On Point-Weighted Designs

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Abstract

A point-weighted structure is an incidence structure with each point assigned an element of some set $\mathbf{W} \subset \mathbb{Z}^+$ as a 'weight'. A point-weighted structure with no repeated blocks and the property that the sum of the weights of the points incident with any one block is a constant k is called a point-weighted design. A $t-(v,k,\lambda;\mathbf{W})$ point-weighted design is such a structure with the sum of the weights of all the points equal to v and the property that every set of t distinct points is incident with exactly λ blocks. This thesis introduces and examines this generalisation of block designs.

The first chapter introduces incidence structures and designs.

Chapter 2 introduces and defines point-weighted designs. Three constructions of families of $t-(v,k,\lambda;\mathbf{W})$ point-weighted designs are given.

Associated with any point-weighted design is the incidence structure on which it is based - the 'underlying' incidence structure (u.i.s.). It is shown in Chapter 3 that any automorphism of the u.i.s. of a $t-(v,k,\lambda;\mathbf{W})$ point-weighted design with more than one block and $t > 1$ preserves weights in the point-weighted design. The u.i.s. of such a point-weighted design is shown to be a block design if and only if every point is assigned the same weight. A necessary and sufficient condition is obtained for the assignment of weights in any point-weighted design to be essentially uniquely determined by the u.i.s.

Chapter 4 considers $t-(v,k,\lambda;\mathbf{W})$ point-weighted designs in which all of the points apart from a 'special' point have the same weight. It is shown that when $v > k$ the weight of the special point is an integer multiple of the weight assigned to all the other points. A class of these point-weighted designs is demonstrated to be equivalent to a class of group-divisible designs with specific parameters.

The final chapter uses the procedure of point-complementing incidence structures to construct point-weighted designs. Trivial point-weighted designs are defined and a necessary and sufficient condition for the existence of a member of a certain class of these is obtained. A correspondence between this class of point-weighted designs and certain trivial block designs is given using point-complementing.

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Contents

List of Figures

Chapter 1

Introduction

This thesis is concerned with examining incidence structures in which each point is assigned a positive integer as a 'weight'. For certain incidence structures it is possible to assign weights to points so that the sum of the weights of the points in each block is a constant k. We call a structure with such an assignment of weights and no repeated blocks a 'Point-Weighted Design'. We use v to denote the sum of the weights of all the points in a point-weighted design and $\mathbf{W}$ to denote the set of integers used as weights. A uniform design with every point assigned a weight of 1 is a point-weighted design and so point-weighted designs can be viewed as a generalisation of uniform designs. We will be concerned with those point-weighted designs in which every set of t distinct points is incident with a fixed number of blocks. These '$t-(v,k,\lambda;\mathbf{W})$ point-weighted designs' are a generalisation of $t-(v,k,\lambda)$ designs.

In this chapter we introduce incidence structures and designs, discuss some of their properties and define some notation. Most of the results in the chapter are covered by [20], [1] and [2].

We call an incidence structure $\mathcal{U}$ with 'block-sizes' from some set K, no repeated blocks and the property that every set of t distinct points is incident with exactly λ blocks a $t-(v,K,\lambda)$ design. Associated with a point-weighted design is its 'underlying' incidence structure, and the underlying incidence structure of a $t-(v,k,\lambda;\mathbf{W})$ point-weighted design is a $t-(v,K,\lambda)$ design. It would be interesting to ascertain which $t-(v,K,\lambda)$ designs can have weights assigned to the points so that the resulting point-weighted structure is a point-weighted design. This is an equivalent problem to determining when a solution $\mathbf{w}$ with entries from the set of positive rationals exists to,

$$A^T\mathbf{w} = \mathbf{1},$$

where A is an incidence matrix of a $t-(v,K,\lambda)$ and $\mathbf{1}$ is the constant vector with every entry equal to $\mathbf{1}$. For a general matrix A, this is a notoriously difficult problem in linear algebra, and it seems that what is known of the structure imposed on A by it being the incidence matrix of a $t-(v,K,\lambda)$ does not appear to make the problem any more soluble. However, in this thesis we make progress in the characterisation of $t-(v,k,\lambda;\mathbf{W})$ point-weighted designs in certain directions.

The second chapter introduces point-weighted designs and three constructions of families of $t-(v,k,\lambda;\mathbf{W})$ point-weighted designs are given.

In the third chapter, we show that the assignment of weights in a point-weighted design $\mathcal{D}$ is uniquely determined up to 'equivalence' by its underlying incidence structure if and only if the rank of the underlying incidence structure is equal to the number of points. We obtain an expression for the number of blocks incident with a $(t-1)$-set of points in a $t-(v,k,\lambda;\mathbf{W})$ point-weighted design. We show that if $\mathcal{D}$ is a $t-(v,k,\lambda;\mathbf{W})$ with $t>1$ and $v>k$ then any automorphism of the underlying incidence structure of $\mathcal{D}$ 'preserves' weights in $\mathcal{D}$.

The final two chapters investigate specific classes of $t-(v,k,\lambda;\{a_1,a_2\})$ point-weighted designs in which exactly one point has weight a_1. It is shown that in any such point-weighted design with $v>k$, a_2 divides a_1 and the point-weighted design is equivalent to a $t-\left(\frac{v}{a_2},\frac{k}{a_2},\lambda;\left\{1,\frac{a_1}{a_2}\right\}\right)$. A class of these is shown to have a certain type of square group divisible design as a substructure of its underlying incidence structure. Trivial point-weighted designs are defined and necessary and sufficient conditions are obtained for the existence of a trivial $t-(v,k,\lambda;\mathbf{W})$ in which all but one of the points have weight 1. The procedure of point-complementing is used to establish a correspondence between the class of trivial $t-(v,k,\lambda;\mathbf{W})$ point-weighted designs in which all but one of the points have weight 1 and a certain class of trivial block designs.

1.1 Incidence Structures and Designs

Definition 1.1 *An* **incidence structure** *is a triple* $(\mathbf{V},\mathbf{B},\mathbf{I})$*, where* $\mathbf{V}$ *is a finite set of* points*,* $\mathbf{B}$ *is a finite set of* blocks*, and* $\mathbf{I}\subseteq\mathbf{V}\times\mathbf{B}$ *is a binary relation between* $\mathbf{V}$ *and* $\mathbf{B}$*. A point,* $P\in\mathbf{V}$*, and block,* $x\in\mathbf{B}$*, are said to be* incident *if the pair* (P,x) *is contained in* $\mathbf{I}$.

A pair $(P,x)\in\mathbf{I}$ is called a **flag**. Instead of $(P,x)\in\mathbf{I}$ we sometimes write PIx or use language such as 'P is on x,' or 'x is on P'. Conventionally, the letter v is used to denote the number of points (i.e., $v=|\mathbf{V}|$) and b to denote the number of blocks (i.e., $b=|\mathbf{B}|$). An incidence structure in which every block is incident with exactly the same number of points is said to be **uniform**, and the letter k is used to denote the number of points incident with any one block. If an incidence structure has the property that every point is incident with exactly the same number of blocks then it is said to be **regular**, and the letter r is used to denote the number of blocks incident with any one point.

The above definition of an incidence structure allows for the possibility that either of $\mathbf{V}$ or $\mathbf{B}$ are empty. It is also possible that a point $P\in\mathbf{V}$ is not incident with any blocks or that a block $x\in\mathbf{B}$ is not incident with any points. We shall assume throughout this thesis that none of these possibilities occur.

If $\mathcal{S}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) is an incidence structure and if $\mathbf{V}_0$ and $\mathbf{B}_0$ are subsets of the point-set and block-set respectively, then the **substructure** of $\mathcal{S}$ defined by $\mathbf{V}_0$ and $\mathbf{B}_0$ is the incidence structure $\mathcal{S}_0 = (\mathbf{V}_0, \mathbf{B}_0, \mathbf{I}_0)$ where,

$$\mathbf{I}_0 = \mathbf{I} \cap (\mathbf{V}_0 \times \mathbf{B}_0).$$

Thus a substructure of $\mathcal{S}$ is a set of points and blocks of $\mathcal{S}$ with incidence 'inherited' from $\mathcal{S}$.

For any structure which is both uniform and regular, we obtain the following well-known result, simply by counting flags.

Result 1.2 *Let $\mathcal{S}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be a uniform, regular structure with every block incident with k points and every point incident with r blocks. Let $v = |\mathbf{V}|$ and $b = |\mathbf{B}|$, then the parameters k, r, v and b are related by, $bk = vr$.* ∎

Let b, k, v and r be positive integers with $v > k$ and $b \leq \binom{v}{k}$ satisfying $bk = vr$. Then it is possible to construct a uniform, regular structure with v points b blocks, k points on any one block and r blocks on any one point with no two blocks incident with exactly the same points (see for example [9] page 105). We say that an incidence structure with two blocks incident with exactly the same points has **repeated blocks** and call an incidence structure with no repeated blocks a **design**. In a design, we can identify a block with the points with which it is incident, and it is often convenient to consider the blocks as subsets of the point-set. When this is the case, we say '$P \in x$' or 'x contains P' if a point P and block x are incident. We also use '$|x|$' to denote the number of points incident with a block x.

Definition 1.3 *Let $\mathcal{S}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be a design with $v = |\mathbf{V}|$ points. Let K be a set of positive integers such that $|x| \in K$ for every $x \in \mathbf{B}$. For positive integers t and λ, we say that $\mathcal{S}$ is a $t-(v, K, \lambda)$ design if it has the property that every set of t distinct points is contained in exactly λ blocks.*

For convenience, we often drop the word 'design' and refer to $\mathcal{S}$ as a $t-(v, K, \lambda)$. Clearly, if $\mathcal{S}$ is a $t-(v, K, \lambda)$ then it is also a $t-(v, K^{'}, \lambda)$ for any $K^{'} \supseteq K$. For simplicity, we shall assume throughout this thesis that in any $t-(v, K, \lambda)$, K is 'minimal' in the sense that for every element k of K, there is a block incident with exactly k points. We also assume that $\min K \geq t$. We say that a block incident with k points 'has block-size k' or 'is a block of size k'.

An incidence structure which has the property that every set of t distinct points is incident with a fixed number of blocks is said to be '$t-$balanced'. A variety of names have been given to $t-$balanced designs with certain properties. A $2-(v, K, \lambda)$ has been called a 'λ-linked design' (see for example [34]), but the more recent convention is to refer to such a structure as a 'pairwise balanced design' (as in [2]). The $t-(v, K, \lambda)$ designs which have been studied most are those which are also uniform, and we turn our attention to these for the remainder of this section.

Definition 1.4 *Let $\mathcal{D}$ =(**V**,**B**,**I**) be a $t-(v,\{k\},\lambda)$ for some $k \geq t$. Then we call $\mathcal{D}$ a $t-(v,k,\lambda)$ design and, for convenience, often refer to $\mathcal{D}$ as a $t-(v,k,\lambda)$.*

Given a set **V** of v points and positive integers $k \leq v$ and $t \leq k$, we can always construct a $t-(v,k,\lambda)$, $\mathcal{T}$, for some λ as follows. Viewing the blocks as sets of points, we define the block-set, **B**, of $\mathcal{T}$ to consist of every possible set of k distinct points from **V**. Then $|\mathbf{B}| = \binom{v}{k}$, and every set of t distinct points is contained in exactly $\binom{v-t}{k-t}$ blocks. We say that such a design is **trivial**. If $\mathcal{T}$ is a trivial $t-(v,k,\lambda)$ then it clearly has the property that for every s in the range $1 \leq s \leq k$, $\mathcal{T}$ is a $s-(v,k,\lambda_s)$, where $\lambda_s = \binom{v-s}{k-s}$. This leads us to note a property common to all $t-(v,k,\lambda)$ designs:

Result 1.5 *Let $\mathcal{D}$ be a $t-(v,k,\lambda_t)$. Then, for every integer s within the range $0 \leq s < t$, $\mathcal{D}$ is a $s-(v,k,\lambda_s)$, where λ_s is given by,*

$$\lambda_s = \lambda_t \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

Proof

Let S be a fixed set of some s points of $\mathcal{D}$. We prove the result by calculating the number of blocks containing S, z say, and showing that this value is independent of the choice of S. Let T be a set of t points of $\mathcal{D}$ containing S, and let x be a block of $\mathcal{D}$ containing T. We count the pairs (T,x) in two ways.

For the fixed set S, there are $\binom{v-s}{t-s}$ ways of choosing a set T of t points containing S, and each of these sets is contained in exactly λ_t blocks (since $\mathcal{D}$ is a $t-(v,k,\lambda_t)$). Hence, the number of pairs (T,x) is $\lambda_t\binom{v-s}{t-s}$.

The set S is contained in z blocks, and each of these blocks contains exactly $k-s$ other points. So, from each block x containing S, there are $\binom{k-s}{t-s}$ ways of choosing a set T of t points containing S. Hence, the number of pairs (T,x) is $z\binom{k-s}{t-s}$.

Equating the two expressions for the number of pairs (T,x) gives,

$$z = \lambda_t \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

Clearly z is independent of the choice of S, and so every set of s points is contained in exactly z blocks. Setting $\lambda_s = z$ gives the result. ■

A $2-(v,k,\lambda)$ is called a **block design with parameters** v,k **and** λ. From the above result we see that the class of block designs is exactly the class of $t-(v,k,\lambda_t)$ designs with $t \geq 2$.

Considering the case $s=0$ in the above result gives an expression for the total number of blocks in a $t-(v,k,\lambda)$. Setting $s=1$ gives an expression for the number of blocks containing any point in a $t-(v,k,\lambda)$ and shows that every $t-(v,k,\lambda)$ is regular. Calculating the expressions for λ_0 and λ_1 from Result 1.5 for a block design with parameters v,k and λ gives the following necessary conditions for the existence of a block design as a corollary.

Corollary 1.6 *Necessary conditions for the existence of a* $2-(v,k,\lambda)$ *are that,*

$$\lambda(v-1) \equiv 0 \pmod{(k-1)}$$

and

$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}.$$

Proof

Let $\mathcal{D}$ be a $2-(v,k,\lambda)$. Then, by Result 1.5, the number of blocks on any one point, $r=\lambda_1$, is given by,

$$r = \lambda\,\frac{(v-1)}{(k-1)}.$$

Also, the total number of blocks, $b=\lambda_0$, is,

$$b = \lambda\,\frac{v(v-1)}{k(k-1)}.$$

The values r and b must be integers, and hence the corollary is proved. ∎

The above necessary conditions for the existence of a block design with parameters v,k and λ are not in general sufficient. Hanani ([13],[14],[15]) has shown that they are however sufficient when $k=3$ or 4 for all λ and, apart from one exception, when $k=5$ (see for example [2]). In [31], [32] and [33], Wilson proves that for given k and λ, there is a value C dependent on k and λ such that the conditions in Corollary 1.6 are necessary and sufficient for $v \geq C$.

1.2 Related Structures

Given an incidence structure $\mathcal{S}$ =(**V**,**B**,**I**), there are a number of incidence structures related to $\mathcal{S}$ and we define four of them here. We first introduce the concepts of isomorphic structures and automorphisms.

Definition 1.7 *Let* $\mathcal{S}$ *and* $\mathcal{U}$ *be incidence structures and let* P *be a point and* x *a block of* $\mathcal{S}$. *An* **isomorphism**, α, *from* $\mathcal{S}$ *to* $\mathcal{U}$ *is a bijection from the point-set of* $\mathcal{S}$ *to the point-set of* $\mathcal{U}$ *and from the block-set of* $\mathcal{S}$ *to the block-set of* $\mathcal{U}$ *such that* P^α *and* x^α *are incident in* $\mathcal{U}$ *if and only if* P *and* x *are incident in* $\mathcal{S}$ *(i.e.,* α *'preserves incidence'). If there exists an isomorphism from* $\mathcal{S}$ *to* $\mathcal{U}$ *then* $\mathcal{S}$ *and* $\mathcal{U}$ *are said to be* **isomorphic**.

An **automorphism** of an incidence structure $\mathcal{S}$ is an isomorphism of $\mathcal{S}$ to itself. Thus, an automorphism of $\mathcal{S}$ is a permutation of the points and a permutation of the blocks which preserves incidence. The set of all automorphisms of an incidence structure $\mathcal{S}$ forms a group whose binary operation is the usual product of mappings, and this group is denoted by Aut$\mathcal{S}$.

Letting $\mathcal{S}$ =(**V**,**B**,**I**) be any incidence structure, we now define some structures related to $\mathcal{S}$.

The **complement** of $\mathcal{S}$, $\mathcal{C}(\mathcal{S}) = (\overline{\mathbf{V}}, \overline{\mathbf{B}}, \bar{\mathbf{I}})$, has the same point-set and block-set as $\mathcal{S}$, but a point P is incident with a block x in $\mathcal{C}(\mathcal{S})$ if and only if P and x are not incident in $\mathcal{S}$. So we have,

$$\begin{aligned}
\overline{\mathbf{V}} &= \mathbf{V}, \\
\overline{\mathbf{B}} &= \mathbf{B}, \\
\bar{\mathbf{I}} &= \{(P,x) \mid P \in \mathbf{V}, x \in \mathbf{B}, (P,x) \notin \mathbf{I}\}.
\end{aligned}$$

The following result (see for example [20] page 24) shows that typically, the complement of a block design is also a block design.

Result 1.8 *Let $\mathcal{D} = (\mathbf{V}_1, \mathbf{B}_1, \mathbf{I}_1)$ be a $2-(v,k,\lambda)$ with $2 \leq k \leq v-2$. Let $b = |\mathbf{B}_1|$ and r be the number of blocks incident with a point of $\mathcal{D}$. Then $\mathcal{C}(\mathcal{D})$ is a $2-(v, v-k, b-2r+\lambda)$.* ∎

The **dual** of $\mathcal{S}$, $\mathcal{S}^{'} = (\mathbf{V}^{'}, \mathbf{B}^{'}, \mathbf{I}^{'})$ is obtained by 'exchanging' points and blocks and 'retaining' incidence. So we have,

$$\begin{aligned}
\mathbf{V}^{'} &= \mathbf{B}, \\
\mathbf{B}^{'} &= \mathbf{V}, \\
\mathbf{I}^{'} &= \{(x,P) \mid x \in \mathbf{V}^{'}, P \in \mathbf{B}^{'}, (P,x) \in \mathbf{I}\}.
\end{aligned}$$

The **derived structure** of $\mathcal{S}$ at a point $P \in \mathbf{V}$, $\mathcal{S}_P = (\mathbf{V}_P, \mathbf{B}_P, \mathbf{I}_P)$, is the structure whose points are the points of $\mathcal{S}$ distinct from P and on a block of $\mathcal{S}$ with P, and whose blocks are those blocks of $\mathcal{S}$ incident with P in $\mathcal{S}$; with incidence 'retained' from $\mathcal{S}$. So we have,

$$\begin{aligned}
\mathbf{B}_P &= \{x \mid x \in \mathbf{B}, (P,x) \in \mathbf{I}\}, \\
\mathbf{V}_P &= \mathbf{V} \setminus (\{P\} \cup \{Q \mid (Q,x) \notin \mathbf{I} \cap (\mathbf{V} \times \mathbf{B}_P), \forall x \in \mathbf{B}_P\}), \\
\mathbf{I}_P &= \mathbf{I} \cap (\mathbf{V}_P \times \mathbf{B}_P).
\end{aligned}$$

Finally, the **point-residue** of $\mathcal{S}$ at a point $P \in \mathbf{V}$, $\mathcal{S}^P = (\mathbf{V}^P, \mathbf{B}^P, \mathbf{I}^P)$, is the structure obtained by removing from $\mathcal{S}$ the point P, all the blocks incident with P and any point which is only on blocks of $\mathcal{S}$ which are also incident with P in $\mathcal{S}$. So we have,

$$\begin{aligned}
\mathbf{B}^P &= \mathbf{B} \setminus \{x \mid x \in \mathbf{B}, (P,x) \in \mathbf{I}\}, \\
\mathbf{V}^P &= \mathbf{V} \setminus \Big(\{P\} \cup \Big\{Q \mid (Q,x) \notin \mathbf{I} \cap \Big(\mathbf{V} \times \mathbf{B}^P\Big), \forall x \in \mathbf{B}^P\Big\}\Big), \\
\mathbf{I}^P &= \mathbf{I} \cap \Big(\mathbf{V}^P \times \mathbf{B}^P\Big).
\end{aligned}$$

Let P be a point of $\mathcal{S}$, then we note that both the derived structure of $\mathcal{S}$ at P and the point-residue of $\mathcal{S}$ at P are substructures of $\mathcal{S}$.

1.3 Incidence Matrices and Fisher's Inequality

An incidence structure can be specified by a $(0,1)-$matrix called an *incidence matrix*, which we define in this section. By considering the rank of the incidence matrix of a pairwise balanced design we show that such a design has the property that the number of blocks is greater than or equal to the number of points. This is a slight generalisation of 'Fisher's Inequality' for block designs. Several authors have given different methods of proof of this inequality - we follow a slightly simplified version of Majumdar's method (see [22]), as in [20].

Definition 1.9 *Let* $\mathcal{S}$ =(**V**,**B**,**I**) *be an incidence structure and let* $v = |\mathbf{V}|$ *and* $b = |\mathbf{B}|$. *Let the points of* $\mathcal{S}$ *be labelled* $P_1, P_2, \ldots, P_v$ *and the blocks of* $\mathcal{S}$ *be labelled* $x_1, x_2, \ldots, x_b$. *Then the* $v \times b$ *matrix* $A = (a_{ij})$ *defined by,*

$$a_{ij} = \begin{cases} 1 & \textit{if } (P_i, x_j) \in \mathbf{I} \\ 0 & \textit{otherwise,} \end{cases}$$

is called the **incidence matrix** *of* $\mathcal{S}$ *with respect to the above labelling of the points and blocks.*

It is clear that different labellings of the points and blocks of an incidence structure $\mathcal{S}$ will give rise to different incidence matrices. However, if A and B are two different incidence matrices of $\mathcal{S}$ then there exist permutation matrices P and Q such that $PAQ = B$. Thus, any two incidence matrices of $\mathcal{S}$ are equivalent and so have the same rank (throughout this thesis, the rank of a matrix will be taken to be its rank over the field of rationals). We define the **rank** of an incidence structure to be the rank of one (and hence all) of its incidence matrices.

Let $\mathcal{S}$ =(**V**,**B**,**I**) be a $2-(v, K, \lambda)$. Let $P_1, P_2, \ldots, P_v$ be a labelling of the points of $\mathcal{S}$ and $x_1, x_2, \ldots, x_b$ be a labelling of the blocks (with $b = |\mathbf{B}|$). Let A be the incidence matrix of $\mathcal{S}$ with respect to this labelling and denote the number of blocks on the point P_i by r_i, for each $i = 1, 2, \ldots, v$. Then,

Result 1.10 $AA^T = N + \lambda J_v$, *where* J_v *is the* $v \times v$ *matrix with every entry equal to 1, and* N *is the* $v \times v$ *diagonal matrix,* $diag(r_1 - \lambda, r_2 - \lambda, \ldots, r_v - \lambda)$.

Proof
The entry in the $(i,j)^{\text{th}}$ position of AA^T is equal to the number of blocks incident with both the points P_i and P_j in $\mathcal{S}$. So when $i = j$ this value is the number of blocks incident with the point P_i, $r_i = r_i - \lambda + \lambda$. When $i \neq j$, the value is λ since $\mathcal{S}$ is a $2-(v, K, \lambda)$. ■

An incidence structure is said to be **proper** if it contains a block which is incident with more than one point but not all of the points. A sufficient condition for a $2-(v, K, \lambda)$ to be proper is clearly that $v > \max K$.

Result 1.11 *If* $\mathcal{S}$ *is a proper* $2-(v, K, \lambda)$ *then* $r_i - \lambda > 0$ *for each* $i = 1, 2, \ldots, v$.

Proof

Clearly $r_i - \lambda \geq 0$ for each i. Suppose there exists a point P_i for which $r_i - \lambda = 0$, i.e., P_i is incident with exactly λ blocks. Let P_j be any other point of $\mathcal{S}$. Then since there are λ blocks incident with both P_i and P_j, every block incident with P_i is also incident with P_j. Hence, every block on P_i is incident with every other point of $\mathcal{S}$. Now let P_l and P_m be any two points of $\mathcal{S}$ distinct from P_i. Then the λ blocks incident with both P_l and P_m are all also incident with P_i and hence every other point of $\mathcal{S}$. So, every block incident with at least two points is incident with every point of $\mathcal{S}$. But this gives a contradiction since $\mathcal{S}$ is proper. Hence, $r_i - \lambda > 0$ for each $i = 1, 2, \ldots, v$. ■

Result 1.12 *If $\mathcal{S}$ is a proper $2-(v, K, \lambda)$ with incidence matrix A defined as above then,*

$$det(AA^T) = \prod_{i=1}^{v} (r_i - \lambda) \left(1 + \lambda \sum_{j=1}^{v} \frac{1}{(r_j - \lambda)} \right).$$

Proof

Consider the matrix AA^T:

$$\begin{pmatrix} r_1 & \lambda & \lambda & \ldots & \lambda & \lambda \\ \lambda & r_2 & \lambda & \ldots & \lambda & \lambda \\ \lambda & \lambda & r_3 & \ldots & \lambda & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda & \lambda & \lambda & \ldots & r_{v-1} & \lambda \\ \lambda & \lambda & \lambda & \ldots & \lambda & r_v \end{pmatrix}.$$

To compute $\det(AA^T)$ we use row and column operations to reduce AA^T to upper triangular form. The determinant is then the product of the entries in the leading diagonal (see for example [10]).

Firstly, subtract the first row from every other row. Then for each $i = 2, 3, \ldots, v$, add a multiple of a_i of the ith column to the first column, where $a_i = \frac{r_1 - \lambda}{r_i - \lambda}$, for each $i = 2, 3, \ldots, v$. This reduces AA^T to the form:

$$\begin{pmatrix} x & \lambda & \lambda & \ldots & \lambda \\ & r_2 - \lambda & & & \\ & & r_3 - \lambda & & \\ & & & \ddots & \\ & & & & r_v - \lambda \end{pmatrix},$$

with all entries off the leading diagonal and first row equal to zero, and x given by,

$$x = r_1 + \lambda \left(\sum_{j=2}^{v} \frac{r_1 - \lambda}{r_j - \lambda} \right).$$

Then $\det(AA^T)$ is given by,

$$\det(AA^T) = \left(r_1 + \lambda\left(\sum_{j=2}^{v} \frac{r_1-\lambda}{r_j-\lambda}\right)\right)\left(\prod_{i=2}^{v}(r_i-\lambda)\right),$$

which simplifies to,

$$\det(AA^T) = \prod_{i=1}^{v}(r_i-\lambda)\left(1+\lambda\sum_{j=1}^{v}\frac{1}{(r_j-\lambda)}\right).$$

■

This expression for the determinant of AA^T is clearly non-zero (by Result 1.11) and so we obtain the corollary:

Corollary 1.13 *Let $\mathcal{S}$ be a proper $2-(v,K,\lambda)$ with incidence matrix A defined as above. Then rank$(A) = v$.*

Proof
Since $\det(AA^T) \neq 0$, the rank of (AA^T) is v. But $\text{rank}(AA^T) \leq \text{rank}(A)$ and so $\text{rank}(A) \geq v$. But A has v rows and so we have $\text{rank}(A) = v$. ■

Corollary 1.14 *Let $\mathcal{S}$ be a proper $2-(v,K,\lambda)$ with b blocks. Then $b \geq v$.*

Proof
Let A be an incidence matrix of $\mathcal{S}$. Then $\text{rank}(A) = v$ by Corollary 1.13. But A has b columns and so $\text{rank}(A) \leq b$. Hence, $b \geq v$. ■

This result is known as 'Fisher's Inequality' in the case when $\mathcal{S}$ is a proper $2-(v,k,\lambda)$. A $2-(v,K,\lambda)$ with $b = v$ is said to be 'square', and these structures have been the subject of much work (see for example [11],[26],[27],[34]). A square $2-(v,k,\lambda)$ is called a 'symmetric' block design. It is to some of these and a few other classes of incidence structures to which we now turn our attention.

