma$ is also a root. In the second case, we now have that $\rho - \alpha$, β, γ are mutually orthogonal, so by Fact 2.8 the same is true of α, $\rho - \beta$, $\rho - \gamma$. The work for the first case thus shows that $2\rho - \alpha - (\rho - \beta) - (\rho - \gamma) = \beta + \gamma - \alpha$ is a root; since it has α-height 1, it follows that $\delta' = \rho - (\beta + \gamma - \alpha)$ is also a root. Note that δ and δ' have α-height 1 and that $\langle \delta, \alpha \rangle = 0$ and $\langle \delta', \alpha \rangle = 1$; thus $x_{\rho-\delta}$ is indeed in B and $x_{\rho-\delta'}$ is in A.

By the definitions of the bilinear product and of the cubic form f_1, we have in the first case

$$\langle a, x_\beta x_\gamma \rangle = 3f_1(a, x_\beta, x_\gamma) = \frac{1}{2}q(x_\alpha, x_\beta, x_\gamma, a)$$

for any $a \in A$. By Corollary 3.3, the only basis element for which the right-hand side is not zero is $a = x_\delta$; the same must be true of the left-hand side, so $x_\beta x_\gamma$ is a scalar multiple of $x_{\rho-\delta}$. Setting $a = x_\delta$ and $x_\beta x_\gamma = k x_{\rho-\delta}$, we find $k = \frac{1}{2}c_{\delta,\rho-\delta}q(x_\alpha, x_\beta, x_\gamma, x_\delta)$.

In the second case, we have

$$\langle x_\beta x_\gamma, b \rangle = 3f_2(x_\beta, x_\gamma, b) = \frac{1}{2}q(x_{\rho-\alpha}, x_\beta, x_\gamma, b)$$

for any $b \in B$. The only basis element for which the right-hand side is not zero is $b = x_{\delta'}$; thus $x_\beta x_\gamma$ is a scalar multiple of $x_{\rho-\delta'}$. We find $x_\beta x_\gamma = k' x_{\rho-\delta'}$, with $k' = \frac{1}{2}c_{\delta',\rho-\delta'}q(x_{\rho-\alpha}, x_\beta, x_\gamma, x_{\delta'})$.

When x_β and x_γ are in A, the bilinear product can be nonzero only if there is a root δ such that $\alpha + \beta + \gamma + \delta = 2\rho$. If β and γ are not orthogonal, then Proposition 3.30 implies that some two of α, β and γ must sum to ρ. However, since β and γ are each orthogonal to α, neither $\alpha + \beta$ nor $\alpha + \gamma$ is a root. Also, $\beta + \gamma \neq \rho$, since $\langle \beta + \gamma, \alpha \rangle = 0$ but $\langle \rho, \alpha \rangle = 1$. Thus $x_\beta x_\gamma$ is zero if β and γ are not orthogonal.

Likewise, if x_β and x_γ are in B, then the bilinear product can be nonzero only if there is a root δ' such that $(\rho - \alpha) + \beta + \gamma + \delta' = 2\rho$. Again, if β and γ are not orthogonal, some two of $\rho - \alpha$, β and γ must sum to ρ. In this case, $\rho - \alpha$ is orthogonal to each of β and γ, so neither $(\rho - \alpha) + \beta$ nor $(\rho - \alpha) + \gamma$ is a root. Finally, $\beta + \gamma \neq \rho$, since $\langle \beta + \gamma, \rho - \alpha \rangle = 0$ but $\langle \rho, \rho - \alpha \rangle = 1$. $\square$

Lemma 4.10. *With the preceding definitions, $(A, B, \langle -, - \rangle, f_1, f_2)$ is an E_6-structure.*

Proof. We need to show that equations (4.8) are satisfied. First, we show that any element $a \in A$ is not in orbit 4. Since the elements of the form x_β

with β having α-height 1 and $\langle \alpha, \beta \rangle = 0$ are a basis for A, we may write a as an F-linear combination of such elements, say $a = \sum \lambda_i x_{\beta_i}$. Thus $q(a)$ is a sum of terms of the form $\lambda q(x_\beta, x_\gamma, x_\delta, x_\epsilon)$ with $\lambda \in F$ and where $\beta, \gamma, \delta, \epsilon$ are roots, not necessarily distinct, of α-height 1 that are orthogonal to α. However, the sum $\beta + \gamma + \delta + \epsilon$ cannot equal 2ρ since $\langle \beta + \gamma + \delta + \epsilon, \alpha \rangle = 0$, but $\langle 2\rho, \alpha \rangle = 2$. Hence all such terms are zero by Corollary 3.3, so we have $q(a) = 0$. By Proposition 4.2, a is thus not in orbit 4, so it is in the closure of orbit 3.

The equation $(aa)(aa) = f_1(a)a$ is preserved by scaling and all the operations being used are defined in terms of the forms that are stabilized by $(G_0)^{\mathrm{ss}}$, so to verify the equation on A it will suffice to choose a to be any representative of orbit 3. Choose β, γ, δ so that $\alpha, \beta, \gamma, \delta$ are mutually orthogonal; by Fact 2.9, $\alpha + \beta + \gamma + \delta = 2\rho$. Now $a = x_\beta + x_\gamma + x_\delta$ is a representative of orbit 3. Using the notation of the preceding lemma, we compute

$$aa = 2x_\beta x_\gamma + 2x_\gamma x_\delta + 2x_\delta x_\beta$$
$$= 2k_{\beta,\gamma} x_{\rho-\delta} + 2k_{\gamma,\delta} x_{\rho-\beta} + 2k_{\delta,\beta} x_{\rho-\gamma},$$

and thus

$$(aa)(aa) = 8k_{\beta,\gamma}k_{\gamma,\delta} x_{\rho-\delta} x_{\rho-\beta} + 8k_{\gamma,\delta}k_{\delta,\beta} x_{\rho-\beta} x_{\rho-\gamma} + 8k_{\delta,\beta}k_{\beta,\gamma} x_{\rho-\gamma} x_{\rho-\delta}$$
$$= 8k_{\beta,\gamma}k_{\gamma,\delta}k'_{\rho-\delta,\rho-\beta} x_\gamma + 8k_{\gamma,\delta}k_{\delta,\beta}k'_{\rho-\beta,\rho-\gamma} x_\delta + 8k_{\delta,\beta}k_{\beta,\gamma}k'_{\rho-\gamma,\rho-\delta} x_\beta.$$

