Yang–Mills theory over surfaces and the Atiyah–Segal theorem

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In this paper we explain how Morse theory for the Yang-Mills functional can be used to prove an analogue, for surface groups, of the Atiyah–Segal theorem. Classically, the Atiyah–Segal theorem relates the representation ring $R(\Gamma)$ of a compact Lie group Γ to the complex K–theory of the classifying space $B\Gamma$. For infinite discrete groups, it is necessary to take into account deformations of representations, and with this in mind we replace the representation ring by Carlsson's deformation K–theory spectrum $K_{\mathrm{def}}(\Gamma)$ (the homotopy-theoretical analogue of $R(\Gamma)$). Our main theorem provides an isomorphism in homotopy $K^*_{\mathrm{def}}(\pi_1\Sigma) \cong K^*(\Sigma)$ for all compact, aspherical surfaces Σ and all $* > 0$. Combining this result with work of Tyler Lawson, we obtain homotopy theoretical information about the stable moduli space of flat unitary connections over surfaces.

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1 Introduction

Associated to any group Γ, one has the (unitary) representation ring $R(\Gamma)$, which consists of "virtual isomorphism classes" of representations. Each representation $\rho\colon \Gamma \to U(n)$ induces a vector bundle E_ρ over the classifying space $B\Gamma$, and this provides a map $R(\Gamma) \xrightarrow{\alpha} K^0(B\Gamma)$. Now assume that Γ is a compact Lie group. By a theorem of Buhstaber and Miscenko [8], $K^0(B\Gamma)$ is isomorphic to the limit of the K–theories of the skeleta $B\Gamma^{(n)}$, and hence has the structure of a complete ring. The classical theorem of Atiyah and Segal [5, 7] states that in this situation the map α becomes an isomorphism after completing $R(\Gamma)$ appropriately.

Extensions and analogues of the Atiyah–Segal theorem have been considered by a number of authors. For infinite discrete groups Γ satisfying appropriate finiteness conditions, Adem [3] studied the relationship between the K–theory of the classifying space $B\Gamma$ and the representation rings of the finite subgroups of Γ. Lück and Oliver considered the case of an infinite discrete group Γ acting properly, i.e. with finite stabilizers, on a space X. They showed that the Γ–equivariant K–theory of X, completed appropriately, agrees with the topological K–theory of the homotopy orbit space $E\Gamma \times_\Gamma X$. When X is a point, properness forces Γ to be finite, and the Lück–Oliver theorem reduces to the Atiyah–Segal theorem. Work of Kasparov–Skandalis [27] and Kasparov–Higson [19] on the Baum-Connes conjecture may be viewed as establishing an isomorphism between an analytical version of the representation ring of an infinite discrete group and the equivariant K–theory of the classifying space for proper actions. This work applies to a wide class of groups, including the surface groups discussed in this paper.

In this article, we will explore a direct homotopical analogue of the Atiyah-Segal theorem for discrete groups Γ admitting finite dimensional classifying spaces. As a motivating example, we consider the simplest case, namely $\Gamma = \mathbb{Z}$. Representations of $\mathbb{Z}$ are simply unitary matrices, and isomorphism classes of representations are conjugacy classes in $U(n)$. By the spectral theorem, these conjugacy classes correspond to points in the symmetric product $\mathrm{Sym}^n(S^1)$, and the natural map $\coprod_n \mathrm{Sym}^n(S^1) \to R(\mathbb{Z})$ is injective. So the discrete representation ring of $\mathbb{Z}$ is quite large, and bears little relation to K–theory of $B\mathbb{Z} = S^1$: every complex vector bundle over S^1 is trivial, and so $K^0(Z)$ is just the integers.

We now observe that in this setting, deformations of representations play an important role. Here by a deformation of a representation $\rho_0 \colon \Gamma \to U(n)$ we simply mean a representation ρ_1 and a continuous path of representations ρ_t connecting ρ_0 and ρ_1. The vector bundle over $B\Gamma$ associated to the representation ρ may be formed using the mixing construction:

$$E_\rho = (E\Gamma \times \mathbb{C}^n)/\Gamma,$$

where Γ acts on $E\Gamma$ by deck transformations and on $\mathbb{C}^n$ via the representation ρ. (The quotient E_ρ is then formed by modding out the diagonal action.) The path ρ_t of representations now induces a family of bundles E_{ρ_t}, which fit together to produce a bundle homotopy between E_{ρ_0} and E_{ρ_1}. Hence the bundle associated to ρ_0 is *isomorphic* to the bundle associated to each of its deformations, and the map associating an element of K–theory to a representation factors through deformation classes.

Returning to the example $\Gamma = \mathbb{Z}$, we observe that since $U(n)$ is connected, the natural map from deformation classes of representations to K–theory of $\mathbb{Z} = S^1$ group completes to an isomorphism. (We remark that for finite groups, this discussion is moot: any deformation of a representation ρ is actually isomorphic to ρ, because the trace of a representation gives a continuous, complete invariant of the isomorphism type, and on representations of a fixed dimension, the trace takes on only finitely many values. Hence when G is finite, deformations are already taken into account by the construction of $R(G)$.)

With this situation understood, one is inclined to look for an analogue of the representation ring which captures deformations of representations, i.e. the topology of representation spaces. The most naive way of taking deformations into account fails rather badly: the monoid of deformation classes $\coprod_n \pi_0 \mathrm{Hom}(\Gamma, U(n))$ admits a well-defined map to $K^0(B\Gamma)$, but (despite the case $\Gamma = \mathbb{Z}$) this map does not usually group-complete to an isomorphism: the representation spaces $\mathrm{Hom}(\Gamma, U(n))$ are compact CW-complexes, so have finitely many components, but there can be infinitely many isomorphism types of n–dimensional bundles over $B\Gamma$. In the case of Riemann surfaces Σ, the spaces $\mathrm{Hom}(\pi_1\Sigma, U(n))$ are always connected (see discussion at the end of Section 2), so the monoid of deformation classes is just $\mathbb{N}$ and its group completion is $\mathbb{Z}$; on the other hand bundles over a Riemann surface are determined by their dimension and first Chern class (and all Chern classes are realized) so $K^0(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$. Note here that $\Sigma = B(\pi_1\Sigma)$ except in the case of the Riemann sphere.

The deformation-theoretical approach is not doomed to failure, though. Let $\mathrm{Rep}(\Gamma)$ denote the topological monoid of unitary representation spaces, and let Gr denote the Grothendieck group functor. Carlsson's deformation K–theory spectrum $K_{\mathrm{def}}(\Gamma)$ [9] is a lifting of the functor $\mathrm{Gr}\,(\pi_0\mathrm{Rep}(\Gamma))$ to the category of spectra, or in fact, **ku**–modules, in the sense that

$$\pi_0 K_{\mathrm{def}}(\Gamma) \cong \mathrm{Gr}\,(\pi_0 \mathrm{Rep}(\Gamma))\,.$$

As we will describe in Section 2, the deformation K–theory may be viewed as the precise homotopy theoretical analogue of the discrete representation ring $R(\Gamma)$. Our main result may now be seen as a correction to the fact that

$$\mathrm{Gr}\,(\pi_0 \mathrm{Rep}(\pi_1\Sigma)) \longrightarrow K^0(\Sigma)$$

fails to be an isomorphism when Σ is a compact, aspherical Riemann surface.

Theorem 5.1 *Let M be a compact, aspherical surface. Then for $* > 0$,*

$$K^*_{\mathrm{def}}(\pi_1(M)) \cong K^*(M).$$

This result will be proven in Section 5. For non-orientable surfaces, there is actually an isomorphism on π_0 as well; this is just a re-interpretation of the results of Ho and Liu [23, 25]. Since the K-theory of a surface is easily computed, Theorem 5.1

gives a complete computation of $K^*_{\text{def}}(\pi_1 M)$ (see Corollary 5.2). We note that when $M = S^1 \times S^1$, Theorem 5.1 follows from T. Lawson's product formula $K_{\text{def}}(\Gamma_1 \times \Gamma_2) \simeq K_{\text{def}}(\Gamma_1) \wedge_{\mathbf{ku}} K_{\text{def}}(\Gamma_2)$ [31] together with his calculation of $K_{\text{def}}(\mathbb{Z})$ as a **ku**–module [30]. As the proof will show, the isomorphism in Theorem 5.1 is functorial for smooth maps between surfaces, and is in particular equivariant with respect to the mapping class group of the surface.

Theorem 5.1 suggests that deformation K–theory is the proper setting in which to study Atiyah–Segal phenomena for groups with finite classifying spaces. In particular, we expect that for any discrete group Γ with a compact classifying space, the deformation K-groups of Γ will agree with $K^*_{\text{def}}(B\Gamma)$ in high degree. We note that the author's excision result for free products [38] and Lawson's product formula [31] indicate that this phenomenon should be stable under both free and direct products of discrete groups.

The failure of Theorem 5.1 in degree zero, and the resulting failure in higher degrees for related groups, reflect an important feature of deformation K-theory. This failure stems from the close ties between this functor and the topology of representation spaces. While K-theory is a stable homotopy invariant of G (i.e. depends only on the stable homotopy type of BG), the representation spaces carry a great deal more information about the group G. Hence deformation K-theory should be viewed as a subtler invariant of G, and its relationship to topological K-theory of BG should be viewed as an important computational tool.

As an application of Theorem 5.1 (and a justification of the preceding paragraph), we obtain homotopy-theoretical information about the stable moduli space of flat unitary connections over a compact, aspherical surface. Ho and Liu [22, 24] have shown that for each n, the moduli space of flat $U(n)$–connections is connected (in fact, their results apply to flat G–connections for any compact, connected Lie group G). In Section 7, we combine our work with T. Lawson's results on the Bott map in deformation K–theory [30] to study these moduli spaces after stabilizing with respect to the rank n. In particular, we prove:

Corollary 1.1 *Let Σ be a compact, aspherical surface. Then the fundamental group of the stable moduli space of flat unitary connections over Σ is isomorphic to $K^1(\Sigma) \cong K^1_{\text{def}}(\pi_1\Sigma)$.*

The proof of Theorem 5.1 relies on Morse theory for the Yang-Mills functional, as devoped by Atiyah and Bott [6], Daskalopoulos [10], and Råde [37]; the key analytical input comes from Uhlenbeck's compactness theorem [43, 44]. In the non-orientable case, we rely on recent work of Ho and Liu [22, 25] regarding representation spaces of non-orientable surface groups and Yang-Mills theory over non-orientable surfaces.

The link between deformation K–theory and Yang-Mills theory is provided by the well-known fact that representations of the fundamental group induce flat connections, which form a critical set for the Yang-Mills functional. To motivate our arguments, we give a proof along these lines of the well-known equivalence

$$(G^{\mathrm{Ad}})_{hG} \simeq LBG = \mathrm{Map}(S^1, BG),$$

where G is a compact, connected Lie group and $(G^{\mathrm{Ad}})_{hG} = EG \times_G (G^{\mathrm{Ad}})$ is the homotopy orbit space of the adjoint action of G on itself. (This result is well-known for any group G, but the only reference of which I am aware is the elegant proof given by Gruher in her thesis [16]).