1.4 Some Special Structures

In this section we define some incidence structures and designs with special properties and mention some results relating to them.

1.4.1 Projective and Affine Planes

Incidence structures with the property that there is at most one block incident with any pair of distinct points can be viewed geometrically with the blocks viewed as lines. In such a case we say that 'the point P lies on the line x,' or 'the line x lies on the point P,' if P and x are incident. Two points are 'collinear' if they lie on a line together, and two lines x and y are said to 'intersect' at the point P if P lies on both x and y.

Definition 1.15 *A* **projective plane** *is an incidence structure with points and lines satisfying:*

(i) any two distinct points lie on a unique line;

(ii) any two lines intersect in a unique point;

(iii) there exist at least four points no three of which are collinear.

An incidence structure satisfying the first two of the above conditions but not the third is called a **degenerate projective plane**. For any $k \geq 2$ there is exactly one degenerate projective plane up to isomorphism with $k+1$ points and more than one line.

Any finite projective plane $\mathcal{P}$ has an associated parameter q called the **order** of $\mathcal{P}$. Counting arguments can be used to show that a projective plane of order q has q^2+q+1 points and q^2+q+1 lines, with every line on $q+1$ points and every point on $q+1$ lines. A projective plane of order q is then clearly a symmetric $2-(q^2+q+1, q+1, 1)$, and the class of projective planes is in fact exactly the class of symmetric $2-(v,k,1)$ designs. A projective plane of order q can be constructed for every prime-power q, with the points co-ordinatised by the finite field of order q, $GF(q)$. Such planes are said to be 'Desarguesian'. There are examples of projective planes of prime-power order which are not isomorphic to the Desarguesian plane of that order, but there are no known examples of projective planes of non-prime-power order (see [19] for further details).

It is known that any projective plane of order 2 is isomorphic to the Desarguesian plane of that order. Such a structure is called a **Fano plane**. Similarly, any plane of order $4 = 2^2$ is isomorphic to the Desarguesian plane of order 4.

Let $\mathcal{P}$ be a projective plane. A substructure $\mathcal{P}_0$ of $\mathcal{P}$ which is itself a projective plane is said to be a **subplane** of $\mathcal{P}$ and is called **proper** if $\mathcal{P}_0 \neq \mathcal{P}$. The possible orders for subplanes of any projective plane are restricted by the following result due to Bruck ([7]):

Result 1.16 *Let $\mathcal{P}$ be a finite projective plane of order n with a proper subplane $\mathcal{P}_0$ of order m. Then either $n = m^2$ or $n \geq m^2 + m$.* ∎

If $n = m^2$ then $\mathcal{P}_0$ is called a **Baer subplane** of $\mathcal{P}$. Every line of $\mathcal{P}$ is incident with either 1 point of $\mathcal{P}_0$ or every point on some line of $\mathcal{P}_0$ ($m+1$ points). Since $GF(q)$ is contained in $GF(q^2)$ for any prime-power q, the Desarguesian plane of order q is a Baer subplane of the Desarguesian plane of order q^2.

An **affine plane** $\mathcal{A}$ can be constructed from a projective plane $\mathcal{P}$ by removing a fixed line l_∞ of $\mathcal{P}$ and all the points on l_∞. The line l_∞ is called the 'line at infinity'. An affine plane constructed from a projective plane of order q is a $2-(q^2, q, 1)$. The lines of such an affine plane can be partitioned into 'parallel classes' of non-intersecting lines. Two lines are in the same parallel class of $\mathcal{A}$ if and only if they intersect at a point on l_∞ in $\mathcal{P}$. Every point of $\mathcal{A}$ is incident with exactly one line of each parallel class.

1.4.2 Projective and Affine Spaces

We now define projective and affine spaces and state how they can be used to obtain families of $2-(v,k,\lambda)$ designs for certain values of v,k and λ. Further details can be found in [17], [20] and [2].

Let V be the vector space of dimension $n+1$ over the field $GF(q)$, for some $q>1$. The set of subspaces of V together with the incidence relation of subspace containment is called the **projective space**, $PG(n,q)$, and n is called the **dimension** of $PG(n,q)$. If W is a subspace of V of dimension $m+1$, where $0 \leq m \leq n$, then the set of subspaces of W together with the incidence relation of subspace containment is called an m-dimensional subspace of $PG(n,q)$. The subspaces of $PG(n,q)$ of dimension 0 are called the **points** of $PG(n,q)$, and subspaces of dimension 1,2 and $n-1$ are called **lines**, **planes** and **hyperplanes** respectively.

Given a projective space $PG(n,q)$ we define an incidence structure whose points are the points of $PG(n,q)$ and blocks are the m-dimensional subspaces of $PG(n,q)$, for some $1 \leq m \leq n$, with incidence 'inherited' from $PG(n,q)$. We denote such a structure by $PG_m(n,q)$, and note that it has the following property,

Result 1.17 *$PG_m(n,q)$ is a $2-\left(\frac{q^{n+1}-1}{q-1}, \frac{q^{m+1}-1}{q-1}, \lambda\right)$, where λ is given by,*

$$\lambda = \frac{\left(q^{n-1}-1\right)\left(q^{n-2}-1\right)\ldots\left(q^{n-m+1}-1\right)}{\left(q^{m-1}-1\right)\left(q^{m-2}-1\right)\ldots(q-1)}.$$

■

An **affine space**, $AG(n,q)$, of **dimension** n is obtained by removing from $PG(n,q)$ a fixed hyperplane (a $(n-1)$-dimensional subspace) and all the subspaces incident with it. The removed hyperplane of $PG(n,q)$ is called the **hyperplane at infinity**, and we denote it by Π_∞. We define the subspaces of dimension m of $AG(n,q)$, for some $0 \leq m \leq n$, to be the subspaces of dimension m of $PG(n,q)$ but with the points of Π_∞ deleted.

Given an affine space $AG(n,q)$, we obtain an incidence structure $AG_m(n,q)$ whose points are the points of $AG(n,q)$ and whose blocks are the m-dimensional subspaces of $AG(n,q)$, for some $1 \leq m \leq n$, with incidence 'inherited' from $AG(n,q)$.

Result 1.18 *$AG_m(n,q)$ is a $2-(q^n,q^m,\lambda)$, where λ is given by,*

$$\lambda = \frac{\left(q^{n-1}-1\right)\left(q^{n-2}-1\right)\ldots\left(q^{n-m+1}-1\right)}{\left(q^{m-1}-1\right)\left(q^{m-2}-1\right)\ldots(q-1)}.$$

■

1.4.3 Resolvable Designs

Let $\mathcal{D}$ be a $t-(v,k,\lambda)$. A **resolution** of $\mathcal{D}$ is a partition of the blocks of $\mathcal{D}$ into classes, called **resolution classes**, such that the points of $\mathcal{D}$ and the blocks of any given resolution class form a $1-(v,k,\sigma)$, for some σ dependent on the resolution class. A $t-(v,k,\lambda)$ which admits a resolution is said to be **resolvable**. If a resolvable design $\mathcal{D}$ has the property that the blocks of any resolution class and the points of $\mathcal{D}$ form a $1-(v,k,\sigma)$ for some fixed σ then the resolution is said to be a σ-resolution.

A 1-resolution is more commonly called a **parallelism**, and in such a case, the resolution classes are called **parallel classes**. We have already seen that an affine plane is an example of a $2-(q^2,q,1)$ which admits a parallelism. In fact, for some $q \geq 2$, $n > 1$ and $1 \leq m \leq n$, the incidence structure $AG_m(n,q)$ defined in Section 1.4.2 admits a parallelism.

Let $\mathcal{D}$ be a $t-(v,k,\lambda)$ which admits a σ-resolution. Then, every point of $\mathcal{D}$ is on r blocks of $\mathcal{D}$, for some integer r, and on exactly σ blocks of each resolution class. Hence, if c is the number of resolution classes, then $r = c\sigma$ and so we have,

Result 1.19 *Let $\mathcal{D}$ be a $t-(v,k,\lambda)$ which admits a σ-resolution. Let r be the number of blocks on any point of $\mathcal{D}$, and c the number of resolution classes. Then $c = \frac{r}{\sigma}$.* ■

1.4.4 Biplanes and Nets

Since there is at least one projective plane of order q for every prime-power q, there are an infinite number of symmetric $2-(v,k,1)$ designs. However, it is conjectured (see for example [8]) that there are only finitely many symmetric $2-(v,k,\lambda)$ designs for any given $\lambda > 1$. A symmetric $2-(v,k,2)$ is called a **biplane**, and the only values of k for which examples of biplanes are known are $k = 3,4,5,6,9,11$ and 13 (see for example [1] and [8]). Furthermore, it is known that there is only one biplane (up to isomorphism) with block-size k for each $k = 3,4,5$ (in the case $k = 3$, the biplane is the trivial design with four points and block-size three), and there are three non-isomorphic biplanes with block-size 6. For given k, a necessary condition for the existence of a symmetric $2-(v,k,2)$ is that $v = \frac{1}{2}\left(k^2-k+2\right)$.

A **net** $\mathcal{N}$ is an incidence structure of points and blocks such that:

(i) there exist points and blocks, and for every point (block) there exist two blocks (points) not incident with it;

(ii) for any two distinct points P and Q there exists at most one block incident with both P and Q;

(iii) if a point P is not incident with a block x then there exists a unique block y incident with P such that x and y do not intersect.

The third condition defines a parallelism in $\mathcal{N}$ (see for example [12] page 141 and [24] page 190), and a net with i parallel classes is often called an $i-$net. An $i-$net with s lines in each parallel class has s^2 points, si blocks, with each block incident with s points and each point incident with i blocks (one from each parallel class). Possible values for the parameter s are bounded below by

$s \geq i-1$. An affine plane is an example of a net; a finite $i-$net $\mathcal{N}$ is an affine plane if and only if the blocks of $\mathcal{N}$ are partitioned into parallel classes of size $i-1$ (see [12]). Nets have been the subject of much investigation, and the reader is referred to [20], [12] and [24] for a more detailed summary of these incidence structures and their properties.

1.4.5 Group Divisible Designs

Group divisible designs were introduced by Bose and others (see for example [4], [3]) as a generalisation of block designs. The following definition is consistent with that of Bose and of Raghavarao ([24]).

Definition 1.20 *Let $\mathcal{G}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be a uniform, regular design with a partition of the point-set into n point-classes, $\mu_1, \mu_2, \ldots, \mu_n$, each of size m. Let $v = mn$ be the number of points and let k be the number of points incident with any one block. Then $\mathcal{G}$ is said to be a* **group divisible** $2-(v, k, (\lambda_1, \lambda_2))$ **design** *if the number of blocks incident with a pair of points is equal to λ_1 if they are in the same point-class and λ_2 if they are in different point-classes, for some integers λ_1 and λ_2.*

We denote a group divisible $2-(v, k, (\lambda_1, \lambda_2))$ design by GD $2-(v, k, (\lambda_1, \lambda_2))$. A block design can be viewed as a group divisible design with point-classes containing only one point (i.e., with $m = 1$). Thus the above definition generalises that of block designs.

The above is a slightly more general definition than that given by Hanani in [16], who considers group divisible $2-(v, k, (0, \lambda_2))$ designs. Bose and Connor ([3]) obtain the following relation between the parameters of a GD $2-(v, k, (\lambda_1, \lambda_2))$ with n point-classes each of size m,

$$(m-1)\,\lambda_1 + m\,(n-1)\,\lambda_2 = r\,(k-1)\,,$$

where r is the number of blocks incident with any one point. In the case $\lambda_1 = 0$, this yields the necessary condition for the existence of a GD $2-(v, k, (0, \lambda_2))$,

$$\lambda_2\,(v-m) \equiv 0 \pmod{(k-1)}. \tag{1.1}$$

Using counting arguments, a further necessary condition for the existence of a GD $2-(v, k, (0, \lambda_2))$ can be obtained,

$$\lambda_2 v\,(v-m) \equiv 0 \pmod{k(k-1)}. \tag{1.2}$$

In [6] it is shown that (1.1) and (1.2) are sufficient conditions for $k = 3$ and $k = 4$ with the two exceptions that there is no GD $2-(8, 4, (0, 1))$ with $m = 2$, and there is no $2-(24, 4, (0, 1))$ with $m = 6$ (see also [2]).

A **transversal design** is a GD $2-(v, k, (0, 1))$ with $v > k$ and the property that each block is incident with exactly one point from each point-class (see for example [1]). With this defintion, a transversal design is precisely the dual of a net, and vice versa. Recall from Section 1.4.4 that for an $i-$net with s blocks in each parallel class, the value of s is at least $i-1$. Then, since a transversal design is the dual of a net and the number of points on any one block of a transversal design is equal to the number of point-classes, we have the following result.

Result 1.21 *Let $\mathcal{T}$ be a transversal design with block-size k and point-classes each of size m. Then, $m \geq k-1$.* ∎

We conclude this subsection by observing that the dual of a $2-(v,k,1)$ which admits a parallelism is a GD $2-(v,k,(0,1))$.

1.5 Balanced Orthogonal Designs

Let v, b, k, r and λ be positive integers and let W be a $v \times b$ matrix with entries from $\{0,1,-1\}$. Define the *underlying matrix* N of W to be the $(0,1)-$matrix obtained by replacing every entry of W equal to -1 by 1. Then W is a **balanced orthogonal design** with parameters v, b, k, r and λ if:

(i) the underlying matrix, N, of W is the incidence matrix of a $2-(v,k,\lambda)$ with r blocks on any one point;

(ii) the inner (dot) product of any two distinct rows of W is zero.

We denote a balanced orthogonal design with parameters v, b, k, r and λ by BOD (v,b,k,r,λ).

Balanced orthogonal designs were introduced by Bhaskar Rao in [25], and can be used to construct group divisible designs. Let W be a BOD (v,b,k,r,λ) for some v, b, r, k and even λ. The incidence matrix of a group divisible design $\mathcal{G}$ can be constructed by viewing the rows of W as point-classes and the columns of W as block-classes. For each $i = 1,2,\ldots v$ and $j = 1,2,\ldots,b$, denote the entry of W in the $(i,j)^{\text{th}}$ position by w_{ij}. If $w_{ij} = 0$ then replace w_{ij} by the 2×2 matrix with every entry equal to zero. If $w_{ij} = 1$, then replace w_{ij} by the 2×2 identity matrix. If $w_{ij} = -1$, then replace w_{ij} by the matrix,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The resulting $(0,1)-$matrix with $2v$ rows and $2b$ columns is the incidence matrix of a GD $2-\left(2v,k,\left(0,\frac{\lambda}{2}\right)\right)$ with point-classes of size two.

Chapter 2

Point-weighted Designs - Introduction and Some Constructions

In this chapter we define point-weighted designs and the notation we will be using when discussing them. We also give some constructions of families of point-weighted designs.

2.1 Basic Definitions

Definition 2.1 *A* **point-weighted structure**, $(\mathbf{V},\mathbf{B},\mathbf{I},w)$ *is an incidence structure with point-set* $\mathbf{V}$*, block-set* $\mathbf{B}$ *and incidence relation* $\mathbf{I}$*, together with a 'weight function',* $w : \mathbf{V} \to \mathbb{Z}^+$*, assigning a 'weight' to every point.*

Definition 2.2 *For any point-weighted structure,* $\mathcal{S} = (\mathbf{V},\mathbf{B},\mathbf{I},w)$*, we define the* **weight-set***,* $\mathbf{W}$*, to be the image of the weight function* w*.*

Since any point-weighted structure is an incidence structure together with a weight function, it is natural to define the *underlying incidence structure* of a point-weighted structure to be that on which it is based, i.e.,

Definition 2.3 *The* **underlying incidence structure** *of a point-weighted structure* $\mathcal{S} = (\mathbf{V},\mathbf{B},\mathbf{I},w)$ *is the incidence structure* $\mathcal{U} = (\mathbf{V},\mathbf{B},\mathbf{I})$*.*

We can now define a point-weighted design and a $t-(v,k,\lambda;\mathbf{W})$ point-weighted design:

Definition 2.4 *A* **point-weighted design** *with parameters* v *and* k *is a finite point-weighted structure with no repeated blocks, the sum of the weights of all the points in the point-set equal to* v*, and the sum of the weights of the points incident with any one block equal to a constant* k*.*

As with incidence structures, we say that a point-weighted structure is **proper** if it contains a block incident with more than one point but not all of the points. Also, as with designs, since no two blocks are incident with exactly the same points, we can identify a block with the points with which it is incident. It is often convenient to consider the blocks to be subsets of the point-set, in which case a point P is said to be 'contained in' a block x if P and x are incident.

Definition 2.5 *A* $t-(v,k,\lambda;\mathbf{W})$ **point-weighted design** *is a point-weighted design with parameters* v *and* k *and weight-set* $\mathbf{W}$*, and the property that every set of* t *distinct points is contained in exactly* λ *blocks, for positive integers* t *and* λ*.*

As with $t-(v,K,\lambda)$ designs we shall assume throughout this thesis that in a $t-(v,k,\lambda;\mathbf{W})$ point-weighted design each block is incident with at least t points. For ease of expression, we shall often write $t-(v,k,\lambda;\mathbf{W})$ to mean $t-(v,k,\lambda;\mathbf{W})$ point-weighted design. We write u for the number of points in the point-set of a point-weighted design, i.e. $u=|\mathbf{V}|$. Then, denoting the point-set by,

$$\mathbf{V}=\{P_i \mid i=1,2,\ldots,u\},$$

we have,

$$v=\sum_{i=1}^{u} w(P_i).$$

We note that a point-weighted structure in which all points have weight 1 has the property that the number of points on any one block is equal to the sum of the weights of those points, and the number of points in the point-set is equal to the sum of their weights (i.e., $v=|\mathbf{V}|=u$). Hence, the underlying incidence structure of a $t-(v,k,\lambda;\{1\})$ is a $t-(v,k,\lambda)$. We also note that the underlying incidence structure of any $t-(v,k,\lambda;\mathbf{W})$ is a $t-(v,K,\lambda)$, for some set of positive integers K, since every set of t distinct points is incident with exactly λ blocks.

Example 2.6 Consider the simple incidence structure in Figure 2.6, a degenerate projective plane, with point-set $\mathbf{V}=\{P_1,P_2,P_3,P_4,P_5\}$ and block-set $\mathbf{B}=\{x_1,x_2,x_3,x_4,x_5\}$. Considering the blocks as subsets of the point-set we have,

$$\begin{aligned}
x_1 &= \{P_1,P_2\},\\
x_2 &= \{P_1,P_3\},\\
x_3 &= \{P_1,P_4\},\\
x_4 &= \{P_1,P_5\},\\
x_5 &= \{P_2,P_3,P_4,P_5\}.
\end{aligned}$$

Define the weight function $w:\mathbf{V}\to\{1,3\}$ by,

$$\begin{aligned}
&w(P_1)=3,\\
&w(P_2)=w(P_3)=w(P_4)=w(P_5)=1.
\end{aligned}$$

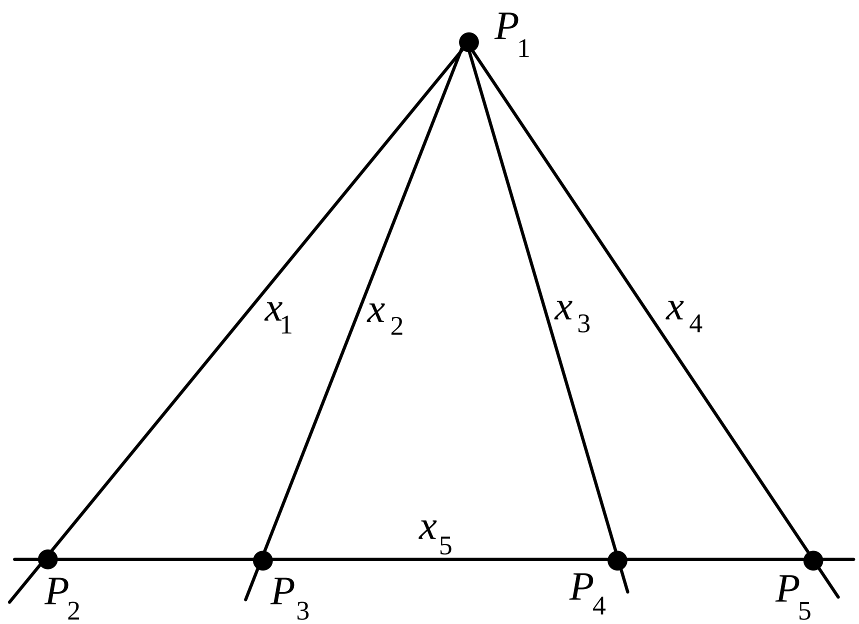

Figure 2.1: A degenerate projective plane

Then, the sum of the weights of the points in every block is equal to 4 and every pair of points occurs in exactly one block. Hence, since the sum of the weights of all the points in $\mathbf{V}$ is 7, we have constructed a $2-(7,4,1;\{1,3\})$.

We have already noted that the underlying incidence structure of a $t-(v,k,\lambda;\{1\})$ is a $t-(v,k,\lambda)$, but in fact we can say slightly more:

Lemma 2.7 *Let $\mathcal{D}$ be a $t-(v,k,\lambda;\{a\})$ for some positive integer a. Then the underlying incidence structure of $\mathcal{D}$ is a $t-\left(\frac{v}{a},\frac{k}{a},\lambda\right)$.*

Proof
Since $\mathbf{W}=\{a\}$, every point of $\mathcal{D}$ must have weight a. Thus the sum of the weights of the points of $\mathcal{D}$, v, must equal ua, where u is the number of points. Similarly, for any block of $\mathcal{D}$, the sum of the weights of the points incident with the block must equal k_1a, where k_1 is the number of points on the block. But the sum of the weights of the points on each block is k, and so each block must be incident with the same number of points, $k_1=\frac{k}{a}$. So, the underlying incidence structure of $\mathcal{D}$ is a $t-(u,k_1,\lambda)$ or, substituting for u and k_1, a $t-\left(\frac{v}{a},\frac{k}{a},\lambda\right)$. ∎

Clearly, given any $t-(v,k,\lambda;\mathbf{W}_1)$, $\mathcal{D}_1$ say, and any positive integer a, we can construct $\mathcal{D}_2$ - a $t-(va,ka,\lambda;\mathbf{W}_2)$ - by setting the underlying incidence structure of $\mathcal{D}_2$ to be the same as that of $\mathcal{D}_1$ and then setting the weight of every point in $\mathcal{D}_2$ to be its weight in $\mathcal{D}_1$ multiplied by a. This leads us to define a notion of equivalence of point-weighted structures:

Definition 2.8 *Let $\mathcal{S}_1 = (\mathbf{V}_1, \mathbf{B}_1, \mathbf{I}_1, w_1)$ and $\mathcal{S}_2 = (\mathbf{V}_2, \mathbf{B}_2, \mathbf{I}_2, w_2)$ be two point-weighted structures and let $\mathcal{U}_1 = (\mathbf{V}_1, \mathbf{B}_1, \mathbf{I}_1)$ and $\mathcal{U}_2 = (\mathbf{V}_2, \mathbf{B}_2, \mathbf{I}_2)$ be their respective underlying incidence structures. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be* **equivalent** *if there exists an isomorphism $\phi : \mathcal{U}_1 \to \mathcal{U}_2$ satisfying,*

$$w_2(\phi(P)) = aw_1(P),$$

for every point $P \in \mathbf{V}_1$ and for some positive rational a.

So, $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent if their underlying incidence structures are isomorphic and one weight function is a 'multiple' of the other. We note that this definition of equivalence does indeed give an equivalence relation as the name suggests. In the case where a is equal to one we say that the two point-weighted structures are isomorphic, or more formally:

Definition 2.9 *Let $\mathcal{S}_1 = (\mathbf{V}_1, \mathbf{B}_1, \mathbf{I}_1, w_1)$ and $\mathcal{S}_2 = (\mathbf{V}_2, \mathbf{B}_2, \mathbf{I}_2, w_2)$ be two point-weighted structures and let $\mathcal{U}_1 = (\mathbf{V}_1, \mathbf{B}_1, \mathbf{I}_1)$ and $\mathcal{U}_2 = (\mathbf{V}_2, \mathbf{B}_2, \mathbf{I}_2)$ be their respective underlying incidence structures. An* **isomorphism** *from $\mathcal{S}_1$ to $\mathcal{S}_2$ is an isomorphism, ϕ, from $\mathcal{U}_1$ to $\mathcal{U}_2$ satisfying,*

$$w_2(\phi(P)) = w_1(P),$$

for every point $P \in \mathbf{V}_1$. Two point-weighted structures are **isomorphic** *if there exists an isomorphism from one to the other.*

Definition 2.10 *An* **automorphism** *of a point-weighted structure $\mathcal{S}$ is an isomorphism of $\mathcal{S}$ onto itself (i.e., an automorphism of the underlying incidence structure which 'preserves weights').*

It is easily verified that the set of all automorphisms of a point-weighted structure $\mathcal{S}$ forms a group whose binary operation is the usual product of mappings, and we denote this group by $\mathrm{Aut}\mathcal{S}$. We note that any automorphism of $\mathcal{S}$ is also an automorphism of its underlying incidence structure $\mathcal{U}$. Hence $\mathrm{Aut}\mathcal{S}$ is a subgroup of $\mathrm{Aut}\,\mathcal{U}$.