Applying the definitions of the scalars k and k', we find that the coefficient of each term is $-c_{\beta,\rho-\beta}c_{\gamma,\rho-\gamma}c_{\delta,\rho-\delta}q_1^2 q_2$, where $q_1 = q(x_\alpha, x_\beta, x_\gamma, x_\delta)$ and $q_2 = q(x_{\rho-\alpha}, x_{\rho-\beta}, x_{\rho-\gamma}, x_{\rho-\delta})$. Thus $(aa)(aa)$ is proportional to a; it remains to show that the coefficient is equal to $f_1(a)$.

By definition, $f_1(a) = \frac{1}{6}q(x_\alpha, a, a, a)$. Since $a = x_\beta + x_\gamma + x_\delta$, the expansion of the 4-linear form includes six terms equal to $q(x_\alpha, x_\beta, x_\gamma, x_\delta)$; all other terms include a repeated argument and thus are seen to be zero by applying

Lemma 3.19. Hence $f_1(a) = q(x_\alpha, x_\beta, x_\gamma, x_\delta)$; that is, $f_1(a) = q_1$. Thus it will suffice to show that $-c_{\beta,\rho-\beta}c_{\gamma,\rho-\gamma}c_{\delta,\rho-\delta}q_1q_2 = 1$.

By Proposition 3.35, we have

$$q_1 = c_{\alpha,\delta-\rho}c_{\beta,\alpha-\rho}c_{\gamma,\delta-\rho}c_{\delta,\alpha-\rho}$$

and

$$q_2 = c_{\rho-\alpha,-\delta}c_{\rho-\beta,-\alpha}c_{\rho-\gamma,-\delta}c_{\rho-\delta,-\alpha}.$$

Some of the factors are equal, and so will cancel in the product:

$$c_{\alpha,\delta-\rho} = c_{\rho-\delta,-\alpha},$$
$$c_{\delta,\alpha-\rho} = c_{\rho-\alpha,-\delta}.$$

The product $-c_{\beta,\rho-\beta}c_{\gamma,\rho-\gamma}c_{\delta,\rho-\delta}q_1q_2$ is thus

$$-c_{\beta,\rho-\beta}c_{\gamma,\rho-\gamma}c_{\delta,\rho-\delta}c_{\beta,\alpha-\rho}c_{\gamma,\delta-\rho}c_{\rho-\beta,-\alpha}c_{\rho-\gamma,-\delta}$$

Introducing two factors of $c_{\alpha,\rho-\alpha}$ and substituting $c_{\rho-\beta,-\alpha} = c_{\alpha,\beta-\rho}$ and $c_{\rho-\gamma,-\delta} = c_{\delta,\gamma-\rho}$, we regroup the resulting nine factors as

$$-c_{\alpha,\rho-\alpha}(c_{\alpha,\rho-\alpha}c_{\beta,\rho-\beta}c_{\beta,\alpha-\rho}c_{\alpha,\beta-\rho})(c_{\gamma,\rho-\gamma}c_{\delta,\rho-\delta}c_{\gamma,\delta-\rho}c_{\delta,\gamma-\rho}).$$

Applying Lemma 3.6 to the orthogonal roots α and β (resp. γ and δ) yields $c_{\alpha,\beta-\rho}c_{\beta,\alpha-\rho}c_{\alpha,-\rho}c_{\beta,-\rho} = 1$ (resp. $c_{\gamma,\delta-\rho}c_{\delta,\gamma-\rho}c_{\gamma,-\rho}c_{\delta,-\rho} = 1$). These are the same as the parenthesized products, so the complete product is $-c_{\alpha,\rho-\alpha}$. Since we are assuming $c_{\rho-\alpha,\alpha} = 1$, this is 1, as required.

For an element $b \in B$, the result $(bb)(bb) = f_2(b)b$ follows in similar fashion. Specifically, we take $b = x_{\rho-\beta} + x_{\rho-\gamma} + x_{\rho-\delta}$ as a representative of orbit 3 and compute

$$bb = 2x_{\rho-\beta}x_{\rho-\gamma} + 2x_{\rho-\gamma}x_{\rho-\delta} + 2x_{\rho-\delta}x_{\rho-\beta}$$
$$= 2k'_{\rho-\beta,\rho-\gamma}x_\delta + 2k'_{\rho-\gamma,\rho-\delta}x_\beta + 2k'_{\rho-\delta,\rho-\beta}x_\gamma$$

and thus

$$
\begin{aligned}
(bb)(bb) &= 8k'_{\rho-\beta,\rho-\gamma}k'_{\rho-\gamma,\rho-\delta}x_\delta x_\beta + 8k'_{\rho-\gamma,\rho-\delta}k'_{\rho-\delta,\rho-\beta}x_\beta x_\gamma \\
&\quad + 8k'_{\rho-\delta,\rho-\beta}k'_{\rho-\beta,\rho-\gamma}x_\gamma x_\delta \\
&= 8k'_{\rho-\beta,\rho-\gamma}k'_{\rho-\gamma,\rho-\delta}k_{\delta,\beta}x_{\rho-\gamma} + 8k'_{\rho-\gamma,\rho-\delta}k'_{\rho-\delta,\rho-\beta}k_{\beta,\gamma}x_{\rho-\delta} \\
&\quad + 8k'_{\rho-\delta,\rho-\beta}k'_{\rho-\beta,\rho-\gamma}k_{\gamma,\delta}x_{\rho-\beta}.
\end{aligned}
$$