To begin, note that $(G^{\mathrm{Ad}})_{hG} = \mathrm{Hom}(\mathbb{Z}, G)_{hG}$. Connections A over the circle are always flat, and hence give rise to holonomy representations of $\pi_1 S^1 = \mathbb{Z}$:

$$A \mapsto (\rho_A \colon \mathbb{Z} \to G).$$

After modding out based gauge transformations, one obtains a homeomorphism (Proposition 3.9)

$$\mathcal{A}(S^1 \times G)/\mathrm{Map}_*(S^1, G) \cong \mathrm{Hom}(\mathbb{Z}, G),$$

and since the based gauge group acts freely, a standard fact about homotopy orbit spaces yields a homotopy equivalence

$$\left(\mathcal{A}(S^1 \times G)/\mathrm{Map}_*(S^1, G)\right)_{hG} \simeq \left(\mathcal{A}(S^1 \times G)\right)_{h\mathrm{Map}(S^1,G)}.$$

But connections form a contractible (affine) space, so the right hand side is the classifying space of the (full) gauge group. Atiyah and Bott have shown that the space $\mathrm{Map}(S^1, BG) = LBG$ is a model for this classifying space, so we conclude that $G^{\mathrm{Ad}}_{hG} \simeq LBG$, as desired.

Our interest in this argument lies in the fact that deformation K–theory (of $\mathbb{Z}$, say) is built from the homotopy orbit spaces

$$EU(n) \times_{U(n)} \mathrm{Hom}(\mathbb{Z}, U(n)) = EU(n) \times_{U(n)} (U(n)^{\mathrm{Ad}})$$

(see Corollary 2.4), and the homotopy groups of $LBU(n) = \mathrm{Map}(S^1, BU(n))$ are precisely the complex K-groups of $S^1 = B\mathbb{Z}$ (in dimensions $0 < k < 2n$). Thus the statement $U(n)^{\mathrm{Ad}}_{hU(n)} \simeq LBU(n)$ may be interpreted as a sort of Atiyah–Segal theorem for the group $\mathbb{Z}$.

When $\mathbb{Z}$ is replaced by the fundamental group of a two-dimensional surface, one can try to mimic this argument. Not all connections are flat in this case, but flat connections do form a critical set for the Yang-Mills functional. Hence one may hope to relate this critical set to the space $\mathcal{A}$ of all connections via the Morse stratification for the Yang-Mills functional, i.e. the stratification of $\mathcal{A}$ by stable manifolds. Råde's work [37] provides deformation retractions from the strata to their critical subsets, and in particular allows us to pass from the critical set of flat connections to its stable manifold. By results of Daskalopoulos [10], the Morse stratification agrees with the Harder-Narasimhan stratification from complex geometry (as was conjectured by Atiyah and Bott) and in particular the stable manifold for the space of flat connections is the space of semi-stable holomorphic structures. We give precise bounds on the codimensions of the Harder-Narasimhan strata, and our main results then follow from an application of Smale's infinite dimensional transversality theorem.

This paper is organized as follows. In Section 2, we introduce deformation K–theory and explain how the McDuff-Segal group completion theorem provides us with a convenient model for the zeroth space of the Ω–spectrum $K_{\mathrm{def}}(\pi_1 M)$ when M is a compact, aspherical surface. The precise passage from representation varieties to spaces of flat connections is discussed in Section 3. In Section 4 we discuss the Harder-Narasimhan stratification on the space of holomorphic structures and the results of Daskalopoulos and Råde which link it to Morse theory for the Yang-Mills functional. This leads to the required connectivity estimates for the space of flat connections. The main theorem is proven in Section 5, using the results of the previous three sections. In Section 6, we explain how the failure of Theorem 5.1 in degree zero leads to a failure of excision (in the sense of Ramras [38]) for connected sum decompositions of Riemann surfaces. In Section 7 we discuss T. Lawson's work on the Bott map and its implications for the homotopy groups of the stable moduli space of flat connections. Finally, we have included an appendix discussing the holonomy representation associated to a flat connection.

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2 Deformation K-theory

In this section, we introduce Carlsson's notion of deformation K–theory [9] and discuss its basic properties. Deformation K–theory is a contravariant functor from discrete groups to spectra, and is meant to capture homotopy-theoretical information about the representation spaces of the group in question. This spectrum may be constructed as the K–theory spectrum associated to a topological permutative category of representations. This approach originated in T. Lawson's thesis [29], and is explained in detail in Ramras [38, Section 2].

Here we will take a more naive, but essentially equivalent, approach. The present viewpoint makes clear the precise analogy between deformation K–theory and the classical representation ring. Its relation to the categorical construction will be described at the end of this section.

For the rest of this section, we fix a discrete group Γ. The construction of the (unitary) representation ring $R(\Gamma)$ may be broken down into several steps: one begins with the *sets* $\mathrm{Hom}(\Gamma, U(n))$, which form a monoid under direct sum; next, one takes isomorphism classes by modding out the actions of the groups $U(n)$ on the sets $\mathrm{Hom}(\Gamma, U(n))$. The monoid structure descends to the quotient, and in fact tensor product now induces the structure of a semi-ring. Finally we form the Grothendieck ring $R(\Gamma)$ of this semi-ring of isomorphism classes. Deformation K–theory (additively, at least) may be constructed simply by replacing each step in this construction by its homotopy theoretical analogue. To be precise, we begin with the *space*

$$\mathrm{Rep}(\Gamma) = \coprod_{n=0}^{\infty} \mathrm{Hom}(\Gamma, U(n)),$$

which is a topological monoid under block sum. Rather than passing to $U(n)$–orbit spaces, we now form the homotopy quotient

$$\mathrm{Rep}(\Gamma)_{hU} = \coprod_{n=0}^{\infty} EU(n) \times_{U(n)} \mathrm{Hom}(\Gamma, U(n)).$$

Block sum of unitary matrices induces maps

$$EU(n) \times EU(m) \longrightarrow EU(n+m)$$

on universal bundles, and together with the monoid structure on $\mathrm{Rep}(\Gamma)$ these give $\mathrm{Rep}(\Gamma)_{hU}$ the structure of a topological monoid (for associativity to hold, we must use a functorial model for $EU(n)$, as opposed to the infinite Stiefel manifolds; see Remark 2.2). Finally, we apply the homotopical version of the Grothendieck construction to this topological monoid and call the resulting space (Γ), the (unitary) deformation K–theory of Γ.

Definition 2.1 *The deformation K–theory of a discrete group Γ is the space*

$$K_{\mathrm{def}}(\Gamma) := \Omega B\left(\mathrm{Rep}(\Gamma)_{hU}\right),$$

whose homotopy groups we denote by $K^*_{\mathrm{def}}(\Gamma) = \pi_* K_{\mathrm{def}}(\Gamma)$.

Here B denotes the simplicial bar construction, namely the classifying space of the category with one object and with $\mathrm{Rep}(\Gamma)_{hU}$ as its space of morphisms. Note that $K_{\mathrm{def}}(\Gamma)$ is a contravariant functor from discrete groups to spaces.

It was shown in [38, Section 2] that the space $K_{\mathrm{def}}(\Gamma)$ as defined here is weakly equivalent to the zeroth space of the connective Ω–spectrum associated to T. Lawson's topological permutative category of unitary representations; in particular the homotopy groups of this spectrum agree with the homotopy groups of the space $K_{\mathrm{def}}(\Gamma)$.

We note that constructing a ring structure in deformation K–theory requires a subtler approach, and this has been carried out by T. Lawson [31].

The first two homotopy groups of $K_{\mathrm{def}}(\Gamma)$ have rather direct meanings: $K^0_{\mathrm{def}}(\Gamma)$ is the Grothendieck group of virtual connected components of representations, i.e. $\mathrm{Gr}\left(\pi_0 \mathrm{Rep}(\Gamma)\right)$ [38, Section 2]. It follows from work of Lawson [30] that the group $K^1_{\mathrm{def}}(\Gamma)$ is a stable version of the group $\pi_1 \mathrm{Hom}(\Gamma, U(n))/U(n)$; a precise discussion will be given in Section 7.

Remark 2.2 *In [38], the simplicial model for $EU(n)$ is used; in this paper we will need to use universal bundles for Sobolev gauge groups, where the simplicial model may not give an actual universal bundle. Hence it is more convenient to use Milnor's model [33], which is functorial and applies to all topological groups. There is a natural zig-zag of weak equivalences connecting these two versions of the classifying space, and this gives a zig-zag connecting the simplicial version of deformation K–theory to the Milnor version.*

2.1 Group completion in deformation K-theory

The starting point for our work on surface groups is an analysis of the consequences of McDuff-Segal Group Completion theorem [32] for deformation K–theory, as carried out in [38]. Here we recall that result and explain its consequences for surface groups. Given a topological monoid M and an element $m \in M$, we say that M is *stably group-like* with respect to m if the submonoid of $\pi_0 M$ generated by the component containing m is cofinal (in $\pi_0 M$). Explicitly, M is stably-group-like with respect to m if for every $x \in M$, there exists an element $x^{-1} \in M$ such that $x \cdot x^{-1}$ is connected by a path to m^n for some $n \in \mathbb{N}$. We then have:

Theorem 2.3 ([38]) *Let Γ be a finitely generated discrete group such that* $\mathrm{Rep}(\Gamma)$ *is stably group-like with respect to* $\rho \in \mathrm{Hom}(\Gamma, U(k))$. *Then there is a weak equivalence*

$$K_{\mathrm{def}}(\Gamma) \simeq \text{telescope}\left(\mathrm{Rep}(\Gamma)_{hU} \xrightarrow{\oplus\rho} \mathrm{Rep}(\Gamma)_{hU} \xrightarrow{\oplus\rho} \cdots\right),$$

where $\oplus\rho$ *denotes block sum with the point* $[*_k, \rho] \in EU(k) \times_{U(k)} \mathrm{Hom}(\Gamma, U(k))$.

Here, and throughout this article, telescope refers to the mapping telescope of a sequence of maps. The novel aspect of this result is that, unlike elsewhere in algebraic K–theory, Quillen's +-construction does not appear. This is due to the fact that the fundamental group on the right-hand side is already abelian, a fact which (in general) depends on rather special properties of the unitary groups.

In low dimensions, this result has the following manifestation:

Corollary 2.4 *Let M be either the circle or an aspherical compact surface. Then there is a weak equivalence between* $K_{\mathrm{def}}(\pi_1(M))$ *and the space*

$$\underset{\overrightarrow{\oplus 1}}{\text{telescope}}(\mathrm{Rep}(\pi_1 M)_{hU}) := \text{telescope}\left(\mathrm{Rep}(\pi_1 M)_{hU} \xrightarrow{\oplus 1} \mathrm{Rep}(\pi_1 M)_{hU} \xrightarrow{\oplus 1} \cdots\right)$$

where $\oplus 1$ *denotes the map induced by block sum with the identity matrix* $1 \in U(1)$.

There are at least two approaches to the problem of showing that $\mathrm{Rep}(\pi_1 M)$ is stably group-like with respect to $1 \in \mathrm{Hom}(\pi_1 M, U(1))$. In Corollaries 4.11 and 4.12 we use Yang-Mills theory to show that $\mathrm{Rep}(\pi_1 M)$ is stably group-like for any compact, aspherical surface M. (In the orientable case, this amounts to showing that the representation spaces are all connected, which is a well-known folk theorem.) This argument is quite close to Ho and Liu's proof of connectivity for the moduli space of flat connections [22, Theorem 5.4]. For most surfaces, other work of Ho and Liu [25] gives an alternative method, depending on Alekseev, Malkin, and Meinrenken's theory of quasi-Hamiltonian moment maps [4]. A version of that argument, adapted to the present situation, appears in the author's thesis [39, Chapter 6].

3 Representations and flat connections

Let M denote an n–dimensional, compact, connected manifold, with a fixed basepoint $m_0 \in M$. Let G be a compact Lie group, and $P \xrightarrow{\pi} M$ be a smooth principal G–bundle, with a fixed basepoint $p_0 \in \pi^{-1}(m_0) \subset P$. Our principal bundles will always have a *right* action of the structure group G.

In this section we explain how to pass from G–representation spaces of $\pi_1(M)$ to spaces of flat connections on principal G–bundles over M, which form critical sets for the Yang-Mills functional. The main result of this section is the following proposition, which we state informally for the moment.