Finally, for any finite point-weighted structure, we define a *weighted incidence matrix* as follows:

Definition 2.11 *Let $S = (\mathbf{V}, \mathbf{B}, \mathbf{I}, w)$ be a finite point-weighted structure and set $u = |\mathbf{V}|$ and $b = |\mathbf{B}|$. Let the points of $\mathcal{S}$ be labelled $P_1, P_2, \ldots, P_u$ and let the blocks be labelled $x_1, x_2, \ldots, x_b$. Then the $u \times b$ matrix $A_W = (a_{ij})$ defined by,*

$$a_{ij} = \begin{cases} w(P_i) & \text{if } P_i \text{ is incident with } x_j \\ 0 & \text{otherwise,} \end{cases}$$

is called the **weighted incidence matrix** *of $\mathcal{S}$ with respect to the above labelling of the points and blocks.*

Let S =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a finite point-weighted structure with underlying incidence structure $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$), and label the points and blocks of $\mathcal{S}$ (and hence $\mathcal{U}$) as in the above definition. Let A be the incidence matrix of $\mathcal{U}$ with respect to such a labelling of the points and blocks. We note that the weighted incidence matrix, A_W, of $\mathcal{S}$ with respect to the same labelling of points and blocks is then given by,

$$A_W = WA,$$

where W is the diagonal matrix, $W = \mathrm{diag}(w(P_1), w(P_2), \ldots, w(P_u))$.

The most convenient way to completely specify a point-weighted structure on paper is often by giving a weighted incidence matrix of it, and all the specific examples of point-weighted structures given in this thesis will be represented in such a way.

Example 2.12 The point-weighted design constructed in Example 2.6 can be represented by the weighted incidence matrix,

$$A_W = \begin{pmatrix} 3 & 3 & 3 & 3 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Example 2.13 A $2-(17,5,1;\{1,2,3\})$ is specified completely by the weighted incidence matrix given in Figure 2.2.

$$\begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Figure 2.2: A weighted incidence matrix of a $2-(17,5,1;\{1,2,3\})$.

2.2 Some Constructions

We now exhibit three constructions of families of point-weighted designs. We start with a construction of point-weighted designs with one point of weight greater than 1 and all other points of weight 1.

2.2.1 Imbedding a Block Design

Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be a $t-(u,k,\lambda)$. Label the points of $\mathcal{D}$ as $P_1, P_2, \ldots, P_u$ and the blocks $x_1, x_2, \ldots, x_b$ (where $b = |\mathbf{B}|$) and let A be the incidence matrix of $\mathcal{D}$ with respect to this labelling. Let $\mathcal{T} = (\mathbf{V}, \mathbf{B}_T, \mathbf{I}_T)$ be the trivial design with block-size $t-1$ on the u points $P_1, P_2, \ldots, P_u$, and let T be the incidence matrix of $\mathcal{T}$ with respect to the above labelling of the points and some labelling of the blocks $z_1, z_2, \ldots, z_{\binom{u}{t-1}}$.

We define a point-weighted structure $\mathcal{D}'$, by adjoining to $\mathcal{D}$ a new point, P_{u+1}, distinct from $P_1, P_2, \ldots, P_u$, and new blocks $y_1, y_2, \ldots, y_{\binom{u}{t-1}}$, distinct from the blocks of $\mathcal{D}$ and $\mathcal{T}$. Put $\mathcal{D}' = (\mathbf{V}', \mathbf{B}', \mathbf{I}', w)$ by setting,

- $\mathbf{V}' = \mathbf{V} \cup \{P_{u+1}\}$,
- $\mathbf{B}' = \mathbf{B} \cup \{y_j \mid j = 1, 2, \ldots, \binom{u}{t-1}\}$,
- $\mathbf{I}' = \mathbf{I} \cup \mathbf{I}_1$,

where $\mathbf{I}_1$ is defined by,

$$\begin{array}{llll} (P_{u+1}, y_j) & \in \mathbf{I}_1, & & j = 1, 2, \ldots \binom{u}{t-1} \\ (P_i, y_j) & \in \mathbf{I}_1 \iff & (P_i, z_j) \in \mathbf{I}_T, & j = 1, 2, \ldots \binom{u}{t-1} \text{ and } i = 1, 2, \ldots, u. \end{array}$$

So, considering the blocks as sets of points, each new block contains the point P_{u+1} together with a set of $t-1$ points from $\mathbf{V}$, and every possible set of $t-1$ points from $\mathbf{V}$ determines such a block.

Define the weight function w on the points of $\mathcal{D}'$ by,

$$w(P_i) = \begin{cases} 1 & \text{if } i \in \{1, 2, ..., u\} \\ k-t+1 & \text{if } i = u+1. \end{cases}$$

Then $\mathcal{D}'$ is a point-weighted structure. Order the blocks of $\mathcal{D}'$ with the new blocks first, i.e. as $y_1, y_2, \ldots, y_{\binom{u}{t-1}}, x_1, x_2, \ldots, x_b$, and the points $P_1, P_2, \ldots, P_u, P_{u+1}$. Then the weighted incidence matrix of $\mathcal{D}'$ with respect to this ordering is,

$$A'_W = \left(\begin{array}{cccc|cccc} & T & & & & A & & \\ \hline (k-t+1) & (k-t+1) & \ldots & (k-t+1) & 0 & 0 & \ldots & 0 \end{array} \right).$$

Lemma 2.14 *If $\mathcal{D}$ is a $t-(u,k,1)$ then $\mathcal{D}'$ is a $t-(u+k-t+1, k, 1; \{1, (k-t+1)\})$.*

Proof

Each block of $\mathcal{D}'$ is incident with either k points of weight 1 or one point of weight $k-t+1$ and $t-1$ points of weight 1. Thus, for every block $x \in \mathbf{B}'$,

$$\sum_{PIx} w(P) = k.$$

Each of the new blocks $y_j,\ j = 1, 2, \ldots, \binom{u}{t-1}$, is incident with the new point, P_{u+1}, and so the new blocks are all distinct from the blocks of $\mathcal{D}$. Thus, since $\mathcal{D}$ and $\mathcal{T}$ have no repeated blocks, $\mathcal{D}^{'}$ has no repeated blocks and is a point-weighted design.

The sum of the weights of all the points in $\mathcal{D}^{'}$ is,

$$\begin{aligned}\sum_{i=1}^{u+1} w(P_i) &= \sum_{i=1}^{u} w(P_i) + w(P_{u+1}) \\ &= u + (k - t + 1).\end{aligned}$$

Since $\mathcal{D}$ is a $t-(u,k,1)$, every t-set of points of $\mathcal{D}$ is incident with exactly one block in $\mathcal{D}$, and hence exactly one block in $\mathcal{D}^{'}$. To check that $\mathcal{D}^{'}$ is the required point-weighted design we need to show that every t-set of points of the form $\{P_{u+1}, P_{i_1}, P_{i_2}, \ldots, P_{i_{t-1}}\}$, $P_{i_1}, P_{i_2}, \ldots, P_{i_{t-1}} \in \mathbf{V}$, is incident with exactly one block. But every $(t-1)$-set of points, $\{P_{i_1}, P_{i_2}, \ldots, P_{i_{t-1}}\}$, $P_{i_1}, P_{i_2}, \ldots, P_{i_{t-1}} \in \mathbf{V}$, is incident with exactly one block in $\mathcal{T}$ and so by construction is incident with exactly one block in $\mathcal{D}^{'}$ together with the point P_{u+1}. ■

This construction is a generalisation of the case when $t = 2$, which was proved as joint work with T.Powlesland.

2.2.2 Imbedding a Resolvable Block Design

Let $\mathcal{A} = (\mathbf{V}_1, \mathbf{B}_1, \mathbf{I}_1)$ be a $2-(u,k,\lambda)$ for some $\lambda \geq 1$ which admits a λ-resolution. Let $c = \frac{u-1}{k-1}$ be the number of resolution classes and label the resolution classes $\beta_1, \beta_2, \ldots, \beta_c$. Suppose there exists a $2-\left(c, \frac{k}{a}+1, \lambda\right)$, $\mathcal{R} = (\mathbf{V}_2, \mathbf{B}_2, \mathbf{I}_2)$, for some positive integer a. Set $b_1 = |\mathbf{B}_1|$ and $b_2 = |\mathbf{B}_2|$, and label the points and blocks of $\mathcal{A}$ and $\mathcal{R}$ as:

$$\begin{aligned}\mathbf{V}_1 &= \{P_i \mid i = 1, 2, \ldots, u\}, \\ \mathbf{B}_1 &= \{x_j \mid j = 1, 2, \ldots, b_1\}, \\ \mathbf{V}_2 &= \{B_i \mid i = 1, 2, \ldots, c\}, \\ \mathbf{B}_2 &= \{y_j \mid j = 1, 2, \ldots, b_2\}.\end{aligned}$$

We construct a point-weighted structure $\mathcal{D} = (\mathbf{V}, \mathbf{B}, \mathbf{I}, w)$, by setting:

- $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$,
- $\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2$,
- $\mathbf{I} = \mathbf{I}_1 \cup \mathbf{I}_2 \cup \mathbf{I}_3$, where $\mathbf{I}_3$ is a subset of $\mathbf{V}_2 \times \mathbf{B}_1$ and is defined by,

$$(B_i, x_j) \in \mathbf{I}_3 \iff x_j \in \beta_i \qquad i = 1, 2, \ldots, c \text{ and } j = 1, 2, \ldots, b_1,$$

- w to be the function defined by:

$$\begin{aligned}w(P_i) &= 1 \qquad i = 1, 2, \ldots, u, \\ w(B_i) &= a \qquad i = 1, 2, \ldots, c.\end{aligned}$$

Lemma 2.15 *The point-weighted structure $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) constructed above is a* $2-(u+ca, k+a, \lambda; \{1, a\})$.

Proof

$\mathcal{D}$ has u points of weight 1 and c points of weight a and so, labelling the points of $\mathbf{V}$ as $Q_1, Q_2, \ldots, Q_{u+c}$,

$$\sum_{i=1}^{u+c} w(Q_i) = u + ca.$$

There are two types of blocks in $\mathcal{D}$ - those from $\mathbf{B}_1$ and those from $\mathbf{B}_2$. The λ-resolution is a partition of $\mathbf{B}_1$ and so each block from $\mathbf{B}_1$ is in exactly one resolution class. Hence, every block from $\mathbf{B}_1$ is incident in $\mathcal{D}$ with exactly k points from $\mathbf{V}_1$, all of weight 1, and one point from $\mathbf{V}_2$, of weight a. So, the sum of the weights of the points incident with a block from $\mathbf{B}_1$ is $k+a$. Every block from $\mathbf{B}_2$ is incident with $\frac{k}{a}+1$ points from $\mathbf{V}_2$, all of weight a, and no points from $\mathbf{V}_1$. Hence, the sum of the weights of the points incident with a block from $\mathbf{B}_2$ is $a\left(\frac{k}{a}+1\right) = k+a$. Clearly, no two blocks of $\mathcal{D}$ are incident with exactly the same points, thus $\mathcal{D}$ is a point-weighted design.

Since $\mathcal{A}$ is a $2-(u,k,\lambda)$, every pair of points from $\mathbf{V}_1$ is incident with exactly λ blocks from $\mathbf{B}_1$ in $\mathcal{D}$. Each block from $\mathbf{B}_2$ is incident with points only from $\mathbf{V}_2$ and so every pair of points from $\mathbf{V}_1$ is incident with exactly λ blocks of $\mathcal{D}$. Since $\mathcal{R}$ is a $2-\left(c, \frac{k}{a}+1, \lambda\right)$, every pair of points from $\mathbf{V}_2$ is incident with exactly λ blocks from $\mathbf{B}_2$ in $\mathcal{D}$. Each block from $\mathbf{B}_1$ is incident with only one point from $\mathbf{V}_2$ and so every pair of points from $\mathbf{V}_2$ is incident with exactly λ blocks of $\mathcal{D}$.

Let P be a point from $\mathbf{V}_1$ and B_i a point from $\mathbf{V}_2$, for some $i \in \{1, 2, \ldots, c\}$. We need to show that there are exactly λ blocks of $\mathcal{D}$ incident with both P and B_i. Clearly, no block from $\mathbf{B}_2$ is incident with both P and B_i, since P is from $\mathbf{V}_1$. Then, by the definition of a λ-resolution, there are exactly λ blocks in the resolution class β_i incident with P, and each of these is also incident with B_i in $\mathcal{D}$. Hence, there are exactly λ blocks of $\mathcal{D}$ incident with both P and B_i. Thus, $\mathcal{D}$ is a $2-(u+ca, k+a, \lambda; \{1, a\})$. ■

If $\mathcal{A}$ is a $2-(u,k,1)$ admitting a parallelism, then setting $a = k$ and $\mathcal{R}$ to be the trivial block design with $c = \frac{u-1}{k-1}$ points and block-size 2, we can construct a $2-\left(\frac{2uk-(u+k)}{k-1}, 2k, 1; \{1, k\}\right)$. A specific example of this is given in Example 2.16. Also, if $\mathcal{A}$ is an affine plane of order n (a $2-(n^2, n, 1)$), then setting $a = 1$ and $\mathcal{R}$ to be the trivial $2-(n+1, n+1, 1)$ with just one block incident with all the points of $\mathcal{R}$ yields a point-weighted design in which all points have weight 1 and the underlying incidence structure is a projective plane of order n.

The case when $\mathcal{A}$ is an affine plane, $a = k$ and $\mathcal{R}$ is a trivial design with block-size 2 was developed jointly with T. Powlesland.

Example 2.16 Letting $\mathcal{A}$ be the affine plane of order 3 in the above construction; setting $x = 3$, and letting $\mathcal{R}$ be the trivial block design with four points and block-size two yields a $2-(21, 6, 1; \{1, 3\})$. A weighted incidence matrix for such a point-weighted design is given in Figure 2.3.

$$\begin{pmatrix}
0 & 0 & 0 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 3 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & 0 & 3 & 0 & 3 & 3 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Figure 2.3: A $2-(21, 6, 1; \{1, 3\})$ constructed from the affine plane of order 3

2.2.3 Construction from a Block Design with a Parallelism

Let $\mathcal{R} = (\mathbf{V}_0, \mathbf{B}_0, \mathbf{I}_0)$ be a proper $2-(u, k, 1)$ which admits a parallelism. Let $c = \frac{u-1}{k-1}$ be the number of parallel classes and set $b = |\mathbf{B}_0|$. Label the points of $\mathcal{R}$ as $P_1, P_2, \dots, P_u$, the blocks $x_{0,1}, x_{0,2}, \dots, x_{0,b}$, and the parallel classes $\beta_1, \beta_2, \dots \beta_c$.

Let λ be an integer in the range $0 \leq \lambda < c$, and let a be a positive integer. Suppose $\mathcal{S} = (\mathbf{V}_\lambda, \mathbf{B}_\lambda, \mathbf{I}_\lambda)$ is a $2-(c, a+1, \lambda)$. Label the points of $\mathcal{S}$ as $B_1, B_2, \dots, B_c$, and setting $b_\lambda = |\mathbf{B}_\lambda|$, label the blocks of $\mathcal{S}$ as $y_1, y_2, \dots, y_{b_\lambda}$. Let σ be a permutation of the set of indices, $C = \{1, 2, \dots, c\}$, such that, for each $i \in C$, $i\sigma^j \neq i$ for each $j = 1, 2, \dots, \lambda - 1$ (where $i\sigma^j$ denotes the image of i under j applications of σ). Then we construct a point-weighted structure $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) as follows:

Set $\mathbf{V} = \mathbf{V}_0 \cup \mathbf{V}_\lambda$.

For each $l = 1, 2, \dots, \lambda - 1$, define a block-set, $\mathbf{B}_l$, to consist of b blocks labelled as,

$$\mathbf{B}_l = \{x_{l,1}, x_{l,2}, \dots, x_{l,b}\}.$$

Then set,

$$\mathbf{B} = \mathbf{B}_0 \cup \mathbf{B}_1 \cup \mathbf{B}_2 \cup \ldots \cup \mathbf{B}_{\lambda-1} \cup \mathbf{B}_\lambda.$$

Define the set $\mathbf{I}^{'} \subset (\mathbf{V}_\lambda \times \mathbf{B}_0)$ by,

$$(B_i, x_{0,j}) \in \mathbf{I}^{'} \iff x_{0,j} \in \beta_i \qquad i = 1, 2, \dots, c \text{ and } j = 1, 2, \dots, b,$$

and set $\mathbf{I}_0^{'} = \mathbf{I}_0 \cup \mathbf{I}^{'}$. For each $l = 1, 2, \ldots, \lambda - 1$, define the set $\mathbf{I}_l \subset (\mathbf{V} \times \mathbf{B}_l)$ by,

$(P_i, x_{l,j}) \in \mathbf{I}_l \iff (P_i, x_{0,j}) \in \mathbf{I}_0 \qquad i = 1, 2, \ldots, c$ and $j = 1, 2, \ldots, b$,

and,

$(B_i, x_{l,j}) \in \mathbf{I}_l \iff x_{0,j} \in \beta_{i\sigma^l} \qquad i = 1, 2, \ldots, c$ and $j = 1, 2, \ldots, b$.

Then define incidence in $\mathcal{D}$ by,

$$\mathbf{I} = \mathbf{I}_0^{'} \cup \mathbf{I}_1 \cup \mathbf{I}_2 \cup \ldots \cup \mathbf{I}_{\lambda-1} \cup \mathbf{I}_\lambda.$$

So essentially, we are using the elements of $\mathbf{V}_\lambda$ to assign a labelling to the parallel classes of $\mathcal{R}$, and then permuting the labels.

Finally, we define the weight function w by,

$$\begin{aligned} w(P_i) &= a \qquad i = 1, 2, \ldots, u, \\ w(B_i) &= k \qquad i = 1, 2, \ldots, c. \end{aligned}$$

Lemma 2.17 *The point-weighted structure* $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) *constructed above is a* $2-(ua+ck, k\,(a+1)\,, \lambda; \{a, k\})$.

Proof

$\mathcal{D}$ has exactly u points of weight a and c points of weight k, so the sum of the weights of all the points of $\mathcal{D}$ is $ua + ck$.

Each block of $\mathcal{R}$ is in exactly one parallel class of $\mathcal{R}$. Hence, each block from $\mathbf{B}_0$ is incident with exactly one point from $\mathbf{V}_\lambda$ in $\mathcal{D}$, the point corresponding to the parallel class of $\mathcal{R}$ containing that block. Each block from $\mathbf{B}_0$ is also incident with exactly k points from $\mathbf{V}_0$. Hence, the sum of the weights of the points incident with a block from $\mathbf{B}_0$ is $k + ak$. By definition, each block from $\mathbf{B}_l$, for each $l \in \{1, 2, \ldots, \lambda - 1\}$, is also incident with exactly one point from $\mathbf{V}_\lambda$ and k points from $\mathbf{V}_0$. Hence, the sum of the weights of the points incident with a block from $\mathbf{B}_l$, for some $l \in \{1, 2, \ldots, \lambda - 1\}$, is also $k + ak$.

Each block from $\mathbf{B}_\lambda$ is incident with exactly $a + 1$ points from $\mathbf{V}_\lambda$ and no points from $\mathbf{V}_0$, and so the sum of the weights of the points incident with a block from $\mathbf{B}_\lambda$ is $(a + 1)k$. So, for $\mathcal{D}$ to be a point-weighted design we need to show that no two blocks of $\mathcal{D}$ are incident with exactly the same points.

Clearly, no two blocks from the same block-set $\mathbf{B}_l$, $l \in \{0, 1, \ldots, \lambda\}$, are incident with exactly the same points since $\mathcal{R}$ and $\mathcal{S}$ are designs. Also, a block from $\mathbf{B}_\lambda$ cannot be incident with exactly the same points as a block from $\mathbf{B}_l$, for any $l \in \{1, 2, \ldots, \lambda - 1\}$, since the blocks from $\mathbf{B}_\lambda$ are incident with points only from $\mathbf{V}_\lambda$.

For some $l, m \in \{0, 1, \ldots, \lambda - 1\}$ with $l \neq m$, let $x_{l,i}$ and $x_{m,j}$ be blocks from $\mathbf{B}_l$ and $\mathbf{B}_m$ respectively, . Then, if $i \neq j$, $x_{l,i}$ and $x_{m,j}$ are not both incident with the same points from $\mathbf{V}_0$. If $i = j$, $x_{l,i}$ and $x_{m,i}$ are incident with the same points from $\mathbf{V}_0$ but, by the definition of σ, each with a different point from $\mathbf{V}_\lambda$. Thus, $\mathcal{D}$ is a point-weighted design.

We now show that every pair of points of $\mathcal{D}$ is on exactly λ blocks of $\mathcal{D}$. Clearly, any pair of points from $\mathbf{V}_\lambda$ is on exactly λ blocks from $\mathbf{B}_\lambda$ and no other blocks of $\mathcal{D}$. Also, since $\mathcal{R}$ is a $2-(u, k, 1)$, every pair of points from $\mathbf{V}_0$ is on exactly one block from $\mathbf{B}_0$. Thus, every pair of points from $\mathbf{V}_0$ is on exactly one block from $\mathbf{B}_i$ for each $i = 1, 2, \ldots, \lambda - 1$. The blocks from $\mathbf{B}_\lambda$ are incident with points only from $\mathbf{V}_\lambda$, hence every pair of points from $\mathbf{V}_0$ is on exactly λ blocks of $\mathcal{D}$.

Let P be a point from $\mathbf{V}_0$. Then P is on exactly one block in each parallel class of $\mathcal{R}$. Thus, if B_i ($i \in \{1, 2, \ldots, c\}$) is a point from $\mathbf{V}_\lambda$, P is on exactly one block from $\mathbf{B}_0$ together with B_i - the block in the parallel class β_i which is incident with P. Also, for some $l \in \{1, 2, \ldots, \lambda - 1\}$, P is on exactly one block of $\mathbf{B}_l$ together with B_i - the block corresponding with the block of $\mathcal{R}$ in the parallel class $\beta_{i\sigma^l}$ which is incident with P. Finally, there are no blocks of $\mathbf{B}_\lambda$ incident with P. Hence there are exactly λ blocks of $\mathcal{D}$ incident with both P and B_i.

Thus, $\mathcal{D}$ is a $2 - (ua + ck, k\,(a+1)\,, \lambda; \{a, k\})$. ■

Given a proper $2 - (u, k, 1)$, $\mathcal{R}$, which admits a parallelism with c parallel classes, it is possible to construct a certain point-weighted design, $\mathcal{D}$, as described above. Setting $a = \lambda = 1$, and $\mathcal{S}$ to be the trivial block design with c points and block-size two yields a $2 - (u + ck, 2k, 1; \{1, k\})$.

Example 2.18 Let $\mathcal{R}$ be the trivial block design with six points and block-size two. Then $\mathcal{R}$ admits a parallelism with $c = 5$ parallel classes, and we label these $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$. Let σ be the cyclic permutation $(1, 2, 3, 4, 5)$ and set $\lambda = 3$ and $a = 3$. Let $\mathcal{S}$ be the trivial block design with five points and block-size two. Then, using the above construction with these parameters, we obtain a $2 - (28, 8, 3; \{2, 3\})$.The transpose of a weighted incidence matrix of such a point-weighted design is given in Figure 2.4.

$$
\left(\begin{array}{cccccc|ccccc}
3 & 3 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
3 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 0 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 2 \\
0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 3 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 3 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 3 & 0 & 3 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 3 & 0 & 2 & 0 & 0 & 0 \\
\hline
3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
3 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 \\
0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 3 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 3 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 3 & 3 & 2 & 0 & 0 & 0 & 0 \\
\hline
3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
3 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 \\
0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 3 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 3 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 3 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 2 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2
\end{array}\right)
$$

Figure 2.4: A transposed weighted incidence matrix of a $2-(28, 8, 3; \{2, 3\})$.

Chapter 3

Some Combinatorial Analysis of Point-Weighted Designs

In this chapter we consider some properties of $t-(v,k,\lambda;\mathbf{W})$ point-weighted designs and their underlying incidence structures. Let $\mathcal{D}$ be a $t-(v,k,\lambda;\mathbf{W})$ with $v>k$ and $t>1$, and let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{D}$. We obtain a necessary and sufficient condition for $\mathcal{U}$ to be a block design, and show that any automorphism of $\mathcal{U}$ is also an automorphism of $\mathcal{D}$. We show that a point-weighted design is uniquely determined by its underlying incidence structure up to equivalence if and only if the rank of its underlying incidence structure is equal to the number of points. The chapter concludes by considering some properties of $2-(v,k,\lambda;\mathbf{W})$ point-weighted designs. We start by obtaining a result for $1-(v,k,r;\mathbf{W})$ point-weighted designs which will be required later.

Lemma 3.1 *Let $\mathcal{D}=(\mathbf{V},\mathbf{B},\mathbf{I},w)$ be a $1-(v,k,r;\mathbf{W})$ and define $b=|\mathbf{B}|$ to be the number of blocks. Then,*

$$bk=vr.$$

Proof

A *flag* in a structure is a pair (P,x), where P is a point and x is a block on P. Let the *flag-weight* of (P,x) be the value of the weight of the point P in a flag. We sum all flag-weights in $\mathcal{D}$ in two ways. Any point P is on exactly r blocks, since $\mathcal{D}$ is a $1-(v,k,r;\mathbf{W})$, and so the sum $\sum_{P\in\mathbf{V}} w(P)r$ simply sums all the flag-weights. But,

$$\begin{aligned}\sum_{P\in\mathbf{V}} w(P)r &= r\sum_{P\in\mathbf{V}} w(P)\\ &= vr.\end{aligned}$$

Also, the weights of the points in any block, x, sum to k and there are b blocks, so the sum of all flag-weights equals bk. Hence, $bk = vr$. ∎

Clearly, in the case where $\mathcal{D}$ is a $t-(v,k,\lambda;\{1\})$, the underlying incidence structure is a $t-(v,k,\lambda)$, and thus a $1-(v,k,r)$ for some r. Then the above lemma yields the well-known Result 1.2 for block designs.

3.1 Properties of a $t-(v,k,\lambda;\mathbf{W})$

Recall from Chapter 1 that if $\mathcal{D}$ is a $t-(v,k,\lambda)$ then for every integer s, such that $1 \leq s < t$, there exists a positive integer λ_s such that $\mathcal{D}$ is a $s-(v,k,\lambda_s)$. In this section we will show that if $\mathcal{D}$ is a $t-(v,k,\lambda;\mathbf{W})$ then there exists an integer λ_{t-1} such that $\mathcal{D}$ is a $(t-1)-(v,k,\lambda_{t-1};\mathbf{W})$ if and only if $|\mathbf{W}| = 1$. We first require some preliminary definitions and lemmas.

