AUSLANDER-REITEN COMPONENTS IN THE BOUNDED DERIVED CATEGORY

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Abstract. Derived Categories of finite dimensional algebras whose Auslander-Reiten quiver has a finite component, a component with Dynkin tree class or a bounded component are classified. Their Auslander-Reiten quiver is determined. We use these results to show that certain algebras are piecewise hereditary. A necessary condition for components of Euclidean tree class is deduced and components that contain shift periodic complexes are determined.

1. introduction

In this paper we analyze the Auslander-Reiten triangles in bounded derived categories of a finite-dimensional algebra A, denoted by $D^b(A)$. The bounded derived category of a finite-dimensional algebra is a triangulated category and Auslander-Reiten triangles are triangles with analogous properties then Auslander-Reiten sequences of finite-dimensional algebras. The conditions of existence of those triangles in $D^b(A)$ have been determined by Happel in [H].

Analogously to the classical Auslander-Reiten theory, which applies to Artin algebras, we can define Auslander-Reiten components in the bounded derived category. Those are locally finite graph, where the vertices correspond to indecomposable complexes in $D^b(A)$. We want to know how and if certain results on the Auslander-Reiten quivers of finite-dimensional algebras extend to the bounded derived category. We call a component stable if the so called translation map is an automorphism when restricted to the elements of the component. Stable components are isomorphic to $\mathbb{Z}[T]/\Gamma$ where T is a tree, and Γ is an admissible group of automorphisms.

The Auslander-Reiten components of $D^b(A)$ for A self-injective have been determined by W. Wheeler in [W]. But very little is known about the components of the bounded derived category of non self-injective algebras.

Date: July 3, 2009.

In the first section we introduce Auslander-Reiten triangles as defined by Happel in [H1] and deduce some properties of Auslander-Reiten triangles that will be used in the other sections.

In the classical Auslander-Reiten theory finite components occur if and only if the algebra is representation type finite. We show in the second section that finite stable Auslander-Reiten components occur for $D^b(A)$ if and only if A is semi-simple. In this case the components of the Auslander-Reiten quiver are all isomorphic to A_1.

We also show that finite components can occur in the bounded derived category of non semi-simple algebras if the Auslander-Reiten component is not stable by constructing an example.

In this section, we introduce bounded Auslander-Reiten components and show that the following are equivalent:

1) $D^b(A)$ has finite representation type;

2) There is a stable component with Dynkin tree class;

3) There is a bounded stable component.

The Auslander-Reiten quiver consists in this case of only one component $\mathbb{Z}[T]$ with $T \neq A_1$ a finite Dynkin diagram or of infinitely many components A_1. In the first case A is derived equivalent to kT, which is a hereditary algebra of finite representation type. In the second case A is semi-simple.

Finally possible tree classes for derived categories with shift periodic modules are deduced and a sufficient condition for components of tree class A_∞.

In the third section irreducible maps in $Comp^{-,b}(A)$ that end in an indecomposable contractible complex are analyzed. Those irreducible maps start in a complex that is isomorphic to a simple module of A embedded into $D^b(A)$. A corollary of this result is that stable Euclidean Auslander-Reiten components all contain a simple A-module.

Finally we analyze the Auslander-Reiten quiver of Nakayama algebras given as a path algebra kA_n/I for the ideal $I \leq kA_n$ gerenated by a path of length $n-1$ in the first case and paths of length 2 in the second case. The Auslander-Reiten quiver is given by one component $\mathbb{Z}[D_n]$ in the first case and $\mathbb{Z}[A_n]$ in the second case. This result provides an alternative proof to [HS] that A is derived equivalent to kA_n and kD_n respectively.

2. Introduction to Auslander-Reiten triangles

In this section we introduce Auslander-Reiten Theory for triangulated categories. We state the existence conditions for Auslander-Reiten triangles in the bounded derived category of a finite-dimensional algebra and prove some properties that will be needed in the other sections.

Let $\mathcal{T}$ be a triangulated category with translation functor T.

Definition 2.1. [H1, 4.1] *[Auslander-Reiten triangles] A distinguished triangle $X \to_u Y \to_v Z \to_w TX$ is called an Auslander-Reiten triangle if the following conditions are satisfied:*

(1) X, Z are indecomposable

(2) $w \neq 0$

(3) If $f : W \to Z$ is not a retraction, then there exists $f' : W \to Y$ such that $v \circ f' = f$.

By [H1, 4.2] we have the following equivalences: the condition (3) is equivalent to

(3') If $f : X \to W$ is not a section, then there exists $f' : Y \to W$ such that $f' \circ u = f$.

The condition (2) is equivalent to

(2') u is not a section.

(2") v is not a retraction.

We say that the Auslander-Reiten triangle $X \to Y \to Z \to TX$ starts in X, has middle term Y and ends in Z. Note also that an Auslander-Reiten triangle is uniquely determined up to isomorphisms of triangles by the isomorphism class of the element it ends or starts with. Analogously to the classical Auslander-Reiten theory we can define irreducible maps, minimal maps, left almost split maps and right almost split maps. We have the same statement than in the case of Artin algebras for irreducible maps:

Lemma 2.2. *Let $f : N \to M$ be an irreducible map.*

(1) Let $N \to_g Q \to E \to TN$ be the Auslander-Reiten triangle, then there is a section $s : Q \to M$ such that $f = s \circ g$.

(2) Let $L \to B \to_h M \to TL$ be an Auslander-Reiten triangle, then there is a retraction $r : N \to B$ such that $f = h \circ r$.

In order to define Auslander-Reiten components we need the following property:

Definition 2.3. [Ben, 1.4.2] *An object M of an additive category $\mathcal{C}$ has the unique decomposition property if*

(1)M has a finite decomposition as direct sum of indecomposable object.

(2)The decomposition is unique up to isomorphism and reordering. So if $M = \oplus_{i=1}^{n} M_i \cong \oplus_{i=1}^{m} N_i$ *for indecomposable objects* M_i *and* N_i*, then* $n = m$ *and there is a permutation* π *of* n *such that* $M_{\pi(i)} \cong N_i$ *for all* $i = 1, \cdots, n$*.*

We call an additive category $\mathcal{C}$ *a Krull-Schmidt category if every object in* $\mathcal{C}$ *has a unique decomposition property.*

If $\mathcal{T}$ is a Krull-Schmidt category we define the Auslander-Reiten quiver to be a labelled graph Λ with vertices the isomorphism classes of indecomposable objects. We denote the set of vertices by Λ_0. The label of an arrow $X \stackrel{(d_{XY}, d'_{XY})}{\to} Y$ is defined as follows: if there is an Auslander-Reiten triangle t starting in X, then d_{XY} is the multiplicity of Y as a direct summand of the middle term of t. Analogously if there is an Auslander-Reiten triangle s ending in Y, then d'_{XY} is the multiplicity of X as a direct summand of the middle term of s.

Let Λ'_0 denote the subset of vertices of Λ such that there is an Auslander-Reiten triangle that ends in this vertex. Let τ be the morphism $\tau : \Lambda'_0 \to \Lambda_0$ such that for all arrows $y \to x$ and $x \in \Lambda'_0$ we have an arrow $\tau(x) \to y$. Then an Auslander-Reiten quiver is a translation quiver with translation τ.

From now on let A be a finite-dimensional algebra and $\mathcal{T} = D^b(A)$. By [K, 2.6] we have that $D^b(A)$ is a Krull-Schmidt category and that the endomorphism ring of indecomposable objects is local. Therefore Auslander-Reiten components of $D^b(A)$ are well-defined.

As the endomorphism spaces of indecomposable elements are local, the following lemma holds:

Lemma 2.4. [H1, 4.3, 4.5] *Let* $X \to_u M \to_v Z \to TX$ *be an Auslander-Reiten triangle and* $M \cong M_1 \oplus M_2$*, where* M_1 *is indecomposable. Let* $i : M_1 \to M$ *be the inclusion and* $p : M \to M_1$ *the projection. Then* $v \circ i : M_1 \to Z$ *and* $p \circ u : X \to M$ *are irreducible maps.*

Let ν_A denote the Nakayama functor of A. We denote by ν the functor that maps a complex $D \in Comp(\mathcal{P})$ to $\nu(D) \in Comp(\mathcal{I})$, where $\nu(D)^i := \nu_A(D^i)$ and $d^i_{\nu(D)} := \nu_A(d^i_D)$ for all $i \in \mathbb{Z}$. As ν maps contractible complexes to contractible complexes, ν is a well-defined functor on $K(\mathcal{P})$ and therefore also on $K^{-,b}(\mathcal{P}) \cong D^b(A)$. Note that ν is the left derived functor of ν_A on $D^b(A)$ and ν^{-1} the right derived funtor of ν_A^{-1}.

The conditions for the existence of Auslander-Reiten triangle in a triangulated category have been determined in [BR, I.2.4]. It is shown that a triangulated category admits Auslander-Reiten triangles, that is for every indecomposable element X there is an Auslander-Reiten triangle that ends in X and one that starts in X, if and only if the category has a Serre functor.

A specialization of this result is given in the next Theorem for the case of $D^b(A)$.

Theorem 2.5. [H, 1.4] *(1) Let $Z \in K^{-,b}(\mathcal{P})$ be indecomposable. Then there exists an Auslander-Reiten triangle ending in Z if and only if $Z \in K^b(\mathcal{P})$. The triangle is then given by $\nu(Z)[-1] \to Y \to Z \to \nu(Z)$ for some $Y \in K^{-,b}(\mathcal{P})$.*

(2) Let $X \in K^{+,b}(\mathcal{I})$ be indecomposable, then there exists an Auslander-Reiten triangle starting in X if and only if $X \in K^b(I)$. The triangle is then given by $X \to Y \to \nu^{-1}(X)[1] \to X[1]$ for some $Y \in K^{-,b}(\mathcal{P})$.

We deduce the following from this result:

(1) The translation τ is given by $\nu[-1]$ and τ is a bijective map from $K^b(\mathcal{P})$ to $K^b(\mathcal{I})$.

(2) Let $N,\ M \in D^b(A)$ be two indecomposable elements and let $f : N \to M$ be an irreducible map. Then there is an arrow from N to M in the Auslander-Reiten quiver representing f if and only if $N \in K^b(\mathcal{I})$ or $M \in K^b(\mathcal{P})$.

(3) $D^b(A)$ admits Auslander-Reiten triangles if and only if A has finite global dimension.

(4) Every Auslander-Reiten triangle is isomorphic to

$$\nu(X)[-1] \to \mathrm{cone}(w)[-1] \to X \to_w \nu(X)$$

for $X \in K^b(\mathcal{P})$ and some map $w : X \to \nu(X)$.

The following lemma determines the relation between irreducible maps, retractions and sections in $K(\mathcal{P})$ and $Comp(\mathcal{P})$. Note that by duality the same is true if we replace $\mathcal{P}$ by $\mathcal{I}$.