Proposition 3.9 *For any n–manifold M and any compact, connected, real algebraic Lie group G, holonomy induces a G–equivariant homeomorphism*

$$\coprod_{[P_i]} \mathcal{A}_{\mathrm{flat}}(P_i)/\mathcal{G}_0(P_i) \xrightarrow{\overline{\mathcal{H}}} \mathrm{Hom}(\pi_1(M), G),$$

where the disjoint union is taken over some set of representatives for the (unbased) isomorphism classes of principal G–bundles over M.

In order to give a precise statement and proof of Proposition 3.9, we need to introduce the relevant Sobolev spaces of connections and gauge transformations. Our notation and discussion follow Atiyah–Bott [6, Section 14], and another excellent reference is the appendix to Wehrheim [44].

We use the notation L^p_k to denote functions with k weak (distributional) derivatives, each in the Sobolev space L^p. We will record the necessary assumptions on k and p as they arise. The reader interested only in the applications to deformation K–theory may safely ignore these issues, noting only that all the results of this section hold in the Hilbert space L^2_k for large enough k. When $n = 2$, our main case of interest, we just need $k \geqslant 2$.

Definition 3.1 *Let $k \geqslant 1$ be an integer, and let $1 \leq p < \infty$. We denote the space of all connections on the bundle P of Sobolev class L^p_k by $\mathcal{A}^{k,p}(P)$. This is an affine space, modeled on the Banach space of L^p_k sections of the vector bundle $T^*M \otimes \mathrm{ad}\, P$ (here $\mathrm{ad}\, P = P \times_G \mathfrak{g}$, and $\mathfrak{g}$ is the Lie algebra of G equipped with the adjoint action). Hence $\mathcal{A}^{k,p}(P)$ acquires a canonical topology, making it homeomorphic to the Banach space on which it is modeled. Flat L^p_k connections are defined to be those with zero curvature. The subspace of flat connections is denoted by $\mathcal{A}^{k,p}_{\mathrm{flat}}(P)$.*

We let $\mathcal{G}^{k+1,p}(P)$ denote the gauge group of all bundle automorphisms of P of class L^p_{k+1}, and (when $(k+1)p > n$) we let $\mathcal{G}^{k+1,p}_0(P)$ denote the subgroup of based automorphisms (those which are the identity on the fiber over $m_0 \in M$). These gauge groups are Banach Lie groups, and act smoothly on $\mathcal{A}^{k,p}(P)$. We will always use the left action, meaning that we let gauge transformations act on connections by pushforward. We denote the group of all continuous gauge transformations by $\mathcal{G}(P)$. Note that so long as $(k+1)p > n$, the Sobolev Embedding Theorem gives a continuous inclusion $\mathcal{G}^{k+1,p}(P) \hookrightarrow \mathcal{G}(P)$, and hence in this range $\mathcal{G}^{k+1,p}_0(P)$ is well-defined. We denote the smooth versions of these objects by $(-)^\infty(P)$.

The following lemma is well-known.

Lemma 3.2 *Assume $(k+1)p > n$. Then the inclusion $\mathcal{G}^{k+1,p}(P) \hookrightarrow \mathcal{G}(P)$ is a weak equivalence.*

Proof Gauge transformations are simply sections of the adjoint bundle $P \times_G \mathrm{Ad}(G)$ (see [6, Section 2]). Hence this result follows from general approximation results for sections of smooth fiber bundles. □

Note the continuous inclusion $\mathcal{G}^{k+1,p}(P) \hookrightarrow \mathcal{G}(P)$ implies that there is a well-defined, continuous homomorphism $r\colon \mathcal{G}^{k+1,p}(P) \to G$ given by restricting a gauge transformation to the fiber over the basepoint $m_0 \in M$. To be precise, $r(\phi)$ is defined by $p_0 \cdot r(\phi) = \phi(p_0)$, and hence depends on our choice of basepoint $p_0 \in P$.

Lemma 3.3 *If G is connected, then the restriction map $r\colon \mathcal{G}^{k+1,p}(P) \longrightarrow G$ is surjective. If we assume further that $(k+1)p > n$, then r induces a homeomorphism $\mathcal{G}^{k+1,p}(P)/\mathcal{G}^{k+1,p}_0(P) \cong G$. The same statements hold for the smooth gauge groups.*

Proof Thinking of gauge transformations as sections of the adjoint bundle, we may deform the identity map $P \to P$ over a neighborhood of m_0 so that it takes any desired value at p_0 (here we use connectivity of G). This proves surjectivity.

By a similar argument, we may construct continuous local sections $s\colon U \to \mathcal{G}^\infty(P)$ of the map r, where $U \subset G$ is any chart. If $\pi\colon \mathcal{G}^\infty(P) \to \mathcal{G}(P)^\infty/\mathcal{G}_0^\infty(P)$ is the quotient map, then the maps $\pi \circ s$ are inverse to $\overline{r}$ on U. Hence $\overline{r}^{-1}$ is continuous. The same argument applies to $\mathcal{G}^{k+1,p}(P)$, although we must require $(k+1)p > n$ so that r is well-defined and continuous. □

I do not know whether Lemma 3.3 holds for non-connected groups; certainly the proof shows that the image of the restriction map is always a union of components of G.

Flat connections are related to representations of $\pi_1 M$ via the holonomy map. Our next goal is to analyze this map carefully in the current context of Sobolev connections. The holonomy of a smooth connection is defined via parallel transport: given a smooth loop γ based at $m_0 \in M$, there is a unique A–horizontal lift $\widetilde{\gamma}$ of γ with $\widetilde{\gamma}(0) = p_0$, and the holonomy representation $\mathcal{H}(A) = \rho_A$ is then defined by the equation $\widetilde{\gamma}(1)\cdot\rho_A([\gamma]) = p_0$. (Since flat connections are locally trivial, a standard compactness argument shows that this definition depends only on the homotopy class $[\gamma]$ of γ.) It is important to note here that the holonomy map depends on the chosen the basepoint $p_0 \in P$. For further details on holonomy, we refer the reader to the Appendix.

Lemma 3.4 *The holonomy map $\mathcal{A}^{k,p}_{\text{flat}}(P) \to \text{Hom}(\pi_1 M, G)$ is continuous if $k \geqslant 2$ and $(k-1)p > n$.*

Proof The assumptions on k and p guarantee a continuous embedding $L^p_k(M) \hookrightarrow C^1(M)$. Hence if $A_i \in \mathcal{A}^{k,p}_{\text{flat}}(P)$ is a sequence of connections converging (in $\mathcal{A}^{k,p}_{\text{flat}}(P)$) to A, then $A_i \to A$ in C^1 as well. We must show that for any such sequence, the holonomies of the A_i converge to the holonomy of A.

It suffices to check that for each loop γ the holonomies around γ converge. These holonomies are defined (continuously) in terms of the integral curves of the vector fields $V(A_i)$ on γ^*P arising from the connections A_i. Since these vector fields converge in the C^1 norm, we may assume that the sequence $||V(A_i) - V(A)||_{C^1}$ is decreasing and less than 1. By interpolating linearly between the $V(A_i)$, we obtain a vector field on $\gamma^*P \times I$ which at time t_i is just $V(A_i)$, and at time 0 is $V(A)$. This is clearly a Lipschitz vector field and hence its integral curves vary continuously in the initial point (Lang [28, Chapter IV]) completing the proof. □

Remark 3.5 *With a bit more care, one can prove Lemma 3.4 under the weaker assumptions $k \geqslant 1$ and $kp > n$. The basic point is that these assumptions give an embedding $L^p_k(M) \hookrightarrow C^0(M)$, and by compactness $C^0(M) \hookrightarrow L^1(M)$ (and similarly* *after restricting to a smooth curve in M). Working in local coordinates, one can deduce continuity of the holonomy map from the fact that limits commute with integrals in $L^1([0,1])$.*

Lemma 3.6 *Assume $p > n/2$ (and if $n = 2$, assume $p \geqslant 4/3$). If G is connected, then each $\mathcal{G}_0^{k+1,p}(P)$–orbit in $\mathcal{A}_{\text{flat}}^{k,p}(P)$ contains a unique $\mathcal{G}_0^{\infty}(P)$–orbit of smooth connections.*

Proof By Wehrheim [44, Theorem 9.4], the assumptions on k and p guarantee that each $\mathcal{G}^{k+1,p}(n)$ orbit in $\mathcal{A}_{\text{flat}}^{k,p}(n)$ contains a smooth connection. Now, say $\phi \cdot A$ is smooth for some $\phi \in \mathcal{G}^{k+1,p}(P)$. By Lemma 3.3, there exists a smooth gauge transformation ψ such that $r(\psi) = r(\phi)^{-1}$. The composition $\psi \circ \phi$ is clearly based, and since ψ is smooth we know that $(\psi \circ \phi) \cdot A$ is still smooth. This proves existence. For uniqueness, say $\phi \cdot A$ and $\psi \cdot A$ are both smooth, where $\phi, \psi \in \mathcal{G}_0^{k+1,p}(P)$. Then $\phi\psi^{-1}$ is smooth by [6, Lemma 14.9], so these connections lie in the same $\mathcal{G}_0^{\infty}$–orbit. □

The following elementary lemma provides some of the compactness we will need.

Lemma 3.7 *If G is a compact, real algebraic group then only finitely many isomorphism classes of principal G–bundles over M admit flat connections.*

Proof As described in the Appendix, any bundle E admitting a flat connection A is isomorphic to the bundle induced by holonomy representation $\rho_A \colon \pi_1 M \to G$. If two bundles E_0 and E_1 arise from representations ρ_0 and ρ_1 in the same path component of $\mathrm{Hom}(\pi_1 M, G)$, then choose a path ρ_t of representations connecting ρ_0 to ρ_1. The bundle

$$E = (\widetilde{M} \times [0,1] \times G) \Big/ (\widetilde{m}, t, g) \sim (\widetilde{m} \cdot \gamma, t, \rho_t(\gamma)^{-1} g)$$

is a principal G–bundle over $M \times [0,1]$ and provides a bundle homotopy between E_0 and E_1; by the Bundle Homotopy Theorem we conclude $E_0 \cong E_1$. Hence the number of isomorphism classes admitting flat connections is at most the number of path components of $\mathrm{Hom}(\pi_1 M, G)$.

Now we use the assumption that G is algebraic. Since $\pi_1 M$ is finitely generated (by k elements, say), $\mathrm{Hom}(\pi_1 M, G)$ is the subvariety of G^k cut out by the relations in $\pi_1 M$. So this space is a real algebraic variety, hence triangulable (for a proof, see Hironaka [20]). Since compact CW complexes have finitely many path components, the proof is complete. □

Remark 3.8 *We note that the previous lemma holds even if G is not compact, by a result of Whitney [45] regarding components of varieties.*

We can now prove the result which connects representation theory with Yang-Mills theory.

Proposition 3.9 *Assume $p > n/2$ (and if $n = 2$, assume $p > 4/3$), $k \geqslant 1$, and $kp > n$. Then for any n–manifold M and any compact, connected, real algebraic Lie group G, the holonomy map induces a G–equivariant homeomorphism*

$$\coprod_{[P_i]} \mathcal{A}_{\text{flat}}^{k,p}(P_i)/\mathcal{G}_0^{k+1,p}(P_i) \xrightarrow{\overline{\mathcal{H}}} \text{Hom}(\pi_1(M), G),$$

where the disjoint union is taken over some set of representatives for the (unbased) isomorphism classes of principal G–bundles over M. (Note that to define $\overline{\mathcal{H}}$ we choose, arbitrarily, a base point in each representative bundle P_i.)