Lemma 3.2 *Let $A = \{a_1, a_2, \ldots, a_u\}$ be a finite multiset of some $u > 1$ integers, not all equal. Then, for all $1 \leq t < u$, there exists a subset $T_1 \subset A$, of size t, with $\sigma_1 = \sum_{a_i \in T_1} a_i$, and a subset $T_2 \subset A$, of size t, with $\sigma_2 = \sum_{a_i \in T_2} a_i$, satisfying,*

$$\sigma_1 > \sigma_2.$$

Proof
Since the elements of A are not all equal, there exist $l, m \in \{1, 2, \ldots, u\}$ such that $a_l > a_m$. Then we note that since $u > t$, $|A \setminus \{a_l, a_m\}| \geq t-1$. Hence, it is possible to choose a set S of size $t-1$ from $A \setminus \{a_l, a_m\}$. Let $T_1 = S \cup \{a_l\}$ and $T_2 = S \cup \{a_m\}$, and define $\sigma_1 = \sum_{a_i \in T_1} a_i$ and $\sigma_2 = \sum_{a_i \in T_2} a_i$. Then clearly $\sigma_1 > \sigma_2$. ∎

We now extend the concept of a derived incidence structure at a point P mentioned in Section 1.2, to that of a derived structure at a set of points:

Definition 3.3 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a point-weighted structure and let $S \subset \mathbf{V}$ be a set of some s points of $\mathcal{D}$, which we label as $S = \{P_1, P_2, \ldots, P_s\}$. The* **derived structure at** *S, $\mathcal{D}_S = (\mathbf{V}_S, \mathbf{B}_S, \mathbf{I}_S, w_S)$, is defined by,*

$$\begin{aligned}
\mathbf{B}_S &= \{x \in \mathbf{B} \mid (P_i, x) \in \mathbf{I} \ , \forall i = 1, 2, \ldots, s\}, \\
\mathbf{V}_S &= \mathbf{V} \setminus (S \cup \{Q \mid (Q, x) \notin \mathbf{I} \cap (\mathbf{V} \times \mathbf{B}_S), \forall x \in \mathbf{B}_S\}), \\
\mathbf{I}_S &= \mathbf{I} \cap (\mathbf{V}_S \times \mathbf{B}_S),
\end{aligned}$$

with w_S being the restriction of w to $\mathbf{V}_S$.

Definition 3.4 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a point-weighted structure and let $S \subset \mathbf{V}$ be a set of points of $\mathcal{D}$. The* **weight-sum** *of S, denoted by $\sigma(S)$, is the sum of the weights of the points contained in S. So,*

$$\sigma(S) = \sum_{P \in S} w(P).$$

Lemma 3.5 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t-(v,k,\lambda;\mathbf{W})$ for some $\lambda \geq 1$, $t > 1$, k, $v \geq k$, and weight-set $\mathbf{W}$. Let $S \subset \mathbf{V}$ be a set of $t-1$ points of $\mathcal{D}$ with weight-sum $\sigma(S)$. Then the derived structure at S, $\mathcal{D}_S = (\mathbf{V}_S, \mathbf{B}_S, \mathbf{I}_S, w_S)$, is a $1-(v-\sigma(S), k-\sigma(S), \lambda; \mathbf{W}_S)$, where $\mathbf{W}_S$ is the image of w_S.*

Proof

It is useful to consider the blocks as sets of points and incidence as containment. Then, since $\mathcal{D}$ is a $t-(v,k,\lambda;\mathbf{W})$, every point of $\mathbf{V}_S$ is contained in exactly λ blocks of $\mathcal{D}$ together with the set S, and hence in exactly λ blocks of $\mathcal{D}_S$. The point-set of $\mathcal{D}_S$ is exactly the point-set of $\mathcal{D}$ with the points of S removed, and so the sum of the weights of all the points in $\mathcal{D}_S$ is $v-\sigma(S)$. Similarly, every block of $\mathcal{D}_S$ is a block of $\mathcal{D}$ with the points of S removed. Thus, the sum of the weights of the points in any block of $\mathcal{D}_S$ is $k-\sigma(S)$. ∎

We are now in a position to obtain an expression for the number of blocks on a set of $t-1$ points in a $t-(v,k,\lambda;\mathbf{W})$ for some $t > 1$, and show that it is dependent on the weight-sum of the set.

Theorem 3.6 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t-(v,k,\lambda;\mathbf{W})$, for some $t > 1$, and let S be a set of $t-1$ distinct points of $\mathcal{D}$, with weight-sum $\sigma(S)$. Then the number of blocks on S, which we denote by λ_S, is given by,*

$$\lambda_S = \lambda \frac{(v-\sigma(S))}{(k-\sigma(S))}.$$

Proof

Consider the derived structure at the set S, $\mathcal{D}_S$. By Lemma 3.5, $\mathcal{D}_S$ is a $1-(v-\sigma(S), k-\sigma(S), \lambda; \mathbf{W}_S)$, where $\mathbf{W}_S$ is the image of the restriction of w to $\mathbf{V} \setminus S$. The total number of blocks in $\mathcal{D}_S$ is just the number of blocks of $\mathcal{D}$ on the set S, λ_S. We apply Lemma 3.1 to $\mathcal{D}_S$ to obtain the equality,

$$\lambda_S(k-\sigma(S)) = \lambda(v-\sigma(S)). \tag{3.1}$$

Since $\mathcal{D}$ is a $t-(v,k,\lambda;\mathbf{W})$, there is at least one block on the set S which is also on at least one other point, and the sum of the weights of the points in this block is k. Hence, $k > \sigma(S)$, and so we divide through by $(k-\sigma(S))$ in (3.1) to obtain the expression for λ_S in the statement of the theorem. ∎

Corollary 3.7 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t-(v,k,\lambda;\mathbf{W})$ with $v > k$. Let S_1 and S_2 be two sets each of $t-1$ points of $\mathcal{D}$ with weight-sums $\sigma(S_1)$ and $\sigma(S_2)$ respectively. Then, if $\sigma(S_1) > \sigma(S_2)$, the number of blocks on S_1 is greater than the number of blocks on S_2.*

Proof

We need to show that for $\sigma(S_1) > \sigma(S_2)$, the inequality,

$$\frac{v-\sigma(S_1)}{k-\sigma(S_1)} > \frac{v-\sigma(S_2)}{k-\sigma(S_2)},$$

holds, and then the result follows from Theorem 3.6. From the statement of the corollary we have,

$$v > k.$$

Now, $\sigma(S_1) > \sigma(S_2)$, and so we multiply through by $(\sigma(S_1) - \sigma(S_2))$,

$$\sigma(S_1)v - \sigma(S_2)v > \sigma(S_1)k - \sigma(S_2)k.$$

Subtracting $(\sigma(S_1)v + \sigma(S_1)k)$ from both sides gives,

$$-\sigma(S_1)k - \sigma(S_2)v > -\sigma(S_1)v - \sigma(S_2)k,$$

and adding $(vk + \sigma(S_1)\sigma(S_2))$ to both sides gives,

$$vk - \sigma(S_1)k - \sigma(S_2)v + \sigma(S_1)\sigma(S_2) > vk - \sigma(S_2)k - \sigma(S_1)v + \sigma(S_1)\sigma(S_2),$$

i.e.,

$$(v - \sigma(S_1))(k - \sigma(S_2)) > (v - \sigma(S_2))(k - \sigma(S_1)).$$

Recall from the proof of Theorem 3.6 that $k > \sigma(S_1)$, and so we divide through by $(k - \sigma(S_1))(k - \sigma(S_2))$ to obtain the required inequality. ■

The following example demonstrates that the condition that v is greater than k in the above corollary is necessary.

Example 3.8 Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be the point-weighted design defined by:

- $\mathbf{V} = \{P_1, P_2, P_3, P_4\}$,
- $\mathbf{B} = \{x\}$,
- $\mathbf{I} = \{(P_1, x), (P_2, x), (P_3, x), (P_4, x)\}$,
- $w(P_1) = 1, \quad w(P_2) = 2, \quad w(P_3) = 3, \quad w(P_4) = 4.$

Then $\mathcal{D}$ is a $4 - (10, 10, 1; \{1, 2, 3, 4\})$ and is an example of a point-weighted design with parameters v and k, with just one block and $v = k$. Set $S_1 = \{P_2, P_3, P_4\}$ and $S_2 = \{P_1, P_2, P_3\}$. Clearly, $\sigma(S_1) > \sigma(S_2)$ but there is exactly one block on S_1 and exactly one block on S_2. So the above corollary does not hold for $v = k$.

Corollary 3.9 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t - (v, k, \lambda; \mathbf{W})$ with $v > k$, $t > 1$, and $|\mathbf{W}| \geq 2$. Then there does not exist a constant λ_{t-1} such that $\mathcal{D}$ is a $(t-1) - (v, k, \lambda_{t-1}; \mathbf{W})$.*

Proof

We need to show that there exist two sets of $t-1$ points of $\mathcal{D}$, S_1 and S_2 say, such that the number of blocks on S_1 is greater than the number of blocks on S_2. We achieve this by exhibiting two sets of $t-1$ points of $\mathcal{D}$, S_1 and S_2, such that the weight-sum of S_1 is greater than the weight-sum of S_2. The number of blocks on S_1 is then greater than the number of blocks on S_2 by Corollary 3.7.

Let $u = |\mathbf{V}|$ and label the points of $\mathcal{D}$ as,

$$\mathbf{V} = \{P_i \mid i = 1, 2, \ldots, u\}.$$

Define the multiset Ω to be,

$$\Omega = \{w(P_i) \mid i = 1, 2, \ldots, u\}.$$

Then Ω is a multiset of u positive integers and, since $|\mathbf{W}| \geq 2$, not all the elements of Ω are equal. Also, since $v > k$, it is clear that $u > t$. By Lemma 3.2, we can then find two subsets of Ω of size $t-1$, C and D say, such that, defining σ_C to be the sum of the elements of C and σ_D to be the sum of the elements of D,

$$\sigma_C > \sigma_D.$$

Setting S_1 and S_2 to be the sets of points such that,

$$\begin{aligned} C &= \{w(P) \mid P \in S_1\}, \\ D &= \{w(P) \mid P \in S_2\}, \end{aligned}$$

gives $\sigma(S_1) > \sigma(S_2)$ as required. Hence, the number of blocks on S_1 is greater than the number of blocks on S_2. ■

Corollary 3.10 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t-(v,k,\lambda;\mathbf{W})$ with $v > k$, $t > 1$, and $|\mathbf{W}| \geq 2$. Let $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be the underlying incidence structure of $\mathcal{D}$. Then $\mathcal{U}$ is not a block design.*

Proof

Suppose $\mathcal{U}$ is a block design. Then there is a constant $k^{'}$ such that every block of $\mathcal{U}$ is on exactly $k^{'}$ points. Since $\mathcal{D}$ is a $t-(v,k,\lambda;\mathbf{W})$, $\mathcal{U}$ has the property that every set of t distinct points is incident with exactly λ blocks. Hence, $\mathcal{U}$ is a $t-(u,k^{'},\lambda)$, and so there exists a constant λ_{t-1} such that $\mathcal{U}$ is a $(t-1)-(u,k^{'},\lambda_{t-1})$. But then $\mathcal{D}$ is a $(t-1)-(v,k,\lambda_{t-1};\mathbf{W})$, contradicting Corollary 3.9. Thus $\mathcal{U}$ is not a block design. ■

Combining this corollary with Lemma 2.7 and noting that, by definition, $|\mathbf{W}| \geq 1$, we have proved the following theorem:

Theorem 3.11 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t-(v,k,\lambda;\mathbf{W})$ with $v > k$, $t > 1$, and let $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be the underlying incidence structure of $\mathcal{D}$. Then there exists a constant $k^{'}$ such that $\mathcal{U}$ is a $t-(u,k^{'},\lambda)$ if and only if $|\mathbf{W}| = 1$.* ■

In Example 3.8 we exhibit a $4-(10,10,1;\{1,2,3,4\})$ whose underlying incidence structure is a $4-(4,4,1)$. Hence the above theorem is clearly not true when $v = k$. In the following example we exhibit a $1-(6,3,2;\{1,2\})$ whose underlying incidence structure is a $1-(4,2,2)$, thus showing that the condition $t > 1$ in the above theorem is necessary.

Example 3.12 Let $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be the incidence structure with point-set $\mathbf{V} = \{P_1, P_2, P_3, P_4\}$, block-set $\mathbf{B} = \{x_1, x_2, x_3, x_4\}$ and incidence matrix,

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then $\mathcal{U}$ is a $1-(4,2,2)$. Define a weight function, w, on $\mathbf{V}$ by,

$$\begin{array}{rcccl} w(P_1) & = & w(P_2) & = & 2, \\ w(P_3) & = & w(P_4) & = & 1, \end{array}$$

and set $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w). Then $\mathcal{D}$ is a $1-(6,3,2;\{1,2\})$ with underlying incidence structure $\mathcal{U}$ - a $1-(4,2,2)$.

3.2 Automorphisms of a $t-(v,k,\lambda;\mathbf{W})$

We now use the results of Section 3.1 to show that the automorphism group of a proper $t-(v,k,\lambda;\mathbf{W})$ with $t>1$ is the same as the automorphism group of its underlying incidence structure.

Let $\mathcal{S}$ be a point-weighted structure with underlying incidence structure $\mathcal{U}$. Let P be any point of $\mathcal{S}$ and let α be an automorphism of $\mathcal{U}$. We denote the image of P under i applications of α by P^{α^i}, for $i \geq 1$, and set $P^{\alpha^1} = P^\alpha$ and $P^{\alpha^0} = P$. Similarly, letting S be a set of some s points ($s \geq 1$) we let S^α denote the image of S under α. So, if $S = \{P_j \mid j = 1,2,\ldots,s\}$ then $S^\alpha = \{P_j^\alpha \mid j = 1,2,\ldots,s\}$.

Lemma 3.13 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t-(v,k,\lambda;\mathbf{W})$ with $v>k$, $t>2$, and let α be an automorphism of the underlying incidence structure of $\mathcal{D}$, $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$). Suppose Q is a point of $\mathcal{D}$ satisfying $w(Q) > w(Q^\alpha)$. Then $w(Q^\alpha) > w(Q^{\alpha^2})$.*

Proof

Since $\mathcal{D}$ is a $t-(v,k,\lambda;\mathbf{W})$ with $v>k$ we have that $u>t$. Let S be a set of some $t-2$ distinct points not containing Q or Q^α. Then $S \cup \{Q\}$ is a set of $t-1$ distinct points. Since α preserves incidence, the number of blocks on $(S\cup\{Q\})^\alpha$ must equal the number of blocks on $S\cup\{Q\}$. Thus, by Theorem 3.6 we have,

$$\begin{aligned} \sigma(S\cup\{Q\}) &= \sigma\left((S\cup\{Q\})^\alpha\right) \\ &= \sigma\left(S^\alpha \cup Q^\alpha\right) \\ &= \sigma\left(S^\alpha\right) + w\left(Q^\alpha\right). \end{aligned}$$

But $\sigma(S\cup\{Q\}) = \sigma(S)+w(Q)$, and $w(Q) > w(Q^\alpha)$. Hence, we have established that $\sigma(S) < \sigma(S^\alpha)$.

Now consider the set of $t-1$ distinct points, $S \cup \{Q^\alpha\}$, and its image under α. The number of blocks on $(S \cup \{Q^\alpha\})^\alpha$ must equal the number of blocks on $S \cup \{Q^\alpha\}$ and so again, by Theorem 3.6 we have,

$$\sigma\left(S \cup \{Q^\alpha\}\right) = \sigma\left((S \cup \{Q^\alpha\})^\alpha\right).$$

Simplifying this gives,

$$\sigma(S) + w(Q^\alpha) = \sigma(S^\alpha) + w(Q^{\alpha^2}).$$

But, $\sigma(S) < \sigma(S^\alpha)$ and so, $w(Q^\alpha) > w(Q^{\alpha^2})$. ∎

Theorem 3.14 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t-(v,k,\lambda;\mathbf{W})$ with $v > k$, $t > 1$ and let $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be the underlying incidence structure of $\mathcal{D}$. Then $Aut\mathcal{D} \cong Aut\mathcal{U}$.*

Proof

It is clear that an automorphism of any point-weighted structure is an automorphism of its underlying incidence structure. We need to show that with $\mathcal{D}$ as above, any automorphism, α say, of its underlying incidence structure $\mathcal{U}$ is also an automorphism of $\mathcal{D}$. So we need to show that $w(P) = w(P^\alpha)$ for any point P of $\mathcal{D}$. This is clearly true when $|\mathbf{W}| = 1$ and $\mathcal{U}$ is a block design and so we consider the case $|\mathbf{W}| > 1$.

The theorem follows as a direct corollary of Theorem 3.6 when $t = 2$, since the number of blocks incident with a point in a $2-(v,k,\lambda;\mathbf{W})$ depends directly on the weight of the point. So, if $\mathcal{D}$ is a $2-(v,k,\lambda;\mathbf{W})$ and α an automorphism of its underlying incidence structure then α preserves incidence and so satisfies $w(P) = w(P^\alpha)$ for every point P of $\mathcal{D}$.

Now let $\mathcal{D}$ be a $t-(v,k,\lambda;\mathbf{W})$ with $v > k$ and $t > 2$, and let α be an automorphism of the underlying incidence structure $\mathcal{U}$. Suppose there is a point P of $\mathcal{D}$ such that $w(P) > w(P^\alpha)$. Consider the set of t points, $T = \{P^{\alpha^i} \mid i = 0,1,\ldots,t-1\}$. For each pair of points of the form $(P^{\alpha^i}, P^{\alpha^{i+1}})$, $i = 0,1,\ldots,t-2$, it is possible to find a set, S_i, of $t-2$ distinct points of $\mathcal{D}$ not containing P^{α^i} or $P^{\alpha^{i+1}}$. We then apply Lemma 3.13 recursively, to each element of T in turn. At the first step set Q to be P, and S to be S_0. At the second step set Q to be P^α and S to be S_1, and so on, so that at the i^{th} step we set Q to be $P^{\alpha^{i-1}}$ and S to be S_{i-1}. Then, given that $w(P) > w(P^\alpha)$, we have,

$$w(P) > w(P^\alpha) > w(P^{\alpha^2}) > \ldots > w(P^{\alpha^{t-2}}) > w(P^{\alpha^{t-1}}).$$

Hence, T is a set of t distinct points. Set T_1 to be the set of $t-1$ distinct points,

$$T_1 = \{P^{\alpha^i} \mid i = 0,1,\ldots,t-2\},$$

then clearly the image of T_1 under α is,

$$T_1^\alpha = \{P^{\alpha^i} \mid i = 1,2,\ldots,t-1\}.$$

Now α preserves incidence and so by Theorem 3.6,

$$\begin{aligned}\sigma(T_1) &= \sigma(T_1^{\alpha}) \\ &= \sigma(T_1) - w(P) + w(P^{\alpha^{t-1}}).\end{aligned}$$

Hence,

$$w(P) = w(P^{\alpha^{t-1}}),$$

giving a contradiction. Hence, it is not possible that $w(P) > w(P^{\alpha})$.

Now suppose there is a point P of $\mathcal{D}$ such that under some automorphism, α, of $\mathcal{U}$, $w(P) < w(P^{\alpha})$. Then, since the automorphisms of $\mathcal{U}$ form a group under composition, there is an automorphism, α^{-1}, of $\mathcal{U}$ such that $(P^{\alpha})^{\alpha^{-1}} = P$. Then, setting $Q = P^{\alpha}$, we have that $w(Q) > w(Q^{\alpha^{-1}})$. But we have shown that this cannot happen, giving a contradiction. Hence, it is not possible that $w(P) < w(P^{\alpha})$.

So we have shown that any automorphism of $\mathcal{U}$ cannot map a point to another point of different weight in $\mathcal{D}$. Hence, any automorphism of $\mathcal{U}$ is also an automorphism of $\mathcal{D}$ and the theorem is proved. ∎

The following two examples demonstrate that the conditions $v > k$ and $t > 1$ are necessary for the above theorem to hold.

Example 3.15 Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be the point-weighted design exhibited in Example 3.8, with $\mathbf{V} = \{P_1, P_2, P_3, P_4\}$ and $\mathbf{B} = \{x\}$. Let $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be the underlying incidence structure of $\mathcal{D}$ and let α be the automorphism of $\mathcal{U}$ defined by,

$$\begin{aligned}\alpha(P_1) &= P_2, \\ \alpha(P_2) &= P_1, \\ \alpha(P_3) &= P_3, \\ \alpha(P_4) &= P_4, \\ \alpha(x) &= x.\end{aligned}$$

Then, $w(\alpha(P_1)) = 2$ and $w(P_1) = 1$. Hence α is not an automorphism of $\mathcal{D}$, despite being an automorphism of $\mathcal{U}$.

Example 3.16 Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be the point-weighted design exhibited in Example 3.12 with underlying incidence structure $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$). Let α be the automorphism of $\mathcal{U}$ defined by,

$$\begin{array}{ll} \alpha(P_1) = P_3, & \alpha(P_3) = P_1, \\ \alpha(P_2) = P_4, & \alpha(P_4) = P_2, \\ \alpha(x_1) = x_1, & \alpha(x_2) = x_3, \\ \alpha(x_3) = x_2, & \alpha(x_4) = x_4. \end{array}$$

Then $w(\alpha(P_1)) = 1$ and $w(P_1) = 2$. Hence, α is an example of an automorphism of $\mathcal{U}$ which is not an automorphism of $\mathcal{D}$.

3.3 Point-Weighted Designs with the same Underlying Incidence Structure

Let $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be an incidence structure. In this section we consider whether it is possible to find two weight functions w_1 and w_2 such that, if we set $\mathcal{D}_1 = (\mathbf{V}, \mathbf{B}, \mathbf{I}, w_1)$ and $\mathcal{D}_2 = (\mathbf{V}, \mathbf{B}, \mathbf{I}, w_2)$, then $\mathcal{D}_1$ and $\mathcal{D}_2$ are both point-weighted designs but are *not* equivalent. So, we examine whether the weight function of a point-weighted design is uniquely determined (up to equivalence) by the underlying incidence structure.

Theorem 3.17 *Let $\mathcal{D}_1 = (\mathbf{V}, \mathbf{B}, \mathbf{I}, w_1)$ be a point-weighted design with parameters v_1 and k_1, with underlying incidence structure $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$). Let A be an incidence matrix of $\mathcal{U}$ and let $u = |\mathbf{V}|$. Then every point-weighted design with underlying incidence structure $\mathcal{U}$ is equivalent to $\mathcal{D}_1$ if and only if rank$(A) = u$.*

Proof

We first show that if rank$(A) = u$ then any point-weighted design with underlying incidence structure $\mathcal{U}$ is equivalent to $\mathcal{D}_1$. Suppose rank$(A) = u$, let $\mathcal{U}$ and $\mathcal{D}_1$ be as above, and set $b = |\mathbf{B}|$. Let w_2 be a weight function on $\mathbf{V}$ such that setting $\mathcal{D}_2 = (\mathbf{V}, \mathbf{B}, \mathbf{I}, w_2)$ gives a point-weighted design with some parameters v_2 and k_2. Label the points of $\mathcal{U}$ as $P_1, P_2, \ldots, P_u$ and the blocks as $x_1, x_2, \ldots, x_b$, such that A is the incidence matrix of $\mathcal{U}$ with respect to such a labelling. Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be the vectors $\left(\frac{w_1(P_1)}{k_1}, \frac{w_1(P_2)}{k_1}, \ldots, \frac{w_1(P_u)}{k_1}\right)^T$ and $\left(\frac{w_2(P_1)}{k_2}, \frac{w_2(P_2)}{k_2}, \ldots, \frac{w_2(P_u)}{k_2}\right)^T$ respectively. Then, $\mathcal{D}_1$ is a point-weighted design with parameters v_1 and k_1 and so,

$$A^T \mathbf{w}_1 = \mathbf{1}, \tag{3.2}$$

where $\mathbf{1}$ is the constant vector of size u with every entry equal to 1. Similarly, since $\mathcal{D}_2$ is a point-weighted design with parameters v_2 and k_2,

$$A^T \mathbf{w}_2 = \mathbf{1}. \tag{3.3}$$

Then, subtracting (3.3) from (3.2) gives,

$$A^T \left(\mathbf{w}_1 - \mathbf{w}_2\right) = \mathbf{0}, \tag{3.4}$$

where $\mathbf{0}$ is the constant vector of size u with every entry equal to zero. Thus, either $(\mathbf{w}_1 - \mathbf{w}_2) = \mathbf{0}$, or otherwise rank$(A^T) < u$ (with some of the columns of A^T linearly dependent - see for example [10]). But rank$(A) = u$ and so rank$(A^T) = u$. Hence, $\mathbf{w}_1 = \mathbf{w}_2$ and so $w_1(P_i) = \frac{k_1}{k_2} w_2(P_i)$ for each $i = 1, 2, \ldots, u$. Thus, $\mathcal{D}_1$ and $\mathcal{D}_2$ are equivalent.

Now let $\mathcal{D}_1 = (\mathbf{V}, \mathbf{B}, \mathbf{I}, w_1)$ be as above, with underlying incidence structure $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$). Label the points and blocks of $\mathcal{U}$ as above, and let A be the incidence matrix of $\mathcal{U}$ with respect to this labelling. Suppose rank$(A) < u$. We exhibit a weight function w_2 on $\mathbf{V}$ such that, setting $\mathcal{D}_2 = (\mathbf{V}, \mathbf{B}, \mathbf{I}, w_2)$

gives a point-weighted design with underlying incidence structure $\mathcal{U}$ which is not equivalent to $\mathcal{D}_1$.

Label the columns of A^T as $c_1, c_2, \ldots, c_u$. Then $\text{rank}(A^T) < u$, so the u columns are linearly dependent and there exist integers $a_1, a_2, \ldots, a_u$, not all zero, such that,

$$a_1c_1 + a_2c_2 + \ldots + a_uc_u = 0.$$

Let $\mathbf{a}$ be the vector $(a_1, a_2, \ldots, a_u)^T$ and let $a_{(1)} = \max\{a_i \mid i = 1, 2, \ldots, u\}$. Define the vector $\mathbf{a}^{'}$ by, $\mathbf{a}^{'} = \left(\frac{a_1}{(a_{(1)}+1)k_1}, \frac{a_2}{(a_{(1)}+1)k_1}, \ldots, \frac{a_u}{(a_{(1)}+1)k_1}\right)^T$. Then $A^T\mathbf{a} = \mathbf{0}$ and so $A^T\mathbf{a}^{'} = \mathbf{0}$. Let $\mathbf{w}_1$ be defined as above, i.e, the vector $\left(\frac{w_1(P_1)}{k_1}, \frac{w_1(P_2)}{k_1}, \ldots, \frac{w_1(P_u)}{k_1}\right)^T$, and recall that $A^T\mathbf{w}_1 = \mathbf{1}$. We note that, for each $i = 1, 2, \ldots, u$, the entry in the i^{th} position of $\mathbf{a}^{'}$ is less than the entry in the i^{th} position of $\mathbf{w}_1$, i.e. for each $i = 1, 2, \ldots, u$,

$$\frac{a_i}{\left(a_{(1)} + 1\right) k_1} < \frac{w_1(P_i)}{k_1}.$$

Define the vector $\mathbf{w}_2$ by,

$$\mathbf{w}_2 = \mathbf{w}_1 - \mathbf{a}^{'}.$$

Clearly, every entry of $\mathbf{w}_2$ is a positive rational number, and furthermore,

$$\begin{aligned} A^T\mathbf{w}_2 &= A^T(\mathbf{w}_1 - \mathbf{a}^{'}) \\ &= \mathbf{1}. \end{aligned}$$

Label the entries of $\mathbf{w}_2$ as $\mathbf{w}_2 = (w_{2,1}, w_{2,2}, \ldots, w_{2,u})^T$ and let k_2 be the lowest common multiple of the denominators of $w_{2,1}, w_{2,2}, \ldots, w_{2,u}$. Then the value $k_2w_{2,i}$ is a positive integer for each $i = 1, 2, \ldots, u$ and so we define the weight function w_2 on $\mathbf{V}$ by,

$$w_2(P_i) = k_2w_{2,i} \qquad \text{for each } i = 1, 2, \ldots, u.$$

Let $\mathcal{D}_2$ be the point-weighted structure $(\mathbf{V}, \mathbf{B}, \mathbf{I}, w_2)$ and set $v_2 = \sum_{i=1}^{u} k_2w_{2,i}$. Then $\mathcal{D}_2$ is a point-weighted design with parameters v_2 and k_2 and underlying incidence structure $\mathcal{U}$. We now show that $\mathcal{D}_2$ is not equivalent to $\mathcal{D}_1$.

Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are equivalent. The entry in the i^{th} position of $\mathbf{w}_2$ is $\frac{w_2(P_i)}{k_2}$ for each $i = 1, 2, \ldots, u$, and so $\mathbf{w}_2 = x\mathbf{w}_1$ for some positive rational x. The value x cannot be equal to one since $\mathbf{a}^{'} = \mathbf{w}_1 - \mathbf{w}_2$, and $\mathbf{a}^{'}$ does not have every entry equal to zero by definition. Then, $\mathbf{w}_1 - \mathbf{w}_2 = (1 - x)\mathbf{w}_1$ and so,

$$A^T(\mathbf{w}_1 - \mathbf{w}_2) = (1 - x)\mathbf{1}.$$

But $\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{a}^{'}$ and $A^T\mathbf{a}^{'} = \mathbf{0}$, giving a contradiction. Hence, $\mathcal{D}_1$ and $\mathcal{D}_2$ are not equivalent and the theorem is proved. ■

Recall from Corollary 1.13 that if $\mathcal{U}$ is a proper $2-(u, K, \lambda)$ and A is an incidence matrix of $\mathcal{U}$, then $\operatorname{rank}(A) = u$. Hence, we have the immediate corollary,

Corollary 3.18 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $2-(v, k, \lambda; \mathbf{W})$ with $v > k$ and underlying incidence structure $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$). Then every point-weighted design with underlying incidence structure $\mathcal{U}$ is equivalent to $\mathcal{D}$.* ■

In the following example we demonstrate that the condition that $v > k$ in the above corollary is necessary.

Example 3.19 Let $\mathcal{U}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be the incidence structure defined by:

- $\mathbf{V} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$,
- $\mathbf{B} = \{x\}$,
- $\mathbf{I} = \mathbf{V} \times \mathbf{B}$.

Define a weight function w on $\mathbf{V}$ by,

$$\begin{aligned}&w(P_1) = 4,\\ &w(P_2) = w(P_3) = w(P_4) = w(P_5) = 2,\\ &w(P_6) = w(P_7) = 1.\end{aligned}$$

Set $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w), then $\mathcal{D}$ is a $2-(14, 14, 1; \{1, 2, 4\})$ with underlying incidence structure $\mathcal{U}$. Define a weight function w_1 by,

$$\begin{aligned}&w(P_1) = w(P_2) = 4,\\ &w(P_3) = 2,\\ &w(P_4) = w(P_5) = w(P_6) = w(P_7) = 1.\end{aligned}$$

Set $\mathcal{D}_1 = (\mathbf{V}, \mathbf{B}, \mathbf{I}, w_1)$, then $\mathcal{D}_1$ is also a $2-(14, 14, 1; \{1, 2, 4\})$ with underlying incidence structure $\mathcal{U}$. However, $\mathcal{D}_1$ is not equivalent to $\mathcal{D}$.

3.4 Properties of a $2-(v, k, \lambda; \mathbf{W})$

We turn now to consider $2-(v, k, \lambda; \mathbf{W})$ point-weighted designs and establish some properties of these structures. The results of this section were obtained as joint work with T. Powlesland. Theorem 3.6 gives us an expression for the number of blocks on a set of $t-1$ distinct points of given weight-sum in a $t-(v, k, \lambda; \mathbf{W})$. In the case $t = 2$ this expression tells us the number of blocks on a single point of given weight in a $2-(v, k, \lambda; \mathbf{W})$. Let $\mathcal{D}$ be a $2-(v, k, \lambda; \mathbf{W})$ and let $P_{(i)}$ be a point of $\mathcal{D}$ of weight i. Let $r_{(i)}$ denote the number of blocks on $P_{(i)}$, then,

$$r_{(i)} = \lambda \frac{(v-i)}{(k-i)}. \tag{3.5}$$

So, a point of some weight i is on exactly the same number of blocks as any other point of the same weight, but two points of different weights are each incident with a different number of blocks.

A necessary condition for the existence of a $t-(v,k,\lambda;\mathbf{W})$ is clearly that the expression in Theorem 3.6 is an integer for every possible $(t-1)$-set of points. In the case $t=2$ this simplifies to:

Corollary 3.20 *A necessary condition for the existence of a $2-(v,k,\lambda;\mathbf{W})$ is,*

$$\lambda(v-i) \equiv 0 \pmod{(k-i)} \qquad \forall i \in \mathbf{W}.$$

■

With an expression for the number of blocks on a $(t-1)$-set of points of given weight-sum in a $t-(v,k,\lambda;\mathbf{W})$ for some $t \geq 2$, it is possible to obtain an expression for the number of blocks on a $(t-2)$-set of points of given weight-sum. In the case $t=2$, this gives us an expression for the total number of blocks in a $2-(v,k,\lambda;\mathbf{W})$:

Lemma 3.21 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $2-(v,k,\lambda;\mathbf{W})$ and let $b=|\mathbf{B}|$ be the number of blocks of $\mathcal{D}$. For each weight $i \in \mathbf{W}$, let $u_{(i)}$ be the number of points of $\mathcal{D}$ of weight i. Then,*

$$b = \frac{\lambda}{k} \sum_{i\in\mathbf{W}} i u_{(i)} \frac{(v-i)}{(k-i)}. \tag{3.6}$$

Proof

As in the proof of Lemma 3.1 we sum all flag-weights in $\mathcal{D}$ in two ways. For any $i \in \mathbf{W}$, the total number of flags containing a point of weight i is $u_{(i)}r_{(i)}$, where $u_{(i)}$ is the number of points of weight i in $\mathcal{D}$ and $r_{(i)}$ is the number of blocks on each point of weight i. Thus, the summation $\sum_{i\in\mathbf{W}} u_{(i)}r_{(i)}$ counts all flags, and the summation $\sum_{i\in\mathbf{W}} i u_{(i)}r_{(i)}$ sums all flag-weights. But, in any point-weighted design the flag-weights sum to bk, and so we have,

$$bk = \sum_{i\in\mathbf{W}} i u_{(i)}r_{(i)},$$

and substituting for $r_{(i)}$ from (3.5),

$$bk = \sum_{i\in\mathbf{W}} i u_{(i)} \lambda \frac{(v-i)}{(k-i)}.$$

Dividing through by k gives the expression for the total number of blocks. ■

Noting that the number of blocks in a point-weighted design is an integer gives us the following immediate corollary:

Corollary 3.22 *A necessary condition for the existence of a* $2-(v,k,\lambda;\mathbf{W})$ *is that,*

$$\frac{\lambda}{k}\sum_{i\in\mathbf{W}} iu_{(i)}\frac{(v-i)}{(k-i)},$$

is an integer. ∎

Given any $2-(v,k,\lambda)$, we can construct a $2-(v,k,\lambda;\{1\})$, $\mathcal{D}$, by assigning each point a weight of 1. But then the number of points of weight 1, $u_{(1)}$, is equal to the total number of points which is also equal to the sum of the weights of all the points in $\mathcal{D}$. Then, applying Corollaries 3.20 and 3.22 to $\mathcal{D}$ gives us the following well-known result for block designs (Corollary 1.6) as a further corollary:

Corollary 3.23 *Two necessary conditions for the existence of a* $2-(v,k,\lambda)$ *are,*

$$\begin{aligned}\lambda(v-1) &\equiv 0 \pmod{(k-1)}\\ \lambda v(v-1) &\equiv 0 \pmod{k(k-1)}.\end{aligned}$$

∎

Chapter 4

Point-Weighted Designs with One 'Special' Point

Let a_1 and a_2 be positive integers with $a_1 \neq a_2$. In this chapter we consider $t-(v,k,\lambda;\{a_1,a_2\})$ point-weighted designs in which one point has weight a_1 and all other points have weight a_2. We begin by showing that any such point-weighted design with more than one block (i.e., with $v > k$) is equivalent to a $t-(v,k,\lambda;\{1,a\})$ in which one point has weight $a > 1$ and all other points have weight 1. We then consider $2-(v,k,\lambda;\{1,a\})$ point-weighted designs with exactly one point of weight $a > 1$. We find a necessary condition on v for their existence, and demonstrate that the class of these point-weighted designs with $\lambda = 1$ is exactly those point-weighted designs produced by a given construction from a certain class of group divisible designs.

For given k, a lower bound for v in a $2-(v,k,1;\{1,k-2\})$ with just one point of weight $k-2$ is obtained. It is shown that any such point-weighted design in which v attains this bound has a certain type of square group divisible design as a substructure of its underlying incidence structure. Such a square group divisible design is shown to exist if and only if there exists a balanced orthogonal design with underlying matrix equal to an incidence matrix of a biplane with block-size k.

4.1 A Specific Class of Point-Weighted Designs

Let a_1 and a_2 be non-equal positive integers. We now consider the class of $t-(v,k,\lambda;\{a_1,a_2\})$ point-weighted designs in which exactly one point has weight a_1 and all other points have weight a_2.

Lemma 4.1 *Let a_1 and a_2 be positive integers greater than one such that a_2 does not divide a_1. Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t-(v,k,\lambda;\{a_1,a_2\})$ with exactly one point of weight a_1, all other points having weight a_2. Then $v = k$.*

Proof

Let P be the point of weight a_1. We show that P is on every block of $\mathcal{D}$. Since $\mathcal{D}$ is a $t-(v,k,\lambda;\{a_1,a_2\})$, P is on at least one block and the weights of the points incident with any one block of $\mathcal{D}$ sum to k. Hence, since every other point of $\mathcal{D}$ has weight a_2,

$$k = a_2 c + a_1, \qquad \text{for some } c \geq 0.$$

Now suppose there is a block $x_1 \in \mathbf{B}$, not on P. Then x_1 is incident with points only of weight a_2. But $\mathcal{D}$ is a point-weighted design and so the sum of the weights of the points incident with x_1 is k. Hence, $a_2 \mid k$. But $k = a_2 c + a_1$ and so $a_2 \mid a_1$, giving a contradiction since a_2 does not divide a_1. Thus P is on every block of $\mathcal{D}$.

In the case $t = 1$ the lemma is proved since every point is on the same number of blocks as the point P. Thus, every point is on every block of $\mathcal{D}$, giving $v = k$. So we consider the case $t > 1$.

Let S be a set of $t-1$ distinct points of $\mathcal{D}$, all of weight a_2, and set $T = S \cup \{P\}$. Then T is a set of t distinct points of $\mathcal{D}$ and so is incident with exactly λ blocks. Since the point P is on every block of $\mathcal{D}$, the set $\mathcal{S}$ is also incident with exactly λ blocks. The weight-sum of S is clearly $(t-1)a_2$, and so by Theorem 3.6, the number of blocks on S, λ_S, is,

$$\lambda_S = \lambda \frac{(v-(t-1)\,a_2)}{(k-(t-1)\,a_2)}.$$

But there are exactly λ blocks on S and so we require $\lambda_S = \lambda$. Hence, $v = k$. ■

Hence we have the following theorem:

Theorem 4.2 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a $t-(v,k,\lambda;\{a_1,a_2\})$ with $v > k$, $a_1 \neq a_2$, and exactly one point of weight a_1 (all other points having weight a_2). Then $a_2 \mid a_1$ and $\mathcal{D}$ is equivalent to a $t-\left(\frac{v}{a_2}, \frac{k}{a_2}, \lambda; \left\{\frac{a_1}{a_2}, 1\right\}\right)$ with exactly one point of weight $\frac{a_1}{a_2}$ (all other points having weight 1).*

Proof

By Lemma 4.1, we have that $a_2 \mid a_1$, and since $a_1 \neq a_2$, we see that $a_1 > a_2$. Then by Lemma 2.7, $\mathcal{D}$ is equivalent to a $t-\left(\frac{v}{a_2}, \frac{k}{a_2}, \lambda; \left\{\frac{a_1}{a_2}, 1\right\}\right)$ with exactly one point of weight $\frac{a_1}{a_2}$, all other points having weight 1. ■

Before discussing $t-(v,k,\lambda;\{1,a\})$ point-weighted designs with exactly one point of weight $a > 1$ and $t = 2$, we briefly consider the case $t = 1$. Recall from Chapter 1 that for any positive integers b, k, v and r with $v > k$ and $b \leq \binom{v}{k}$, a necessary and sufficient condition for the existence of a $1-(v,k,r)$ is that $bk = vr$. By Lemma 3.1 the condition that $bk = vr$ is necessary for the existence of a $1-(v,k,\lambda;\mathbf{W})$ with b blocks and any weight-set $\mathbf{W}$. This condition is clearly sufficient when $\mathbf{W} = \{1\}$ since assigning every point of a

$1-(v,k,r)$ the weight 1 will always give a $1-(v,k,r;\{1\})$. We now give an example to show that the condition $bk = vr$ is not in general sufficient for the existence of a $1-(v,k,\lambda;\{1,a\})$ with b blocks and exactly one point of specified weight a.

Consider the parameters $v = 7$, $b = 7$, $k = 5$, $r = 5$. Then $bk = vr$ and so there exists a $1-(7,5,5)$ with seven blocks. Therefore there exists a $1-(7,5,5;\{1\})$ with seven blocks. Suppose there exists a $1-(7,5,5;\{1,3\})$ with seven blocks and exactly one point, P say, of weight 3. Then there are four points of weight 1 and so the point P is on every block. Hence, $r = b$ and we have a contradiction. Thus, there does not exist a $1-(7,5,5;\{1,3\})$ with seven blocks and exactly one point, P say, of weight 3.

4.2 The Case $t = 2$

In this section we consider $2-(v,k,\lambda;\{1,a\})$ point-weighted designs with exactly one point of weight $a > 1$. Corollaries 3.20 and 3.22 give us necessary conditions for the existence of such a point-weighted design, although these are not in general sufficient. From Corollary 3.20 we obtain the condition on v in the following lemma, for any $2-(v,k,\lambda;\{1,a\})$. Using standard notation, we use (a_1,a_2) to denote the greatest common divisor of positive integers a_1 and a_2, and $[a_1,a_2]$ to denote the lowest common multiple of a_1 and a_2.

Lemma 4.3 *Let $\mathcal{D}$ be a $2-(v,k,\lambda;\{1,a\})$ for some $a > 1$. Then v satisfies,*

$$v = k + c\left[\frac{k-a}{(k-a,\lambda)}, \frac{k-1}{(k-1,\lambda)}\right],$$

for some integer $c \geq 0$.

Proof
Corollary 3.20 gives,

$$\lambda v \equiv \lambda a \pmod{(k-a)}, \tag{4.1}$$

and also,

$$\lambda v \equiv \lambda \pmod{(k-1)}. \tag{4.2}$$

Then (4.1) and (4.2) hold if and only if (see [23], page 49),

$$v \equiv a \quad \left(\mathrm{mod}\frac{k-a}{(k-a,\lambda)}\right), \tag{4.3}$$

and,

$$v \equiv 1 \quad \left(\mathrm{mod}\frac{k-1}{(k-1,\lambda)}\right). \tag{4.4}$$

Clearly, $v = k$ is a solution to (4.3) and (4.4) and is the smallest possible solution for v (since $v \geq k$). Then, by the Chinese Remainder Theorem, every solution for v to (4.3) and (4.4) is of the form,

$$v = k + c\left[\frac{k-a}{(k-a,\lambda)}, \frac{k-1}{(k-1,\lambda)}\right],$$

for some integer $c \geq 0$. ■

The expression for v in the above corollary with $c = 0$ obviously corresponds to a point-weighted design with $v = k$ with one block incident with all the points. Observing that if the point-weighted design in the above corollary is proper then $v > k$ gives us a lower bound for the sum of the weights of the points in a proper $2-(v, k, \lambda; \{1, a\})$:

Corollary 4.4 *Let $\mathcal{D}$ be a proper $2-(v, k, \lambda; \{1, a\})$, for some $a > 1$. Then,*

$$v = k + \left[\frac{k-a}{(k-a,\lambda)}, \frac{k-1}{(k-1,\lambda)}\right],$$

■

This bound applies to the sum of the weights of the points in any proper $2-(v, k, \lambda; \{1, a\})$ with $a > 1$. In this chapter we are considering those point-weighted designs in which exactly one point has weight a. In the following example we give a construction of such a point-weighted design for $\lambda = 1$ and any $k > 2$, with a specific value of a, in which v attains the bound.

Example 4.5 We generalise here the construction of a point-weighted design given in Example 2.6 to give a $2-(2k-1, k, 1; \{1, k-1\})$ for any $k > 2$. Let $\mathcal{U}$ =(**V**,**B**,**I**) be the degenerate projective plane with $k+1$ points, for some $k > 2$. Label the points of $\mathcal{U}$ as $P_1, P_2, \ldots, P_{k+1}$, and the blocks of $\mathcal{U}$ as $x_1, x_2, \ldots, x_{k+1}$, such that, considering the blocks as subsets of the point-set,

$$\begin{aligned} x_i &= \{P_i, P_{k+1}\} && \text{for each } i = 1, 2, \ldots, k, \\ x_{k+1} &= \{P_i \mid i = 1, 2, \ldots, k\}. \end{aligned}$$

Define a weight function w on the points of $\mathcal{U}$ by,

$$\begin{aligned} w(P_i) &= 1, && \text{for each } i = 1, 2, \ldots, k, \\ w(P_{k+1}) &= k-1. \end{aligned}$$

Set $\mathcal{D}$ =(**V**,**B**,**I**,w) to be the point-weighted structure with $\mathcal{U}$ as its underlying incidence structure and weight function w defined as above. Then clearly $\mathcal{D}$ is a point-weighted design since the sum of the weights of the points in any block is k. Furthermore, since $\mathcal{U}$ is a degenerate projective plane, each pair of points is contained in exactly one block and so $\mathcal{D}$ is a $2-(v, k, 1; \{1, k-1\})$, with $v = 2k-1$. It is easy to verify that this value of v attains the bound given in Corollary 4.4.

4.3 The Case $t = 2$ with $\lambda = 1$

In this section we give a construction of $2-(v, k, 1; \{1, a\})$ point-weighted designs with exactly one point of weight a, for some $a > 1$, from a specific class of group divisible designs. We show that every proper $2-(v, k, 1; \{1, a\})$ with exactly one point of weight $a > 1$ can be constructed in such a way.

Let $\mathcal{G}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$) be a GD $2-(u, k, (0, 1))$ for some u and k, with point-classes of size m, for some $m \leq k-2$. Let $n = \frac{u}{m}$ be the number of point-classes, and label the point-classes $\mu_1, \mu_2, \ldots, \mu_n$. Label the points of $\mathcal{G}$ as $P_1, P_2, \ldots, P_u$ and, setting $b = |\mathbf{B}|$, label the blocks of $\mathcal{G}$ as $x_1, x_2, \ldots, x_b$. We define a point-weighted structure $\mathcal{D} = (\mathbf{V}_1, \mathbf{B}_1, \mathbf{I}_1, w)$ by firstly adjoining to $\mathcal{G}$ a new point, P_{u+1}, and n new blocks, $y_1, y_2, \ldots, y_n$, distinct from the blocks of $\mathcal{G}$. So we put,

- $\mathbf{V}_1 = \mathbf{V} \cup \{P_{u+1}\}$,
- $\mathbf{B}_1 = \mathbf{B} \cup \{y_1, y_2, \ldots, y_n\}$.

We define incidence in $\mathcal{D}$ by,

- $\mathbf{I}_1 = \mathbf{I} \cup \mathbf{I}^{'}$,

where $\mathbf{I}^{'}$ is defined by,

$$\begin{array}{llll} (P_{u+1}, y_j) & \in \mathbf{I}^{'}, & & j = 1, 2, \ldots n \\ (P_i, y_j) & \in \mathbf{I}^{'} \iff & P_i \in \mu_j, & j = 1, 2, \ldots n \text{ and } i = 1, 2, \ldots, u. \end{array}$$

So each new block is incident with exactly the point P_{u+1} together with all the points in one point-class of $\mathcal{G}$, and each point-class of $\mathcal{G}$ defines such a block. Finally, we define the weight function w by,

$$w(P_i) = \begin{cases} 1 & \text{if } i \in \{1, 2, ..., u\} \\ k-m & \text{if } i = u+1. \end{cases}$$

Lemma 4.6 *The point-weighted structure $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) constructed above is a $2-(u+k-m, k, 1; \{1, k-m\})$.*

Proof

Since $k-2 \geq m$ we note that $k-m > 1$ and so the two elements of the weight-set of $\mathcal{D}$ are distinct.

The points $P_1, P_2, \ldots, P_u$ all have weight 1 in $\mathcal{D}$ and the point P_{u+1} has weight $k-m$, and so,

$$\sum_{i=1}^{u+1} w(P_i) = u + k - m.$$

Each block of $\mathcal{D}$ is incident with either k points of weight 1, or m points of weight 1 together with the one point of weight $k-m$. Thus, the sum of the weights of the points incident with any one block of $\mathcal{D}$ is k.

We now need to show that there is exactly one block incident with each pair of points of $\mathcal{D}$. We first note that the point-classes of $\mathcal{G}$ partition the point-set $\mathbf{V} = \{P_1, P_2, \ldots, P_u\}$. Hence, each point P_i, $i = 1, 2, \ldots, u$, is in exactly one point-class of $\mathcal{G}$, μ_j, for some $j \in \{1, 2, \ldots, n\}$, and thus incident with exactly one block of $\mathcal{D}$ together with the point P_{u+1}, namely y_j. So, for each $i = 1, 2, \ldots, u$ there is exactly one block of $\mathcal{D}$ incident with both P_i and P_{u+1}.

Now let $i, l \in \{1, 2, \ldots, u\}$ with $i \neq l$. We consider the points P_i and P_l. Either P_i and P_l are in the same point-class of $\mathcal{G}$, μ_j, for some $j \in \{1, 2, \ldots, n\}$, or they are in different point-classes. In the case where they are both in the point-class μ_j, there is no block of $\mathcal{G}$ incident with both P_i and P_l. The remaining blocks of $\mathcal{D}$ are all incident with P_{u+1} and exactly one of these blocks is incident with both P_i and P_l, namely the block y_j. Thus, there is exactly one block of $\mathcal{D}$ incident with both P_i and P_l.

In the case where P_i and P_l are in distinct point-classes of $\mathcal{G}$, there is no block of $\mathcal{D}$ incident with P_{u+1} which is also incident with P_i and P_l. Then, there is exactly one block of $\mathcal{G}$ incident with both P_i and P_l, and thus exactly one block of $\mathcal{D}$ incident with both P_i and P_l. ∎

We note that in the case $m = 1$, with $\mathcal{G}$ a $2-(v, k, 1)$, the construction above is the same as the construction given in Section 2.2.1 with $t = 2$.

We now show that any proper $2-(v, k, 1; \{1, a\})$ with exactly one point of weight a has a group divisible design as a substructure of its underlying incidence structure.

Lemma 4.7 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a proper $2-(v, k, 1; \{1, a\})$ with exactly one point P of weight $a > 1$, and let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{D}$. Let $\mathcal{G}$ be the point-residue of $\mathcal{U}$ at P. Then, $\mathcal{G}$ is a GD $2-(v-a, k, (0, 1))$ with point-classes of size $k-a$ defined by the blocks on P in $\mathcal{U}$.*

Proof

The points of $\mathcal{G}$ are precisely those points of $\mathcal{U}$ which are assigned weight 1 in $\mathcal{D}$. By Theorem 3.6, every point of weight 1 is incident with exactly $r_{(1)} = \frac{(v-1)}{(k-1)}$ blocks of $\mathcal{D}$ (and thus $\mathcal{U}$). $\mathcal{D}$ is a $2-(v, k, 1; \{1, a\})$ and so there is exactly one block of $\mathcal{D}$ (and thus $\mathcal{U}$) incident with both P and a given point of weight 1. Hence, every point of $\mathcal{G}$ is incident with exactly $r_{(1)} - 1$ blocks of $\mathcal{G}$, and so $\mathcal{G}$ is regular. The blocks of $\mathcal{G}$ are those blocks of $\mathcal{D}$ which are not incident with P, and so are those blocks of $\mathcal{D}$ which are only incident with points of weight 1. Thus, every block of $\mathcal{G}$ is incident with exactly k points, and so $\mathcal{G}$ is uniform. The number of points of $\mathcal{G}$ is equal to the sum of their weights in $\mathcal{D}$ (since they all have weight 1 in $\mathcal{D}$) and so is $v - a$.

Let Q and R be distinct points of $\mathcal{G}$. We define point-classes in $\mathcal{G}$ by setting Q and R to be in the same point-class if and only if there is a block of $\mathcal{D}$ incident with P,Q and R. Since every point of weight 1 is incident with exactly one block of $\mathcal{D}$ together with P, the point-classes partition the points of $\mathcal{G}$. Each block of $\mathcal{D}$ which is incident with P is also incident with exactly $k - a$ points of weight 1. Thus, the point-classes partition the points of $\mathcal{G}$ into sets of size $m = k - a$.

Since Q and R are distinct points of weight 1 in $\mathcal{D}$, there is exactly one block of $\mathcal{D}$, x say, incident with both Q and R. This block x is either incident with P or not. If x is incident with P, then Q and R are in the same point-class and there is no block of $\mathcal{G}$ incident with both Q and R. If x is not incident with P, then x is a block of $\mathcal{G}$ and so there is exactly one block of $\mathcal{G}$ incident with both Q and R. There is then no block of $\mathcal{D}$ incident with P, Q and R, and so Q and R are in different point-classes.