Lemma 2.6. *Let $B, C \in Comp(\mathcal{P})$ be indecomposable complexes that are not contractible. Let $f : B \to C$ be a map of complexes.*

(1) f is irreducible in $Comp(\mathcal{P})$ if and only if f is irreducible in $K(\mathcal{P})$.

(2) f is a retraction in $Comp(\mathcal{P})$ if and only if f is a retraction in $K(\mathcal{P})$.

(3) f is a section in $Comp(\mathcal{P})$ if and only if f is a section in $K(\mathcal{P})$.

Proof. We first give a proof of (2). Let $f : B \to C$ be a retraction in $Comp(\mathcal{P})$. Then f is clearly a retraction in $K(\mathcal{P})$. Let f be a retraction in $K(\mathcal{P})$, then there is a map $g : C \to B$ such that $f \circ g$ is homotopic to

id_C. Therefore $f \circ g - \mathrm{id}_C$ factors through a contractible complex P via $s : C \to P$ and $t : P \to C$. Then $\binom{t}{h} \circ (-s, g) = \mathrm{id}_C$. As C does not have a contractible summand, we have that $f \circ g$ is an isomorphism. The proof of (3) is analogous.

We prove (1). Let $f : B \to C$ be an irreducible map in $K(\mathcal{P})$, then by (2) f is also an irreducible map in $Comp(\mathcal{P})$. Suppose now that f is irreducible in $Comp(\mathcal{P})$ and let $g \circ h$ be homotopic to f for some $g : D \to C$ and $h : B \to D$. Then $g \circ h - f$ factors through a contractible complex P via $s : B \to P$ and $t : P \to C$. So $f = \binom{g}{-t} \circ (h, s)$ factors through $D \oplus P$ in $Comp(\mathcal{P})$. Therefore (h, s) is a section in $Comp^b(\mathcal{P})$ or $\binom{g}{-t}$ is a retraction in $Comp(\mathcal{P})$. This means that h is a section in $K(\mathcal{P})$ or g is a retraction in $K(\mathcal{P})$. Therefore f is irreducible in $K(\mathcal{P})$. □

The next theorem determines the relation between Auslander-Reiten triangles in $D^b(A)$ and Auslander-Reiten sequences in $Comp^{-,b}(\mathcal{P})$. The analogous statement for self-injective algebras was given by [W, 2.3, 2.2].

Theorem 2.7. *Let $P \in Comp^b(\mathcal{P})$ be an indecomposable complex that is not contractible. Let $w : P \to \nu(P)$ be a map of complexes. Then*

$$0 \to \nu P[-1] \to \mathrm{cone}(w)[-1] \to P \to 0$$

is an Auslander-Reiten sequence in $Comp^{-,b}(\mathcal{P})$ if and only if w induces an Auslander-Reiten triangle in $D^b(A)$.

Proof. Let $w : P \to \nu(P)$ induce an Auslander-Reiten triangle in $D^b(A)$. The sequence

$$0 \to \nu P[-1] \to \mathrm{cone}(w)[-1] \to_\sigma P \to 0$$

is exact. Furthermore $\nu P[-1]$ and P are indecomposable. Let P_1 be an indecomposable complex in $Comp^{-,b}(\mathcal{P})$ and $f : P_1 \to P$ be a non-split map in $Comp^{-,b}(\mathcal{P})$. Then by 2.6, f is not a retraction in $K^{-,b}(\mathcal{P})$. Therefore there is a map $f_1 : P_1 \to \mathrm{cone}(w)[-1]$ such that $\sigma f_1 = f$ in $K^{-,b}(\mathcal{P})$. Therefore $f - \sigma f_1$ factors through a contractible complex P_2. Let $f - \sigma f_1 = g \circ h$ where $h : P_1 \to P_2$ and $g : P_2 \to P$. As P_2 is projective in $Comp^{-,b}(\mathcal{P})$ there is a map $s : P_2 \to Q$ such that $g = \sigma \circ s$. We set $f' = f_1 + sh$, then $\sigma f' = \sigma f_1 + \sigma sh = f$. The converse follows immediately by the definitions. □

All indecomposable contractible complexes in $Comp(\mathcal{P})$ have the form

$$\cdots \to 0 \to P \to_{\mathrm{id}} P \to 0 \to \cdots$$

for an indecomposable projective module P of A. We denote such a complex where P occurs in degree 0 and -1 by $\bar{P}$. Note that if $\bar{P} \in Comp^b(\mathcal{P})$ is contractible, then there is no Auslander-Reiten sequence in $Comp^{-,b}(\mathcal{P})$ that ends in $\bar{P}$, as contractible complexes are projective object in $Comp^{-,b}(\mathcal{P})$ by [W, 1].

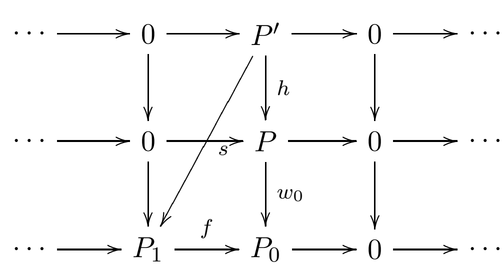
Let N be a left A-module and $N \to I_0 \to_{d_I^0} I_1 \to_{d_I^1} \cdots$ its minimal injective resolution and by $\cdots \to_{d_P^2} P_1 \to_{d_P^1} P_0 \to N$ its minimal projective resolution. We define pN to be the complex with $(pN)^i = P_{-i}$ and $d^i := d_P^{-i}$ for $i \leq 0$ and $(pN)^i = 0$ for $i > 0$. Similarly we define iN to be the complex with $(iN)^n = I_n$ and $d^n := d_I^n$ for $n \geq 0$ and $(iN)^n = 0$ for $n < 0$.

Finally we define for a complex X, the complex $\sigma^{\leq n}(X)$ to be the complex with $\sigma^{\leq n}(X)^i = X^i$ for $i \leq n$ and $d^i_{\sigma^{\leq n}(X)} = d^i_X$ for $i < n$ and $\sigma^{\leq n}(X)^i = 0$ for $i > n$. We define $\sigma^{\geq n}(X)$ analogously.

We can determine the homology of the middle term of an Auslander-Reiten triangle ending in a projective indecomposable module.

Lemma 2.8. *Let P be a projective indecomposable module and let M be the middle term of the Auslander-Reiten triangle ending in P. Then $H^1(M) = I/\operatorname{soc} I$ where $I := \nu(P)$, $H^0(M) = \operatorname{rad} P$ and $H^i(M) = 0$ for all $i \neq 0, 1$.*

Proof. By 2.5 the Auslander-Reiten triangle can be written as $I[-1] \to \operatorname{cone}(w)[-1] \to P \to_w I$, where $I := \nu(P)$. Then $M = \operatorname{cone}(w)[-1]$. Let $\cdots \to P_2 \to_g P_1 \to_f P_0$ be a minimal projective resolution of I. Then I is isomorphic to pI in $D^b(A)$. Using this isomorphism, we view w as a map of complexes in $K^{-,b}(\mathcal{P})$. Then M is given by $\cdots \to P_2 \to_g P \oplus P_1 \to_{(f,w_0)} P_0 \to 0 \to \cdots$. Let $h : P' \to P$ be a projective cover of $\operatorname{rad} P$. We identify h with the corresponding map of complexes $P' \to P$. As h is not a retraction we have $w \circ h = 0$ in $K^{-,b}(\mathcal{P})$ and $w \circ h$ is therefore homotopic to zero. Then there is a map $s : P' \to P_1$ such that $w_0 \circ h = f \circ s$. We visualize this in the following diagram:



Therefore $w_0(\operatorname{rad} P) \subset \operatorname{Im} f$ and $w_0(P) \not\subset \operatorname{Im} f$. As $P/\operatorname{rad} P$ is simple, we have $H^1(M) = I/\operatorname{soc} I$. Furthermore $\ker((f, w_0)) = \operatorname{rad} P \oplus \operatorname{Im}(g)$. Therefore $H^0(M) = \operatorname{rad}(P)$. Clearly $H^i(M) = 0$ for all $i \neq 0, 1$. □

We call an Auslander-Reiten component Λ stable, if τ is an automorphism on Λ. By 2.5 this is equivalent to the fact that all vertices in Λ are in $K^b(\mathcal{I})$ and $K^b(\mathcal{P})$. By [XZ, 2.2.1], the Auslander-Reiten quiver of $D^b(A)$ does not contain loops. Therefore we can apply Riedtmanns structure theorem [Ben, 4.15.6].

Corollary 2.9. *Let Γ be a stable Auslander-Reiten component of $D^b(A)$. Then $\Gamma \cong \mathbb{Z}[T]/I$, where $\bar{T}$ is a tree and I is an admissible ideal of aut$(\mathbb{Z}[T])$.*

We can determine when all components are stable.

Lemma 2.10. *The Auslander-Reiten components of $D^b(A)$ containing at least one elements of $K^b(\mathcal{P})$ or $K^b(\mathcal{I})$ are all stable if and only if A has finite injective dimension.*

Proof. A has finite injective dimension if and only if A^* has finite projective dimension. This is equivalent to the fact that $K^b(\mathcal{P}) = K^b(\mathcal{I})$ as subcategories of $D^b(A)$. By 2.5 this is equivalent to the fact that all Auslander-Reiten components are stable. □

The next result is then immediate.

Corollary 2.11. *Let A be a self-injective algebra or an algebra of finite global dimension. Then the Auslander-Reiten quiver is a stable translation quiver.*

Lemma 2.12 (Irreducible maps that do not appear in Auslander-Reiten triangles)**.** *Let $f : B \to C$ be an irreducible map in $D^b(A)$ that does not appear in an Auslander-Reiten triangle. Then $B, C \notin K^b(\mathcal{P})$ and $B, C \notin K^b(\mathcal{I})$.*

Proof. By 2.5 it is clear that $B \notin K^b(\mathcal{I})$ and $C \notin K^b(\mathcal{P})$. Let us assume that $B \in K^b(\mathcal{P})$ and let $n \in \mathbb{N}$ be minimal such that $B^n \neq 0$. Then f factorizes through $\sigma^{\geq n-1}(C)$, where C is represented as an element of $K^{-,b}(\mathcal{P})$. Let $f = h \circ g$ be this factorization, then g is not a section, as f is not a section and h is not a retraction as $\sigma^{\geq n-1}(C) \ncong C$. This is a contradiction to the fact that f is irreducible. Therefore $B \notin K^b(\mathcal{P})$. Analogously, we can show $C \notin K^b(\mathcal{I})$. □

3. Finite and bounded Auslander-Reiten Components

In this section, we determine the tree class of bounded and finite stable Auslander-Reiten components. We show that finite stable components can only appear if A is semi-simple. Bounded stable components appear if and only the representation type of $D^b(A)$ is finite. This is also equivalent to the fact that the Auslander-Reiten quiver has a component with tree class finite Dynkin. We describe the Aulander-Reiten quiver concretely in these cases.