The G–action on the left is induced by the actions of $\mathcal{G}^{k+1,p}(P_i)$ together with the homeomorphisms $\mathcal{G}^{k+1,p}(P_i)/\mathcal{G}_0^{k+1,p}(P_i) \cong G$, which again depends on the chosen basepoints in the bundles P_i.

Proof The assumptions on k and p allow us to employ all previous results in this section (note Remark 3.5). It is well-known that the holonomy map

$$H\colon \coprod_{[P_i]} \mathcal{A}_{\text{flat}}^{\infty}(P_i) \longrightarrow \text{Hom}(\pi_1(M), G)$$

is invariant under the action of the based gauge group and induces an equivariant bijection

$$\bar{H}\colon \coprod_{[P_i]} \mathcal{A}_{\text{flat}}^{\infty}(P_i)/\mathcal{G}_0^{\infty}(P_i) \longrightarrow \text{Hom}(\pi_1(M), G).$$

For completeness we have included a proof of this result in the Appendix. By Lemma 3.6, the left hand side is unchanged (set-theoretically) if we replace $\mathcal{A}_{\text{flat}}^{\infty}$ and $\mathcal{G}_0^{\infty}$ by $\mathcal{A}_{\text{flat}}^{k,p}$ and $\mathcal{G}_0^{k+1,p}$, and hence Lemma 3.4 tells us that we have a continuous equivariant bijection

$$\bar{\mathcal{H}}\colon \coprod_{[P_i]} \mathcal{A}_{\text{flat}}^{k,p}(P_i)/\mathcal{G}_0^{k+1,p}(P_i) \longrightarrow \text{Hom}(\pi_1(M), G).$$

We will show that for each P, $\mathcal{A}_{\text{flat}}^{k,p}(P)/\mathcal{G}_0^{k+1,p}(P)$ is sequentially compact. Since, by Lemma 3.7, only finitely many isomorphism types of principal G–bundle admit flat connections, this implies that

$$\coprod_{[P_i]} \mathcal{A}_{\text{flat}}^{k,p}(P_i)/\mathcal{G}_0^{k+1,p}(P_i)$$

is sequentially compact. A continuous bijection from a sequentially compact space to a Hausdorff space is a homeomorphism, so this will complete the proof.

The Strong Uhlenbeck Compactness Theorem [44] (see also Daskalopoulos [10, Proposition 4.1]) states that the space $\mathcal{A}^{k,p}_{\text{flat}}(P)/\mathcal{G}^{k+1,p}(P)$ is sequentially compact. Now, given a sequence $\{A_i\}$ in $\mathcal{A}^{k,p}_{\text{flat}}(P)$, there exists a sub-sequence $\{A_{i_j}\}$ and a sequence $\phi_j \in \mathcal{G}^{k+1,p}(n)$ such that $\phi_j \cdot A_{i_j}$ converges in $\mathcal{A}^{k,p}$ to a flat connection A. Let $g_j = r(\phi_j)$. Since G is compact, passing to a sub-sequence if necessary we may assume that the g_j converge to an element $g \in G$. The proof of Lemma 3.3 shows that we may choose a convergent sequence $\psi_j \in \mathcal{G}^{k+1,p}(P)$ such that $r(\psi_j) = g_j^{-1}$; we let $\psi = \lim \psi_j$, so $r(\psi) = g^{-1}$. Now continuity of the action implies that the sequence $(\psi_j \circ \phi_j) \cdot A_{i_j}$ converges to $\psi \cdot A$. Since $\psi_j \circ \phi_j \in \mathcal{G}_0^{k+1,p}(P)$, this completes the proof. □

Remark 3.10 *It is worth noting that point-set considerations alone show that sequential compactness of the quotient space $\mathcal{A}^{k,p}_{\text{flat}}(P)/\mathcal{G}_0^{k+1,p}(P)$ suffices to prove its compactness: specifically, $\mathcal{A}^{k,p}_{\text{flat}}(P)$ is second countable, since it is a subspace of a separable Banach space. The quotient map of a group action is open, so $\mathcal{A}^{k,p}_{\text{flat}}(P)/\mathcal{G}_0^{k+1,p}(P)$ is second countable as well. Now, any second countably, sequentially compact space is compact. (The necessary point-set topology can be found in Wilansky [46, 5.3.2, 7.3.1, 5.4.1].)*

More interesting is that Proposition 3.9 implies that the based gauge orbits in $\mathcal{A}^{k,p}_{\text{flat}}(P)$ are closed (the quotient embeds in $\text{Hom}(\pi_1 M, G)$*). Since G is compact, one also concludes that the full gauge orbits are closed.*

4 Morse theory for the Yang-Mills Functional and the Harder-Narasimhan stratification

In the previous section, we explained how to pass from spaces of representations to spaces of flat connections. We now focus on the case where M is a compact Riemann surface and $G = U(n)$. As explained in the introduction, we wish to compare the space of flat connections on a $U(n)$–bundle P over M to the *contractible* space of all connections on P, and in particular we want to understand what happens as the rank of P tends to infinity. Atiyah and Bott made such a comparison (for a fixed bundle P) using computations in equivariant cohomology. We will instead work directly with homotopy groups, by employing Smale's infinite dimensional transversality theorem.

A (co)homological approach could be used in the orientable case, but there are some technical difficulties (related to equivariant Thom isomorphisms) in extending such an argument to the non-orientable case. These issues are currently being studied in joint work with Ho and Liu.

The main result of this section is a connectivity estimate for the space of flat connections:

Proposition 4.9 *Let $M = M^g$ denote a compact Riemann surface of genus g, and let $n > 1$ be an integer. Then the space $\mathcal{A}^k_{\text{flat}}(n)$ of flat connections on a trivial rank n bundle over M is $2g(n-1)$–connected, and if Σ is a non-orientable surface with double cover M^g, then the space of flat connections on any principal $U(n)$–bundle over Σ is $(g(n-1)-1)$–connected.*

We will work in the Hilbert space of L^2_k connections, and we will assume $k \geqslant 2$ so that the results of Section 3 apply. We will now suppress $p = 2$ from the notation, writing simply $\mathcal{A}^k$, $\mathcal{G}^k$, and so on. Over a Riemann surface, any principal $U(n)$–bundle admitting a flat connection is trivial (see Corollary 4.11), and hence we restrict our attention to the case $P = M \times U(n)$ and use the notation $\mathcal{A}^k(n) = \mathcal{A}^k(M \times U(n))$, etc.

For any smooth principal $U(n)$–bundle $P \to M$, the Yang-Mills functional

$$L\colon\ \mathcal{A}^k(P) \to \mathbb{R}$$

is defined by the formula

$$L(A) = \int_M ||F(A)||^2 d\text{vol}$$

where $F(A)$ denotes the curvature form of the connection A and the volume of M is normalized to be 1. The space $\mathcal{A}^k_{\text{flat}}(n)$ of flat connections forms a critical set for the L [6, Proposition 4.6], and so one hopes to employ Morse-theoretic ideas to compare the topology of this critical set to the topology of $\mathcal{A}^k(P)$. In particular, the gradient flow of L should allow one to define stable manifolds associated to critical sets of M, which should deformation retract to those critical subsets. The necessary analytical work has been done by Daskalopoulos [10] and Råde [37], and furthermore Daskalopoulos has explicitly identified the Morse stratification of $\mathcal{A}^k(P)$ (proving a conjecture of Atiyah and Bott). We now explain this situation.

Over a Riemann surface, there is a correspondence between connections on a principal $U(n)$–bundle P and holomorphic structures on the C^∞ vector bundle $E = P \times_{U(n)} \mathbb{C}^n$. The set $\mathcal{C}(E)$ of holomorphic structures may be viewed as an affine space, modeled on the vector space $\Omega^{0,1}(M; \operatorname{End} E)$ of endomorphism-valued $(0,1)$–forms. Since this is the space of (smooth) sections of a vector bundle on M, we may define Sobolev spaces $\mathcal{C}^k(E) = \mathcal{C}^{k,2}(E)$ of holomorphic structures simply by taking L^2_k–sections of this bundle. If we fix a Hermitian metric on E, then to each holomorphic structure there corresponds a unique compatible (metric) connection [15, p. 73]. When M is a Riemann surface, this induces an isomorphism of affine spaces, which extends to an isomorphism $\mathcal{A}^k(P) \cong \mathcal{C}^k(P \times_{U(n)} \mathbb{C}^n)$. For further details, see [6, Sections 5, 7] or [10, Section 2].

There is a natural algebraic stratification of $\mathcal{C}^k(E)$ called the Harder-Narasimhan stratification, which turns out to agree with the Morse stratification of $\mathcal{A}^k(P)$. We now describe this stratification in the case $E = M \times \mathbb{C}^n$.

Definition 4.1 *Let E be a holomorphic bundle over M. Let* $\deg(E)$ *denote its first Chern number and let* $\mathrm{rk}(E)$ *denote its dimension. We call E semi-stable if for every proper holomorphic sub-bundle $E' \subset E$, one has*

$$\frac{\deg(E')}{\mathrm{rk}(E')} \leqslant \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Replacing the $\leqslant$ by $<$ in this definition, one has the definition of a stable *bundle.*

Given a (smooth) holomorphic structure $\mathcal{E}$ on the bundle $M \times \mathbb{C}^n$, there is a unique filtration (the Harder-Narasimhan filtration [17])

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \mathcal{E}_r = \mathcal{E}$$

of $\mathcal{E}$ by holomorphic sub-bundles with the property that each quotient $\mathcal{D}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable ($i = 1, \ldots, r$) and $\mu(\mathcal{D}_1) > \mu(\mathcal{D}_2) > \cdots > \mu(\mathcal{D}_r)$, where $\mu(\mathcal{D}_i) = \frac{\deg(\mathcal{D}_i)}{\mathrm{rank}(\mathcal{D}_i)}$, and $\deg(\mathcal{D}_i)$ is the first Chern number of the vector bundle $\mathcal{D}_i$. Letting $n_i = \mathrm{rank}(D_i)$ and $k_i = \deg(D_i)$, we call the sequence

$$\mu = ((n_1, k_1), \ldots, (n_r, k_r))$$

the *type* of $\mathcal{E}$. Note that since ranks and degrees add in exact sequences, we have $\sum_i n_i = n$ and $\sum_i k_i = 0$. By the results of [6, Section 14], each orbit of the *complex* gauge group on $\mathcal{C}^k(E)$ contains a unique isomorphism type of smooth holomorphic structure, so we may define $\mathcal{C}^k_\mu = \mathcal{C}^k_\mu(n) \subset \mathcal{C}^k(n)$ to be the subspace of all holomorphic structures gauge-equivalent to a smooth structure of type μ, and the $\mathcal{C}^k_\mu$ partition $\mathcal{C}^k(n)$. Note that the semi-stable stratum corresponds to $\mu = ((n, 0))$,

It is a basic fact that every flat connection on E corresponds to a semi-stable bundle: the Narasimhan-Seshardri Theorem [6, (8.1)] says that irreducible representations induce *stable* bundles. By Proposition 3.9, every flat connection comes from some unitary representation, which is a sum of irreducible representations, and hence the holomorphic bundle associated to any representation, i.e. any flat connection, is a sum of stable bundles. Finally, an extension of stable bundles of the same degree is always semi-stable.

We can now state the result we will need.

Theorem 4.2 (Daskalopoulos, Rade) *Let M be a compact Riemann surface. Then the gradient flow of the Yang-Mills functional is well-defined for all positive time, and defines continuous deformation retractions from the Harder-Narasimhan strata $\mathcal{C}^k_\mu$ to their critical subsets. Moreover, these strata are locally closed submanifolds of $\mathcal{C}^k(n)$ of complex codimension*

$$c(\mu) = \left(\sum_{i>j} n_i k_j - n_j k_i\right) + (g-1)\left(\sum_{i>j} n_i n_j\right).$$

In particular, there is a continuous deformation retraction (defined by the gradient flow of L) from the space $\mathcal{C}^k_{ss}(n)$ of all semi-stable L^2_k holomorphic structures on $M \times \mathbb{C}^n$ to the subspace $\mathcal{A}^k_{\text{flat}}(n)$ of flat (unitary) connections.