Once again, the coefficients are equal; each is $-c_{\beta,\rho-\beta}c_{\gamma,\rho-\gamma}c_{\delta,\rho-\delta}q_1 q_2^2$, where q_1 and q_2 are as before. This must be the same as $f_2(b) = \frac{1}{6}q(x_{\rho-\alpha}, b, b, b)$; here the resulting nonzero terms are the six equal to $q(x_{\rho-\alpha}, x_{\rho-\beta}, x_{\rho-\gamma}, x_{\rho-\delta})$, so $f_2(b) = q_2$. Hence it suffices to show $-c_{\beta,\rho-\beta}c_{\gamma,\rho-\gamma}c_{\delta,\rho-\delta}q_1 q_2 = 1$, exactly as in the previous case. $\qquad\square$

The following lemma allows us to move from an element that stabilizes the quartic form and fixes v to one that preserves even more structure. It will be used again in the next section.

Lemma 4.11. *If $g \in \mathrm{GL}(\mathfrak{g}_1)$ is an element that stabilizes the quartic form and fixes $v = x_\alpha + x_{\rho-\alpha}$, then there is an element g' that preserves the spaces A and B and stabilizes $\langle -, - \rangle$ and the cubic forms defined on A and B such that $g'g^{-1} \in \langle (G_0)^{ss}, \mu_4 \rangle$.*

Proof. Let g be an element that stabilizes q and fixes v. By Lemma 4.3, the action of g takes strictly regular elements to strictly regular elements, so $g \cdot x_\alpha$ and $g \cdot x_{\rho-\alpha}$ are strictly regular. Since g fixes v, we have $v = g \cdot v = g \cdot x_\alpha + g \cdot x_{\rho-\alpha}$. However, by Lemma 3.23, the expression of v as a sum of two strictly regular elements is unique, so g must either fix both x_α and $x_{\rho-\alpha}$ or interchange them. By §12.10 in [15], there is an element $z \in i(G_0)^{ss}$ that interchanges x_α and $x_{\rho-\alpha}$; of course, such an element also stabilizes q. Hence either g or zg is an element that stabilizes q and fixes x_α and $x_{\rho-\alpha}$; call whichever element does so g'. Thus we have $g'g^{-1} \in \langle (G_0)^{ss}, \mu_4 \rangle$.

Let W be the subspace of $\mathfrak{g}_1$ consisting of elements orthogonal to both x_α and $x_{\rho-\alpha}$; by Corollary 4.5, W is invariant under g'. All the basis elements x_β with β of α-height 1 except for x_α and $x_{\rho-\alpha}$ are in W, and they form a basis of W. Thus W is the direct sum of the $+1$ and -1 eigenspaces of Proposition 4.1, the spaces we have named A and B.

Let A' be the subspace of elements $x \in W$ such that $q(x_{\rho-\alpha}, x, y, z) = 0$ for all $y, z \in W$, and define a cubic form on A' by $\frac{1}{6}q(x_\alpha, x, x, x)$. Clearly g' preserves A' and stabilizes the cubic form. We claim A' is in fact A, the $+1$ eigenspace of Proposition 4.1.

On the one hand, if x_β is a basis element of the $+1$ eigenspace, then we have $\langle \rho - 2\alpha, \beta \rangle = 1$. Since $\langle \rho, \beta \rangle = 1$, it follows that $\langle \alpha, \beta \rangle = 0$. By writing elements $y, z \in W$ as linear combinations of the basis elements, $q(x_{\rho-\alpha}, x_\beta, y, z)$ expands into a linear combination of terms of the form $q(x_{\rho-\alpha}, x_\beta, x_\gamma, x_\delta)$ with γ, δ such that $\langle \gamma, \alpha \rangle$ and $\langle \delta, \alpha \rangle$ are each either 0 or 1. But then we cannot have $(\rho - \alpha) + \beta + \gamma + \delta = 2\rho$, since $\langle (\rho - \alpha) + \beta + \gamma + \delta, \alpha \rangle = -1 + 0 + \langle \gamma, \alpha \rangle + \langle \delta, \alpha \rangle$ is at most 1, but $\langle 2\rho, \alpha \rangle = 2$. Hence all the terms $q(x_{\rho-\alpha}, x_\beta, x_\gamma, x_\delta)$ are zero, so x_β is in A'. Thus $A \subseteq A'$.

Conversely, if $x \in W$ is not in the $+1$ eigenspace, then it has a nonzero component involving some basis element x_β with $\langle \beta, \alpha \rangle = 1$. Thus $\langle \rho - \alpha, \beta \rangle = 0$, so $\rho - \alpha$ and β are orthogonal roots of α-height 1. It follows from Lemma 2.4 in [24] that any such pair of roots can be extended to a set of four mutually orthogonal roots, say $\rho - \alpha, \beta, \gamma, \delta$. By Lemma 3.12, $q(x_{\rho-\alpha}, x_\beta, x_\gamma, x_\delta)$ is then nonzero, and thus $q(x_{\rho-\alpha}, x, x_\gamma, x_\delta)$ is also nonzero, since no other component of x contributes to the value of the form. Thus x is not in A'. Therefore $A' \subseteq A$.