Throughout the rest of this chapter, let A be an indecomposable finite-dimensional k-algebra. We start with the following easy Lemma

Lemma 3.1. *The following are equivalent:*

(1) There is an Auslander-Reiten component of $D^b(A)$ isomorphic to A_1;

(2) $D^b(A)$ has an Auslander-Reiten triangle with middle term zero;

(3) A is semi-simple;

(4) The Auslander-Reiten quiver of $D^b(A)$ is the union of infinitely many components A_1.

Proof. Clearly from (1) follows (2). If $\nu(x)[-1] \to 0 \to x \to_w \nu(x)$ is an Auslander-Reiten triangle, then w is an isomorphism. Also for all indecomposable $m \in D^b(A)$ with $m \neq x$ we have $\mathrm{Hom}_{D^b(A)}(m,x) = 0$ and $\mathrm{rad}\,\mathrm{End}(x) = 0$ by the third Auslander-Reiten triangle axiom. Therefore there is a simple projective module $S \in \mathrm{mod}\, A$ such that $x = S$. Furthermore $\nu(S) = S$ as w is an isomorphism. Therefore S is injective. By [Ben, 1.8.5] A is semi-simple and S is the only simple module in A up to isomorphism. Therefore (3) follows from (2). Clearly (3) implies (4) and (4) implies (1). □

In $Comp^{-,b}(\mathcal{P})$ we have

Lemma 3.2. *The following are equivalent:*

(1) There is an Auslander-Reiten component of $Comp^{-,b}(\mathcal{P})$ isomorphic to A_∞^∞;

(2) $Comp^{-,b}(\mathcal{P})$ has an Auslander-Reiten triangle with contractible middle term;

(3) A is semi-simple;

(4) The Auslander-Reiten quiver of $Comp^{-,b}(\mathcal{P})$ is isomorphic to A_∞^∞.

Proof. Use 3.1 and 2.7. □

Let S be the simple A-module and let $\bar{S}$ be a contractible complex. Then the Auslander-Reiten quiver of $Comp^{-,b}(\mathcal{P})$ is given by $\cdots \to S[-1] \to \bar{S}[-1] \to S \to \bar{S} \to \cdots$

Lemma 3.3. *Let $\tau(C) \to_f B \to_g C \to_w \tau(C)[1]$ be an Auslander-Reiten triangle in $D^b(A)$ and M an indecomposable element in $D^b(A)$. Then*

$$\mathrm{Hom}(M, \tau(C)) \to_{f^*} \mathrm{Hom}(M, B) \to_{g^*} \mathrm{Hom}(M, C)$$

and

$$\mathrm{Hom}(C, M) \to_{\bar{g}} \mathrm{Hom}(B, M) \to_{\bar{f}} \mathrm{Hom}(\tau(C), M)$$

are exact. Furthermore

(1) f^* is injective if and only if $M[1] \ncong C$;
(2) $\bar{g}$ is injective if and only if $M[-1] \ncong \tau(C)$;
(3) g^* is surjective if and only if $M \ncong C$;
(4) $\bar{f}$ is surjective if and only if $M \ncong \tau(C)$.

Proof. The sequence $\mathrm{Hom}(M, \tau(C)) \to_{f^*} \mathrm{Hom}(M, B) \to_{g^*} \mathrm{Hom}(M, C)$ is exact as by T5 and T4 there exists for any map $s : M \to B$ with $g \circ s = 0$ a map t such that the following diagram of distinguished triangles commutes:

$$\begin{array}{ccccccc} M & \xrightarrow{\mathrm{id}} & M & \longrightarrow & 0 & \longrightarrow & M[1] \\ \downarrow t & & \downarrow s & & \downarrow & & \downarrow t[1] \\ \tau(C) & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{w} & \tau(C)[1]. \end{array}$$

Suppose there is an $h : M \to \tau(C)$ such that $f \circ h = 0$. Then by T5 there is an $j : M[1] \to C$ such that the following diagram of distinguished triangles commutes:

$$\begin{array}{ccccccc} M & \longrightarrow & 0 & \longrightarrow & M[1] & \xrightarrow{\mathrm{id}} & M[1] \\ \downarrow h & & \downarrow & & \downarrow j & & \downarrow h[1] \\ \tau(C) & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{w} & \tau(C)[1]. \end{array}$$

If $M[1] \ncong C$ then $w \circ j = 0$ by Auslander-Reiten axioms. This forces $h[1] = 0$ and therefore $h = 0$. So in this case f^* is injective. If $M[1] \cong C$, then f^* is not injective as $w[-1] : M \to \tau(C)$ is mapped to zero. By Auslander-Reiten axioms it is clear that g^* is surjective if and only if $M \ncong C$. This proves (1) and (4).

The remaining cases are proven analogously. □

The following version of Lemma [Ben, 4.13.4] holds for our setup.

Lemma 3.4. *(1) Suppose that C and B are indecomposable complexes in $K^b(\mathcal{P})$ that are not contractible. Let $t : C \to B$ be a map that is not an isomorphism. Suppose there is no chain of irreducible maps in $D^{-,b}(A)$ from C to B of length less than n.*

Assume B lies in a component Λ of the Auslander-Reiten quiver of $D^b(A)$ such that all elements of Λ are in $K^b(\mathcal{P})$. Then there exists a chain of irreducible maps in Λ

$$P^0 \to_{g^1} P^1 \to_{g^2} \cdots \to P^{n-1} \to_{g^n} B$$

and a map $h : C \to P^0$ with $g^n \cdots g^1 h \neq 0$.

(2) Suppose that C and B are indecomposable complexes in $K^b(\mathcal{I})$. Let $t : C \to B$ be a map that is not an isomorphism. Suppose there is no chain of irreducible maps in $D^b(A)$ from C to B of length less then n.

Assume C lies in a component Λ of the Auslander-Reiten quiver of $D^b(A)$ such that all elements of Λ are in $K^b(\mathcal{I})$. Then there exists a chain of irreducible maps in Λ

$$C \to_{g^1} P^1 \to_{g^2} \cdots \to P^{n-1} \to_{g^n} P^n$$

and a map $h : P^n \to B$ with $hg^n \cdots g^1 \neq 0$.

Proof. We give a proof of (1) as the proof of (2) is analogous. The proof follows by induction on n. Assume first $n = 1$. Then there is no irreducible map from C to B. Let $v(B)[-1] \to_f E \to_l B \to \nu(B)$ be the Auslander-Reiten triangle ending in B. As t is not an isomorphism, there is some $\sigma : C \to E$ such that $t = l \circ \sigma$. Let $E = \oplus_{i=1}^m E_i$. Then $t = \sum_{i=1}^m l_i \circ \sigma_i$ for some maps $l_i : E_i \to B$ and $\sigma_i : C \to E_i$. Clearly there exists an $1 \leq s \leq m$ such that $l_s \circ \sigma_s \neq 0$ in $K^b(\mathcal{P})$ as $t \neq 0$ in $K^b(\mathcal{P})$. By 2.2 $g_1 := l_s$ is irreducible and E_s lies in Λ. Furthermore by the induction assumption σ_s is not an isomorphism as there is no irreducible map from C to B. We can therefore use the same argument in the induction step on the map σ_s. □

We denote by l_c the function that maps an element $B \in Comp^b(A)$ to the length of composition series of $\oplus_{i \in \mathbb{Z}} B^i$. For $X \in K^b(\mathcal{P})$ we denote by $l_p(X)$:=min$\{l(Y) | Y \in Comp^b(\mathcal{P})$ and $Y \cong= X$ in $K^b(\mathcal{P})\}$. We define analogously l_i for elements of $K^b(\mathcal{I})$.

With exactly the same proof as in [Ben, 4.14.1] we have

Lemma 3.5. *Let $P_0, \ldots, P_{2^n-1} \in Comp^b(A)$ be indecomposable. If $l_c(P_i) \leq n$ for all i and $f_i : P_{i-1} \to P_i$ is not an isomorphism for $1 \leq i \leq 2^n - 1$, then $f_{2^n-1} \cdots f_2 f_1 = 0$.*

By 2.6 every irreducible map $f : C \to B$ in $K^b(\mathcal{P})$ where C and B are indecomposable complexes in $Comp^b(\mathcal{P})$, is an irreducible map in $Comp(\mathcal{P})$. Therefore f is a non-isomorphism, when seen as a map in $Comp^b(A)$. Also C and B are indecomposable complexes in $Comp^b(A)$.

We therefore have the following result.

Corollary 3.6. *Let $P_0, \ldots, P_{2^n-1} \in K^b(\mathcal{P})$ be indecomposable. If $l_c(P_i) \leq n$ for all i and $f_i : P_{i-1} \to P_i$ irreducible maps for $1 \leq i \leq 2^n - 1$, then $f_{2^n-1} \cdots f_2 f_1 = 0$.*

We call an Auslander-Reiten component Λ bounded if l_p and l_i take bounded values on Λ. We can now determine some properties of bounded components.

Theorem 3.7 (bounded components)**.** *Let Λ be a stable bounded Auslander-Reiten component of $D^b(A)$. We assume that A is not semi-simple. Then Λ is the only component of the Auslander-Reiten quiver. Furthermore A has finite global dimension and is representation type finite.*

Proof. There is an $n \in \mathbb{N}$ such that $l_p(M) \leq n$ and $l_i(M) \leq n$ for all complexes $M \in \Lambda$. Let $R \in K^b(\mathcal{P})$ be an indecomposable non-contractible complex such that there is a map $g : R \to N$ that is not an isomorphism and not homotopic to zero for some $N \in \Lambda$. Let $u := \max(l_p(R), n) * \dim A$ then by 3.4 part (1) and 3.6 there exists a chain of irreducible maps of length at most 2^u from R to N. Therefore $R \in \Lambda$. As A is not semi-simple, there is a non-isomorphism from M to $\tau(M)[1]$ that is not homotopic to zero for any complex $M \in \Lambda$ by 3.1. Therefore we have that $\tau(M), \tau(M)[1] \in \Lambda$. Thus the [1] shift acts on the component.