Remark 4.3 *This result holds for any C^∞ vector bundle. Daskalopoulos proved convergence of the Yang-Mills flow modulo gauge transformations, and established continuity in the limit on (the gauge quotient of) each Harder-Narasimhan stratum (which he proved to be submanifolds). Råde later proved the full convergence result stated above. We will discuss the analogue of this situation in the non-orientable case in the proof of Proposition 4.9. We note that one could ask for a result slightly stronger than Theorem 4.2: since the gradient flow of L converges at time $+\infty$ to give a continuous retraction from each $\mathcal{C}^k_\mu$ to its critical set, this stratum is certainly a disjoint union of Morse strata. However, I do not know in general whether $\mathcal{C}^k_\mu$ is connected. Hence the Morse stratification may be finer than the Harder-Narasimhan stratification.*

The following definition will be useful.

Definition 4.4 *Consider a sequence of pairs of integers $((n_1, k_1), \ldots, (n_r, k_r))$. We call such a sequence* admissible *of total rank n (and total Chern class 0) if $n_i > 0$ for each i, $\sum n_i = n$, $\sum_i k_i = 0$, and $\frac{k_1}{n_1} > \cdots > \frac{k_r}{n_r}$. Hence admissible sequences of total rank n and total Chern class 0 are precisely those describing Harder-Narasimhan strata in $\mathcal{C}(n)$.*

We denote the collection of all admissible sequences of total rank n and total Chern class 0 by $\mathcal{I}(n)$.

We now compute the minimum codimension of a non semi-stable stratum. In particular, this computation shows that this minimum tends to infinity with n, so long as the genus g is positive.

Lemma 4.5 *The minimum (real) codimension of a non semi-stable stratum in* $\mathcal{C}^k(n)$ *(*$n > 1$*) is precisely* $2n + 2(n-1)(g-1) = 2g(n-1) + 2$*.*

Proof Let $\mu = ((n_1, k_1), \ldots, (n_r, k_r)) \in \mathcal{I}(n)$ be any admissible sequence with $r > 1$. Then from Theorem 4.2, we see that it will suffice to show that

$$\sum_{i>j} n_i k_j - n_j k_i \geqslant n \tag{1}$$

and

$$\sum_{i>j} n_i n_j \geqslant n - 1. \tag{2}$$

To prove (1), we begin by noting that since $\sum k_i = 0$ and the ratios $\frac{k_i}{n_i}$ are strictly decreasing, we must have $k_1 > 0$ and $k_n < 0$. Moreover, there is some $l_0 \in \mathbb{R}$ such that $k_l \geqslant 1$ for $l < l_0$ and $k_l \leqslant -1$ for $l > l_0$. We allow l_0 to be an integer if and only if $k_l = 0$ for some l; then this integer l is unique, and in this case we define $l_0 := l$. Since $r \geqslant 2$, we know that $1 < l_0 < r$.

Now, if $i > l_0 > j$ we have $k_j \geqslant 1$ and $k_i \leqslant -1$, so,

$$n_i k_j - n_j k_i \geqslant n_i + n_j.$$

If $i > l_0$ and $j = l_0$, we have $k_j = 0$ and $k_i \leqslant -1$, so

$$n_i k_{l_0} - n_{l_0} k_i \geqslant 0 + n_{l_0} = n_{l_0}.$$

Finally, if $i = l_0$ and $j < l_0$, then $k_i = 0$ and $k_j \geqslant 1$ so we have

$$n_{l_0} k_j - n_j k_{l_0} \geqslant n_{l_0} - 0 = n_{l_0}.$$

Now, since $n_i k_j - n_j k_i = n_i n_j (k_j/n_j - k_i/n_i)$ and the k_l/n_l are strictly decreasing, we know that each term in the sum $\sum_{i>j} n_i k_j - n_j k_i$ is positive. Dropping terms and applying the above bounds gives

$$\begin{aligned}\sum_{i>j} n_i k_j - n_j k_i &\geqslant \sum_{i>l_0>j} (n_i k_j - n_j k_i) + \sum_{l_0>j} (n_{l_0} k_j - n_j k_{l_0}) + \sum_{i>l_0} (n_i k_{l_0} - n_{l_0} k_i) \\ &\geqslant \sum_{i>l_0>j} (n_i + n_j) + \sum_{l_0>j} n_{l_0} + \sum_{i>l_0} n_{l_0}.\end{aligned}$$

(In the second and third expressions, the latter sums are taken to be empty if l_0 is not an integer.) Since $\sum n_i = n$, to check that the above expression is at least n it suffices to check that each n_i appears in the final sum. But since $1 < l_0 < r$, each n_l with $l \neq l_0$ appears in the first term, and if $l_0 \in \mathbb{N}$ then n_{l_0} appears in both of the latter terms. This completes the proof of (1).

To prove (2), we fix $r \in \mathbb{N}$ ($r \geqslant 2$) and consider partitions $\overrightarrow{\mathbf{p}} = (p_1, \ldots, p_r)$ of n. We will minimize the function $\phi_r(\overrightarrow{\mathbf{p}}) = \sum_{i>j} p_i p_j$, over all length r partitions of n.

Consider a partition $\overrightarrow{\mathbf{p}} = (p_1, \ldots, p_r)$ with $p_m \geqslant p_l > 1$ ($l \neq m$), and define another partition $\overrightarrow{\mathbf{p}'}$ by setting

$$p_i' = \begin{cases} p_i, & i \neq l,\ m, \\ p_l - 1, & i = l, \\ p_m + 1 & i = m. \end{cases}$$

It is easily checked that $\phi_r(\overrightarrow{\mathbf{p}}) > \phi_r(\overrightarrow{\mathbf{p}'})$.

Now, if we start with any partition $\overrightarrow{\mathbf{p}}$ such $p_i > 1$ for more than one index i, the above argument shows that $\overrightarrow{\mathbf{p}}$ cannot minimize ϕ_r. Thus ϕ_r is minimized by the partition $\overrightarrow{\mathbf{p}_0} = (1, \ldots, 1, n-r-1)$, and $\phi_r(\overrightarrow{\mathbf{p}_0}) = \binom{r-1}{2} + (r-1)(n-r-1)$. The latter is an increasing function for $r \in (0, n)$ and hence $\sum_{i>j} p_i p_j$ is minimized by the partition $(1, n-1)$. This completes the proof of (2).

Finally, note that the sequence $((1,1),(n-1,-1))$ has complex codimension $n + (n-1)(g-1)$. □

Remark 4.6 *It is interesting to note that the results in the next section clearly fail in the case when M has genus 0. From the point of view of homotopy theory, the problem is that S^2 is not the classifying space of its fundamental group, and so one should not expect a relationship between K–theory of S^2 and representations of $\pi_1 S^2 = 0$. But the only place where our argument breaks down is the previous lemma, which tells us that there are strata of complex codimension 1 in the Harder-Narasimhan stratification of $\mathcal{C}^k(S^2 \times \mathbb{C}(n))$, and in particular the minimum codimension does not tend to infinity with the rank. Thus there appears to be a relationship between the codimensions of these strata and the contractibility of the universal cover of M.*

The main result of this section will be an application of the following infinite-dimensional transversality theorem, due to Smale [2, Theorem 19.1] (see also [1]). Recall that a residual set in a topological space is a countable intersection of open, dense sets. By the Baire category theorem, any residual subset of a Banach space is dense, and since any Banach manifold is *locally* a Banach space, any residual subset of a Banach manifold is dense as well.

Theorem 4.7 (Smale) *Let A, X, and Y be second countable C^r Banach manifolds, with X of finite dimension k. Let $W \subset Y$ be a (locally closed) submanifold of Y, of finite codimension q. Assume that $r > \max(0, k-q)$. Let $\rho\colon A \to C^r(X,Y)$ be a C^r–representation, that is, a function for which the evaluation map $\mathrm{ev}_\rho\colon A \times X \to Y$ given by $\mathrm{ev}_\rho(a,x) = \rho(a)x$ is of class C^r.*

For $a \in A$, let $\rho_a\colon X \to Y$ be the map $\rho_a(x) = \rho(a)x$. Then $\{a \in A | \rho_a \pitchfork W\}$ is residual in A, provided that $\mathrm{ev}_\rho \pitchfork W$.

Corollary 4.8 *Let Y be a second countable Banach space, and let $\{W_i\}_{i\in I}$ be a countable collection of (locally closed) submanifolds of Y with finite codimension. Then if $U = Y - \bigcup_{i\in I} W_i$ is a non-empty submanifold, it has connectivity at least $\mu - 2$, where*

$$\mu = \min\{\text{codim } W_i \ : \ i \in I\};$$

equivalently the inclusion $U \hookrightarrow Y$ is $(\mu-2)$–connected.

Proof To begin, consider a continuous map $f\colon S^{k-1} \to U$, with $k-1 \leqslant \mu - 2$. We must show that f is null-homotopic in U; note that our homotopy need not be based. First we note that since U is a manifold, f may be smoothed, i.e. we may replace f by a C^{k+1} map $f'\colon S^{k-1} \to U$ which is homotopic to f inside U.

Choose a smooth function $\phi\colon \mathbb{R} \to \mathbb{R}$ with the property that $\phi(t) = 1$ for $t \geqslant 1/2$ and $\phi(t) = 0$ for all $t \leqslant 1/4$. Let $D^k \subset \mathbb{R}^k$ denote the closed unit disk, so $\partial D^k = S^{k-1}$. The formula $H^+(x) = \phi(||x||)f(x/||x||)$ now gives a C^{k+1} map $D^k \to Y$ which restricts to f on each shell $\{x \in D^k \mid \ ||x|| = r\}$ with $r \geqslant 1/2$. Gluing two copies of H^+ now gives a C^{k+1} "null-homotopy" of f defined on the closed manifold S^k.

We now define

$$A = \{F \in C^{k+1}(S^k, Y) \mid \ F(x) = 0 \ \text{for} \ x \in S^{k-1} \subset S^k\}.$$

Note that A is a Banach space: since S^k is compact, [1, Theorem 5.4] implies that $C^{k+1}(S^k, Y)$ is a Banach space, and A is a closed subspace of $C^{k+1}(S^k, Y)$. (This is the reason for working with C^{k+1} maps rather than smooth ones.)

Next, we define $\rho\colon A \to C^{k+1}(S^k, Y)$ by setting $\rho(F) = F + H$. The evaluation map $\mathrm{ev}_\rho\colon A \times S^k \to Y$ is given by $\mathrm{ev}_\rho(F,x) = F(x) + H(x)$. Since both $(F,x) \mapsto F(x)$ and

$(F,x) \mapsto x \mapsto H(x)$ are of class C^{k+1}, so is their sum (the fact that the evaluation map $(F,x) \mapsto F(x)$ is of class C^{k+1} follows from [1, Lemma 11.6]).

We are now ready to apply the transversality theorem. Setting $X = S^k$, $W = W_i$ (for some $i \in I$) and with A as above, all the hypotheses of Theorem 4.7 are clearly satisfied, except for the final requirement that $\mathrm{ev}_\rho \pitchfork W_i$. But this is easily seen to be the case. In fact, the derivative of ev_ρ surjects onto T_yY for each y in the image of ev_ρ, because given a C^{k+1} map $F\colon S^k \to Y$ with $F(x) = y$ and a vector $v \in T_yY$, we may adjust F in a small neighborhood of x so that the map remains C^{k+1} and its derivative hits v.