Interchanging the roles of x_α and $x_{\rho-\alpha}$, we similarly define B' to be the subspace of elements $x \in W$ such that $q(x_\alpha, x, y, z) = 0$ for all $y, z \in W$, and define a cubic form on B' by $\frac{1}{6}q(x_{\rho-\alpha}, x, x, x)$. As before, g' preserves B' and stabilizes the cubic form. We show that B' is actually B, the -1 eigenspace,

by an argument parallel to that for A' and A.

On the one hand, if x_β is a basis element of the -1 eigenspace, then we have $\langle \rho - 2\alpha, \beta \rangle = -1$ and thus $\langle \alpha, \beta \rangle = 1$. Writing elements $y, z \in W$ as linear combinations of the basis elements, $q(x_\alpha, x_\beta, y, z)$ expands into terms of the form $q(x_\alpha, x_\beta, x_\gamma, x_\delta)$ with γ, δ such that $\langle \gamma, \alpha \rangle$ and $\langle \delta, \alpha \rangle$ are each either 0 or 1. But then $\alpha + \beta + \gamma + \delta$ is not 2ρ, since $\langle \alpha + \beta + \gamma + \delta, \alpha \rangle = 2 + 1 + \langle \gamma, \alpha \rangle + \langle \delta, \alpha \rangle$ is at least 3, but $\langle 2\rho, \alpha \rangle = 2$. Hence all the terms $q(x_\alpha, x_\beta, x_\gamma, x_\delta)$ are zero, so x_β is in B'. Thus $B \subseteq B'$.

Conversely, if $x \in W$ is not in the -1 eigenspace, then it has a nonzero component involving some basis element x_β with $\langle \beta, \alpha \rangle = 0$. The pair of orthogonal roots α and β can be extended to a set of four mutually orthogonal roots, say $\alpha, \beta, \gamma, \delta$. Then $q(x_\alpha, x_\beta, x_\gamma, x_\delta)$ is nonzero, so $q(x_\alpha, x, x_\gamma, x_\delta)$ is also nonzero, since no other component of x contributes to the value of the form. Thus x is not in B'. Therefore $B' \subseteq B$.

As in the proof of Corollary 4.5, since g' stabilizes the quartic form, it preserves the bilinear form up to a scalar factor of ± 1. However, since g' fixes x_α and $x_{\rho - \alpha}$ and $\langle x_\alpha, x_{\rho - \alpha} \rangle \neq 0$, the scalar factor is 1; thus g' preserves $\langle -, - \rangle$. $\qquad \square$

Lemma 4.12. *The group that stabilizes the quartic form and fixes the element $v = x_\alpha + x_{\rho - \alpha}$ is contained in the group generated by E_7 and μ_4.*

Proof. As the reader will have anticipated, we are going to apply Lemma 4.10, which says that A and B equipped with $\langle -, - \rangle$ and the cubic forms defined above is an E_6-structure. By Lemma 4.11, if g is an element that stabilizes q and fixes v, there is a $g' \in g\langle E_7, \mu_4 \rangle$ such that A and B are invariant under g' and $\langle -, - \rangle$ and the cubic forms are stabilized by g'. In other words, g' is an automorphism of the E_6-structure. Proposition 1.6 of [26] shows that the automorphism group of an E_6-structure is, as the name suggests, E_6. Thus $g' \in E_6$; and therefore g is in $\langle E_7, \mu_4 \rangle$. $\qquad \square$

4.5 The stabilizer of the quartic form: $G = D_4$

In this section, we again consider the group stabilizing the quartic form and the group stabilizing both the quartic and the bilinear forms on $\mathfrak{g}_1$, this time in the case $G = D_4$.

Although we will not make use of it in the proof, we present a convenient matrix representation of the Lie algebra D_4, derived from the representation of the group D_4 given in [22], §5.IV. This will serve to provide a concrete example of a Freudenthal triple system embedded in a Lie algebra in a form conducive to making explicit calculations.

D_4 may be represented by the set of 8×8 matrices $A = (a_{ij})$ such that A is negated when it is reflected in the anti-diagonal; i.e., $a_{ij} = -a_{9-j,9-i}$. This condition may also be expressed as $JA^tJ = -A$, where J is the matrix (sometimes called the *exchange matrix*) with ones on the anti-diagonal and zeros elsewhere. The diagonal matrices in this set form a 4-dimensional commuting subalgebra which we take as the Cartan subalgebra $\mathfrak{h}$. Writing e_{ij} for the unit matrix with a 1 in the i,j-entry, the 24 root subspaces are represented by matrices of the form $e_{ij} - e_{9-j,9-i}$ for $i \neq j$ and $i + j \leq 8$; we take those with $i < j$ (that is, with entries above the main diagonal) to correspond to the positive roots. The representatives of the corresponding negative roots are obtained by interchanging i and j. This set of representatives is extended to form a Chevalley basis by setting $h_i = [x_{\alpha_i}, x_{-\alpha_i}]$ for $1 \leq i \leq 4$ (note that these are not the obvious basis elements of $\mathfrak{h}$ of the form $e_{ii} - e_{9-i,9-i}$). The root subspaces for the simple roots are then represented by $x_{\alpha_1} = e_{12} - e_{78}$, $x_\alpha = x_{\alpha_2} = e_{23} - e_{67}$, $x_{\alpha_3} = e_{34} - e_{56}$ and $x_{\alpha_4} = e_{35} - e_{46}$; here the numbering of α_1, α_3 and α_4 is arbitrary because of the symmetry of the Dynkin diagram of D_4.