Thus, $\mathcal{G}$ is a uniform, regular design with $v-a$ points, block-size k, and a partition of the point-set into point-classes of size $m = k-a$. The number of blocks incident with two given points is zero if they are in the same point-class and one otherwise. Therefore, $\mathcal{G}$ is a GD $2-(v-a, k, (0,1))$. ∎

Combining this lemma with the construction immediately preceding it gives the following result:

Lemma 4.8 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a point-weighted structure with exactly one point, P, of some weight $a > 1$, all other points having weight 1. Let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{D}$ and let $\mathcal{G}$ be the point-residue of $\mathcal{U}$ at P. Then $\mathcal{D}$ is a proper $2-(v, k, 1; \{1, a\})$ if and only if $\mathcal{G}$ is a GD $2-(v-a, k, (0,1))$ with point-classes of size $k-a$.* ∎

In Chapter 1 we noted that a block design is a group divisible design with point-classes of size one. So we have as a corollary,

Corollary 4.9 *Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a point-weighted structure with exactly one point, P, of weight $k-1$ (for some $k > 2$), all other points having weight 1. Let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{D}$ and let $\mathcal{G}$ be the point-residue of $\mathcal{U}$ at P. Then the point-weighted structure $\mathcal{D}$ is a proper $2-(v, k, 1; \{1, k-1\})$ if and only if $\mathcal{G}$ is a $2-(v-k+1, k, 1)$.* ∎

To construct an example of a proper $2-(v, k, 1; \{1, a\})$ with exactly one point of weight a, for some a in the range $1 < a < k-1$, we look for an example of a GD $2-(u, k, (0,1))$ with point-classes of size m, for some m in the range $1 < m < k-1$. Perhaps the best known examples of GD $2-(u, k, (0,1))$ designs are transversal designs. However, we recall from Result 1.21 that a transversal design with k points on a block and point-classes of size m has $m \geq k-1$, with $m = k-1$ if and only if the design is the dual of an affine plane. So, a transversal design cannot be used in the above construction to obtain a proper $2-(v, k, 1; \{1, a\})$ with exactly one point of weight $a > 1$. Examples of group divisible $2-(u, k, (0,1))$ designs with point-classes of size m, for some m in the range $1 < m < k-1$, do exist (see for example [24]) and we now give a construction of a family of such group divisible designs due to Sprott ([28]).

4.3.1 Sprott's Construction

Let $\mathcal{P}_1$ be a projective plane of order q^2 which contains a Baer subplane $\mathcal{P}_0$ of order q. Recall from Section 1.4.1 that $\mathcal{P}_1$ has the same number of lines as points, $b_1 = v_1 = q^4+q^2+1$, and has $k_1 = q^2+1$ points on a line. Similarly, $\mathcal{P}_0$ has the same number of lines as points, $b_0 = v_0 = q^2+q+1$, and has $k_0 = q+1$

points on a line. Also, every line of $\mathcal{P}_1$ is incident with either 1 or $q+1$ collinear points of $\mathcal{P}_0$.

Define an incidence structure $\mathcal{G}$ to be that whose points are the points of $\mathcal{P}_1$ which are not also points of $\mathcal{P}_0$ and whose blocks are the lines of $\mathcal{P}_1$ which are incident with only one point of $\mathcal{P}_0$ (i.e., those lines of $\mathcal{P}_1$ which are not also lines of $\mathcal{P}_0$). A point and block of $\mathcal{G}$ are incident if and only if the point and corresponding line are incident in $\mathcal{P}_1$.

Result 4.10 (Sprott) *The incidence structure $\mathcal{G}$ constructed above is a square GD $2-\big(q^4-q, q^2, (0,1)\big)$ with point-classes of size $m = q^2 - q$.*

Proof

The number of points of $\mathcal{G}$, u, is equal to the number of blocks and is given by,

$$\begin{aligned} u &= q^4+q^2+1-\left(q^2+q+1\right) \\ &= q^4-q. \end{aligned}$$

Each block of $\mathcal{G}$ is a line of $\mathcal{P}_1$ with one point 'removed', and so is incident with q^2 points. Thus $\mathcal{G}$ is uniform.

Label the set of lines of $\mathcal{P}_1$ which are also lines of $\mathcal{P}_0$ as $\mathbf{B}_0$. We now show that every point of $\mathcal{G}$ is on exactly one line of $\mathbf{B}_0$. Since $\mathcal{P}_0$ is a projective plane, any two lines of $\mathbf{B}_0$ intersect in a point of $\mathcal{P}_0$. Thus, no two lines of $\mathbf{B}_0$ intersect in a point of $\mathcal{G}$. Therefore, there are in total $(k_1-k_0)b_0$ distinct points of $\mathcal{G}$ incident with a line of $\mathbf{B}_0$ and each of these points is incident with exactly one line of $\mathbf{B}_0$. But,

$$\begin{aligned} (k_1-k_0)\,b_0 &= \left(q^2-q\right)\left(q^2+q+1\right) \\ &= q^4-q, \end{aligned}$$

which is the total number of points in $\mathcal{G}$. Hence, every point of $\mathcal{G}$ is on exactly one line of $\mathbf{B}_0$. Thus, the lines of $\mathbf{B}_0$ define a partition of the points of $\mathcal{G}$ into point- classes, with two points in the same point-class if and only if they are incident with a common line of $\mathbf{B}_0$. The number of points in a point-class is given by the number of points of $\mathcal{G}$ on any one line of $\mathbf{B}_0$, i.e.,

$$\begin{aligned} m &= k_1-k_0 \\ &= \left(q^2+1\right)-(q+1) \\ &= q^2-q. \end{aligned}$$

Every point of $\mathcal{P}_1$ is on q^2+1 lines. Then, since the blocks of $\mathcal{G}$ are the lines of $\mathcal{P}_1$ which are not in $\mathbf{B}_0$, every point of $\mathcal{G}$ is incident with exactly q^2 blocks of $\mathcal{G}$. Therefore $\mathcal{G}$ is regular.

Let Q and R be any two points of $\mathcal{G}$. Then they are incident with exactly one common line, l say, in $\mathcal{P}_1$. If $l \in \mathbf{B}_0$ then Q and R are in the same point-class and there is no block of $\mathcal{G}$ incident with both Q and R. If $l \notin \mathbf{B}_0$ then Q and R are not in the same point-class and l is a block of $\mathcal{G}$. So there is exactly one block of $\mathcal{G}$ incident with both Q and R.

Hence, $\mathcal{G}$ is a square GD $2-\big(q^4-q, q^2, (0,1)\big)$ with point-classes of size q^2-q. ∎

In [12], Dembowski proves what is effectively the converse to the above result:

Result 4.11 *Let $\mathcal{G}$ be a square GD $2-\big(q^4-q, q^2, (0,1)\big)$ with point-classes of size $m = q^2-q$, then $\mathcal{G}$ is isomorphic to a projective plane of order q^2 from which the points and lines of a subplane of order q are removed.* ∎

Example 4.12 Let $\mathcal{P}_1$ be the Desarguesian projective plane of order four, and let $\mathcal{P}_0$ be the Desarguesian projective plane of order two. Then $\mathcal{P}_0$ is a Baer subplane of $\mathcal{P}_1$. Using these in Sprott's construction gives a GD $2-(14,4,(0,1))$ with point-classes of size $m=2$. By adjoining to this group divisible design a point of weight $4-2=2$ and some new blocks, a $2-(16,4,1;\{1,2\})$ point-weighted design can be constructed using the method described at the beginning of this section. A weighted incidence matrix of this point-weighted design is given in Figure 4.1.

$$\left(\begin{array}{ccccccc|cccccccccccccc}
2&2&2&2&2&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ \hline
1&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0&1&1&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&1&0&1&1&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&1&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&1&0&1&1&0&1&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&1\\
0&1&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&1&0\\
0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&1&0&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1&1&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&1&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0&1&1
\end{array}\right)$$

Figure 4.1: A weighted incidence matrix of a 2-(16,4,1;{1,2})

Recall from Chapter 1 that any projective plane of order four is isomorphic to the Desarguesian plane of that order, and any projective plane of order two is isomorphic to the Desarguesian plane of order two. An immediate consequence of this together with Lemma 4.8 and Result 4.11 is:

Lemma 4.13 *Let $\mathcal{D}$ be a $2-(16,4,1;\{1,2\})$ with exactly one point of weight 2. Then $\mathcal{D}$ is isomorphic to the point-weighted design whose weighted incidence matrix is given in Figure 4.1.* ∎

4.3.2 The Case $k - a = 2$

Proper $2-(v, k, 1; \{1, k-1\})$ point-weighted designs with just one point of weight $k-1$ are classified in Corollary 4.9. We now consider the class of proper $2-(v, k, 1; \{1, k-2\})$ point-weighted designs with one point of weight $k-2$. We obtain a lower bound on v for given k which improves that given in Corollary 4.4 and show that the example of a point-weighted design whose weighted incidence matrix is given in Figure 4.1 attains this bound.

We note that for $k-2 > 1$ we require $k \geq 4$, and we shall assume that this is so. Suppose $\mathcal{D}$ is a proper $2-(v, k, 1; \{1, k-2\})$ with exactly one point P of weight $k-2$. Let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{D}$ and let $\mathcal{G}$ be the point-residue of $\mathcal{U}$ at P. Lemma 4.8 states that $\mathcal{D}$ is a proper $2-(v, k, 1; \{1, k-2\})$ if and only if $\mathcal{G}$ is a GD $2-(v-k+2, k, (0, 1))$ with point-classes of size two. So we consider the incidence structure $\mathcal{G}$ and suppose it is a GD $2-(v-k+2, k, (0, 1))$ with point-classes of size two. The condition that $\mathcal{D}$ is proper specifies that $\mathcal{G}$ must have at least four points.

To define some notation, let Q be an arbitrary point of $\mathcal{G}$. Then there is a unique point denoted $Q^{'}$, in the same point-class as Q and distinct from Q (since the point-classes are of size 2) We call $Q^{'}$ the *partner point* of Q.

Lemma 4.14 *Let $\mathcal{G}$ be a GD $2-(v-k+2, k, (0, 1))$ with $k \geq 4$ and point-classes of size two. Let Q be a point of $\mathcal{G}$, with partner point $Q^{'}$, then there is a block of $\mathcal{G}$ not incident with either Q or $Q^{'}$.*

Proof

Since $\mathcal{G}$ has at least four points, there is at least one block on Q and at least one block on $Q^{'}$. Let x be a block on Q and let y be a block on $Q^{'}$. Now two points of $\mathcal{G}$ are on at most one block, thus there can be at most one point incident with both x and y. Since $k \geq 4$ there are at least two points on x which are not also on y or equal to Q. Label one of these points as R_1. There are also at least two points on y which are not also on x or equal to $Q^{'}$ and at least one of these is not the partner point of R_1 (i.e., $R_1^{'}$). Label this point R_2. Then R_1 and R_2 are not in the same point-class and so there is a block z incident with both R_1 and R_2. But the block x is incident with both Q and R_1; the block y is incident with both $Q^{'}$ and R_2, and so the block z cannot be incident with either Q or $Q^{'}$. ∎

We use this result to obtain a lower bound on the number of points of $\mathcal{G}$, $u_{(1)}$ say, for given $k \geq 4$.

Lemma 4.15 *Let $\mathcal{G}$ be a GD $2-\left(u_{(1)}, k, (0, 1)\right)$ with $k \geq 4$ and point-classes of size two. Then $u_{(1)} \geq k^2 - k + 2$.*

Proof

For any point Q and its partner point $Q^{'}$, there is a block z not incident with Q or $Q^{'}$. This block is incident with exactly k points, none of which are in the same point-class as Q or $Q^{'}$, and we label these points $R_1, R_2, \ldots, R_k$. For each

$i = 1, 2, \ldots, k$, there is exactly one block incident with both R_i and Q and we label this block x_i. Any two blocks x_i, x_j $(i, j \in \{1, 2, \ldots, k\}, i \neq j)$ are both incident with Q and so there is no other point of $\mathcal{G}$ incident with both x_i and x_j. For each $i = 1, 2, \ldots, k$, the block x_i is incident with k points and so the total number of points incident with the blocks $x_1, x_2, \ldots, x_k$ is $k(k-1)+1$. Since each of these blocks is incident with Q, none of them are incident with $Q^{'}$ and so there are at least $k^2 - k + 2$ points. ■

Corollary 4.16 *Let $\mathcal{D}$ be a proper $2-(v, k, 1; \{1, k-2\})$ with $k \geq 4$ and exactly one point P of weight $k-2$. Then $v \geq k^2$.*

Proof

Let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{D}$. Since $\mathcal{D}$ has exactly one point P of weight $k-2$ the number of points of weight 1 in $\mathcal{D}$ is given by the number of points in the point-residue of $\mathcal{U}$ at P, $\mathcal{G}$ say. By Lemma 4.8, $\mathcal{G}$ is a GD $2-\left(u_{(1)}, k, (0,1)\right)$ with $k \geq 4$ and point-classes of size two, and so by Lemma 4.15, $u_{(1)} \geq k^2 - k + 2$. Then $v = u_{(1)} + k - 2$, and so $v \geq k^2$. ■

We note that the point-weighted design whose weighted incidence matrix is given in Figure 4.1 attains this bound for $k = 4$. We now show that any proper $2-(v, k, 1; \{1, k-2\})$ with exactly one point of weight $k-2$ in which v attains the above bound has a *square* group divisible design as a substructure of its underlying incidence structure.

Lemma 4.17 *Let $\mathcal{D}$ be a proper $2-\left(k^2, k, 1; \{1, k-2\}\right)$ with $k \geq 4$ and exactly one point P of weight $k-2$. Let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{D}$ and let $\mathcal{G}$ be the point-residue of $\mathcal{U}$ at P. Then $\mathcal{G}$ is a square GD $2-\left(k^2-k+2, k, (0,1)\right)$.*

Proof

By Lemma 4.8 $\mathcal{G}$ is a GD $2-\left(k^2-k+2, k, (0,1)\right)$ with point-classes of size two. The number of blocks in $\mathcal{G}$, b_G say, is simply the number of blocks of $\mathcal{D}$ which are not incident with P. We show that this is equal to the number of points in $\mathcal{G}$, k^2-k+2.

Applying Lemma 3.21 gives the total number of blocks in $\mathcal{D}$, b_D say, to be,

$$\begin{aligned} b_D &= \frac{1}{k}\left(\frac{(k-2)\left(k^2-k+2\right)}{k-(k-2)} + \frac{\left(k^2-k+2\right)\left(k^2-1\right)}{k-1}\right) \\ &= \frac{\left(k^2-k+2\right)}{k}\left(\frac{k-2}{2} + k + 1\right) \\ &= \frac{3}{2}\left(k^2-k+2\right). \end{aligned}$$

The number of blocks of $\mathcal{D}$ which are incident with P, r_P say, is given by Equation (3.5) on page 42 to be,

$$r_P = \frac{k^2-k+2}{k-(k-2)}$$

$$= \frac{1}{2}\left(k^2 - k + 2\right).$$

Thus, the number of blocks of $\mathcal{D}$ which are not incident with P is $k^2 - k + 2$, and this is the number of blocks in $\mathcal{G}$. So we have shown that $\mathcal{G}$ is square. ∎

Combining this lemma with Lemma 4.8 gives the corollary,

Corollary 4.18 *A proper $2 - (k^2, k, 1; \{1, k-2\})$ with $k \geq 4$ and exactly one point of weight $k-2$ exists if and only if there exists a square GD $2 - \left(k^2 - k + 2, k, (0,1)\right)$ with point-classes of size two.* ∎

To obtain conditions for the existence of a square GD $2-\left(k^2 - k + 2, k, (0,1)\right)$ with point-classes of size two, we establish a property of such a design:

Lemma 4.19 *Let $\mathcal{G}$ be a square GD $2-\left(k^2 - k + 2, k, (0,1)\right)$ with point-classes of size two. Then every point of $\mathcal{G}$ is incident with k blocks, and the dual of $\mathcal{G}$ is also a square GD $2 - \left(k^2 - k + 2, k, (0,1)\right)$ with point-classes of size two.*

Proof

Since $\mathcal{G}$ is a group divisible design it is both uniform and regular. Result 1.2 and the fact that $\mathcal{G}$ is square then gives that the number of blocks on any one point is equal to the number of points on any one block. Hence, every point of $\mathcal{G}$ is incident with exactly k blocks.

Since $\mathcal{G}$ is square, the dual of $\mathcal{G}$ is also square. To show that the dual of $\mathcal{G}$ is a GD $2 - \left(k^2 - k + 2, k, (0,1)\right)$ with point-classes of size two we need to show that the blocks of $\mathcal{G}$ can be partitioned into classes of size two so that any two blocks in the same class do not intersect, and any two blocks from different classes intersect in exactly one point. Since any two blocks of $\mathcal{G}$ intersect in at most one point, it is sufficient to show that for any block x of $\mathcal{G}$, there is a unique block $x^{'}$ which does not intersect x.

Consider a block x of $\mathcal{G}$. There are k points incident with x, and each of these points is incident with a further $k-1$ blocks. Given any two points on x, P and Q say, the blocks other than x which are incident with P are distinct from the blocks other than x which are incident with Q (since there is only one block, x, incident with both P and Q). Thus, there are in total $k(k-1)$ blocks of $\mathcal{G}$ which intersect x and so there is exactly one block $x^{'}$ which does not intersect x. ∎

For any block x of a square GD $2 - \left(k^2 - k + 2, k, (0,1)\right)$ with point-classes of size two we call the unique block $x^{'}$ which does not intersect x the *partner block* of x. We define block-classes in $\mathcal{G}$ by setting distinct blocks x and y to be in the same block-class if and only if y is the partner block of x.

Lemma 4.20 *Let $\mathcal{G}$ be a square GD $2-\left(k^2 - k + 2, k, (0,1)\right)$ with point-classes of size two. Let P be a point of $\mathcal{G}$ with partner point $P^{'}$, and let x be a block of $\mathcal{G}$ with partner block $x^{'}$. Then P is incident with x if and only if $P^{'}$ is incident with $x^{'}$.*

Proof

Let x be any block of $\mathcal{G}$ and let Q be an arbitrary point incident with x. Label the remaining points on x as $P_1, P_2, \ldots, P_{k-1}$. Then, for each $i = 1, 2, \ldots, k-1$ there is exactly one block, x_i say, incident with both P_i and the partner point of Q, $Q^{'}$. The blocks $x_1, x_2, \ldots, x_{k-1}$ are distinct since the only block of $\mathcal{G}$ incident with any two points from $\{P_1, P_2, \ldots, P_{k-1}\}$ is x. Now $Q^{'}$ is incident with k blocks in total and so there is one further block, y say, on $Q^{'}$ which is not incident with any of $P_1, P_2, \ldots, P_{k-1}$. Furthermore, y is not incident with Q since there is no block on both Q and $Q^{'}$. Thus, y does not intersect x and so is the partner block of x. Now x is an arbitrary block of $\mathcal{G}$ and Q is an arbitrary point on x. So we have shown that for any block x of $\mathcal{G}$ with partner block $x^{'}$, and any point P of $\mathcal{G}$ with partner point $P^{'}$, P is on x if and only if $P^{'}$ is on $x^{'}$. ∎

Hence, given a square GD $2-\left(k^2-k+2, k, (0,1)\right)$ with point-classes of size two we may define a new structure $C(\mathcal{G})$ whose points are the point-classes of $\mathcal{G}$ and whose blocks are the block-classes of $\mathcal{G}$. A point-class μ is incident with a block-class β if and only if the points of μ are each incident with a block of β. In [29], Wild gives a similar construction of symmetric designs from GD $2-(v, k, (0,1))$ designs with $\lambda \geq 2$.

Lemma 4.21 *Let $\mathcal{G}$ be a square GD $2-\left(k^2-k+2, k, (0,1)\right)$ with point-classes of size two. Then the structure $C(\mathcal{G})$ is a symmetric $2-\left(\frac{1}{2}\left(k^2-k+2\right), k, 2\right)$.*

Proof

Each block of $C(\mathcal{G})$ is incident with k points of $C(\mathcal{G})$; the k classes containing the k points on a block of the block-class. $\mathcal{G}$ has k^2-k+2 points with point-classes of size two, and so $\mathcal{G}$ has $\frac{1}{2}\left(k^2-k+2\right)$ points. Similarly, $C(\mathcal{G})$ has $\frac{1}{2}\left(k^2-k+2\right)$ blocks and so is square.

Consider two point-classes μ_1 and μ_2, and let P be a point of μ_1. P is on exactly one block with each of the points of μ_2 and these two blocks are distinct. Furthermore, the two blocks represent two block-classes incident with μ_1 and μ_2 and there are no other block-classes incident with both μ_1 and μ_2 (since any such block-class contains a block on P). Thus, $C(\mathcal{G})$ is a symmetric $2-\left(\frac{1}{2}\left(k^2-k+2\right), k, 2\right)$. ∎

So we have established that a necessary condition for the existence of a GD $2-\left(k^2-k+2, k, (0,1)\right)$ with point-classes of size two is that there exists a biplane with block-size k. Furthermore, let $\mathcal{G}$ be a GD $2-\left(k^2-k+2, k, (0,1)\right)$ with point-classes of size two and label the points of $\mathcal{G}$ as $P_1, P_2, \ldots, P_{k^2-k+2}$ so that, for each $i = 1, 2, \ldots, \frac{1}{2}\left(k^2-k+2\right)$, the point P_{2i} is in the same point-class as the point P_{2i-1}. Label the blocks of $\mathcal{G}$ as $x_1, x_2, \ldots, x_{k^2-k+2}$ so that, for each $i = 1, 2, \ldots, \frac{1}{2}\left(k^2-k+2\right)$, the block x_{2i} is in the same block-class as the block x_{2i-1}. Let $A = (a_{ij})$ be the incidence matrix of $\mathcal{G}$ with respect to this labelling. Then, for each $i = 1, 2, \ldots, \frac{1}{2}\left(k^2-k+2\right)$ and each $j = 1, 2, \ldots, \frac{1}{2}\left(k^2-k+2\right)$, every 2×2 submatrix of A of the form,

$$A_{ij} = \begin{pmatrix} a_{2i-1,2j-1} & a_{2i-1,2j} \\ a_{2i,2j-1} & a_{2i,2j} \end{pmatrix},$$

must be one of,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We define a square matrix $W = (w_{ij})$ of size $\frac{1}{2}\left(k^2 - k + 2\right)$ by,

$$w_{ij} = \begin{cases} 0 & \text{if } A_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 1 & \text{if } A_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ -1 & \text{if } A_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{cases}$$

for each $i = 1, 2, \ldots, \frac{1}{2}\left(k^2 - k + 2\right)$ and each $j = 1, 2, \ldots, \frac{1}{2}\left(k^2 - k + 2\right)$. Replacing every negative entry of W by 1 gives an incidence matrix of $C(\mathcal{G})$, the structure constructed above. Now $C(\mathcal{G})$ is a biplane and so the dot product of any two distinct rows of an incidence matrix of $C(\mathcal{G})$ is equal to two. However, the dot product of any two rows of the matrix A is at most 1 and so it follows that W is a BOD $\left(\frac{1}{2}\left(k^2 - k + 2\right), \frac{1}{2}\left(k^2 - k + 2\right), k, k, 2\right)$ (using the notation defined in Section 1.5). Hence we have proved,

Lemma 4.22 *A necessary condition for the existence of a group divisible* $2 - \left(k^2 - k + 2, k, (0, 1)\right)$ *design with point-classes of size two is that there exists a BOD* $\left(\frac{1}{2}\left(k^2 - k + 2\right), \frac{1}{2}\left(k^2 - k + 2\right), k, k, 2\right)$. ∎

Suppose we are given a BOD $\left(\frac{1}{2}\left(k^2 - k + 2\right), \frac{1}{2}\left(k^2 - k + 2\right), k, k, 2\right)$, which necessarily has underlying matrix equal to an incidence matrix of a biplane with block-size k. Recall from Section 1.5 that it is then possible to construct a GD $2 - \left(k^2 - k + 2, k, (0, 1)\right)$ with point-classes of size two. Combining this with Lemma 4.22 and Corollary 4.18 gives:

Theorem 4.23 *A necessary and sufficient condition for the existence of a proper* $2 - (k^2, k, 1; \{1, k - 2\})$ *with* $k \geq 4$ *and exactly one point of weight* $k - 2$ *is that there exists a BOD* $\left(\frac{1}{2}\left(k^2 - k + 2\right), \frac{1}{2}\left(k^2 - k + 2\right), k, k, 2\right)$. ∎

Recall from Section 1.4.4 that examples of biplanes with block-size k are only known for $k = 3, 4, 5, 6, 9, 11$ and 13. Although the number of known biplanes is small, it is difficult in general to ascertain which biplanes have an incidence matrix which is the underlying matrix of a BOD $\left(\frac{1}{2}\left(k^2 - k + 2\right), \frac{1}{2}\left(k^2 - k + 2\right), k, k, 2\right)$. Example 4.12 exhibits a proper $2 - (16, 4, 1; \{1, 2\})$ and so there exists a BOD $(7, 7, 4, 4, 2)$ whose underlying matrix is an incidence matrix of the symmetric $2 - (7, 4, 2)$. The following result due to Bhaskar Rao (in [25]) demonstrates that there does not exist a BOD $(11, 11, 5, 5, 1)$, nor does there exist a BOD $(79, 79, 13, 13, 2)$.

Result 4.24 (Rao) *When v is odd, a necessary condition for the existence of a BOD (v, v, k, k, λ) is that k is a perfect square.* ∎

We now prove directly that there does not exist a BOD $(16, 16, 6, 6, 2)$.

Lemma 4.25 *There does not exist a BOD* $(16, 16, 6, 6, 2)$.

Proof

The underlying matrix of a BOD $(v, v, k, k, 2)$ is a biplane with block-size k, and so we show that it is not possible to replace the non-zero entries of an incidence matrix of a symmetric $2-(16, 6, 2)$ with ± 1 such that the resulting matrix is a BOD $(16, 16, 6, 6, 2)$.