We now conclude that $\{F \in A | \rho_a \pitchfork W_i\}$ is residual in A, for each stratum W_i. Since the intersection of countably many residual sets is (by definition) residual, we in fact see that

$$\{F \in A \mid \rho_F \pitchfork W_i \;\; \forall\, i \in I\}$$

is residual, hence dense, in A. In particular, since A is non-empty, there exists a map $F\colon S^k \to Y$ such that $F|_{S^{k-1}} = f$ and $\rho_F = F + H$ is transverse to each W_i. Since $k < \mu = \mathrm{codim}(W_i)$, this implies that the image of $F + H$ must be disjoint from each W_i. Hence $(F + H)(S^k) \subset U$, and f represents the zero element in $\pi_{k-1}U$. □

We can now prove the main result of this section. This result extends work of Ho and Liu, who showed that spaces of flat connections over surfaces are connected [22, Theorem 5.4]. We note, though, that their work applies to general structure groups G. We also note that in the orientable case this result is closely related to work of Daskalopoulos and Uhlenbeck [11, Corollary 2.4], which concerns the less-highly connected space of *stable* bundles.

Proposition 4.9 *Let $M = M^g$ denote a compact Riemann surface of genus g, and let $n > 1$ be an integer. Then the space $\mathcal{A}^k_{\mathrm{flat}}(n)$ of flat connections on a trivial rank n bundle over M is $2g(n-1)$–connected, and if Σ is a non-orientable surface with double cover M^g, then the space of flat connections on any principal $U(n)$–bundle over Σ is $(g(n-1)-1)$–connected.*

Proof We begin by noting that Sobolev spaces (of sections of fiber bundles) over compact manifolds are always second countable; this follows from Bernstein's proof of the Weierstrass theorem since we may approximate any function by smooth functions, and locally we may approximate smooth functions (uniformly up to the kth derivative for any k) by Bernstein polynomials. Since the inclusion $\mathcal{A}^k_{\mathrm{flat}}(n) \hookrightarrow \mathcal{C}^k_{ss}(n)$ is a homotopy equivalence (Theorem 4.2), the orientable case now follows by applying Corollary 4.8 (and Lemma 4.5) to the Harder-Narasimhan stratification.

For the non-orientable case, we work in the set-up of non-orientable Yang-Mills theory, as developed by Ho and Liu [22]. Let Σ be a non-orientable surface with double cover M^g, and let P be a principal $U(n)$–bundle over Σ. Let $\pi\colon M^g \to \Sigma$ be the projection, and let $\widetilde{P} = \pi^* P$. Then the deck transformation $\tau\colon M^g \to M^g$ induces an involution $\widetilde{\tau}\colon \widetilde{P} \to \widetilde{P}$, and $\widetilde{\tau}$ acts on the space $\mathcal{A}^k(\widetilde{P})$ by pullback. Connections on P pull back to connections on $\widetilde{P}$, and in fact, the image of the pullback map is precisely the set of fixed points of τ (see, for example, Ho [21]). Hence we have a homeomorphism $\mathcal{A}^k(P) \cong \mathcal{A}^k(\widetilde{P})^{\widetilde{\tau}}$, which we treat as an identification. The Yang-Mills functional L is invariant under $\widetilde{\tau}$, and hence its gradient flow restricts to a flow on $\mathcal{A}^k(P)$.

Assume for the moment that $\mathcal{A}^k_{\text{flat}}(P) \neq \emptyset$. The flat connections on P pull back to flat connections on $\widetilde{P}$, and again the image of $\mathcal{A}^k_{\text{flat}}(P)$ in $\mathcal{A}(\widetilde{P})$ is precisely $\mathcal{A}^k_{\text{flat}}(\widetilde{P})^{\widetilde{\tau}}$. If we let $\mathcal{C}^k_{ss}(P)$ denote the fixed set $\mathcal{C}^k_{ss}(\widetilde{P})^{\widetilde{\tau}}$, then the gradient flow of L restricts to give a deformation retraction from $\mathcal{C}^k_{ss}(P)$ to $\mathcal{A}^k_{\text{flat}}(P)$. The complement of $\mathcal{C}^k_{ss}(P)$ in $\mathcal{A}^k(P)$ may be stratified as follows: for each Harder-Narasimhan stratum $\mathcal{C}^k_{\mu}(\widetilde{P}) \subset \mathcal{A}^k(\widetilde{P}) \cong \mathcal{C}^k\left(\widetilde{P} \times_{U(n)} \mathbb{C}^n\right)$, we consider the fixed set $\mathcal{C}^k_{\mu}(P) := \left(\mathcal{C}^k_{\mu}(\widetilde{P})\right)^{\widetilde{\tau}}$. By Ho and Liu [22, Proposition 5.1], $\mathcal{C}^k_{\mu}(P)$ is a real submanifold of $\mathcal{A}^k(P)$, and if it is non-empty then its real codimension in $\mathcal{A}^k(P)$ is half the real codimension of $\mathcal{C}^k_{\mu}(\widetilde{P})$ in $\mathcal{A}^k(\widetilde{P})$. The codimensions of the non semi-stable strata $\mathcal{C}^k_{\mu}(P)$ are hence at least $g(n-1)+1$ (by Lemma 4.5). It now follows from Corollary 4.8 that $\mathcal{A}^k_{\text{flat}}(P)$ has the desired connectivity.

To complete the proof, we must show that all bundles P over Σ actually admit flat connections. This was originally proven by Ho and Liu [25, Theorem 5.2], and in the current context may be seen as follows. There are precisely two isomorphism types of principal $U(n)$–bundles over any non-orientable surface. (A map from Σ into $BU(n)$ may be homotoped to a cellular map, and since the 3-skeleton of $BU(n)$ is a 2-sphere, the classification of $U(n)$–bundles is independent of n. Hence it suffices to note that the relative K-group $\widetilde{K}_0(\Sigma)$ has order 2.) Except in the case $g=1, n=2$, we have just shown that the space of flat connections on each bundle is either empty or connected, so Proposition 3.9 gives a bijection between connected components of $\text{Hom}(\pi_1\Sigma, U(n))$ and bundles admitting a flat connection. So it suffices to show that the representation space has at least two components. This follows easily from the obstruction defined by Ho and Liu [op. cit.].

The case $g=1$, $n=2$ can be handled by an argument similar to the proof of Corollary 4.12, using the facts that there are exactly two bundles over any non-orientable surface, classified by their first Chern classes in $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}/2$, and there are exactly two components in $\mathrm{Hom}(\pi_1\Sigma, U(n))$, classified by Ho and Liu's additive obstruction (for an elementary algebraic proof of the latter fact, see [39, Proposition 6.1.11]). Since we will not need this case, we leave details to the reader. □

Remark 4.10 *In the non-orientable case, some improvement to Theorem 4.9 is possible. The results of Ho and Liu show that many of the Harder-Narasimhan strata for the double cover of non-orientable surface contain no fixed points, and hence the above lower bound on the minimal codimension of the Morse strata is not tight in the non-orientable case. In the orientable case, the tightness of Lemma 4.5 shows that the bound on connectivity of* $\mathcal{A}^k_{\mathrm{flat}}(n)$ *is tight. This can be proven using the Hurewicz theorem and a homological calculation; the latter requires constructing good tubular neighborhoods for the locally closed submanifolds* $\mathcal{C}^k_\mu$ *(such a construction will appear in joint work with Ho and Liu).*

As discussed in Section 2.1, the following results are quite close to the work of Ho and Liu.

Corollary 4.11 *For any compact Riemann surface* M *and any* $n \geqslant 1$*, the representation space* $\mathrm{Hom}(\pi_1(M), U(n))$ *is connected. In particular,* $\mathrm{Rep}(\pi_1 M)$ *is stably group-like.*

Proof The genus 0 case is trivial. When $n = 1$, $U(1) = S^1$ is abelian and all representations factor through the abelianization of $\pi_1 M$. Hence $\mathrm{Hom}(\pi_1 M, U(1))$ is a wedge of circles. When $g, n \geqslant 1$ we have $2g(n-1) \geqslant 0$, so Proposition 4.9 implies that $\mathcal{A}^k_{\mathrm{flat}}(n)$ is connected. Connectivity of $\mathrm{Hom}(\pi_1(M), U(n))$ follows from Proposition 3.9, because any $U(n)$ bundle over a Riemann surface which admits a flat connection is trivial (Thaddeus [42] has given a beautiful and elementary proof of this fact). □

Corollary 4.12 *Let* Σ *be a compact, non-orientable, aspherical surface. Then for any* $n \geqslant 1$*, the representation space* $\mathrm{Hom}(\pi_1\Sigma, U(n))$ *has two connected components, and if* $\rho \in \mathrm{Hom}(\pi_1\Sigma, U(n))$ *and* $\psi \in \mathrm{Hom}(\pi_1\Sigma, U(m))$ *lie in the non-identity components, then* $\rho \oplus \psi$ *lies in the identity component of* $\mathrm{Hom}(\pi_1\Sigma, U(n+m))$*. In particular,* $\mathrm{Rep}(\pi_1\Sigma)$ *is stably group-like.*

Proof First we consider connected components. The case $n = 1$ follows as in Corollary 4.11. When $n > 1$, it follows immediately from Proposition 4.9 that the space of flat connections on any principal $U(n)$–bundle over Σ is connected, unless $n = 2$ and the genus of the universal cover of Σ is 1, i.e. Σ is the Klein bottle. For any n, the case of the Klein bottle may be handled by an elementary algebraic argument [39, Proposition 6.1.11]. Note, however, that to prove that $\mathrm{Rep}(\pi_1\Sigma)$ is stably group-like, we are free to ignore the structure of $\mathrm{Hom}(\pi_1\Sigma, U(n))$ for small n.

As discussed in the proof of Proposition 4.9, there are precisely two bundles over Σ, classified by their first Chern classes, and there is a bijection between components of the representation space and isomorphism classes of bundles. Hence the components of $\mathrm{Hom}(\pi_1\Sigma, U(n))$ are classified by the Chern classes of their induced bundles, and since Chern classes are additive, the sum of two representations in the non-identity components of $\mathrm{Hom}(\pi_1\Sigma, U(-))$ lies in the identity component. □

5 Proof of the main theorem

We can now prove our surface-group analogue of the Atiyah–Segal theorem.

Theorem 5.1 *Let M be a compact, aspherical surface (in other words, $M \neq S^2, \mathbb{R}P^2$). Then for $* > 0$,*

$$K^*_{\mathrm{def}}(\pi_1(M)) \cong K^*(M),$$

where $K^*(M)$ denotes the complex K–theory of M. In the non-orientable case, this in fact holds in degree 0 as well; in the orientable case, we have $K^0_{\mathrm{def}}(\pi_1(M)) \cong \mathbb{Z}$.

We note that the isomorphism in Theorem 5.1 is functorial for *smooth* maps between surfaces, as will be apparent from the proof. In particular, the isomorphism is equivariant with respect to the mapping class group of M.

The K–theory of surfaces is easily computed (using the Mayer-Vietoris sequence or the Atiyah-Hirzebruch spectral sequence), so Theorem 5.1 gives a complete computation of the deformation K-groups.