This representation of D_4 is shown schematically in Figure 4.1. The eight root subspaces making up $\mathfrak{g}_1$ appear in the rectangle which has x_α and $x_{\rho - \alpha}$

at opposite corners.

$$
\begin{pmatrix}
* & x_{\alpha_1} & \cdot & \cdot & \cdot & x_{\rho-\alpha} & x_\rho & 0 \\
 & * & x_\alpha & \cdot & \cdot & \cdot & 0 & -x_\rho \\
 & & * & x_{\alpha_3} & x_{\alpha_4} & 0 & \cdot & -x_{\rho-\alpha} \\
 & & & * & 0 & -x_{\alpha_4} & \cdot & \cdot \\
 & & & 0 & -* & -x_{\alpha_3} & \cdot & \cdot \\
 & & 0 & & & -* & -x_\alpha & \cdot \\
 & 0 & & & & & -* & -x_{\alpha_1} \\
0 & & & & & & & -*
\end{pmatrix}
$$

Figure 4.1: A matrix representation for the Lie algebra D_4

From another viewpoint, the diagram that results when $\alpha = \alpha_2$ is removed from the Dynkin diagram of D_4 consists of three unconnected vertices; that is, it represents the Lie algebra which is the product of three copies of $\mathfrak{sl}_2$. Thus $\mathfrak{g}_0$ is 10-dimensional, generated by the three pairs of roots $x_{\alpha_i}, x_{-\alpha_i}$ for $i = 1, 3, 4$ and the four-dimensional Cartan subalgebra of D_4; $(G_0)^{\mathrm{ss}}$ is thus SL_2^3. Since D_4 has dimension 28, there are 18 other roots; setting aside ρ and $-\rho$, we again see that $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are eight-dimensional. Here is a list of the roots β of α-height 1, sorted according to eigenspace decomposition of Proposition 4.1:

β	$\langle \rho - 2\alpha, \beta \rangle$	$\langle \alpha, \beta \rangle$
$\rho - \alpha$	3	-1
$\alpha + \alpha_1 + \alpha_3, \alpha + \alpha_1 + \alpha_4, \alpha + \alpha_3 + \alpha_4$	1	0
$\alpha + \alpha_1, \alpha + \alpha_2, \alpha + \alpha_3$	-1	1
α	-3	2

To establish the stabilizer of the quartic form, we follow a similar strategy to that employed in the proof of Theorem 4.6: We define the spaces A and B and cubic forms on them as in the previous section. We adjust an element $g \in \mathrm{GL}(\mathfrak{g}_1)$ that stabilizes the quartic form to obtain an element that also fixes $x_\alpha + x_{\rho-\alpha}$, then apply Lemma 4.11 to obtain a g' that preserves the spaces A and B and stabilizes the cubic forms on them. In this case A and B are simple enough so that we can give the cubic forms explicitly and determine a suitable subgroup of $\mathrm{GL}(\mathfrak{g}_1)$ that contains g'.

Theorem 4.13. *The stabilizer of the quartic form on $\mathfrak{g}_1$ when $G = D_4$ is $\langle \mathrm{SL}_2^3, \mu_4 \rangle \rtimes S_3$, where S_3 is the symmetric group corresponding to the diagram automorphisms of D_4.*

Proof. Since $(G_0)^{\mathrm{ss}} = \mathrm{SL}_2^3$ and μ_4 both stabilize the quartic form, $\langle \mathrm{SL}_2^3, \mu_4 \rangle$ is in $\mathrm{Stab}(q)$. We will now show that the diagram automorphisms also stabilize the quartic form.

It will suffice to show that a diagram automorphism fixes x_ρ and $x_{-\rho}$. By Corollaire 5.5 bis in [10], an outer automorphism of $\mathfrak{g}$ may be taken to act on the Chevalley basis elements x_{α_i} corresponding to the simple roots by permuting the subscripts, and to act on the elements $h_i = [x_{\alpha_i}, x_{-\alpha_i}]$ by applying the same permutation to the subscripts; thus the elements $x_{-\alpha_i}$ are also permuted in the same way. We will write x_ρ in terms of the x_{α_i}, and show that this expression is unaltered by a permutation of the subscripts 1, 3 and 4; the same argument with the negatives of the roots will show that $x_{-\rho}$ is fixed as well.

The highest root of D_4 is $\rho = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. We write this as $\rho = \alpha_2 + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_2$; in this expression each partial sum is also a root. Thus we have

$$x_\rho = c[x_{\alpha_2}, [x_{\alpha_4}, [x_{\alpha_3}, [x_{\alpha_1}, x_{\alpha_2}]]]], \tag{4.14}$$

where c is a constant (in fact, $c = \pm 1$ since all the roots are long and thus the

structure constants are ± 1). Our claim is that this expression is unaltered when the factors x_{α_1}, x_{α_3}, x_{α_4} are permuted.