Let $A = \bigoplus_{i=1}^n P_i$ be a decomposition of A into indecomposable projective summands P_i. Let C be an element of Λ. Then there exists a map f from P_i to C for some $1 \leq i \leq n$ that is not homotopic to zero and not an isomorphism. Therefore $P_i \in \Lambda$ and as A is indecomposable we have $P_j \in \Lambda$ for all $1 \leq j \leq n$. For all indecomposable elements X in $K^b(\mathcal{P})$ there is an $s_x \in \mathbb{Z}$ such that there is a non-zero map $P_i \to X[s_x]$. Therefore $X[s_x] \in \Lambda$ using the first part of the proof. The proof for $X \in K^b(\mathcal{I})$ is analogous. As the [1] shift acts on the component, every indecomposable complex in $K^b(\mathcal{I})$ and $K^b(\mathcal{P})$ is in Λ. Therefore Λ is the Auslander-Reiten quiver. Furthermore A has finite global dimension as Λ is bounded. As the dimension of the indecomposable A-modules are bounded, we know by [**?**, 1.5] that A has finite representation type. □

We can now determine finite components.

Theorem 3.8 (finite components)**.** *Let Λ be a finite Auslander-Reiten component of $D^b(A)$ such that all elements in Λ belong to $K^b(\mathcal{P})$. Then A is semi-simple and Λ is isomorphic to A_1.*

Proof. Suppose that A is not semi-simple. As Λ is a finite component and all vertices of Λ are in $K^b(\mathcal{P})$, the translation τ is an automorphism on Λ. Therefore the component Λ is stable and bounded. By 3.7, [1] acts on Λ which is a contradiction, as Λ contains only finitely many vertices. Therefore A is semi-simple and Λ is isomorphic to A_1 by 3.1. □

This theorem together with 2.5 give the next corollary.

Corollary 3.9. *Let A be a finite-dimensional indecomposable algebra of finite global dimension. Suppose that the Auslander-Reiten quiver of A has a finite component Λ, then A is semi-simple.*

In the case of $Comp^{-,b}(\mathcal{P})$ there are no finite components:

Corollary 3.10. *There is no finite Auslander-Reiten component Λ, such that all elements in Λ are in $Comp^b(\mathcal{P})$.*

Proof. Suppose Λ is a finite component such that all elements in Λ are in $Comp^b(\mathcal{P})$. By 2.7 we have that the corresponding Auslander-Reiten component in $D^b(A)$ is finite. Therefore A is semi-simple by 3.8. But by 3.2 we have $\Lambda \cong A_\infty^\infty$ which is a contradiction to the finiteness of Λ. □

For hereditary algebras the Auslander-Reiten quiver has already been determined by [H, IV]. We give an alternative proof. Let $T_r(A)$ denote the regular component of the Auslander-Reiten quiver of an hereditary algebra A.

Theorem 3.11. *Let Q be an oriented tree. Then the Auslander-Reiten quiver of kQ has a component $\mathbb{Z}[Q]$ containing a shift of every indecomposable projective and every indecomposable injective module. If Q is a Dynkin diagram, then the Auslander-Reiten quiver is $\mathbb{Z}[Q]$. Otherwise the Auslander-Reiten quiver consists of infinitely many copies of $\mathbb{Z}[Q][i]$ and $T_r(A)[i]$ for $i \in \mathbb{Z}$.*

Proof. Let i be a vertex in Q with set of predecessors S and R the set of successors. Then the predecessors of P_i in the Auslander-Reiten quiver of $D^b(A)$ are P_j for $j \in I$ and $I_t[-1] = \tau(P_t)$ for $t \in R$ by triangle hereditary. Therefore the Auslander-Reiten component containing the projectives is $\mathbb{Z}[Q]$. As $D^b(kQ)$ has finite representation type if and only if Q is a Dynkin diagram, we have that the Auslander-Reiten quiver is $\mathbb{Z}[Q]$ by 3.13. If Q is not a Dynkin diagram, then by 3.7, the Auslander-Reiten quiver does not have bounded components. Therefore $[i]$ does not act on $\mathbb{Z}[Q]$ for any $i \in \mathbb{Z}$. So there are infinitely many copies $\mathbb{Z}[Q]$ in the Auslander-Reiten quiver and infinitely many component isomorphic to $T_r(A)$ by [H, 4.7]. □

Note also that this result provides us with Auslander-Reiten components of Euclidean tree class.

We say that $D^b(A)$ has finite representation type, if all indecomposable complexes are shifts of finitely many complexes.

We call an indecomposable complex X in a stable Auslander-Reiten component τ-periodic, if there are $n, m \in \mathbb{Z}$ such that $\tau^n(X) = X[m]$. If we have τ-periodic modules in a component, we can construct subadditive functions.

Theorem 3.12. *Let C be a stable component. Suppose there is a complex $X \in C$ that is τ-periodic. Then C has tree class T a finite Dynkin diagram or A_∞.*

(a) If T is a finite Dynkin diagram, then the Auslander-Reiten quiver is equal to C and $D^b(A)$ has finite representation type.

(b) Suppose Q is a stable component of the Auslander-Reiten quiver of $D^b(A)$, that is not a shift of C. If the set $\operatorname{Hom}_{D^b(A)}(C, Q[i])$ or $\operatorname{Hom}_{D^b(A)}(Q[i], C)$ is non empty for some $i \in \mathbb{Z}$, then the tree class of Q is either Euclidean or infinite Dynkin.

Proof. Let $n \in \mathbb{N}$, $m \in \mathbb{Z}$ such that $\tau^n(X) = X[m]$. We consider the following subadditive function for all $M \in C$:

$$d(M) := \sum_{i=1}^{n} \sum_{j \in \mathbb{Z}} \dim \operatorname{Hom}(\tau^i(X), M[j]).$$

By 3.3 this is a subadditive function that is not additive. Therefore T is a finite Dynkin diagram or A_∞. So $[m]$ induces an automorphism of finite order on T and we have $M[l] = \tau^t(M)$ for some $t,\ l \in \mathbb{N}$ and all complexes $M \in C$. If T is finite, then C is bounded and by 3.7, C is the only component of the Auslander-Reiten quiver. Furthermore $D^b(A)$ has only finitely many complexes up to shifts.

Without loss of generality, let $\mathrm{Hom}_{D^b(A)}(X, L) \neq 0$ for some $L \in Q$. Then we define $d(M)$ as above for all $M \in Q$. Then d is an additive function by 3.3 on Q and therefore its tree class is Euclidean or infinite Dynkin. □

We can describe Auslander-Reiten quivers that have bounded components and are not finite.

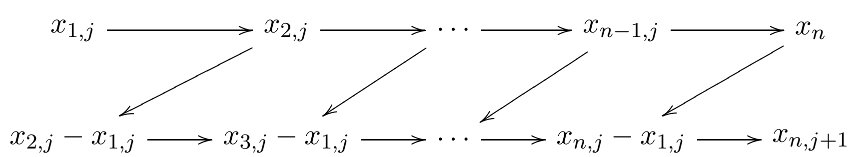
Theorem 3.13. *The following are equivalent:*

1. *The Auslander-Reiten quiver of $D^b(A)$ has a bounded component.*
2. *The representation type of $D^b(A)$ is finite.*
3. *The Auslander-Reiten quiver of $D^b(A)$ has a component of Dynkin tree class.*

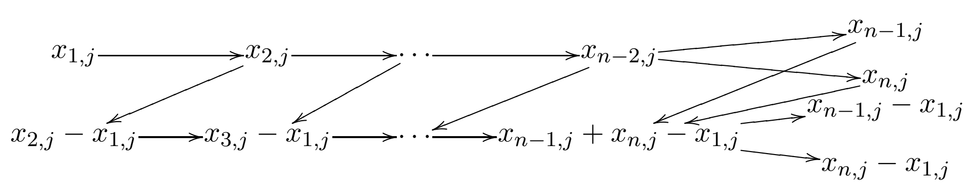
Proof. Let C be a bounded component. Then l_c takes values smaller equal $n \in \mathbb{N}$ for all elements in C. Let $M \in C$ such that M^0 contains the projective summand P and $M^i = 0$ for all $i > 0$. If $P \neq M$ there is a map $\psi : P \to M$ that is not an isomorphism and non-zero in $D^b(A)$. As $P \in C$ by 3.8, we know by 3.4 and 3.6, that there is a chain of irreducible maps of length at most 2^n that connects P and M. Therefore there are only finitely many complexes L in $D^b(A)$ such that L^0 contains P as a summand and $L^i = 0$ for $i > 0$. Therefore $D^b(A)$ has finite representation type.

Let $D^b(A)$ have finite representation type. Then A has finite global dimension and all Auslander-Reiten components are bounded and stable. If there is a finite component then by 3.8 the Auslander-Reiten component consists of copies of A_1. Otherwise the Auslander-Reiten quiver has only one component, that contains a periodic module. Therefore the tree class is finite Dynkin or A_∞ by 3.12. As [1] acts as the identity on A_∞ such a component can only occur if the representation type of $D^b(A)$ is not finite. Therefore the Auslander-Reiten quiver consists of one component $\mathbb{Z}[T]$ where T is a finite Dynkin diagram.

Suppose now that $D^b(A)$ has an Auslander-Reiten component with tree class Dynkin. If this component is finite, then A is semi-simple and $D^b(A)$ has finite representation type by 3.8. Assume now that there is a component $\mathbb{Z}[T]$ with tree class T a finite Dynkin diagram. We index the vertices in $\mathbb{Z}[T]$ by pairs (t, i) where $i \in \mathbb{Z}$ denotes the ith copy of T and t denotes the vertex of T. We assume that l_p is subadditive for only finitely many Auslander-Reiten sequences. Then we can choose an $l \in \mathbb{Z}$ such that l_p is additive for all vertices (t, j) with $j > l$ and t a vertex of T. Choose $T = A_n$ and denote $x_{t,i} := l_p((t, i))$. Let the values $x_{t,j}$ be given for a fix $j > l$ and all vertices t of A_n. Then we can calculate the values of $x_{t,j+1}$ from the left to the right as follows:



Clearly this gives a contradiction as l_p cannot be additive on the Auslander-Reiten sequence ending in $(n, j+1)$. Therefore l_p is not additive for infinitely many Auslander-Reiten sequences. We can use a similar argument for all Dynkin diagrams. If $T = D_n$ we have the following:



Then the values $x_{n,i}$ are strictly decreasing for strictly increasing $i > j$. This is a contradiction as they have to be positive integers for all $i \in \mathbb{Z}$.

In the case E_6 we have by the same argument that $x_{6,j+4} = -x_{1,j}$, in the case E_7 we have that $x_{3,j+20} = -x_{3,j} + x_{4,j}$ and for E_8 we have $x_{1,j+14} = -x_{1,j}$. Those are negative values and we obtain a contradiction to the assumption that l_p is additive on all but finitely many Auslander-Reiten sequences in the component.

As l_p is subadditive for infinitely many Auslander-Reiten sequences in C, there have to be infinitely many complexes that are homotopic to zero in the Auslander-Reiten component in $Comp^b(\mathcal{P})$ that is associated to C.