Let I denote the 2×2 identity matrix, and denote by K the matrix,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are three non-isomorphic biplanes with block-size six (see [21]), which we call $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$. Let N be an incidence matrix of one of $\mathcal{B}_1$, $\mathcal{B}_2$ or $\mathcal{B}_3$. Then the rows of N can be permuted, and the columns of N permuted to give the matrix (see for example [30] page 181):

$$\left(\begin{array}{cccccccc|cccccccc}
0&1&1&0&1&0&1&0&1&1&0&0&0&0&0&0\\
1&0&0&1&0&1&0&1&1&1&0&0&0&0&0&0\\
1&0&0&1&1&0&1&0&0&0&1&1&0&0&0&0\\
0&1&1&0&0&1&0&1&0&0&1&1&0&0&0&0\\
1&0&1&0&0&1&1&0&0&0&0&0&1&1&0&0\\
0&1&0&1&1&0&0&1&0&0&0&0&1&1&0&0\\
1&0&1&0&1&0&0&1&0&0&0&0&0&0&1&1\\
0&1&0&1&0&1&1&0&0&0&0&0&0&0&1&1\\
\hline
1&1&0&0&0&0&0&0&0&1&1&0&1&0&1&0\\
1&1&0&0&0&0&0&0&1&0&0&1&0&1&0&1\\
0&0&1&1&0&0&0&0&1&0&\multicolumn{2}{c}{\multirow{2}{*}{A}}&\multicolumn{2}{c}{\multirow{2}{*}{B}}&\multicolumn{2}{c}{\multirow{2}{*}{C}}\\
0&0&1&1&0&0&0&0&0&1&&&&&&\\
0&0&0&0&1&1&0&0&1&0&\multicolumn{2}{c}{\multirow{2}{*}{D}}&\multicolumn{2}{c}{\multirow{2}{*}{E}}&\multicolumn{2}{c}{\multirow{2}{*}{F}}\\
0&0&0&0&1&1&0&0&0&1&&&&&&\\
0&0&0&0&0&0&1&1&1&0&\multicolumn{2}{c}{\multirow{2}{*}{G}}&\multicolumn{2}{c}{\multirow{2}{*}{H}}&\multicolumn{2}{c}{\multirow{2}{*}{L}}\\
0&0&0&0&0&0&1&1&0&1&&&&&&
\end{array}\right),$$

where each of the 2×2 matrices A, B, C, D, E, F, G, H and L is equal to either K or I. There are three possibilities corresponding to the three known biplanes:

(i) if N is an incidence matrix of $\mathcal{B}_1$ then,

$$\begin{array}{l} A = E = L = K, \\ B = C = D = F = G = H = I; \end{array}$$

(ii) if N is an incidence matrix of $\mathcal{B}_2$ then,

$$\begin{array}{l} B = F = G = K, \\ A = C = D = E = H = L = I; \end{array}$$

(iii) if N is an incidence matrix of $\mathcal{B}_3$ then,

$$\begin{aligned} C &= D = H = K, \\ A &= B = E = F = G = L = I. \end{aligned}$$

Consider the matrix N above. We consider all possible ways of replacing the non-zero entries of the first ten rows and the first ten columns of N with ± 1 so that the dot product of any (distinct) two of the first ten rows is zero. We show that when N is an incidence matrix of either $\mathcal{B}_1$, $\mathcal{B}_2$ or $\mathcal{B}_3$ it is not possible to then replace the remaining non-zero entries of the eleventh row with ± 1 so that the dot product of any two distinct rows of N is zero.

Given a balanced orthogonal design W, it is possible to multiply any row or column by -1 and the resulting matrix is a balanced orthogonal design. Suppose W is a balanced orthogonal design with underlying matrix N. Then we may assume without loss of generality that the first entry in each column and the first entry in each row is 1.

We follow the usual convention of denoting the $(i,j)^{\text{th}}$ entry of a matrix labelled with an upper-case letter by the corresponding lower-case letter with subscript ij (i.e., we set $A = (a_{ij})$, $B = (b_{ij})$, etc.). For clarity, we now use the symbol α for -1 and the symbol $-$ for 0.

Assuming that the first entry in each row and the first entry in each column of W is 1 determines that W is of the form,

$$W = \left(\begin{array}{cccccccc|cccccccc}
- & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & - \\
1 & - & - & 1 & - & 1 & - & 1 & q & & - & - & - & - & - & - \\
1 & - & - & \alpha & p & - & & - & - & - & 1 & 1 & - & - & - & - \\
- & 1 & \alpha & - & - & & - & & - & - & & & - & - & - & - \\
1 & - & & - & - & \alpha & & - & - & - & - & - & 1 & 1 & - & - \\
- & 1 & - & & \alpha & - & - & & - & - & - & - & & & - & - \\
1 & - & & - & & - & - & \alpha & - & - & - & - & - & - & 1 & 1 \\
- & 1 & - & & - & & \alpha & - & - & - & - & - & - & - & & \\
\hline
1 & & - & - & - & - & - & - & - & & \alpha & - & \alpha & - & \alpha & - \\
1 & & - & - & - & - & - & - & & - & - & \alpha & - & \alpha & - & \alpha \\
- & - & 1 & & - & - & - & - & \alpha & - & a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} \\
- & - & 1 & & - & - & - & - & - & \alpha & a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} \\
- & - & - & - & 1 & & - & - & \alpha & - & d_{11} & d_{12} & e_{11} & e_{12} & f_{11} & f_{12} \\
- & - & - & - & 1 & & - & - & - & \alpha & d_{21} & d_{22} & e_{21} & e_{22} & f_{21} & f_{22} \\
- & - & - & - & - & - & 1 & & \alpha & - & g_{11} & g_{12} & h_{11} & h_{12} & l_{11} & l_{12} \\
- & - & - & - & - & - & 1 & & - & \alpha & g_{21} & g_{22} & h_{21} & h_{22} & l_{21} & l_{22}
\end{array}\right),$$

where a blank entry denotes an entry which is either 1 or α, but is as yet undetermined. Similarly, each of p and q is either 1 or α, but is as yet undetermined. The values of p and q determine the values of all the remaining non-zero entries in the first ten columns and the first ten rows. There are four cases to consider:

(i) $p = q = 1$;

(ii) $p = 1$, $q = \alpha$;

(iii) $p = \alpha$, $q = 1$;

(iv) $p = q = \alpha$.

We consider each case in turn. Label the rows of W in order from the top as $r_1, r_2, \ldots, r_{16}$. For some $i, j \in \{1, 2, \ldots, 16\}$, we say that r_i and r_j 'pass' if the dot product of r_i and r_j is zero.

Case (i) : $p = q = 1$ In this case, W is determined to be of the form:

$$W = \left(\begin{array}{cccccccc|cccccccc}
- & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & - \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & \alpha & - & - & - & - & - & - \\
1 & - & - & \alpha & 1 & - & \alpha & - & - & - & 1 & 1 & - & - & - & - \\
- & 1 & \alpha & - & - & 1 & - & \alpha & - & - & \alpha & 1 & - & - & - & - \\
1 & - & \alpha & - & - & \alpha & 1 & - & - & - & - & - & 1 & 1 & - & - \\
- & 1 & - & \alpha & \alpha & - & - & 1 & - & - & - & - & \alpha & 1 & - & - \\
1 & - & 1 & - & \alpha & - & - & \alpha & - & - & - & - & - & - & 1 & 1 \\
- & 1 & - & 1 & - & \alpha & \alpha & - & - & - & - & - & - & - & \alpha & 1 \\
\hline
1 & \alpha & - & - & - & - & - & - & - & 1 & \alpha & - & \alpha & - & \alpha & - \\
1 & 1 & - & - & - & - & - & - & \alpha & - & - & \alpha & - & \alpha & - & \alpha \\
- & - & 1 & 1 & - & - & - & - & \alpha & - & a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} \\
- & - & 1 & \alpha & - & - & - & - & - & \alpha & a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} \\
- & - & - & - & 1 & 1 & - & - & \alpha & - & d_{11} & d_{12} & e_{11} & e_{12} & f_{11} & f_{12} \\
- & - & - & - & 1 & \alpha & - & - & - & \alpha & d_{21} & d_{22} & e_{21} & e_{22} & f_{21} & f_{22} \\
- & - & - & - & - & - & 1 & 1 & \alpha & - & g_{11} & g_{12} & h_{11} & h_{12} & l_{11} & l_{12} \\
- & - & - & - & - & - & 1 & \alpha & - & \alpha & g_{21} & g_{22} & h_{21} & h_{22} & l_{21} & l_{22}
\end{array}\right).$$

If N (the underlying matrix of W) is an incidence matrix of $\mathcal{B}_1$ then b_{11} is non-zero. For r_5 and r_{11} to pass we require $b_{11} = 1$. However, for r_6 and r_{11} to pass we require $b_{11} = \alpha$, giving a contradiction. Thus, W cannot be a balanced orthogonal design.

If N is an incidence matrix of either $\mathcal{B}_2$ or $\mathcal{B}_3$ then a_{11} is non-zero. For r_3 and r_{11} to pass we require $a_{11} = 1$. However, for r_4 and r_{11} to pass we require $a_{11} = \alpha$, giving a contradiction. Thus, W cannot be a balanced orthogonal design.

Case (ii) : $p = 1$, $q = \alpha$ In this case, W is determined to be of the form:

$$W = \left(\begin{array}{cccccccc|cccccccc}
- & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & - \\
1 & - & - & 1 & - & 1 & - & 1 & \alpha & 1 & - & - & - & - & - & - \\
1 & - & - & \alpha & 1 & - & \alpha & - & - & - & 1 & 1 & - & - & - & - \\
- & 1 & \alpha & - & - & 1 & - & \alpha & - & - & 1 & \alpha & - & - & - & - \\
1 & - & \alpha & - & - & \alpha & 1 & - & - & - & - & - & 1 & 1 & - & - \\
- & 1 & - & \alpha & \alpha & - & - & 1 & - & - & - & - & 1 & \alpha & - & - \\
1 & - & 1 & - & \alpha & - & - & \alpha & - & - & - & - & - & - & 1 & 1 \\
- & 1 & - & 1 & - & \alpha & \alpha & - & - & - & - & - & - & - & 1 & \alpha \\
\hline
1 & 1 & - & - & - & - & - & - & - & \alpha & \alpha & - & \alpha & - & \alpha & - \\
1 & \alpha & - & - & - & - & - & - & 1 & - & - & \alpha & - & \alpha & - & \alpha \\
- & - & 1 & \alpha & - & - & - & - & \alpha & - & a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} \\
- & - & 1 & 1 & - & - & - & - & - & \alpha & a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} \\
- & - & - & - & 1 & \alpha & - & - & \alpha & - & d_{11} & d_{12} & e_{11} & e_{12} & f_{11} & f_{12} \\
- & - & - & - & 1 & 1 & - & - & - & \alpha & d_{21} & d_{22} & e_{21} & e_{22} & f_{21} & f_{22} \\
- & - & - & - & - & - & 1 & \alpha & \alpha & - & g_{11} & g_{12} & h_{11} & h_{12} & l_{11} & l_{12} \\
- & - & - & - & - & - & 1 & 1 & - & \alpha & g_{21} & g_{22} & h_{21} & h_{22} & l_{21} & l_{22}
\end{array}\right).$$

If N is an incidence matrix of $\mathcal{B}_1$ then b_{11} is non-zero. For r_5 and r_{11} to pass we require $b_{11} = 1$. However, for r_6 and r_{11} to pass we require $b_{11} = \alpha$, giving a contradiction. Thus, W cannot be a balanced orthogonal design.

If N is an incidence matrix of either $\mathcal{B}_2$ or $\mathcal{B}_3$ then a_{11} is non-zero. For r_3 and r_{11} to pass we require $a_{11} = \alpha$. However, for r_4 and r_{11} to pass we require

$a_{11} = 1$, giving a contradiction. Thus, W cannot be a balanced orthogonal design.

Case (iii) : $p = \alpha$, $q = 1$ In this case, W is determined to be of the form:

$$W = \left(\begin{array}{cccccccc|cccccccc}
- & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & - \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & \alpha & - & - & - & - & - & - \\
1 & - & - & \alpha & \alpha & - & 1 & - & - & - & 1 & 1 & - & - & - & - \\
- & 1 & \alpha & - & - & \alpha & - & 1 & - & - & \alpha & 1 & - & - & - & - \\
1 & - & 1 & - & - & \alpha & \alpha & - & - & - & - & - & 1 & 1 & - & - \\
- & 1 & - & 1 & \alpha & - & - & \alpha & - & - & - & - & \alpha & 1 & - & - \\
1 & - & \alpha & - & 1 & - & - & \alpha & - & - & - & - & - & - & 1 & 1 \\
- & 1 & - & \alpha & - & 1 & \alpha & - & - & - & - & - & - & - & \alpha & 1 \\
\hline
1 & \alpha & - & - & - & - & - & - & - & 1 & \alpha & - & \alpha & - & \alpha & - \\
1 & 1 & - & - & - & - & - & - & \alpha & - & - & \alpha & - & \alpha & - & \alpha \\
- & - & 1 & 1 & - & - & - & - & \alpha & - & a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} \\
- & - & 1 & \alpha & - & - & - & - & - & \alpha & a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} \\
- & - & - & - & 1 & 1 & - & - & \alpha & - & d_{11} & d_{12} & e_{11} & e_{12} & f_{11} & f_{12} \\
- & - & - & - & 1 & \alpha & - & - & - & \alpha & d_{21} & d_{22} & e_{21} & e_{22} & f_{21} & f_{22} \\
- & - & - & - & - & - & 1 & 1 & \alpha & - & g_{11} & g_{12} & h_{11} & h_{12} & l_{11} & l_{12} \\
- & - & - & - & - & - & 1 & \alpha & - & \alpha & g_{21} & g_{22} & h_{21} & h_{22} & l_{21} & l_{22}
\end{array}\right).$$

If N is an incidence matrix of $\mathcal{B}_1$ then b_{11} is non-zero. For r_5 and r_{11} to pass we require $b_{11} = \alpha$. However, for r_6 and r_{11} to pass we require $b_{11} = 1$, giving a contradiction. Thus, W cannot be a balanced orthogonal design.

If N is an incidence matrix of either $\mathcal{B}_2$ or $\mathcal{B}_3$ then a_{11} is non-zero. For r_3 and r_{11} to pass we require $a_{11} = 1$. However, for r_4 and r_{11} to pass we require $a_{11} = \alpha$, giving a contradiction. Thus, W cannot be a balanced orthogonal design.

Case (iv) : $p = q = \alpha$ In this case, W is determined to be of the form:

$$W = \left(\begin{array}{cccccccc|cccccccc}
- & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & - \\
1 & - & - & 1 & - & 1 & - & 1 & \alpha & 1 & - & - & - & - & - & - \\
1 & - & - & \alpha & \alpha & - & 1 & - & - & - & 1 & 1 & - & - & - & - \\
- & 1 & \alpha & - & - & \alpha & - & 1 & - & - & 1 & \alpha & - & - & - & - \\
1 & - & 1 & - & - & \alpha & \alpha & - & - & - & - & - & 1 & 1 & - & - \\
- & 1 & - & 1 & \alpha & - & - & \alpha & - & - & - & - & 1 & \alpha & - & - \\
1 & - & \alpha & - & 1 & - & - & \alpha & - & - & - & - & - & - & 1 & 1 \\
- & 1 & - & \alpha & - & 1 & \alpha & - & - & - & - & - & - & - & 1 & \alpha \\
\hline
1 & 1 & - & - & - & - & - & - & - & \alpha & \alpha & - & \alpha & - & \alpha & - \\
1 & \alpha & - & - & - & - & - & - & 1 & - & - & \alpha & - & \alpha & - & \alpha \\
- & - & 1 & \alpha & - & - & - & - & \alpha & - & a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} \\
- & - & 1 & 1 & - & - & - & - & - & \alpha & a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} \\
- & - & - & - & 1 & \alpha & - & - & \alpha & - & d_{11} & d_{12} & e_{11} & e_{12} & f_{11} & f_{12} \\
- & - & - & - & 1 & 1 & - & - & - & \alpha & d_{21} & d_{22} & e_{21} & e_{22} & f_{21} & f_{22} \\
- & - & - & - & - & - & 1 & \alpha & \alpha & - & g_{11} & g_{12} & h_{11} & h_{12} & l_{11} & l_{12} \\
- & - & - & - & - & - & 1 & 1 & - & \alpha & g_{21} & g_{22} & h_{21} & h_{22} & l_{21} & l_{22}
\end{array}\right).$$

If N is an incidence matrix of $\mathcal{B}_1$ then b_{11} is non-zero. For r_5 and r_{11} to pass we require $b_{11} = \alpha$. However, for r_6 and r_{11} to pass we require $b_{11} = 1$, giving a contradiction. Thus, W cannot be a balanced orthogonal design.

If N is an incidence matrix of either $\mathcal{B}_2$ or $\mathcal{B}_3$ then a_{11} is non-zero. For r_3 and r_{11} to pass we require $a_{11} = \alpha$. However, for r_4 and r_{11} to pass we require $a_{11} = 1$, giving a contradiction. Thus, W cannot be a balanced orthogonal design.

So, we have shown that it is not possible to replace the non-zero entries of an incidence matrix of a biplane with block-size six with ± 1 so that the resulting matrix is a balanced orthogonal design. Thus, we have shown that there does not exist a BOD $(16, 16, 6, 6, 2)$. ∎

We have shown that a proper $2-(k^2, k, 1; \{1, k-2\})$ with $k \geq 4$ and exactly one point of weight $k-2$ exists if and only if there exists a biplane with block-size k and a BOD $\left(\frac{1}{2}\left(k^2-k+2\right), \frac{1}{2}\left(k^2-k+2\right), k, k, 2\right)$ with underlying matrix equal to an incidence matrix of the biplane. We have demonstrated the existence of a proper $2-(16, 4, 1; \{1, 2\})$ with exactly one point of weight 2. However, the required balanced orthogonal design does not exist when $k = 5, 6$ or 13, despite the existence of biplanes with block-size five, six and thirteen. Hence, the existence of a biplane with block-size $k \geq 4$ alone is necessary but not sufficient for the existence of a proper $2-(k^2, k, 1; \{1, k-2\})$ with exactly one point of weight $k-2$. It remains an open problem whether a BOD $(37, 37, 9, 9, 2)$ or a BOD $(56, 56, 11, 11, 2)$ exists.

Chapter 5

Point-Complementing and Trivial Point-Weighted Designs

We begin this chapter by introducing the procedure of *point-complementing* incidence structures and use it to construct point-weighted designs from certain block designs. We define *trivial point-weighted designs* and consider some specific families of such structures. When considering these it is convenient to view the blocks of a design as subsets of the point-set, with incidence as containment, and we shall do so throughout this chapter.

Definition 5.1 *A* **trivial point-weighted design** *with parameters v and k is a point-weighted design with those parameters, where the block-set consists of every possible subset of the point-set which has weight-sum k.*

With this definition, a trivial $t-(v,k,\lambda;\{1\})$ clearly has as its underlying incidence structure a trivial $t-(v,k,\lambda)$ - a trivial block design, as defined in Chapter 1. We conclude the chapter by using point-complementing to establish a correspondence between the underlying incidence structures of certain trivial point-weighted designs and a class of trivial block designs.

5.1 Point-Complementing

Let $\mathcal{D}$ be a $2-(v,K,\lambda)$ with $v > \max K$. In 1970, Woodall ([34]) and Bridges ([5]) introduced a method of obtaining from such a design another (possibly isomorphic) $2-(v,K',\lambda')$ for some K' and λ'. We refer to this method as **point-complementing** (Woodall referred to this method as 'point-complementation, point un-changed' and Bridges referred to the resulting $2-(v,K',\lambda')$ as a 'type-I λ-design'). Following the notation of [2], we denote by $\mathcal{S} = (\mathbf{V}, \mathbf{B}, \in)$

an incidence structure with point-set $\mathbf{V}$ and block-set $\mathbf{B}$ whose blocks are to be viewed as sets of points with incidence in $\mathcal{S}$ given by containment.

Let $\mathcal{S} = (\mathbf{V}, \mathbf{B}, \in)$ be an incidence structure. For a given point P, define a block-set $\mathbf{B}_{(P)}$ by,

$$\mathbf{B}_{(P)} = \{x \mid x \in \mathbf{B} \text{ with } P \notin x\} \cup \{(\mathbf{V} \setminus y) \cup \{P\} \mid y \in \mathbf{B} \text{ with } P \in y\}.$$

Then we call the incidence structure $(\mathbf{V}, \mathbf{B}_{(P)}, \in)$ the **point-complement** of $\mathcal{S}$ at P, and denote this structure by $\mathcal{S}_{(P)}$.

Result 5.2 (Woodall, Bridges) *Let $\mathcal{D} = (\mathbf{V}, \mathbf{B}, \in)$ be a $2-(v, K, \lambda)$ with $v > maxK$. Let P be a point of $\mathcal{D}$ and denote the number of blocks containing P by r_P. Then the point-complement of $\mathcal{D}$, $\mathcal{D}_{(P)} = (\mathbf{V}, \mathbf{B}_{(P)}, \in)$, defined as above, is a $2-(v, K', r_P - \lambda)$ for some set K'.*

Proof

It is clear that since $\mathcal{D}$ is a design, no two blocks of $\mathcal{D}_{(P)}$ contain exactly the same points, and so $\mathcal{D}_{(P)}$ is also a design. We show that any pair of points of $\mathcal{D}_{(P)}$ is contained in exactly $r_P - \lambda$ blocks. Firstly, let Q be a point not equal to P. There are λ blocks of $\mathcal{D}$ containing both P and Q, and hence $r_P - \lambda$ blocks of $\mathcal{D}$ containing P and not Q, and hence $r_P - \lambda$ blocks of $\mathcal{D}_{(P)}$ containing both P and Q.

Secondly, let Q and R be distinct points, not equal to P. The number of blocks of $\mathcal{D}_{(P)}$ containing both Q and R is equal to the number of blocks of $\mathcal{D}$ containing both Q and R but not P plus the number of blocks of $\mathcal{D}$ containing P but neither Q or R. Let λ_{PQR} denote the number of blocks of $\mathcal{D}$ containing all of P, Q and R. Then the number of blocks of $\mathcal{D}$ containing both Q and R but not P is $\lambda - \lambda_{PQR}$. The number of blocks of $\mathcal{D}$ containing P but neither Q or R is given by $r_P - 2\lambda + \lambda_{PQR}$. Hence, the number of blocks of $\mathcal{D}_{(P)}$ containing both Q and R is $r_P - \lambda$.

Thus, $\mathcal{D}_{(P)}$ is a $2-(v, K', r_P - \lambda)$ for some set K'. ∎

As remarked upon by Woodall, it is easily seen that for any incidence structure $\mathcal{S}$, $\left(\mathcal{S}_{(P)}\right)_{(P)} = \mathcal{S}$ for any point P of $\mathcal{S}$. Also, for any other point Q of $\mathcal{S}$, $\left(\mathcal{S}_{(P)}\right)_{(Q)}$ is the same as $\mathcal{S}_{(Q)}$ but with the points P and Q interchanged.

We now show a method of using the technique of point-complementing to construct certain point-weighted designs from block designs.

5.1.1 Constructing Point-Weighted Designs

Let $\mathcal{D} = (\mathbf{V}, \mathbf{B}, \in)$ be a proper $2-(u, k, \lambda)$ (i.e., a block design) with $k > \frac{u+1}{2}$ and let r be the number of blocks containing any one point of $\mathcal{D}$. Let P be a point of $\mathcal{D}$ and let $\mathcal{D}_{(P)} = (\mathbf{V}, \mathbf{B}_{(P)}, \in)$ be the point-complement of $\mathcal{D}$ at P. We construct a point-weighted structure, $\mathcal{S} = (\mathbf{V}, \mathbf{B}_{(P)}, \in, w)$, with underlying incidence structure $\mathcal{D}_{(P)}$, by defining the weight function w on $\mathbf{V}$ as follows.

Let Q be any element of $\mathbf{V}$, then,

$$w(Q) = \begin{cases} 2k - u & \text{if } Q = P \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 5.3 *Let $\mathcal{D}$ be a proper $2-(u, k, \lambda)$ with $k > \frac{u+1}{2}$ and r blocks containing any one point, and let P be a point of $\mathcal{D}$. Then the structure $\mathcal{S}$, constructed from $\mathcal{D}$ as above, is a $2-(2k-1, k, r-\lambda; \{1, 2k-u\})$.*

Proof

By Result 5.2, for any point P of $\mathcal{D}$, the point-complement of $\mathcal{D}$ at P is a $2-(u, K, r-\lambda)$, for some set K. Since the underlying incidence structure of $\mathcal{S}$ is the point-complement of $\mathcal{D}$ at some point P, every pair of points of $\mathcal{S}$ is thus contained in exactly $r - \lambda$ blocks.

Since $k > \frac{u+1}{2}$, the weight of the point P in $\mathcal{S}$ is greater than 1. Thus, all the points of $\mathcal{S}$ have weight 1, apart from P which has weight $2k - u$. So the weight-set of $\mathcal{S}$ is $\{1, 2k-u\}$ and the sum of the weights of all the points of $\mathcal{S}$ is $2k - 1$.

We now need to show that the sum of the weights of the points in any block of $\mathcal{S}$ is k. There are two types of blocks in $\mathcal{S}$. The first type does not contain the point P, but contains exactly k other points, all with weight 1. The second type contains the point P plus exactly $u - k$ points of weight 1. Since P has weight $2k - u$, it is clear that the sum of the weights of the points in any block of $\mathcal{S}$ is k.

Thus, $\mathcal{S}$ is a $2-(2k-1, k, r-\lambda; \{1, 2k-u\})$. ■

Example 5.4 Let $\mathcal{D} = (\mathbf{V}, \mathbf{B}, \in)$ be the $2-(9, 6, 5)$ whose incidence matrix is

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1
\end{pmatrix}$$

Figure 5.1: A $2-(9, 6, 5)$.

given in Figure 5.1. Label the points $P_1, P_2, \ldots, P_9$ and the blocks $x_1, x_2, \ldots, x_{12}$ so that the matrix in Figure 5.1 is the incidence matrix of $\mathcal{D}$ with respect to such a labelling. By taking the point-complement of $\mathcal{D}$ at the point P_1 and assigning weights to points as specified in the above construction, we construct a $2-(11, 6, 3; \{1, 3\})$ whose weighted incidence matrix is given in Figure 5.2.

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{pmatrix}$$

Figure 5.2: A $2-(11,6,3;\{1,3\})$.

Consider a block design with parameters u,k and λ such that $k>\frac{u+1}{2}$. We show as a corollary to Lemma 5.3 that the point-complement of such a block design at any point cannot be $3-$balanced (that is, cannot have the property that every set of three distinct points is contained in the same number of blocks).

Corollary 5.5 *Let $\mathcal{D}=(\mathbf{V},\mathbf{B},\in)$ be a $2-(u,k,\lambda)$ with $k>\frac{u+1}{2}$ and let P be a point of $\mathcal{D}$. Let $\mathcal{D}_{(P)}=(\mathbf{V},\mathbf{B}_{(P)},\in)$ be the point-complement of $\mathcal{D}$ at P. Then $\mathcal{D}_{(P)}$ is not $3-$balanced.*

Proof

Suppose $\mathcal{D}_{(P)}$ does have the property that every set of three distinct points is contained in the same number of blocks. Then, letting K be the set of block-sizes in $\mathcal{D}_{(P)}$, there exists a value λ_3 such that $\mathcal{D}_{(P)}$ is a $3-(u,K,\lambda_3)$.

We construct the point-weighted structure $\mathcal{S}=(\mathbf{V},\mathbf{B}_{(P)},\in,w)$ with $\mathcal{D}_{(P)}$ as its underlying incidence structure as above, by setting the weight of P to be $2k-u$ and setting the weight of all other points to be 1. Then, by Lemma 5.3, $\mathcal{S}$ is a $2-(2k-1,k,r-\lambda;\{1,2k-u\})$, where r is the number of blocks of $\mathcal{D}$ containing any one point. But since $\mathcal{D}_{(P)}$ - the underlying incidence structure of $\mathcal{S}$ - has the property that every set of three points is contained in exactly λ_3 blocks, $\mathcal{S}$ is also a $3-(2k-1,k,\lambda_3;\{1,2k-u\})$. But this directly contradicts Corollary 3.9. Hence, $\mathcal{D}_{(P)}$ does not have the property that every set of three distinct points is contained in the same number of blocks. ■

5.2 Trivial $t-(v,k,\lambda;\mathbf{W})$ Point-Weighted Designs

In this section we consider trivial point-weighted designs and examine when a trivial point-weighted design is a $t-(v,k,\lambda;\mathbf{W})$ for some parameters t,v,k and λ and weight-set $\mathbf{W}$ (i.e., when it has the property that every set of t distinct points is contained in a fixed number of blocks). It is generally a hard problem to establish explicit conditions on the parameters so that this is the case but in Section 5.2.3 we do so for trivial point-weighted designs in which all but one of the points have the same weight.