To verify the claim for the permutation that interchanges 1 and 3, we must show that

$$[x_{\alpha_3}, [x_{\alpha_1}, x_{\alpha_2}]] = [x_{\alpha_1}, [x_{\alpha_3}, x_{\alpha_2}]];$$

this is equivalent to the following structure constant equation:

$$c_{\alpha_1,\alpha_2} c_{\alpha_3,\alpha_1+\alpha_2} = c_{\alpha_3,\alpha_2} c_{\alpha_1,\alpha_2+\alpha_3}. \tag{4.15}$$

To obtain (4.15), we apply (2.6) with $\beta = \alpha_1 + \alpha_2$, $\gamma = \alpha_2 + \alpha_3$, $\delta = -\alpha_2$ and $\epsilon = -\alpha_1 - \alpha_2 - \alpha_3$; this yields

$$c_{\alpha_1+\alpha_2,\alpha_2+\alpha_3} c_{-\alpha_2,-\alpha_1-\alpha_2-\alpha_3} +$$

$$c_{\alpha_2+\alpha_3,-\alpha_2} c_{\alpha_1+\alpha_2,-\alpha_1-\alpha_2-\alpha_3} + c_{-\alpha_2,\alpha_1+\alpha_2} c_{\alpha_2+\alpha_3,-\alpha_1-\alpha_2-\alpha_3} = 0.$$

The sum $\alpha_1 + 2\alpha_2 + \alpha_3$ has α-height 2 but is not equal to ρ, so it is not a root; thus the first term is zero. Applying Fact 2.5, we have

$$c_{\alpha_2+\alpha_3,-\alpha_2} = c_{-\alpha_2,-\alpha_3}$$

$$= c_{\alpha_3,\alpha_2},$$

$$c_{\alpha_1+\alpha_2,-\alpha_1-\alpha_2-\alpha_3} = c_{\alpha_3,\alpha_1+\alpha_2},$$

$$c_{-\alpha_2,\alpha_1+\alpha_2} = c_{-\alpha_1,-\alpha_2}$$

$$= -c_{\alpha_1,\alpha_2},$$

$$c_{\alpha_2+\alpha_3,-\alpha_1-\alpha_2-\alpha_3} = c_{\alpha_1,\alpha_2+\alpha_3}.$$

Thus we have $c_{\alpha_3,\alpha_2} c_{\alpha_3,\alpha_1+\alpha_2} - c_{\alpha_1,\alpha_2} c_{\alpha_1,\alpha_2+\alpha_3} = 0$. Since all the structure constants involved are ± 1, this is equivalent to the statement that their product is 1; this in turn is equivalent to (4.15).

By permuting the roots in the expression for ρ, the same argument applies to any transposition of two of the subscripts 1, 3 and 4. Since all the transpositions fix x_ρ and $x_{-\rho}$, all the diagram automorphisms do. Thus $\langle \mathrm{SL}_2^3, \mu_4 \rangle \rtimes S_3$ is contained in the stabilizer of the quartic form.

We now consider the reverse inclusion. Let $v = x_\alpha + x_{\rho-\alpha}$. As in the proof of Theorem 4.6, given some $g \in \mathrm{GL}(\mathfrak{g}_1)$ which stabilizes q, there exists some $z \in (G_0)^{\mathrm{ss}}$ such that $zg \cdot v$ is a scalar multiple of v, and there is some $k \in \mu_4$ such that $g'' = kzg$ fixes v and still stabilizes q.

Applying Lemma 4.11 to g'', we obtain an element g' that preserves A and B and stabilizes $\langle -, - \rangle$ and the cubic forms on A and B.

By definition, the subspace A is generated by the root subspaces corresponding to roots orthogonal to α; examining the list of roots in $\mathfrak{g}_1$, these are $\beta = \alpha + \alpha_1 + \alpha_3$, $\gamma = \alpha + \alpha_1 + \alpha_4$ and $\delta = \alpha + \alpha_3 + \alpha_4$. Either by checking orthogonality directly or by using the formulas given in Section 2.5, we find that α, β, γ and δ are mutually orthogonal. For an arbitrary element $x = \lambda_1 x_\beta + \lambda_2 x_\gamma + \lambda_3 x_\delta$ of A, we find that the cubic form is

$$\frac{1}{6} q(x_\alpha, x, x, x) = \lambda_1 \lambda_2 \lambda_3 q(x_\alpha, x_\beta, x_\gamma, x_\delta),$$

since the terms with a repeated argument are zero by Lemma 3.19. By Proposition 3.35, this is $\epsilon \lambda_1 \lambda_2 \lambda_3$, where $\epsilon = \pm 1$ is a product of structure constants.

Let $T = (a_{ij})$, $1 \leq i, j \leq 3$, be the matrix of the linear transformation on A given by $x \mapsto g' \cdot x$ with respect to the basis x_β, x_γ, x_δ. The value of the cubic form is the same for $x = \lambda_1 x_\beta + \lambda_2 x_\gamma + \lambda_3 x_\delta$ and $g' \cdot x$, so we have

$$\lambda_1 \lambda_2 \lambda_3 = (a_{11}\lambda_1 + a_{12}\lambda_2 + a_{13}\lambda_3)(a_{21}\lambda_1 + a_{22}\lambda_2 + a_{23}\lambda_3)(a_{31}\lambda_1 + a_{32}\lambda_2 + a_{33}\lambda_3)$$

for all $\lambda_1, \lambda_2, \lambda_3 \in F$. By unique factorization in $F[\lambda_1, \lambda_2, \lambda_3]$, the three factors on the right-hand side are (up to units) $\lambda_1, \lambda_2, \lambda_3$, say $c_1\lambda_1, c_2\lambda_2, c_3\lambda_3$, with $c_1 c_2 c_3 = 1$. If the factors occur in that order, then T is diagonal, with the third entry determined by the first two; each such T corresponds to an element (c_1, c_2, c_3) of $\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m$ for which the product of the three components is 1. However, the order of the factors may be different, so in general T may be an element of $(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m) \rtimes S_3$.