As there are only finitely many indecomposable complexes homotopic to zero in $Comp^b(\mathcal{P})$ up to shift, we deduce that a shift $[m]$ induces an automorphism on $\mathbb{Z}[T]$ for some $m \in \mathbb{N}$. Therefore $\mathbb{Z}[T]$ is a bounded component. $\square$

Note that we only require one component to be bounded or of Dynkin tree class in order to deduce that the reprsentation type of $D^b(A)$ is finite.

We can describe the Auslander-Reiten quiver and derived category more precisely in the case of the previous Theorem.

Theorem 3.14. *Let one of the condition of 3.13 be true. Then A is either semi-simple and the Auslander-Reiten quiver of $D^b(A)$ consists of $\mathbb{Z}$ copies of A_1 or the Auslander-Reiten quiver consists of one component $\mathbb{Z}[D]$ where $D \neq A_1$ is a finite Dynkin diagram and A is derived equivalent to kD.*

Proof. Suppose the bounded stable component is finite, then the first case holds by 3.3. If the bounded component is not finite then by 3.7 the Auslander-Reiten quiver consists of only one component which needs to be $\mathbb{Z}[D]$ for D a finite Dynkin diagram $D \neq A_1$ by 3.13. As $D^b(A)$ is of finite representation type, it is discrete in the sense of [**?**, 1.1]. By [BGS, Theorem A,B] A is derived equivalent to kQ where Q is a Dynkin diagram. By 3.11 we have $\bar{Q} = \bar{D}$. Then $D^b(kQ) \cong D^b(kD)$, which proves the Theorem. □

Knowing that there is only one component in the second case we have by [XZ, 3.6.1] that locally finite $D^b(A)$ is in fact equivalent to A is derived equivalent to an hereditary algebra of finite reprsentation type. We can generalize [W, 3.2]

Theorem 3.15. *Let $X \in K^b(\mathcal{P})$ be an indecomposable complex with $\nu^i(X^j)$ is projective and injective for all $i, j \in \mathbb{Z}$. Then X is in an Auslander-Reiten component $\mathbb{Z}[A_\infty]$ or A_1.*

Proof. Let C be the Auslander-Reiten component containing X. The function l_p is constant on all τ-orbits and therefore a subadditive function on C. Suppose l_p is bounded. Then by 3.13 A is derived equivalent to $D^b(kQ)$ for a Dynkin diagram Q or A is semi-simple. As no hereditary algebra has a complex that satisfies the assumption, we have that A is semi-simple. So $C \cong A_1$. Otherwise l_p is unbounded and C has therefore tree class A_∞. Assume without loss of generality that $X^i \neq 0$ if and only if $0 \leq i \leq n$. Then $\tau(X)$ viewed as complex in $K^b(\mathcal{P})$ satisfies $\tau(X)^i \neq 0$ if and only if $1 \leq i \leq n+1$. Therefore X is not periodic and $C \cong \mathbb{Z}[A_\infty]$. □

By the next example we see that it is not sufficient to assume that the τ-orbits are bounded or that there are τ-periodic modules in a component C if we want to deduce that C is bounded.

Example 3.16. *Let k be a field and let G be a quiver*

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

Then let $A := kG/R$ where kG is the path algebra of G, and where R is the ideal generated by $\{\alpha\beta\}$. We denote by S_1 and S_2 the simple modules corresponding to the vertices 1 and 2. Let P_i be the projective covers of S_i. Then P_1 has basis $\{e_1, \beta, \beta\alpha\}$; and P_2 has basis $\{e_2, \alpha\}$. We have $S_1 = P_1/P_2$ and $P_2/S_1 = S_2$. Here e_i is as usual the path of length zero at vertex i. We define $f : P_1 \to P_1$ to be a non-isomorphism and non-zero map that maps top P_1 to $\operatorname{soc} P_1$. We define

$I_n^m = P_1$ for all $0 \leq m \leq n-1$ and $d^l = f$ for $0 \leq l \leq n-2$ for all $n \in \mathbb{N}$.

As $I_1 = P_1$ we have by 3.15 that all I_n are the elements of a component $\mathbb{Z}[A_\infty]$.

As we have $\tau(S_2) = S_2[1]$ the element S_2 belongs to a component $\mathbb{Z}[A_\infty]$ by 3.12 and 3.13.

Finally we note that the τ-orbit of P_2 is given by

$$\tau^n(P_2)_i = \begin{cases} P_1, \text{ for } -n \leq i \leq n \\ P_2, \text{ for } i = -n-1 \\ 0 \text{ else.} \end{cases}$$

Therefore the value of l_p is strictly increasing on τ-orbits of the component containing P_2. The predecessors of P_2 are S_1 and $S_1[-1]$. Therefore the component is $\mathbb{Z}[A_\infty^\infty]$.

Note that the Auslander-Reiten quiver for this classes of example has been determined in [BGS, Theorem A].

The goal of the rest of this section is to construct an example showing that finite components can occur in Auslander-Reiten components that are not stable.

For the next lemmas of this section let P be a projective indecomposable module of A and I an indecomposable injective A-module.

We want to determine the irreducible maps ending in P and starting from P.

For the rest of the section we assume that all indecomposable complexes in $K(\mathcal{P})$ and $K(\mathcal{I})$ are minimal.

Note that the next Lemma is true if we exchange P by I, $K(\mathcal{P})$ by $K(\mathcal{I})$ and P' by an injective module I'.

Lemma 3.17. *Let M be an indecomposable complex in $K(\mathcal{P})$.*

(1) Let $f : P \to M$ be an irreducible map in $K(\mathcal{P})$ and suppose that $M^{-1} \neq 0$. Then f^0 is a section and $M^0 \cong P \oplus P'$ for a projective A-module P'. Furthermore $M^i = 0$ for $i \leq -2$.

(2) Let $f : M \to P$ be an irreducible map in $K(\mathcal{P})$ and suppose that $M^1 \neq 0$. Then f^0 is a retraction and $M^0 \cong P \oplus P'$ for some projective A-module P'. Furthermore $M^i = 0$ for $i \geq 2$.

Proof. Let

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow & & \\
\cdots & \longrightarrow & M^{-1} & \xrightarrow[d]{} & M^0 & \xrightarrow[e]{} & M^1 & \longrightarrow & \cdots
\end{array}$$

be an irreducible map. We can factorize this map through the stupid truncation $\sigma^{\geq 0}M$ as follows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M^0 & \xrightarrow[e]{} & M^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & M^{-1} & \xrightarrow[d]{} & M^0 & \xrightarrow[e]{} & M^1 & \longrightarrow & \cdots .
\end{array}$$

The map between the first two rows has to be a section. Therefore f^0 has to be a section and $M^0 \cong P \oplus P'$. We can also factorize f through $\sigma^{\geq -1}M : \cdots \to 0 \to M^{-1} \to_d P \oplus P' \to_e M^1 \to \cdots$. Then the map between the first two rows is not a section as f is not a section. Therefore the map between the two last rows has to be a retraction and we have $M^i = 0$ for all $i \leq -2$.

Let now f be an irreducible map given as

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M^{-1} & \xrightarrow[d]{} & M^0 & \xrightarrow[e]{} & M^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

We can factorize f through the element $\sigma^{\leq 0}M$ as follows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M^{-1} & \xrightarrow{d} & M^0 & \xrightarrow{e} & M^1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
\cdots & \longrightarrow & M^{-1} & \longrightarrow & M^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The map given by the first two rows is not a section, therefore the map given by the last two rows has to be a retraction. So f_0 is a retraction and we can write $M^0 = P \oplus P'$. As above, we can factorize f through $\sigma^{\leq 1}M$. Then the map between the last two rows is not a retraction and therefore the map between the first two rows has to be a section. It follows that $M^i = 0$ for $i \geq 2$.

□

Lemma 3.18. *Let A be an algebra such that $\mathrm{Hom}_A(S, A) \neq 0$ for all simple A-modules S. Let $f : M \to W$ be an irreducible map from $M \in K^-(\mathcal{P})$ to $W \in K^-(\mathcal{P})$, where M and W are indecomposable. Without loss of generality let $M^i = 0$ for $i \geq 1$. Then $W^i = 0$ for all $i \geq 1$.*

Proof. Let n be the maximal integer such that $W^n \neq 0$. Suppose that $n \geq 1$. As W is indecomposable, $w^n : W^{n-1} \to W^n$ is not surjective. Let S be a simple module in the top of $W^n/\mathrm{Im}\, w^n$, say $S \cong W^n/R$ where $\mathrm{Im}\, w^n \subset R \subset W^n$. Then there is a projective module W' such that there exists a non-zero map $\pi : W^n \to W'$ with $\pi(R) = 0$. We can then factor f as follows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & W^{n-1} & \longrightarrow & W^n & \xrightarrow{\pi} & W & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
\cdots & \longrightarrow & W^{n-1} & \longrightarrow & W^n & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The map between the two first rows is not a split monomorphism, as f is not a split monomorphism and the map between the last two rows is not a split epimorphism. This is a contradiction to the irreducibility of f and therefore $n \leq 0$. □

Lemma 3.19. *Suppose that $P/\operatorname{rad} P \subset \operatorname{soc} A$. Let $f : D \to P$ be an irreducible map with $D \in K^-(\mathcal{P})$ indecomposable. Then $D^0 = P \oplus P'$ for some projective A-module P' and f^0 is a split epimorphism. Furthermore $D^i = 0$ for $i \geq 2$ and $D^1 \neq 0$.*

Proof. Suppose that $D^1 = 0$. Then f^0 is not surjective, as f is not a retraction. As f^0 is not surjective, there exist a map $t : P \to P'$ such that $t \circ f^0 = 0$ (we can for instance map the top of P into the socle of another projective module P'). Then we can factorize f as

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & D^{-1} & \longrightarrow & D^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{t} & P' & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & \cdots .
\end{array}$$

As the map between the last two rows is not a retraction, the map between the first two rows has to be a section. But then f^0 is a section and therefore f is a section. This gives a contradiction.

Therefore $D^1 \neq 0$ and part 2 of 3.17 proves the rest of the statement. □

This lemma proves the following

Corollary 3.20. *Suppose that $P/\operatorname{rad} P \subset \operatorname{soc} A$. If P has infinite injective dimension, then there is no irreducible map from any element of $K^b(\mathcal{I})$ to P, and there is no irreducible map from P to any element of $K^b(\mathcal{P})$.*

Proof. Let $f : D \to P$ be an irreducible map, where D is indecomposable. If P has infinite injective dimension, then D has infinitive injective dimension by 3.19. Therefore $D \notin K^b(\mathcal{I})$.

Suppose there is an irreducible map from P to some object in $K^b(\mathcal{P})$, then by 2.2, there exists an Auslander-Reiten triangle that has P as direct summand of its middle term. This means that there is an irreducible map from $\nu(M)[-1] \in K^b(\mathcal{I})$ to P which contradicts the first part of this corollary. □

This corollary implies that in this case P does not appear in the middle term of an almost split sequence.