Corollary 5.2 *Let M^g be a compact Riemann surface of genus $g > 0$. Then*

$$K^*_{\mathrm{def}}(\pi_1 M^g) = \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}^{2g}, & * \text{ odd} \\ \mathbb{Z}^2, & * \text{ even}, * > 0. \end{cases}$$

Let M be a compact, non-orientable surface of the form $M = M^g \# N_j$ ($g \geqslant 0$), where $j = 1$ or 2 and $N_1 = \mathbb{R}P^2$, $N_2 = \mathbb{R}P^2 \# \mathbb{R}P^2$ (so N_2 is the Klein bottle). So long as $M \neq \mathbb{R}P^2$, we have:

$$K^*_{\text{def}}(\pi_1 M^g \# N_j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & * \text{ even} \\ \mathbb{Z}^{2g+j-1}, & * \text{ odd.} \end{cases}$$

Proof of Theorem 5.1 I. The orientable case: Let M be a Riemann surface of genus $g > 0$. We will show that for any $* \geq 0$,

$$K^*_{\text{def}}(\pi_1(M)) \cong \pi_*(\text{Map}^0(M, BU)),$$

where Map^0 denotes the connected component of the constant map. This will clearly suffice. In fact, we will exhibit a zig-zag of weak equivalences between the zeroth space of the deformation K–theory spectrum and the space $\mathbb{Z} \times \text{Map}^0(M, BU)$.

By Corollary 2.4 and Proposition 3.9 (and the fact that any bundle over a Riemann surface admitting a flat connection is trivial), the zeroth space of the spectrum $K_{\text{def}}(\pi_1 M)$ is weakly equivalent to

$$\underset{\oplus 1}{\overrightarrow{\text{telescope}}}\, \text{Rep}(\pi_1 M)_{hU} \cong \underset{\oplus[\tau]}{\overrightarrow{\text{telescope}}} \coprod_n EU(n) \times_{U(n)} \left(\mathcal{A}^k_{\text{flat}}(n)/\mathcal{G}^{k+1}_0(n)\right) \tag{3}$$

where the maps are induced by direct sum with the trivial connection τ on the trivial line bundle. Since the based gauge groups $\mathcal{G}^{k+1}_0(n)$ act freely on $\mathcal{A}^k(n)$, and (by Mitter–Viallet [34]) the projection maps are locally trivial principal $\mathcal{G}^{k+1}_0(n)$–bundles, a basic result about homotopy orbit spaces [6, 13.1] shows that we have a weak equivalence

$$E\mathcal{G}^{k+1}(n) \times_{\mathcal{G}^{k+1}(n)} \mathcal{A}^k_{\text{flat}}(n) \xrightarrow{\simeq} EU(n) \times_{U(n)} \left(\mathcal{A}^k_{\text{flat}}(n)/\mathcal{G}^k_0(n)\right) \tag{4}$$

It now follows from (4) that the mapping telescope (3) is weakly equivalent to

$$\underset{\oplus \tau}{\overrightarrow{\text{telescope}}} \coprod_n \left(\mathcal{A}^k_{\text{flat}}(n)\right)_{h\mathcal{G}^{k+1}(n)} \simeq \mathbb{Z} \times \underset{\oplus \tau}{\overrightarrow{\text{telescope}}}\, \mathcal{A}^k_{\text{flat}}(n)_{h\mathcal{G}^{k+1}(n)}.$$

Proposition 4.9 shows that the connectivity of the projections $\mathcal{A}^k_{\text{flat}}(n)_{h\mathcal{G}^{k+1}(n)} \to B\mathcal{G}^{k+1}(n)$ tends to infinity, and since the homotopy groups of a mapping telescope may be described as colimits, these maps induce a weak equivalence

$$\mathbb{Z} \times \underset{n\to\infty}{\text{telescope}}\, \mathcal{A}^k_{\text{flat}}(n)_{h\mathcal{G}^{k+1}(n)} \longrightarrow \mathbb{Z} \times \underset{n\to\infty}{\text{telescope}}\, B\mathcal{G}^{k+1}(n). \tag{5}$$

By Lemma 3.2, the inclusion $\mathcal{G}^{k+1}(n) \hookrightarrow \mathcal{G}(n)$ is a weak equivalence, so we may replace $\mathcal{G}^{k+1}(n)$ with $\mathcal{G}(n)$ on the right.

We have been using Milnor's functorial model $E(-)$ for classifying spaces (see Remarks 2.2). Atiyah and Bott have shown [6, Section 2] that the natural map

$$\mathrm{Map}(M, EU(n)) \to \mathrm{Map}^0(M, BU(n))$$

is a universal principal $\mathrm{Map}(M, U(n)) = \mathcal{G}(n)$ bundle, where Map^0 denotes the connected component of the constant map. We now have weak equivalences

$$B\mathcal{G}(n) \longleftarrow (E\mathcal{G}(n) \times \mathrm{Map}(M, EU(n)))\,/\mathrm{Map}(M, U(n)) \longrightarrow \mathrm{Map}^0(M, BU(n)),$$

which are natural in n and hence induce weak equivalences on mapping telescopes (taken with respect to the maps induced by the standard inclusions $U(n) \hookrightarrow U(n+1)$). The space $\mathrm{telescope}_{n\to\infty}\, \mathrm{Map}^0(M, BU(n))$ is weakly equivalent to the colimit $\mathrm{Map}^0(M, BU)$, since maps from compact sets into a colimit land in some finite piece. Hence we have a zig-zag of weak equivalences connecting the zeroth space of $K_{\mathrm{def}}(\pi_1 M)$ to $\mathbb{Z} \times \mathrm{Map}^0(M, BU)$, as desired.

II. The non-orientable case: Let M be a non-orientable surface. Once again, Proposition 2.4 and Proposition 3.9 tell us that the zeroth space of $K_{\mathrm{def}}(\pi_1 M)$ is weakly equivalent to

$$\underset{\overrightarrow{\oplus\tau}}{\mathrm{telescope}} \coprod_{[P_i]} \left(\mathcal{A}^k_{\mathrm{flat}}(P_i)\right)_{h\mathcal{G}^{k+1}(P_i)},$$

where the disjoint union is taken over all n and over all isomorphism types of principal $U(n)$–bundles. By Proposition 4.9 we know that $\mathcal{A}^k_{\mathrm{flat}}(P_i)$ is $(g(\widetilde{M})(n_i - 1) - 1)$–connected, where $n_i = \dim(P_i)$ and $g(\widetilde{M})$ denotes the genus of the double cover of M. Since we have assumed $M \neq \mathbb{R}P^2$, we know that $g(\widetilde{M}) > 0$, and hence the connectivity of $\mathcal{A}^k_{\mathrm{flat}}(P_i)$ tends to infinity with n_i. This shows that the natural map

$$\underset{\overrightarrow{\oplus\tau}}{\mathrm{telescope}} \coprod_{[P_i]} \left(\mathcal{A}^k_{\mathrm{flat}}(P_i)\right)_{h\mathcal{G}^{k+1}(P_i)} \longrightarrow \underset{\overrightarrow{\oplus 1}}{\mathrm{telescope}} \coprod_{[P_i]} B\mathcal{G}^{k+1}(P_i) \tag{6}$$

is a weak equivalence (on the right hand side, 1 denotes the identity element in $\mathcal{G}^{k+1}(1)$). As in the orientable case, we may now switch to the Atiyah-Bott models for $B\mathcal{G}(P_i)$, obtaining the space

$$\underset{\overrightarrow{\oplus 1}}{\mathrm{telescope}} \coprod_{[P_i]} \mathrm{Map}^{P_i}(M, BU(n_i)),$$

where Map^{P_i} denotes the component of the mapping space consisting of those maps $f : M \to BU(n_i)$ with $f^*(EU(n_i))$ isomorphic to P_i. But since the union is taken over *all* isomorphism classes, this space is simply

$$\mathbb{Z} \times \underset{n\to\infty}{\mathrm{telescope}}\, \mathrm{Map}(M, BU(n))$$

(up to homotopy) and as before is weakly equivalent to

$$\mathbb{Z} \times \mathrm{Map}(M, BU) = \mathrm{Map}(M, \mathbb{Z} \times BU). \qquad \square$$

We briefly discuss the spectrum-level version of Theorem 5.1. The space level constructions used in the proof of Theorem 5.1 can be lifted to spectrum-level constructions. This involves constructing a variety of spectra (and maps between them) including, for instance, a spectrum arising from a topological category of flat connections and gauge transformations. Each spectrum involved can be constructed from a Γ–space in the sense of Segal [40], and the space-level constructions above essentially become weak equivalences between the group completions of the monoids underlying these Γ–spaces. Since these group completions are weakly equivalent to the zeroth spaces of these Ω-spectra, the space-level weak equivalences lift to weak-equivalences of spectra. In the non-orientable case, one concludes that $K_{\mathrm{def}}(\pi_1 M)$ is weakly equivalent to the function spectrum $F(M, \mathbf{ku})$. The end result in the orientable case is somewhat uglier, due to the failure of Theorem 5.1 on π_0. In this case, $K_{\mathrm{def}}(\pi_1 M)$ is weakly equivalent to a subspectrum of $F(M, \mathbf{ku})$, essentially consisting of those maps homotopic to a constant map. One may ask whether the intermediate spectra are **ku**–algebra spectra and whether the maps between them preserve this structure. More basically, one may ask whether the isomorphisms in Theorem 5.1 come from a homomorphism of graded rings. Recall that T. Lawson [31] has constructed a **ku**–algebra structure on the spectrum $K_{\mathrm{def}}(G)$. Constructing an analogous structure for the spectrum arising from flat connections appears to be a subtle problem. This problem, and the full details of the spectrum-level constructions, will be considered elsewhere.

We now make the following conjecture regarding the homotopy type of the spectrum $K_{\mathrm{def}}(\pi_1 M)$, as a algebra over the connective K–theory spectrum **ku**. Note that it is easy to check (using Theorem 5.1) that the homotopy groups of the proposed spectrum are the same as $K^*_{\mathrm{def}}(\pi_1(M))$.

Conjecture 5.3 *For any Riemann surface M^g, the spectrum $K_{\mathrm{def}}(\pi_1 M)$ is weakly equivalent, as a* **ku***–algebra, to*

$$\mathbf{ku} \vee \left(\bigvee_{2g} \Sigma \mathbf{ku} \right) \vee \Sigma^2 \mathbf{ku}.$$

6 Connected sum decompositions

In this section we consider the behavior of deformation K–theory on connected sum decompositions of Riemann surfaces.

Given an amalgamation diagram of groups, applying deformation K-theory results in a pull-back diagram of spectra. An excision theorem states that the natural map

$$\Phi\colon\ K_{\mathrm{def}}(G *_K H) \longrightarrow \operatorname{holim}\left(K_{\mathrm{def}}(G) \longrightarrow K_{\mathrm{def}}(K) \longleftarrow K_{\mathrm{def}}(H)\right)$$

is an isomorphism, where holim denotes the homotopy pullback.

Associated to a homotopy cartesian diagram of spaces

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{k} & Z \end{array} \tag{7}$$

there is a long exact "Mayer–Vietoris" sequence of homotopy groups

$$\ldots \longrightarrow \pi_k(W) \xrightarrow{f_*\oplus g_*} \pi_k(X)\oplus\pi_k(Y) \xrightarrow{h_*-k_*} \pi_k(Z) \xrightarrow{\partial} \pi_{k-1}(W) \longrightarrow \ldots \tag{8}$$

(this follows from Hatcher [18, p. 159], together with the fact that the homotopy fibers of the vertical maps in a homotopy cartesian square are weakly equivalent). If the diagram (7) is a diagram in the category of group-like H–spaces, then all of the maps in the sequence (8) (including the boundary map) are homomorphisms in dimension zero. Hence whenever deformation K-theory is excisive on an amalgamation diagram, one obtains a long-exact sequence in K^*_{def}.

Deformation K–theory can fail to satisfy excision in low dimensions, and in particular the failure of Theorem 5.1 in degree zero leads to a failure of excision for connected sum decompositions of Riemann surfaces. We briefly describe this situation.