5.2.1 Improper Trivial Designs

Let $\mathbf{V}$ be a set of u points, w a weight function on $\mathbf{V}$, and set $\mathbf{W} = \text{image}(w)$ and $v = \sum_{P \in \mathbf{V}} w(P)$. Clearly, for any such $\mathbf{V}$ and w, we can construct a trivial point-weighted design, $\mathcal{D}$, which is a $t-(v, v, 1; \mathbf{W})$ for any $1 \leq t \leq u$, by defining the block-set of $\mathcal{D}$ to consist of exactly one block containing all the points of $\mathbf{V}$. This type of point-weighted structure is clearly not proper and so we call such an object an **improper** point-weighted design.

5.2.2 Proper Trivial Designs

Let $\mathcal{D} = (\mathbf{V}, \mathbf{B}, \in)$ be a trivial point-weighted design with parameters v and k, weight-set $\mathbf{W}$ and $u = |\mathbf{V}|$. We have already seen that $\mathcal{D}$ is a $t-(v, k, 1; \mathbf{W})$ for any $1 \leq t \leq u$ if $v = k$ and $\mathcal{D}$ is improper. So we assume that $v > k$ and $\mathcal{D}$ is proper. Let $\mathbf{T} = \{T_i \mid i = 1, 2, \ldots, \binom{u}{t}\}$ be the set of all unordered sets of t distinct points from $\mathbf{V}$ for some $t \geq 1$. For each T_i $(i = 1, 2, \ldots, \binom{u}{t})$ let λ_{T_i} denote the number of blocks of $\mathcal{D}$ containing the set of t points, T_i. Then $\mathcal{D}$ is a $t-(v, k, \lambda; \mathbf{W})$ for some $\lambda > 0$ if and only if $\lambda_{T_i} = \lambda$ for every $i = 1, 2, \ldots, \binom{u}{t}$.

Suppose there exist $j, l \in \{1, 2, \ldots, \binom{u}{t}\}$ such that the points of T_j and T_l can be labelled $P_1, P_2, \ldots, P_t$ and $Q_1, Q_2, \ldots, Q_t$ respectively, with $w(P_i) = w(Q_i)$ for each $i = 1, 2, \ldots, t$. Then, since $\mathcal{D}$ is a trivial point-weighted design, the number of blocks containing T_j is the same as the number of blocks containing T_l, and is equal to the number of subsets of points from $\mathbf{V} \setminus T_j$ whose weights sum to $k - \sigma(T_j)$.

To establish whether a given trivial point-weighted design $\mathcal{D}$ has the property that every set of t distinct points is contained in a fixed number of blocks for some t, it is necessary to count the number of blocks containing a set of t points of given weights for every possible multiset of weights of t distinct points of $\mathcal{D}$. In general this is prohibitively complicated, but in the following section we do so for trivial point-weighted designs in which all but one of the points have the same weight.

5.2.3 Trivial Designs with One 'Special' Point

Let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a trivial point-weighted design with weight-set $\mathbf{W} = \{1, a\}$, where a is a positive integer greater than 1, such that exactly one point has weight a and all other points have weight 1. The block-set $\mathbf{B}$ of $\mathcal{D}$ is determined by a and its parameters v and k. In this section we establish necessary and sufficient conditions for $\mathcal{D}$ to be a $t-(v, k, \lambda; \{1, a\})$ for some t, and determine an expression for the corresponding value of λ.

One necessary condition for $\mathcal{D}$ to be a $t-(v, k, \lambda; \{1, a\})$ for some t is that,

$$k \geq a + t - 1, \tag{5.1}$$

since every set of t distinct points of $\mathcal{D}$ must be contained in λ blocks. Before considering the case when $\mathcal{D}$ is proper, we obtain a necessary and sufficient condition for $\mathcal{D}$ to be improper.

Lemma 5.6 *Let a be an integer greater than 1 and let* $\mathcal{D}$ *be a* $t-(v,k,\lambda;\{1,a\})$ *with exactly one point of weight a, all other points having weight 1. Then* $\mathcal{D}$ *is improper if and only if* $v-a<k$.

Proof

Suppose $\mathcal{D}$ is improper. Then $v=k$ and it immediately follows that $v-a<k$.

We note that the value $v-a$ is precisely the number of points of weight 1 in $\mathcal{D}$, which we denote by $u_{(1)}$. We show that if $u_{(1)}<k$ then $v=k$ and so $\mathcal{D}$ is improper.

Let $u_{(1)}<k$. There is only one point, P say, not of weight 1, and the sum of the weights of the points in any one block is k. So it follows that the point P is contained in every block of $\mathcal{D}$. As in the proof of Lemma 4.1 we show that this gives $v=k$. In the case $t=1$ this is clearly true since every point is contained in the same number of blocks as the point P. Thus, every point is in every block of $\mathcal{D}$ and so $v=k$. We now consider the case $t>1$.

Let S be a set of $t-1$ points of $\mathcal{D}$, all of weight 1, and set $T=S\cup\{P\}$. Then T is a set of t distinct points of $\mathcal{D}$ and so is contained in exactly λ blocks. Since the point P is contained in every block of $\mathcal{D}$, the set S is also contained in exactly λ blocks. The weight-sum of S is $t-1$, and so by Theorem 3.6, the number of blocks containing S, λ_S, is,

$$\lambda_S=\lambda\,\frac{(v-t+1)}{(k-t+1)}.$$

But there are exactly λ blocks containing S and so we require $\lambda_S=\lambda$. Hence, $v=k$ and $\mathcal{D}$ is improper. ∎

Now let $\mathcal{D}=(\mathbf{V},\mathbf{B},\mathbf{I},w)$ be a proper trivial point-weighted design with weight-set $\mathbf{W}=\{1,a\}$, where a is a positive integer greater than 1, such that exactly one point has weight a and all other points have weight 1. Since all but one of the points of $\mathcal{D}$ have weight 1, any set of t distinct points of $\mathcal{D}$ will either contain t points of weight 1 or $t-1$ points of weight 1 together with the point of weight a. Let $\lambda_{a,\mathbf{1}}$ denote the number of blocks of $\mathcal{D}$ containing a set of t distinct points of which exactly one has weight a and all the others have weight 1, and let $\lambda_{1,\mathbf{1}}$ denote the number of blocks of $\mathcal{D}$ containing a set of t distinct points which all have weight 1. Then a set of t distinct points of $\mathcal{D}$ will be contained in either $\lambda_{1,\mathbf{1}}$ blocks or $\lambda_{a,\mathbf{1}}$ blocks. $\mathcal{D}$ is a $t-(v,k,\lambda;\{1,a\})$ if and only if every set of t distinct points is contained in exactly λ blocks, i.e., if and only if,

$$\lambda_{1,\mathbf{1}}=\lambda_{a,\mathbf{1}}=\lambda. \tag{5.2}$$

We now obtain expressions for $\lambda_{1,\mathbf{1}}$ and $\lambda_{a,\mathbf{1}}$. As before, denote the number of points of weight 1 by $u_{(1)}$. Then $v=a+u_{(1)}$, and the total number of points, $u=|\mathbf{V}|$, is equal to $u_{(1)}+1$. The number of blocks containing a set of t distinct points, one of which has weight a with all the others having weight 1, is then equal to the number of ways of choosing an unordered set of $k-(a+t-1)$ points from the remaining $u_{(1)}-(t-1)$ points of weight 1 (recall from (5.1) that $k \geq a+t-1$). So,

$$\lambda_{a,\mathbf{1}} = \binom{u_{(1)}-(t-1)}{k-(a+t-1)}. \tag{5.3}$$

We note that if $k = a+t-1$ then the number of blocks containing a set of t distinct points, one of which has weight a with all the others having weight 1, must be one, since $\mathcal{D}$ is a point-weighted design and does not therefore have repeated blocks. Then the expression for $\lambda_{a,\mathbf{1}}$ in (5.3) is valid if $k = a+t-1$ since the combination is by definition equal to 1.

If $k \geq a+t$, the number of blocks containing a set of t distinct points, all of weight 1, is equal to the number of ways of choosing an unordered set of $k-t$ points from the $u_{(1)}-t$ remaining points of weight 1 plus the number of ways of choosing an unordered set of $k-(a+t)$ points from the $u_{(1)}-t$ remaining points of weight 1 (recall from Lemma 5.6 that $u_{(1)} \geq k$ since $\mathcal{D}$ is proper). So,

$$\lambda_{1,\mathbf{1}} = \binom{u_{(1)}-t}{k-t} + \binom{u_{(1)}-t}{k-(a+t)}. \tag{5.4}$$

We note that this expression for $\lambda_{1,\mathbf{1}}$ is also valid if $k = a+t-1$ since the second term would then by definition be zero. This corresponds to the fact that if $k = a+t-1$ there could be no block containing a set of t points of weight 1 together with the point of weight a. Similarly, the expression is valid if $k = a+t$, since then the second term would by definition be equal to 1. This corresponds to the fact that for a given set of t distinct points of weight 1, T say, there must be exactly one block containing both T and the point of weight a, since $\mathcal{D}$ is a trivial point-weighted design.

Before equating the expressions for $\lambda_{a,\mathbf{1}}$ and $\lambda_{1,\mathbf{1}}$ in (5.3) and (5.4) respectively, to determine t for which $\lambda_{a,\mathbf{1}} = \lambda_{1,\mathbf{1}}$, we require the following result concerning combinations:

Lemma 5.7 *Let a, b and c be positive integers satisfying $\binom{a}{c} = \binom{b}{c}$. Then either,*

(i) $a < c$ in which case $b < c$,

or,

(ii) $a \geq c$ in which case $b \geq c$ and $a = b$.

Proof

(i) Let $a < c$. Then, by definition, $\binom{a}{c} = 0$ and so $\binom{b}{c} = 0$. But b and c are both positive and so $b < c$.

(ii) Let $a \geq c$. Then $\binom{a}{c} > 0$ and so $\binom{b}{c} > 0$. Hence, $b \geq c$.

Now suppose $a \neq b$ and, without loss of generality let $a < b$ (if $a > b$ we re-label a as b and b as a). From the statement of the lemma, $\binom{a}{c} = \binom{b}{c}$, and we express the combinations in this equality as quotients of factorials:

$$\frac{a!}{c!\,(a-c)!} = \frac{b!}{c!\,(b-c)!}.$$

Multiplying through by $c!$ and simplifying gives,

$$a(a-1)(a-2)\ldots(a-c+1) = b(b-1)(b-2)\ldots(b-c+1),$$

or,

$$\left(\frac{a}{b}\right)\left(\frac{a-1}{b-1}\right)\left(\frac{a-2}{b-2}\right)\cdots\left(\frac{a-c+1}{b-c+1}\right) = 1.$$

But $a < b$ and so each bracketed term in the product on the left hand side of the equation is less than 1. Hence, the entire product on the left hand side is less than 1 - giving a contradiction. Thus, $a = b$. ■

We use this lemma and Lemma 5.6 to obtain a necessary and sufficient condition for the existence of trivial $t-(v,k,\lambda;\{1,a\})$ point-weighted designs with exactly one point having weight a.

Theorem 5.8 *Let a be a positive integer greater than 1 and let $\mathcal{D}$ =($\mathbf{V}$,$\mathbf{B}$,$\mathbf{I}$,w) be a trivial point-weighted design with parameters v and k, in which exactly one point has weight a and all other points have weight 1. Let $u = |\mathbf{V}|$ and let t be an integer in the range,*

$$1 \leq t \leq k-a+1.$$

Then a necessary and sufficient condition for $\mathcal{D}$ to be a $t-(v,k,\lambda;\{1,a\})$ for some value of λ is that either,

(i) $v = k$ *and $\mathcal{D}$ is improper,*

or,

(ii) $u = 2k-(a+t-2)$*, in which case* $v = 2k-t+1$.

If $\mathcal{D}$ is a $t-(v,k,\lambda;\{1,a\})$ for some t then either $\lambda = 1$ if $\mathcal{D}$ is improper or,

$$\lambda = \binom{v-a-t+1}{k-a-t+1},$$

if $\mathcal{D}$ is proper.

Proof

Recall from Section 5.2.1 that if $v = k$ (i.e., if $\mathcal{D}$ is improper) then $\mathcal{D}$ is a $t-(v,v,1;\{1,a\})$ for every t in the range $1 \leq t \leq u$. Furthermore, if $v = k$ then we see that $u = k-a+1$, and so $\mathcal{D}$ is a $t-(v,v,1;\{1,a\})$ for every t in the range $1 \leq t \leq k-a+1$. So we now consider the case when $\mathcal{D}$ is proper.

Let $\lambda_{a,\mathbf{1}}$ and $\lambda_{1,\mathbf{1}}$ be defined as above, and let $u_{(1)}$ be the number of points of $\mathcal{D}$ of weight 1. $\mathcal{D}$ is proper and so we recall from Lemma 5.6 that $u_{(1)} \geq k$. Clearly, given t in the desired range, $\mathcal{D}$ will be a $t-(v,k,\lambda;\{1,a\})$ for some value of λ if and only if $\lambda_{a,\mathbf{1}} = \lambda_{1,\mathbf{1}}$. Equating the expressions for $\lambda_{a,\mathbf{1}}$ and $\lambda_{1,\mathbf{1}}$ in (5.3) and (5.4) respectively gives,

$$\binom{u_{(1)}-(t-1)}{k-(a+t-1)} = \binom{u_{(1)}-t}{k-t} + \binom{u_{(1)}-t}{k-(a+t)}.$$

Expressing each term as a quotient of factorials (with 0! defined to be 1), we get,

$$\begin{aligned}\frac{\left(u_{(1)}-(t-1)\right)!}{\left(u_{(1)}-k+a\right)!\,(k-(a+t-1))!} &= \frac{\left(u_{(1)}-t\right)!}{\left(u_{(1)}-k\right)!\,(k-t)!} \\ &\quad + \frac{\left(u_{(1)}-t\right)!}{\left(u_{(1)}-k+a\right)!\,(k-(a+t))!},\end{aligned}$$

and we look to solve this expression for $u_{(1)}$. Since $t < k$, we have $u_{(1)} > t$ and so we multiply through by $\frac{(u_{(1)}-k+a-1)!}{(u_{(1)}-t)!}$ to give,

$$\frac{\left(u_{(1)}-t+1\right)}{\left(u_{(1)}-k+a\right)(k-a-t+1)!} = \frac{\left(u_{(1)}-k+a-1\right)!}{\left(u_{(1)}-k\right)!\,(k-t)!} + \frac{\left(u_{(1)}-k+a-1\right)!}{\left(u_{(1)}-k+a\right)!\,(k-a-t)!}.$$

Subtracting $\frac{1}{\left(u_{(1)}-k+a\right)(k-a-t)!}$ from both sides gives,

$$\frac{\left(u_{(1)}-k+a\right)}{\left(u_{(1)}-k+a\right)(k-a-t+1)!} = \frac{\left(u_{(1)}-k+a-1\right)!}{\left(u_{(1)}-k\right)!\,(k-t)!}.$$

Multiplying through by $(k-t)!$ gives,

$$\frac{(k-t)!}{(k-t-a+1)!} = \frac{\left(u_{(1)}-k+a-1\right)!}{\left(u_{(1)}-k\right)!}.$$

Or, expressing as combinations,

$$(a-1)!\binom{k-t}{a-1} = (a-1)!\binom{u_{(1)}-k+a-1}{a-1} \tag{5.5}$$

Recall from (5.1) that $k-t \geq a-1$. Hence, by Lemma 5.7, (5.5) holds if and only if,

$$k-t = u_{(1)}-k+a-1,$$

i.e.,

$$u_{(1)} = 2k-(a+t-1).$$

Hence, $\mathcal{D}$ will be a $t-(v,k,\lambda;\{1,a\})$ for some value of λ if and only if $\mathcal{D}$ is improper or $u_{(1)} = 2k-(a+t-1)$. Then, $u = u_{(1)}+1$ and $v = u_{(1)}+a$, giving the values for u and v in the statement of the theorem.

If $\mathcal{D}$ is a proper $t-(v,k,\lambda;\{1,a\})$ for some t in the range $1 \leq t \leq k-a+1$, then the value of λ is given in (5.3) to be:

$$\lambda = \binom{u_{(1)}-(t-1)}{k-(a+t-1)}.$$

Observing that $u_{(1)} = v - a$ then gives the expression for λ in the statement of the theorem. ∎

Corollary 5.9 *For any $t > 0$ and $a > 1$ there does not exist a proper trivial $t - (v, k, \lambda; \{1, a\})$ with $v > 2k$ in which exactly one point has weight a.*

Proof

Let $\mathcal{D}$ be a proper trivial $t - (v, k, \lambda; \{1, a\})$ for some $t > 0$ and $a > 1$ in which exactly one point has weight a. Suppose $v > 2k$. Then by Theorem 5.8, t, v and k are related by $v = 2k - t + 1$. But $t > 0$ and so $v < 2k + 1$, giving a contradiction. Hence, it is not possible that $v > 2k$. ∎

Noting that the condition on v, k and t in Theorem 5.8 is independent of the value of a gives one further corollary,

Corollary 5.10 *Let t, v and k be positive integers satisfying $v > k > t$, such that $v = 2k - t + 1$. Then for any a in the range $1 < a \leq k - t + 1$ there exists a $t - (v, k, \lambda; \{1, a\})$ in which exactly one point has weight a.* ∎

Example 5.11 Suppose we wish to find a proper trivial $3 - (v, 5, \lambda; \{1, 2\})$ for some values of v and λ in which exactly one point has weight 2 and all other points have weight 1. Then, by Theorem 5.8, the only example of such a structure has six points of weight 1 and $v = 8$. The value of λ is given by either of the expressions (5.3) or (5.4) to be four. A weighted incidence matrix of this trivial $3 - (8, 5, 4; \{1, 2\})$ is given in Figure 5.3.

$$\begin{pmatrix}
2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0\\
1&1&1&1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&1&1&1&1&0&1\\
1&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&1&1&1&0&1&1&1&0&1&1\\
0&1&0&0&1&0&0&1&1&0&1&0&0&1&1&0&1&1&0&1&1&1&0&1&1&1\\
0&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&1&1&0&1&1&1&1\\
0&0&0&1&0&0&1&0&1&1&0&0&1&0&1&1&0&1&1&1&0&1&1&1&1&1
\end{pmatrix}$$

Figure 5.3: The trivial $3 - (8, 5, 4; \{1, 2\})$ with exactly one point of weight 2.

Example 5.12 Another small example of a trivial point-weighted design which in this case is the only proper trivial $2 - (v, 5, \lambda; \{1, 3\})$ with exactly one point of weight 3 is given in Figure 5.4.

5.3 Trivial Point-Weighted Designs and Point-Complementing

We now use the preceding results in this chapter and consider the class of proper $2-(v,k,\lambda;\{1,a\})$ trivial point-weighted designs with exactly one point of weight $a>1$ and all other points of weight 1. We demonstrate that point-complementing the underlying incidence structures of these trivial point-weighted designs at the point of weight a gives a correspondence between these structures and trivial block designs with certain parameters. For ease of expression, we shall use the title **type I trivial point-weighted design with parameters** k **and** a to refer to a proper $2-(v,k,\lambda;\{1,a\})$ trivial point-weighted design with exactly one point of weight $a>1$ and all other points of weight 1. Recall from Theorem 5.8 that for given k and a, there is exactly one value of v for which there exists a type I trivial point-weighted design with parameters k and a, namely $v=2k-1$. The corresponding value of λ is then given by $\binom{v-a-1}{k-a-1}$ (putting $t=2$ and $u_{(1)}=v-a$ in (5.3)). We first show that the point-complement of the underlying incidence structure of a type I trivial point-weighted design at the point of weight a is a block design.

Lemma 5.13 *Let $\mathcal{D}$ be a type I trivial point-weighted design with parameters k and a and underlying incidence structure $\mathcal{U}$. Let P be the point of weight a in $\mathcal{D}$ and denote by $\mathcal{U}_{(P)}$ the point-complement of $\mathcal{U}$ at P. Then $\mathcal{U}_{(P)}$ is a $2-(2k-a,k,\lambda)$, for some value of λ.*

Proof

Since $\mathcal{D}$ is a type I trivial point-weighted design with parameters k and a, its underlying incidence structure $\mathcal{U}$ is a $2-(u,K,\lambda^{'})$ for some values of u and $\lambda^{'}$ and set K. The value of u - the number of points in $\mathcal{D}$ (and therefore in $\mathcal{U}$) - is given in Theorem 5.8 to be $2k-a$. Hence, the number of points in $\mathcal{U}_{(P)}$, the point- complement of $\mathcal{U}$ at P, is also $2k-a$.

Let r_P denote the number of blocks of $\mathcal{U}$ containing the point P. Then, by Result 5.2, the point-complement of $\mathcal{U}$ at P is a $2-(u,K^{'},r_P-\lambda^{'})$. We need to show that every block of $\mathcal{U}_{(P)}$ contains exactly k points.

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

Figure 5.4: The trivial $2-(9,5,5;\{1,3\})$ with exactly one point of weight 3.

There are two 'classes' of blocks of $\mathcal{U}$. A block of the first class does not contain the point P and contains exactly k other points, all of which have weight 1 in $\mathcal{D}$. A block of the second class does contain the point P together with exactly $k-a$ other points, which have weight 1 in $\mathcal{D}$. The first class is 'un-changed' by point-complementing at P - i.e., the blocks of $\mathcal{U}_{(P)}$ which do not contain P are exactly the blocks of $\mathcal{U}$ which do not contain P. The remaining blocks of $\mathcal{U}_{(P)}$ correspond exactly with the second class of blocks of $\mathcal{U}$. For each block x of $\mathcal{U}$ containing P, there is a unique block y of $\mathcal{U}_{(P)}$ containing exactly the points not contained in x plus the point P. Conversely, for each block y of $\mathcal{U}_{(P)}$ containing P, there is a unique block x of $\mathcal{U}$ containing exactly the points not contained in y, plus the point P. Thus, the blocks of $\mathcal{U}_{(P)}$ containing P each contain a total of $(2k-a)-(k-a)=k$ points.

So, we have shown that every block of $\mathcal{U}_{(P)}$ which does not contain P contains exactly k points and that every block of $\mathcal{U}_{(P)}$ which does contain P contains exactly k points. Hence, every block of $\mathcal{U}_{(P)}$ contains exactly k points, and so $\mathcal{U}_{(P)}$ is a $2-(2k-a,k,r_P-\lambda^{'})$. Setting $\lambda = r_P - \lambda^{'}$ gives the statement of the lemma. ∎

We now show that the point-complement of the underlying incidence structure of a type I trivial point-weighted design with parameters k and a at the point of weight a is in fact a trivial block design.

Lemma 5.14 *Let $\mathcal{D}$ be a type I trivial point-weighted design with parameters k and a and underlying incidence structure $\mathcal{U}$. Let P be the point of weight a in $\mathcal{D}$ and denote by $\mathcal{U}_{(P)}$ the point-complement of $\mathcal{U}$ at P. Then $\mathcal{U}_{(P)}$ is the trivial block design with $2k-a$ points and block-size k.*

Proof

Recall from Lemma 5.13 that $\mathcal{U}_{(P)}$ is a block design with block-size k and $2k-a$ points. Hence we need to show that the block-set of $\mathcal{U}_{(P)}$ consists of every possible set of k points chosen from the $2k-a$ points in total. We note that this will be the case if and only if two conditions are satisfied. Firstly, the set of blocks of $\mathcal{U}_{(P)}$ which do not contain P must be the set of all possible sets of size k of points chosen from the $2k-a-1$ points of $\mathcal{U}_{(P)}$ which have weight 1 in $\mathcal{D}$. Secondly, the derived structure of $\mathcal{U}_{(P)}$ at P must be the trivial block design with $2k-a-1$ points and block-size $k-1$.

The first of the above two conditions is clearly satisfied since $\mathcal{D}$ is a type I trivial point-weighted design. The blocks of $\mathcal{D}$ (and hence blocks of $\mathcal{U}$) which do not contain P form the block-set of the trivial block design with $2k-a-1$ points and block-size k. These blocks are also precisely the blocks of $\mathcal{U}_{(P)}$ which do not contain P.

For the second condition, we consider the derived structure, $\mathcal{R}$ say, of $\mathcal{U}$ at P. Because $\mathcal{D}$ is a type I trivial point-weighted design, $\mathcal{R}$ is just the trivial block design with $2k-a-1$ points and block-size $k-a$. Then, the complement of $\mathcal{R}$ is the trivial block design with $2k-a-1$ points and block-size,

$$(2k-a-1)-(k-a)=k-1.$$

But, by definition, the complement of $\mathcal{R}$ is the derived structure of $\mathcal{U}_{(P)}$ at P. Hence, the second condition is also satisfied and $\mathcal{U}_{(P)}$ is the trivial block design with block size k and $2k-a$ points. ∎

As we have already noted, $\left(\mathcal{U}_{(P)}\right)_{(P)} = \mathcal{U}$, and so the point-complement of the above trivial block design, $\mathcal{U}_{(P)}$, at P is the underlying incidence structure of the type I trivial point-weighted design, $\mathcal{D}$ with parameters k and a. But, since $\mathcal{U}_{(P)}$ is a trivial block design, the point-complement of $\mathcal{U}_{(P)}$ at any other point, Q say, will be isomorphic to the point-complement of $\mathcal{U}_{(P)}$ at P. Hence we have shown:

Lemma 5.15 *Let $\mathcal{T}$ be a trivial block design with block-size $k > 2$ and $u = 2k-a$ points, for some $1 < a < k$. Then the point-complement of $\mathcal{T}$ at any point is the underlying incidence structure of a type I trivial point-weighted design with parameters k and a.* ∎

Combining Lemmas 5.14 and 5.15 gives the following theorem:

Theorem 5.16 *Let $\mathcal{T}$ be a trivial block design with block-size $k > 2$ and $u = 2k-a$ points, for some $1 < a < k$. Then the point-complement of $\mathcal{T}$ at any point is the underlying incidence structure of a type I trivial point-weighted design with parameters k and a. Conversely, let $\mathcal{U}$ be the underlying incidence structure of a type I trivial point-weighted design with parameters k and a, $\mathcal{D}$ say. Then the point-complement of $\mathcal{U}$ at the point of weight a in $\mathcal{D}$ is the trivial block design with block-size k and $2k-a$ points.* ∎

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