Lemma 3.21. *Let $f : M \to W$ be an irreducible map with $M \in K^-(\mathcal{P})$ and $W \in K^b(\mathcal{P})$. Let n be the minimal integer such that $W^n \neq 0$. Suppose* *that* $\cdots \to M^{n-1} \to M^n \to \operatorname{Im} d_M^n$ *is an infinite projective resolution. Then* f^n *is a split epimorphism.*

Proof. Note first that f^n is not injective as $f^n \circ d_M^{n-1} = 0$. Let

$$\cdots \to L^{n-1} \to M^n \to_{f^n} \operatorname{Im} f^n$$

be a projective resolution of $\operatorname{Im} f^n$, then we can factorize f as follows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & L^{n-1} & \longrightarrow & M^n & \xrightarrow[d_W^n f^n]{} & W^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & W^n & \longrightarrow & W^{n+1} & \longrightarrow & \cdots
\end{array}$$

Suppose first that the map between the two first rows is a section. Then all f_i for $i \geq n+1$ are sections. Furthermore $L^i = M^i$ for all $i \leq n-1$. Therefore $\operatorname{Im} d_M^{n-1} \cong \ker f^n$. It follows $\operatorname{Im} d_M^n \cong M^n / \operatorname{Im} d_M^{n-1} \cong \operatorname{Im} f^n$.

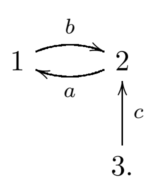
We also have $\operatorname{Im}(f^{n+1} \circ d_M^n) = \operatorname{Im} d_M^n$ as f^{n+1} is a monomorphism. This means that $\mathrm{id}_{\operatorname{Im} d_M^n}$ factorizes through W^n as $\operatorname{Im} f^n \to W^n \overset{d_W^n}{\to} \operatorname{Im} d_M^n$. This holds as $\operatorname{Im} f^n = \operatorname{Im} d_M^n$ is a submodule of W^n and by the fact that $d_W^n(\operatorname{Im} f_n) = \operatorname{Im} f^{n+1} \circ d_M^n = \operatorname{Im} d_M^n$. Therefore $\operatorname{Im} d_M^n$ is a direct summand of W^n and is projective. This is a contradiction to the fact that $\operatorname{Im} d_M^n$ has an infinite projective resolution.

Therefore the map between the last two rows is a retraction. Then f^n is a retraction and $M^n \cong W^n \oplus P'$ for some projective module P'.

□

We give now an example of a finite-dimensional algebra, whose bounded derived category has a finite Auslander-Reiten component.

Example 3.22. *Let* k *be a field and let* G *be a quiver*



Then let $A := kG/R$ *where* kG *is the path algebra of* G*, and where* R *is the ideal generated by all paths of length* ≥ 2*. We denote by* S_1*,* S_2 *and* S_3 *the simple modules corresponding to the vertices* 1*,* 2 *and* 3*. Let* P_i *be the projective covers of* S_i*. Then* P_1 *has basis* $\{1, a\}$*; and* P_2 *has basis* $\{2, b, c\}$*;*

and P_3 is simple spanned by e_3. Here e_i is as usual the path of length zero at vertex i. Clearly $\operatorname{Hom}(S_i, X) \neq 0$ for $i = 1, 2, 3$. Let I_i be the injective hulls of S_i. Each I_i is uniserial of length two, and $I_1/S_1 \cong S_2$, $I_2/S_2 \cong S_1$ and $I_3/S_3 \cong S_2$.

Theorem 3.23. *We use the notation from 3.22. The Auslander-Reiten component of $D^b(X)$ containing P_3 consists only of the triangle $I_3[-1] \to M \to P_3 \to I_3$. Therefore it is finite.*

Proof. We have $\nu(P_3) = I_3$ and P_3 has an infinite injective resolution

$$\begin{array}{ccccccccccc} P_3 & \longrightarrow & I_3 & \longrightarrow & I_2 & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow & \cdots . \\ & & & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow & & \\ & & & & S_2 & & S_1 & & S_2 & & \end{array}$$

Therefore there is no Auslander-Reiten triangle starting in P_3 by 2.5. By Corollary 3.20 there is no Auslander-Reiten triangle that has P_3 as summand of its middle part.

Let M be the middle term of the Auslander-Reiten triangle ending in P_3. Then $M \in K^{-,b}(\mathcal{P})$ is given by

$$\cdots \to P_2 \to P_1 \oplus P_3 \to P_2 \to P_1 \oplus P_3 \to P_2 \to 0 \to \cdots .$$

Clearly M is indecomposable, as $H^1(M) = S_2$ and $H^i(M) = 0$ for $i \neq 1$. Note that $M \cong S_2$ in $D^b(A)$. As S_2 has infinite projective and injective dimension, there is no Auslander-Reiten sequence starting or ending in M. Therefore we only need to show that M does not appear as a direct summand of a middle term of an Auslander-Reiten triangle.

Let therefore $f : M \to W$ be an irreducible map where $W \in K^b(\mathcal{P})$. Let n be minimal so that $W^n \neq 0$. By 3.21, we know that f^n is a retraction. Therefore $M^n = P_1 \oplus P_3$, $W^n \cong P_3$ and either d_W^{n+1} is an injection into a direct summand of W^{n+1} isomorphic to P_2, or $W^{n+1} = 0$. If $W^{n+1} \neq 0$ we have that $f^{n+1} : P_2 \to W^{n+1}$ has $\operatorname{Im} f^{n+1} \cong P_2/S_1$. Such a map does not exist as P_2/S_1 is not a submodule of any projective A-module. Therefore $W^{n+1} = 0$. Suppose that $n < 0$. We can factorize f through $\sigma^{\leq n+1} M$ as follows:

$$\begin{array}{ccccccccccccc}
P_2 & \longrightarrow & P_1 \oplus P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 \oplus P_3 & \longrightarrow & \cdots & \longrightarrow & P_2 & \longrightarrow & 0 \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_2 & \longrightarrow & P_1 \oplus P_3 & \longrightarrow & P_2 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & {\scriptstyle f_n}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_3 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

The map between the first two rows is not a split monomorphism and the map between the last two rows is not a split epimorphism. This proves that $n = 0$ and $M \to P_3$ is the only irreducible map.

The module I_3 has an infinite projective resolution

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 \oplus P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & I_3\ . \\
 & & & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow & & \\
 & & & & S_2 & & S_1 \oplus P_3 & & S_2 & & S_1 & &
\end{array}$$

Therefore I_3 does not appear at the end of an Auslander-Reiten sequence. We therefore only need to show that I_3 does not appear as a direct summand of a middle term of an Auslander-Reiten triangle. We need to determine the irreducible maps $f : I_3 \to L$, where $L \in K^b(\mathcal{P})$. We view L as en element in $K^{+,b}(\mathcal{I})$. By Lemma 3.17, we can assume that $L^i = 0$ for $i \leq -1$ because otherwise $L \notin K^b(\mathcal{P})$. As f^0 is not injective, we have $\operatorname{Im} f^0 \cong S_2$. Therefore L^0 has a direct summand isomorphic to I_2. The simple module S_2 has the periodic injective resolution $S_2 \to I_2 \to I_1 \to I_2 \to \cdots$

Then we can factorize f through this resolution

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & I_3 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & I_2 & \longrightarrow & I_1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & L^0 & \longrightarrow & L^1 & \longrightarrow & \cdots\ .
\end{array}$$

The map between the first two rows is not a split monomorphism. Then the map between the last two rows has to be a split epimorphism. As S_2 is indecomposable, $L \cong S_2$ in $D^b(A)$. As S_2 is Ω-periodic, it has an infinite projective resolution. Therefore $L \notin K^b(\mathcal{P})$. This shows that I_3 does not appear as summand of the middle term of an Auslander-Reiten triangle.

This shows that the Auslander-Reiten component of $D^b(A)$ containing P_3 consists only of the triangle $I_3[-1] \to M \to P_3 \to I_3$.

□

Note that in the previous example the map between the complexes $S_2 \to S_1[-1]$ is an irreducible map that does not appear in an Auslander-Reiten component as $S_1, S_2 \notin K^b(\mathcal{P}),\ K^b(\mathcal{I})$.

4. Irreducible maps ending in contractible complexes

Next we analyze under which conditions a contractible complex can appear as direct summand of $\operatorname{cone}(w)[-1] \in Comp^{-,b}(\mathcal{P})$ for a map w that induces an Auslander-Reiten triangle $\nu(Z)[-1] \to Y \to Z \to_w \nu(Z)$ in $D^b(A)$.

We first introduce a new definition.

Definition 4.1. *Let $P_1,\ P_2 \in \mathcal{P}$ and $f : P_1 \to P_2$ be a map. Then f is p-irreducible if f is not a section and not a retraction and if for any $P \in \mathcal{P}$ and maps $f_1 : P_1 \to P$ and $f_2 : P \to P_2$ such that $f = f_2 \circ f_1$ we have that f_1 is a section or f_2 is a retraction.*

Throughout this section let P be an indecomposable projective module.

Lemma 4.2. *Let $f : Q \to \bar{P}$ be an irreducible map, where $Q \in Comp^b(\mathcal{P})$ is indecomposable and not contractible. Then there exists a map $d : P_0 \to P$ that is p-irreducible, such that $Q = p(\operatorname{Coker}(d))$.*

Proof. Let f be an irreducible map given by the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow[d]{} & Q^1 & \longrightarrow & Q^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow[\mathrm{id}]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

We can factorize f through $\sigma^{\leq 1}Q$ as

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow[d]{} & Q^1 & \longrightarrow & Q^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
\cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow[d]{} & Q^1 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow[\mathrm{id}]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The map given by the last two rows is not a retraction, as f is not a retraction. Therefore the map between the first two rows is a section and $Q^i = 0$ for all $i \geq 2$.

We can factorize f as

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow[d]{} & Q^1 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & Q^1 & \xrightarrow[\mathrm{id}]{} & Q^1 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & {\scriptstyle f^1}\downarrow & & {\scriptstyle f^1}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow[\mathrm{id}]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

If the map between the first two rows is a section, Q is contractible, which is a contradiction. Therefore the map between the last two rows is a retraction. We can write $Q^1 \cong P \oplus P'$ for a projective module P' and f^1 is a retraction. But then we can factorize f as follows:

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow[d]{} & P \oplus P' & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle f^1}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow[f^1 d]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & {\scriptstyle f^0}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow[\mathrm{id}]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

If the map in the last two rows is a retraction, then f is the identity. Therefore the map between the two upper rows has to be a section and $P' = 0$, $f^0 = d$ and $f^1 = \mathrm{id}$.