The subspace B is generated by the root subspaces corresponding to the roots $\rho - \beta = \alpha + \alpha_4$, $\rho - \gamma = \alpha + \alpha_3$ and $\rho - \delta = \alpha + \alpha_1$. As $\alpha, \beta, \gamma, \delta$ are mutually orthogonal, so are $\rho - \alpha, \rho - \beta, \rho - \gamma, \rho - \delta$. The cubic form on B is given by $\frac{1}{6}q(x_{\rho-\alpha}, x, x, x)$; for $x = \lambda_1 x_{\rho-\beta} + \lambda_2 x_{\rho-\gamma} + \lambda_3 x_{\rho-\delta}$ this is, as in the previous case, $\pm \lambda_1 \lambda_2 \lambda_3$. As before, g' must map $x_{\rho-\beta}$, $x_{\rho-\gamma}$ and $x_{\rho-\delta}$ to scalar multiples of the same basis elements, possibly permuted.

However, since g' stabilizes $\langle -, - \rangle$, the action of g' on B can be computed given its action on A. Suppose, for example, that g' maps x_β to cx_γ in A, then $\langle x_\beta, x_{\rho-\beta} \rangle = \langle cx_\gamma, g' \cdot x_{\rho-\beta} \rangle$; since this must be $c_{\beta,\rho-\beta}$, we have that $g' \cdot x_{\rho-\beta}$ is necessarily $c_{\beta,\rho-\beta} c_{\gamma,\rho-\gamma} c^{-1} x_{\rho-\gamma}$. In general, β and γ may be replaced by any of β, γ or δ, with a similar result. Hence the action of g' on B is determined by its action on A; in particular, if acts diagonally on A, it also does so on B.

It remains only to show that an element g' that corresponds to element of $\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m$ is an element of SL_2^3. We will consider the action of an element of SL_2^3 that corresponds to an element of $\mathfrak{h}$ of the form $t_1 h_{\alpha_1} + t_3 h_{\alpha_3} + t_4 h_{\alpha_4}$. By Lemma 19(c) in [27], the action of the element corresponding to $t_1 h_{\alpha_1}$ takes x_β to $t_1^{\langle \beta, \alpha_1 \rangle} x_\beta$, which is $t_1 x_\beta$ since $\langle \beta, \alpha_1 \rangle = 1$. Similarly, it takes x_γ to $t_1 x_\gamma$ since $\langle \gamma, \alpha_1 \rangle = 1$ and takes x_δ to $t_1^{-1} x_\delta$ since $\langle \delta, \alpha_1 \rangle = -1$; thus its action on A is that of the element (t_1, t_1, t_1^{-1}) in $\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m$. In the same fashion, we find that $t_3 h_{\alpha_3}$ corresponds to (t_3, t_3^{-1}, t_3) and $t_4 h_{\alpha_4}$ to (t_4^{-1}, t_4, t_4). Since these classes of elements are multiplicatively independent, they generate $\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m$; the elements with the product of the components equal to 1 come from elements of the form $t_1 h_{\alpha_1} + t_3 h_{\alpha_3} + t_4 h_{\alpha_4}$ with $t_1 t_3 t_4 = 1$. Since $\langle \alpha, \alpha_i \rangle = -1$ for $i = 1, 3, 4$, this element takes x_α to $t_1^{-1} t_3^{-1} t_4^{-1} x_\alpha = x_\alpha$, so it fixes x_α just as g' does. The action on the remaining basis elements, namely $x_{\rho-\alpha}$ and those of B, must also correspond to that of g' because an element of SL_2^3 stabilizes the bilinear form.

Thus g' is in $\mathrm{SL}_2^3 \rtimes S_3$, from which it follows that the original $g \in \mathrm{GL}(\mathfrak{g}_1)$

stabilizing the quartic form is in $\langle \mathrm{SL}_2^3, \mu_4 \rangle \rtimes S_3$. $\qquad\square$

The determination of the group that stabilizes both q and the bilinear form $\langle -, - \rangle$ is parallel to Corollary 4.7.

Corollary 4.16. *In the case $G = D_4$, the subgroup of $\mathrm{GL}(\mathfrak{g}_1)$ stabilizing both the quartic form and the skew-symmetric bilinear form, $\mathrm{Stab}(q, \langle -, - \rangle)$, is $\mathrm{SL}_2^3 \rtimes S_3$*

Proof. The previous theorem and the fact that SL_2^3 and the diagram automorphism stabilize both forms yield the following containments:

$$\mathrm{SL}_2^3 \rtimes S_3 \subseteq \mathrm{Stab}(q, \langle -, - \rangle) \subseteq \mathrm{Stab}(q) = \langle \mathrm{SL}_2^3, \mu_4 \rangle \rtimes S_3.$$

Since $-1 \in \mathrm{SL}_2$, we also have $-1 \in \mathrm{SL}_2^3$. Thus $\mathrm{SL}_2^3 \rtimes S_3$ is an index 2 subgroup of $\langle \mathrm{SL}_2^3, \mu_4 \rangle \rtimes S_3$. However, the coset containing i, a primitive fourth root of unity, is not in $\mathrm{Stab}(q, \langle -, - \rangle)$ since $\langle ix, iy \rangle = -\langle x, y \rangle$ for any $x, y \in \mathfrak{g}_1$. Therefore $\mathrm{Stab}(q, \langle -, - \rangle) = \mathrm{SL}_2^3 \rtimes S_3$. $\qquad\square$

Chapter 5

Conclusion

In this final chapter, we summarize the preceding results and suggest directions for further research.