Let $\cdots \to L_{-2} \to L_{-1} \to \ker d$ be a minimal projective resolution of $\ker d$. Then f factorize through $\operatorname{Coker} d$ as follows:

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow[d]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L_{-1} & \longrightarrow & Q^0 & \xrightarrow[d]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow[\mathrm{id}]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

Clearly the bottom diagram is not a retraction, because else f would be the identity. Therefore the upper diagram is a section and Q is isomorphic to the complex consisting of a minimal projective resolution of $\operatorname{Coker} d$ and

0 elsewhere. Suppose d is not p-irreducible. Then we can factorize $d = s \circ t$ where $t : Q^0 \to \tilde{P}$ is not a section and $s : \tilde{P} \to P$ is not a retraction for some projective module $\tilde{P}$. But then we can factorize f as follows:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow[d]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
\cdots & \longrightarrow & Q^{-1} & \longrightarrow & \tilde{P} & \xrightarrow[t]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow[\mathrm{id}]{} & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

This is a contradiction to the irreducibility of f as the map between the first two rows is not a section and the map between the two bottom rows is not a retraction. Therefore d is p-irreducible. □

We therefore determine p-irreducible maps between indecomposable projective modules.

Lemma 4.3. *Let P_1 and P_2 be indecomposable projective modules. Then $d : P_1 \to P_2$ is p-irreducible if and only if P_1 is a direct summand of the projective cover of* $\operatorname{rad} P_2$.

Proof. Let $d : P_1 \to P_2$ be as in the statement. Suppose that d factors as $d = g \circ f$ where $f : P_1 \to \tilde{P}$ and $g : \tilde{P} \to P_2$, and g is not a retraction. Then $\operatorname{Im} d \subset \operatorname{Im} g \subset \operatorname{rad} P_2$. Let $P_1 \oplus P_1'$ be the projective cover of $\operatorname{rad} P_2$. Then there is a map $e : \tilde{P} \to P_1$ such that the following diagram commutes:

$$\begin{array}{ccccc}
P_1 & \xrightarrow[f]{} & \tilde{P} & \xrightarrow[e]{} & P_1 \oplus P_1' \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle g} & & \downarrow \\
\operatorname{Im} d & \longrightarrow & \operatorname{Im} g & \longrightarrow & \operatorname{rad} P_2
\end{array}$$

Therefore $e \circ f = \mathrm{id}_{P_1}$. So f is a section and d is p-irreducible.

Conversely, let $s : P' \to P_2$ be a p-irreducible map. Then s is not surjective and therefore $\operatorname{Im} s \subset \operatorname{rad} P$. Then s factors through $i : P_0 \to P_2$ and $h : P' \to P_0$, where i is the projection onto $\operatorname{rad} P_2$. As s is p-irreducible, h is a section and P' is a direct summand of P_0. □

Next we determine which p-irreducible maps induce an irreducible map in $Comp^{-,b}(\mathcal{P})$.

Theorem 4.4. *Let P_0 be a projective module and let $d : P_0 \to P$ be a p-irreducible map. Then there is an irreducible map in $Comp^{-,b}(\mathcal{P})$ from the complex $p(\mathrm{Coker}(d))$ to the contractible complex $\bar{P}$ if and only if P_0 is the projective cover of* $\mathrm{rad}\, P$.

Proof. Let $p(\mathrm{Coker}(d)) := \cdots \to P_{-1} \to P_0 \xrightarrow{d} P \to 0 \to \cdots$, where P_0 is the projective cover of $\mathrm{rad}\, P$. We claim that the map h given by

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_{-1} & \longrightarrow & P_0 & \xrightarrow{d} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow d & & \downarrow \mathrm{id} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\mathrm{id}} & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

is irreducible.

Suppose the given map factors through a complex $X \in Comp^{-,b}(\mathcal{P})$ via maps $f : \mathrm{Coker}(d) \to X$ and $g : X \to \bar{P}$. As $\mathrm{id}_P = g^1 \circ f^1$, we have that f^1 is a section and g^1 is a retraction. We have that $X^1 = P \oplus \tilde{P}$. Suppose that $\mathrm{Im}\, g^1 d_X^1 = P$. Then g is a retraction. If $\mathrm{Im}\, g^1 d_X^1 = \mathrm{rad}\, P$, then f^0 is a section, $X^0 = P_0 \oplus P'$ and $g^1 d_X^1(P') = 0$. We visualize this in the next diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_{-1} & \longrightarrow & P_0 & \xrightarrow{d} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow & & \\
\cdots & \longrightarrow & X^{-1} & \longrightarrow & P_0 \oplus P' & \xrightarrow{\left(\begin{smallmatrix} d,0 \\ 0,x \end{smallmatrix}\right)} & P \oplus \tilde{P} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow (d,0) & & \downarrow (\mathrm{id},0) & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\mathrm{id}} & P & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

But then we can factorize the map g through $s : X \to p\,\mathrm{Coker}(d)$ and $h : p\,\mathrm{Coker}(d) \to \bar{P}$. So $s \circ f$ is an isomorphism. Therefore f is a section.

Conversely, if $\tilde{P}$ is a direct summand of P_0 but $P_0 \ncong \tilde{P}$, then the map of complexes $p(\mathrm{Coker}(r)) \to \bar{P}$ induced by the p-irreducible map $r : \tilde{P} \to P$ factors through $p(\mathrm{Coker}(d))$ and is therefore not irreducible.

□

Let P_1 be a summand of the projective cover of $\mathrm{rad}\, P$ and $d : P_1 \to P$ the induced map. Then $p(\mathrm{Coker}(d)) \in K^{-,b}(\mathcal{P})$ is isomorphic to the element $P/\mathrm{Im}\, d \in D^b(A)$ and is therefore indecomposable. By the previous Theorem a contractible complex P appears in the middle term of an Auslander-Reiten sequence in $Comp^{-,b}(\mathcal{P})$ if and only if $P/\operatorname{rad} P$ has finite injective dimension.

Corollary 4.5. *Let θ be a stable Auslander-Reiten component. Then l_p is not additive if and only if there exists an indecomposable projective module P such that $P/\operatorname{rad} P$ has finite projective and finite injective dimension and $P/\operatorname{rad} P \in \theta$.*

Proof. The function l_p is not additive if and only if there exists an indecomposable object $L \in \theta$ and a connecting map $w : L \to p(\nu(L))$ of the Auslander-Reiten triangle in $K^b(\mathcal{P})$ such that $\operatorname{cone}(w)[-1] \in Comp^b(\mathcal{P})$ contains a contractible summand.

So there is a complex $\bar{P}$ that is a direct summand of $\operatorname{cone}(w)[-1]$ and an irreducible map $f : \nu(L)[-1] \to \bar{P}$. By Lemma 4.3, $\nu(L)[-1]$ is isomorphic in $D^b(A)$ to $P/\operatorname{rad} P$. As θ is a stable Auslander-Reiten component, we have $p(P/\operatorname{rad} P) \in K^b(\mathcal{P})$ and $i(P/\operatorname{rad} P) \in K^b(\mathcal{I})$. □

¿From this corollary it follows that if A is self-injective, then l is an additive function on the Auslander-Reiten components of $D^b(A)$.

We can deduce that Euclidean components always contain a simple module.

Theorem 4.6. *Let C be a stable component of the Auslander-Reiten quiver of $D^b(A)$ with tree class an Euclidean diagram. Then C contains a simple module.*

Proof. The function l_p is subadditive on C. Suppose l_p is additive. By [Web][2.4], the function l_p takes bounded values on C. This is a contradiction to 3.13. Therefore l_p is not additive. By 4.5, this means that C contains a simple module. □

5. Auslander-Reiten triangles of Nakayama algebras

In this section we analyze the Auslander-Reiten quiver of certain Nakayama algebras A with finite global dimension and show that they are derived equivalent to hereditary algebras of finite representation type. The author learned that the results are already known by [HS]. As the methods are different and give an application of section 2, they were included.

Let $A := kA_n/I$ where kA_n is the path algebra of A_n, and where I is an ideal of kA_n. Then A is a Nakayama algebra of finite global dimension. We denote by J the ideal generated by the paths of length one in kA_n.

Let $A_n : 1 \to 2 \to \cdots \to n-1 \to n$, and we consider left modules. We denote by S_i the simple modules at the vertex i, by P_i the projective indecomposable modules with $P_i/\operatorname{rad} P_i = S_i$ and by I_i the injective indecomposable modules with $\operatorname{soc}(I_i) = S_i$.

By [ASS, 3.5] every indecomposable module M of A is given by $P_i/\operatorname{rad}^t(P_i)$ for a uniquely determined i and t. We denote by $d_i^j : I_i \to I_j$ the canonical maps. Let $l := l(P)$ and $\bar{l} := l(I_{i-t+1})$. Let $\cdots \to P_{i-l} \to P_{i-t} \to P_i$ be the minimal projective resolution of M and $I_{i-t+1} \to I_{i+1} \to I_{i+\bar{l}-t+1} \to \cdots$ be the minimal injective resolution of M. We thereby set $I_k = 0$ and $P_k = 0$ if $k \leq 0$.

We introduce three conditions:
(1) $d_{i-l}^{i-t+1} = 0$ or $\operatorname{pdim} M \leq 1$.
(2) $d_i^{i+\bar{l}-t+1} = 0$ or $\operatorname{idim} M \leq 1$.
(3) $d_{i-t}^{i+1} = 0$ or $\operatorname{pdim} M = 0$ or $\operatorname{idim} M = 0$.

The next lemma will be used to calculate concrete examples.

Lemma 5.1. *Let $w : pM \to \nu(pM)[-1]$ define an Auslander-Reiten triangle terminating in M.*

(a) Suppose (1) and (3) hold, then $\operatorname{cone}(w)$ *has a direct summand* $v(p(\Omega(M)))[1]$.

(b) Suppose (2) and (3) hold, then $\operatorname{cone}(w)$ *has a direct summand* $i(\Omega^{-1}(M))$.

(c) Suppose (1) and (2) hold, then $\operatorname{cone}(w)$ *has a direct summand* $\cdots \to 0 \to I_{i-t+1} \to I_i \to 0 \cdots$.

(d) Suppose (1), (2) and (3) hold, then $\operatorname{cone}(w)$ *decomposes as sum of the indecomposable complexes* $i(\Omega^{-1}(M))$, $\cdots \to 0 \to I_{i-t+1} \to I_i \to 0 \cdots$, *and* $v(p(\Omega(M)))[1]$.

(e) If at most one condition (1)-(3) holds, then $\operatorname{cone}(w)$ *is indecomposable.*

Proof. We will write d instead of d_i^j for an easier presentation. The connecting map w of the Auslander-Reiten quiver ending in M is given as follows:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I_{i-t+1} & \xrightarrow[d]{} & I_{i+1} & \xrightarrow[d]{} & I_{i+\bar{l}-t+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle d} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & I_{i-l} & \xrightarrow[d]{} & I_{i-t} & \xrightarrow[d]{} & I_i & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Then $\operatorname{cone}(w)$ is given by the sequence

$$\cdots \longrightarrow I_{i-l} \xrightarrow[(d,0)]{} I_{i-t} \oplus I_{i-t+1} \xrightarrow[\binom{d,0}{d,d}]{} I_i \oplus I_{i+1} \xrightarrow[\binom{0}{d}]{} I_{i+\bar{l}-t+1} \longrightarrow \cdots$$

The following diagram is a retraction if and only if condition (1) and (3) are satisfied.

$$\begin{array}{ccccccc}
I_{i-l} & \xrightarrow[(d,0)]{} & I_{i-t}\oplus I_{i-t+1} & \xrightarrow[\binom{d,0}{d,d}]{} & I_i\oplus I_{i+1} & \xrightarrow[\binom{0}{d}]{} & I_{i+l-t+1} \\
\mathrm{id}\downarrow & & \binom{\mathrm{id}}{0}\downarrow\uparrow \mathrm{id}-d & & \downarrow\uparrow & & \downarrow\uparrow \\
I_{i-l} & \xrightarrow[d]{} & I_{i-t} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The bottom row is then $\nu(p(\Omega(M)))[1]$. This is an indecomposable complex, as $\Omega(M)$ is indecomposable.

The following diagram is a retraction if and only if condition (2) and (3) are satisfied:

$$\begin{array}{ccccccc}
I_{i-l} & \xrightarrow[(d,0)]{} & I_{i-t}\oplus I_{i-t+1} & \xrightarrow[\binom{d,0}{d,d}]{} & I_i\oplus I_{i+1} & \xrightarrow[\binom{0}{d}]{} & I_{i+l-t+1} \\
\downarrow & & \downarrow\uparrow & & \binom{-d}{\mathrm{id}}\downarrow\uparrow \mathrm{id} & & \downarrow\uparrow \mathrm{id} \\
0 & \longrightarrow & 0 & \longrightarrow & I_{i+1} & \longrightarrow & I_{i+l-t+1}
\end{array}$$

The bottom row is then $i(\Omega^{-1}(M))$. As A is a Nakayama algebra, $\Omega^{-1}(M)$ is indecomposable. Therefore the complex $i(\Omega^{-1}(M))$ is indecomposable.

The following diagram is a retraction if and only if (1) and (2) are satisfied:

$$\begin{array}{ccccccc}
I_{i-l} & \xrightarrow[(d,0)]{} & I_{i-t}\oplus I_{i-t+1} & \xrightarrow[\binom{d,0}{d,d}]{} & I_i\oplus I_{i+1} & \xrightarrow[\binom{0}{d}]{} & I_{i+l-t+1} \\
\downarrow & & \mathrm{id}\downarrow\uparrow (\mathrm{id},d) & & \binom{d}{\mathrm{id}}\downarrow\uparrow \mathrm{id} & & \downarrow\uparrow \\
0 & \longrightarrow & I_{i-t+1} & \xrightarrow[d]{} & I_i & \longrightarrow & 0
\end{array}$$

The bottom row is clearly indecomposable as a complex. This proves part (a)-(d).

The complexes $\sigma^{\geq 1}(\mathrm{cone}(w))$ and $\sigma^{\leq -1}(\mathrm{cone}(w))$ are indecomposable in $Comp^{+,b}(\mathcal{I})$, as the first one is a minimal injective resolution of an indecomposable module and the second one is v applied to a minimal projective resolution of an indecomposable module. Therefore the retractions presented in the previous three diagrams are the only possiblities for a direct summand. So if at most one condition is satisfied, then $\mathrm{cone}(w)$ has to be indecomposable, which proves part (e).

□

We determine the number of predecessor of the simple modules.

Lemma 5.2. *Let S_i be simple and assume that S_i is not projective and not injective. Then S_i has two predecessors if and only if $d_{i-1}^{i+1} = 0$. Otherwise it has only one predecessor.*

Proof. With the notation of 5.1, we have that $t = 1$ and $\bar{l} = l(I_i)$. Clearly condition (1) and condition (2) are always satisfied. Therefore S_i has two predecessors if and only if (3) is satisfied. This is the case if and only if $d_{i-1}^{i+1} = 0$. Then we have the two predecessor $\nu(p(\operatorname{rad} P_i))$ and $i(I_i/S_i)[-1]$. In all other cases, we have only one predecessor. □

We can determine the Auslander-Reiten quiver for a class of examples.

Theorem 5.3. *Let $A := kA_n/I$ with $n \geq 4$ and I is generated by the path of length n. Then the Auslander-Reiten quiver of $D^b(A)$ is isomorphic to $\mathbb{Z}[D_n]$. If n is even, then $[-1]$ acts as the identity on D_n. If n is odd $[-1]$ acts as the involution on D_n. Also A is derived equivalent to kD_n.*

Proof. Let $1 < i < n$, then the projective resolution of S_i is given by $0 \to P_{i-1} \to P_i$ and the injective resolution by $I_i \to I_{i+1} \to 0$. Therefore $\tau(S_i) = S_{i-1}$ for $2 < i < n$, $\tau(S_2) = \cdots \to 0 \to I_1 \to I_2 \to 0 \to \cdots$ and $\tau^2(S_2) = S_{n-1}[-1]$. Therefore all non-injective and non-projective simples are in the same τ-orbit and $[-1]$ operates on that orbit. By 5.2 the S_i have exactly one predecessor given by $\cdots \to 0 \to I_{i-1} \to I_{i+1} \to 0 \to \cdots$ which is isomorphic to $P_i/\operatorname{rad}^2(P_i)$ for $i > 2$.

We note that I_i has projective resolution $0 \to P_1 \to P_{i-1} \to P_n$ for $i > 2$. Therefore $\tau(I_i) = P_{i-2}[1]$ for $i > 2$, $\tau(I_2) = I_n[-1]$ and $\tau(I_1) = I_{n-1}[-1]$.

Suppose now that n is even, then the orbit of $S_1 = P_1$ is given by $P_1, I_1[-1], I_{n-1}[-2], P_{n-3}[-1], \cdots, I_{n-(2k+1)}[-2], P_{n-(2k+3)}[-1], \cdots, P_1[-1]$. In this case the orbit of $S_n = I_n$ is given by $S_n, P_{n-2}[1], I_{n-2}, \cdots, I_{n-2k}, P_{n-2k-2}[1], \cdots, I_2, S_n[-1]$.

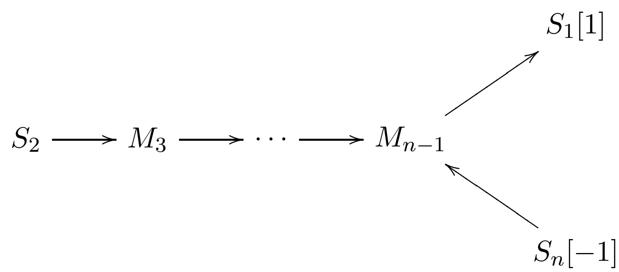
Suppose now that n is odd. Then the orbit of P_1 is given by $P_1, I_1[-1], I_{n-1}[-2], P_{n-3}[-1], \cdots, I_{n-(2k+1)}[-2], P_{n-(2k+3)}[-1], \cdots, I_2[-2], S_n[-3], P_{n-2}[-2], I_{n-2}[-3], \cdots, I_{n-2k}[-3], P_{n-2k-2}[-2], \cdots, P_1[-2]$ and contains the odd shifts of S_n.

So S_n and S_1 lie in different orbits. If n is even [-1] operates on each orbits. If n is odd, $[-2]$ operates on the orbits and [-1] maps the orbit of S_1 onto the orbit of S_n.

The only predecessor of S_1 is given by $P_{n-1}/\operatorname{rad}^{n-2}(P_{n-1})[-1]$.

Next we investigate the predecessors of modules $M_s := P_s/\operatorname{rad}^{s-1}(P_s)$ for $2 \geq s \geq n-1$. The projective resolution of M_s is given by $0 \to P_1 \to P_s$

and the injective resolution of M_s is given by $I_2 \to I_{s+1} \to 0$. Therefore (1) and (2) are always satisfied. We have $d_1^{s+1} = 0$ if and only if $s = n-1$. So M_s has three predecessor if and only if $s = n-1$ and else it has only two predecessor. By the proof of 5.1, the predecessors of M_{n-1} are $S_n[-1]$, $\tau(S_1[1])$ and M_{n-2}. The predecessors of M_s for $2 < s < n-1$ are M_{s-1} and $\tau^{-1}(M_{s+1})$. Therefore a sectional path of the Auslander-Reiten component looks as follows:



The the component is isomorphic to $\mathbb{Z}[D_n]$ by 3.8. Furthermore we know that $[-2]$ acts on the τ-orbit of S_i, S_n and S_1. Therefore $[2]$ acts as the identity on all τ-orbits of the component. By 3.7 this component is the only Auslander-Reiten component of $D^b(A)$.

□

We investigate another cass of examples.

Theorem 5.4. *Let A be of global dimension $n-1$. Then the Auslander-Reiten quiver is isomorphic to $\mathbb{Z}[A_n]$. In particular $[-1]$ is an involution on A_n. Also A is derived equivalent to A_n.*

Proof. If A has global dimension $n-1$, then I is generated by all paths of length 2. Let $1 < i < n$, then S_i is non-projective and non-injective. Furthermore by 5.2 S_i has the two predecessors $\tau(S_{i-1})[1]$ and $S_{i+1}[-1]$, S_1 has the only predecessor $S_2[-1]$ and S_n has the only predecessor $\tau(S_{n-1})[1]$. A sectional path is therefore given by:

$$S_n[-n+1] \longrightarrow \cdots \longrightarrow S_2[-1] \longrightarrow S_1.$$

By direct computation we have that $\tau^s(S_i) = \cdots \to 0 \to I_s \to \cdots \to I_{i+s-1} \to 0 \to \cdots [i-s-1]$. Therefore $\tau^s(S_i)$ has two non-zero homologies except for $s = n-i+1$ where $\tau^{n-i+1}(S_i) = S_{n-i+1}[-n+2i-2]$ and $s = n+1$ where $\tau^{n+1}(S_i) = S_i[-2]$. The τ-orbit of S_1 is given by $\tau^s(S_1) = I_s[-s]$. Therefore a shift of all projective indecomposables and injective indecomposables are in the orbit of S_1. Also $\tau^n(S_1) = S_n[-n]$ and $\tau(S_n) =$

$S_1[n-2]$. Therefore $\tau^{n+1}(S_i) = S_i[-2]$ for all $1 \leq i \leq n$. It is therefore clear that the Auslander-Reiten component is isomorphic to $\mathbb{Z}[A_n]$ given as above.

As the shift [2] operates on τ-orbits, we can see that all elements in the component are of bounded length. By 3.7 all indecomposable elements are part of the component. □

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