5.1 Summary of results

For $\mathfrak{g}$ the Lie algebra of an algebraic group G of type B, D, E or F and rank ≥ 4 over a field of characteristic not 2 or 3, we define a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $x \in \mathfrak{g}_k$ when $[h_\rho, x] = kx$. The subspaces $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ are one-dimensional, containing the basis elements $x_{-\rho}$ and x_ρ, respectively. We define a nondegenerate skew-symmetric bilinear form $\langle -, - \rangle$ on $\mathfrak{g}_1$ by $[x, y] = \langle x, y \rangle x_\rho$ and a quartic form $q(-)$ by $(\operatorname{ad} x)^4(x_{-\rho}) = q(x)x_\rho$. Using the linearization of the quartic form determined by $q(x, x, x, x) = q(x)$, we define a triple product xyz on $\mathfrak{g}_1$ by $q(w, x, y, z) = \langle w, xyz \rangle$.

We find $q(x_\beta, x_\beta, x_{\rho-\beta}, x_{\rho-\beta}) = 1$ and $q(x_\beta, x_\gamma, x_{\rho-\beta}, x_{\rho-\gamma}) = -\frac{1}{2} c_{\beta, -\rho} c_{\gamma, -\rho}$, where β and γ are orthogonal long roots of α-height 1,

An element $x \in \mathfrak{g}_1$ is defined to be strictly regular if $xx\mathfrak{g}_1$ is the linear subspace spanned by x. The basis elements x_β with β a long root of α-height 1 are strictly regular. The following conditions are equivalent:

- $x \in \mathfrak{g}_1$ is strictly regular,

- x is in the smallest nonzero G_0-orbit,

- $xx\mathfrak{g}_1$ is a one-dimensional subspace (i.e., x is rank one),

- $x \neq 0$, $xxx = 0$ and $x \in xx\mathfrak{g}_1$,

- $q(x, x, y, z) = 0$ for all $y \in \mathfrak{g}_1$ and all z in a codimension-1 subspace.

The strictly regular elements span $\mathfrak{g}_1$. Any element in the dense orbit is uniquely the sum of two strictly regular elements. For x strictly regular and $y, z \in \mathfrak{g}_1$, $xxy = \langle y, x \rangle x$ and $q(x, x, y, z) = \langle y, x \rangle \langle z, x \rangle$.

The vector space $\mathfrak{g}_1$ with the operations defined above is a Freudenthal triple system.

The 4-linear form $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$ is zero unless $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\rho$. If the roots are long, this condition is met if and only if the roots consist of two pairs that each sum to ρ or the roots are mutually orthogonal. In the latter case, $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4}) = c_{\beta_1, \beta_4 - \rho} c_{\beta_2, \beta_1 - \rho} c_{\beta_3, \beta_4 - \rho} c_{\beta_4, \beta_1 - \rho}$.

If $\mathfrak{g}$ is of type D or E, then $\mathfrak{g}_1$ is the direct sum of four eigenspaces under $\mathrm{ad}(h_{\rho - \alpha} - h_\alpha)$:

- The -3-eigenspace is spanned by x_α,

- The -1-eigenspace is spanned by the x_β with $\langle \alpha, \beta \rangle = 1$,

- The $+1$-eigenspace is spanned by the x_β with $\langle \alpha, \beta \rangle = 0$,

- The $+3$-eigenspace is spanned by $x_{\rho - \alpha}$.

The -1- and $+1$-eigenspaces have the same dimension.

For $\mathfrak{g}$ of type E_6, E_7 or E_8,

- x is in orbit 0 iff $x = 0$,

- x is in the closure of orbit 1 iff $xx\mathfrak{g}_1 \subset Fx$,

- x is in the closure of orbit 2 iff $xxx = 0$,

- x is in the closure of orbit 3 iff $q(x) = 0$, and

- x is in orbit 4 iff $q(x) \neq 0$.

For $\mathfrak{g}$ of type D_n, a similar result holds except that the elements for which $xxx = 0$ form two ($n > 4$) or three ($n = 4$) orbits.

Any element of $\mathrm{GL}(\mathfrak{g}_1)$ that preserves the quartic form up to a scalar factor must likewise preserve the bilinear form, and thus orthogonality.

When $G = E_8$, the subgroup of $\mathrm{GL}(\mathfrak{g}_1)$ stabilizing the quartic form is $\mathrm{Stab}(q) = \langle E_7, \mu_4 \rangle$. $\mathrm{Stab}(q, \langle -, - \rangle) = E_7$.

When $G = D_4$, $\mathrm{Stab}(q) = \langle \mathrm{SL}_2^3, \mu_4 \rangle \rtimes S_3$ and $\mathrm{Stab}(q, \langle -, - \rangle) = \mathrm{SL}_2^3 \rtimes S_3$.

5.2   Future work

We have a decomposition of $\mathfrak{g}_1$ as the direct sum of four eigenspaces, two of which are one-dimensional. In light of other well-known constructions of Freudenthal triple systems and Lie algebras, it seems likely that the remaining eigenspaces can be described as Jordan algebras or as a Jordan pair. It would be desirable to describe a natural Jordan structure on them and thus complete the process of starting with the Lie algebras and reversing the known constructions.

Some of our results apply only to simple Lie algebras with simply-laced root systems (or, only to the long roots of a Lie algebra meeting the other hypotheses); for example, the explicit determination of the 4-linear form in Proposition 3.35, and the fact that a nonzero element of a root subspace corresponding to a long root is strictly regular (Corollary 3.10). The situation in the presence of short roots is more complicated: there are then more cases

than those listed in Proposition 3.35, and x_β need not be strictly regular if β is a short root. Further study is needed to better illuminate these issues.

The stabilizer subgroups for the forms have been determined in two important cases. It would be of interest to work out other cases as well; in particular, the case $G = E_6$ appears to both manageable and of interest.

All of the results assume the characteristic of the underlying field is not 2 or 3. Finding an appropriate way to generalize these findings to low characteristic would certainly be desirable, but is likely more challenging than the other questions.

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