Letting $M = M^{g_1+g_2}$ denote the surface of genus $g_1 + g_2$ and F_k the free group on k generators, if we think of M as a connected sum then the Van Kampen Theorem gives us an amalgamation diagram for $\pi_1 M$. The long exact sequence coming from excision would end with

$$\begin{aligned} K^1_{\mathrm{def}}(F_{2g_1}) \oplus K^1_{\mathrm{def}}(F_{2g_2}) &\xrightarrow{c_1^*-c_2^*} K^1_{\mathrm{def}}(\mathbb{Z}) \longrightarrow K^0_{\mathrm{def}}(\pi_1 M) \\ &\longrightarrow K^0_{\mathrm{def}}(F_{2g_1}) \oplus K^0_{\mathrm{def}}(F_{2g_2}) \twoheadrightarrow K^0_{\mathrm{def}}(\mathbb{Z}). \end{aligned}$$

The groups appearing in this sequence are all known, and so the sequence would have the form

$$K^1_{\mathrm{def}}(F_{2g_1}) \oplus K^1_{\mathrm{def}}(F_{2g_2}) \xrightarrow{c_1^* - c_2^*} K^1_{\mathrm{def}}(\mathbb{Z}) = \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z}.$$

We claim, however, that the maps $c_i^*\colon K^1_{\mathrm{def}}(F_{2g_i}) \to K^1_{\mathrm{def}}(\mathbb{Z})$ are zero. This leads immediately to a contradiction, meaning that no such exact sequence can exist and excision is not satisfied in degree zero.

If we write the generators of F_{2g_i} as $a_1^i, b_1^i, \ldots, a_{g_i}^i, b_{g_i}^i$, then the map $c_i : \mathbb{Z} \to F_{2g_i}$ is the multiple-commutator map, sending $1 \in \mathbb{Z}$ to $\prod_{j=1}^{g_i} [a_j^i, b_j^i]$. Since the representation spaces of F_k are always connected, $\mathrm{Rep}(F_k)$ is stably group-like with respect to $1 \in \mathrm{Hom}(F_k, U(1))$. Hence (using Theorem 2.3) one finds that the induced map $c_i^*\colon K^*_{\mathrm{def}}(F_{2g_i}) \to K^*_{\mathrm{def}}(\mathbb{Z})$ may be identified with the map

$$\pi_*(\mathbb{Z} \times (U^{2g_i})_{hU}) \to \pi_*(\mathbb{Z} \times U_{hU})$$

induced by the multiple commutator map $C\colon U^{2g_i} \to U$ (here the actions of U are via conjugation). The induced map C_* on homotopy is always zero, and from the diagram of fibrations

$$\begin{array}{ccccc} U^{2g_i} & \longrightarrow & EU \times_U U^{2g_i} & \longrightarrow & BU \\ \downarrow{\scriptstyle C} & & \downarrow & & \downarrow{\scriptstyle ||} \\ U & \longrightarrow & EU \times_U U & \longrightarrow & BU \end{array}$$

one now concludes (using Bott Periodicity) that c_i^* is zero for $*$ odd.

This shows that deformation K–theory is not excisive on π_0 for connected sum decompositions. However, based on Theorem 5.1 we expect that excision will hold in all higher degrees.

7 The stable moduli space of flat connections

In this section we study the moduli space of flat unitary connections over a surface, after stabilizing with respect to rank. By definition, the moduli space of flat connections over a compact manifold, with structure group G, is the space

$$\coprod_{[P_i]} \mathcal{A}^{k,p}_{\mathrm{flat}}(P)/\mathcal{G}^{k+1,p} \cong \mathrm{Hom}(\pi_1 M, G)/G,$$

where the disjoint union is taken over isomorphism classes of prinicpal G–bundles. This homeomorphism follows immediately from Proposition 3.9 (so long as k and p and G satisfy the hypotheses of that result). In particular, the moduli space of flat unitary connections is simply $\mathrm{Hom}(\pi_1 M, U(n))/U(n)$, and the inclusions $U(n) \hookrightarrow U(n+1)$ allow us to stabilize with respect to rank. The colimit of these spaces is just $\mathrm{Hom}(\pi_1 M, U)/U$, where $U = \operatorname{colim} U(n)$ is the infinite unitary group. We call this space the stable moduli space of flat unitary connections.

In order to apply our results on deformation K–theory to the stable moduli space, we need to employ results of T. Lawson regarding the Bott map in deformation K–theory. For the remainder of this section, we think of $K_{\mathrm{def}}(\Gamma)$ as the spectrum described in Section 2. This is a connective spectrum, and we have computed its homotopy groups in Theorem 5.1.

Lawson has shown [30] that for any finitely generated group Γ, there is a homotopy cofiber sequence of spectra

$$\Sigma^2 K_{\mathrm{def}}(\Gamma) \longrightarrow K_{\mathrm{def}}(\Gamma) \longrightarrow R^{\mathrm{def}}(\Gamma), \tag{9}$$

where $R^{\mathrm{def}}(\Gamma)$ denotes the “deformation representation ring" of Γ, as we will explain. The first map in this sequence is the Bott map in deformation K–theory, and is obtained from the Bott map in connective K–theory **ku** by smashing with $K_{\mathrm{def}}(\Gamma)$; this requires the **ku**–module structure in deformation K–theory constructed by Lawson [31]. Since $K_{\mathrm{def}}(\Gamma)$ is connective, the first two homotopy groups of $\Sigma^2 K_{\mathrm{def}}(\Gamma)$ are zero, and hence the long exact sequence in homotopy associated to (9) immediately gives isomorphisms

$$K^i_{\mathrm{def}}(\Gamma) \cong \pi_i R^{\mathrm{def}}(\Gamma) \tag{10}$$

for $i = 0, 1$.

Remark 7.1 *In Section 2, we described $K_{\mathrm{def}}(\Gamma)$ as the spectrum associated to a permutative category of representations. Lawson works with a different model [31], built from the H–space $\coprod_n V(n) \times_{U(n)} \mathrm{Hom}(\Gamma, U(n))$. Here $V(n)$ denotes the infinite Stiefel manifold of n–frames in $\mathbb{C}^\infty$. One may interpolate between the two models by using a spectrum built from the H–space*

$$(EU(n) \times V(n)) \times_{U(n)} \mathrm{Hom}(\Gamma, U(n)),$$

and hence Theorem 5.1 computes the homotopy groups of Lawson's deformation K–theory spectrum as well.

Given any topological abelian monoid A (for which the inclusion of the identity is a cofibration), one may apply Segal's infinite loop space machine [40] to produce a connective Ω–spectrum; equivalently the bar construction BA is again an abelian topological monoid and one may iterate. In particular, the zeroth space of this spectrum is exactly ΩBA. The deformation representation ring $R^{\text{def}}(\Gamma)$ is the spectrum associated to the abelian topological monoid

$$\overline{\text{Rep}(\Gamma)} = \coprod_{n=0}^{\infty} \text{Hom}(\Gamma, U(n))/U(n),$$

so we have

$$\pi_* R^{\text{def}}(\Gamma) \cong \pi_* \Omega B\left(\overline{\text{Rep}(\Gamma)}\right).$$

It is in general rather easy to identify the group completion ΩBA when A is an abelian monoid. In the case of surface groups, we have the following result.

Proposition 7.2 *Let Γ be a finitely generated discrete group, and assume that* $\text{Rep}(\Gamma)$ *stably group-like with respect to the trivial representation* $1 \in \text{Hom}(\Gamma, U(1))$ *(e.g. $\Gamma = \pi_1 M$ with M a compact, aspherical surface). Then the zeroth space of $R^{\text{def}}(\Gamma)$ is weakly equivalent to $K^0_{\text{def}}(\Gamma) \times \text{Hom}(\Gamma, U)/U$. Hence we have*

$$\pi_* \text{Hom}(\Gamma, U)/U \cong \pi_* R^{\text{def}}(\Gamma)$$

for $* > 0$, and in particular $\pi_1 \text{Hom}(\Gamma, U)/U \cong K^1_{\text{def}}(\Gamma)$.

Proof If $\text{Rep}(\Gamma)$ is stably group-like with respect to the trivial representation $1 \in \text{Hom}(\Gamma, U(1))$, the same is true for the monoid of isomorphism classes $\overline{\text{Rep}(\Gamma)}$. We can now apply Ramras [38, Theorem 3.6], because the additional hypothesis (that the representation 1 must be “anchored") is trivially satisfied for abelian monoids. Hence

$$\Omega B\left(\overline{\text{Rep}(\Gamma)}\right) \simeq \underset{\overrightarrow{\oplus 1}}{\text{telescope}}\, \overline{\text{Rep}(\Gamma)},$$

and since $\Omega B(\text{Rep}(\Gamma))$ is a group-like H–space, all components of these spaces are homotopy equivalent. To complete the proof we just need to check that one of these components, say

$$\underset{n\to\infty}{\text{telescope}}\, \text{Hom}(\Gamma, U(n))/U(n),$$

is weakly equivalent to the colimit

$$\underset{n\to\infty}{\text{colim}}\, \text{Hom}(\Gamma, U(n))/U(n).$$

But the natural projection from the mapping telescope to the colimit is a weak equivalence, because in each case compact sets land in some finite piece (for the colimit, this requires that points are closed in $\text{Hom}(\Gamma, U(n))/U(n)$; this space is in fact Hausdorff because the orbits of $U(n)$ are compact, hence closed in $\text{Hom}(\Gamma, U(n))$, which is a metric space, hence normal). □

Combining Proposition 7.2 with (10) and Theorem 5.1 yields:

Corollary 7.3 *For any compact, aspherical surface M, the fundamental group of the stable moduli space of flat unitary connections on M is isomorphic to the complex K-group $K^1(M)$. Equivalently, if M^g is a Riemann surface of genus g,*

$$\pi_1\left(\mathrm{Hom}(\pi_1 M^g, U)/U\right) \cong \mathbb{Z}^{2g},$$

and in the non-orientable cases (letting K denote the Klein bottle) we have

$$\pi_1\left(\mathrm{Hom}(\pi_1 M^g \# \mathbb{R}P^2, U)/U\right) \cong \mathbb{Z}^{2g} \text{ and } \pi_1\left(\mathrm{Hom}(\pi_1 M^g \# K, U)/U\right) \cong \mathbb{Z}^{2g+1}.$$

We note that the proof of this result requires not only the Yang-Mills theory used to prove Theorem 5.1 (which includes deep analytical results like Uhlenbeck compactness and convergence of the Yang-Mills flow) but also the modern stable homotopy theory underlying Lawson's cofiber sequence. His results require, for example, the model categories of module and algebra spectra studied by Elmendorf, Kriz, Mandell, and May [12], Elmendorf and Mandell [13], and Hovey, Shipley, and Smith [26].

Assuming Conjecture 5.3, we know that for Riemann surfaces, $K_{\mathrm{def}}(\pi_1 M^g)$ is free as a **ku**–module. Hence the Bott map is easily calculated, and one may compute the homotopy groups of $\mathrm{Hom}(\pi_1(M^g), U)/U$. It is interesting to note that they vanish above dimension 2; the reader should note the similarity between this calculation (and the previous theorem) and the main result of Lawson's paper [30], which states that U^k/U, the space of isomorphism classes of representations of a *free* group, has the homotopy type of $\mathrm{Sym}^\infty(S^1)^k = \mathrm{Sym}^\infty B(F_k)$. (Of course this space is homotopy equivalent to $(S^1)^k$.)

8 Appendix: Holonomy of flat connections

We now give a careful discussion of the holonomy representation associated to a flat connection on a principal G–bundle over a connected manifold M. We show that holonomy induces a bijection from the set of all such (smooth) connections to the set of representations of $\pi_1 M$ into G, after taking the action of the based gauge group into account (Proposition 8.4). This is essentially well-known, but there does not appear to be a complete reference. Some of the results to follow may be found in Morita's books [36, 35], and a close relative of the main result is stated in the introduction to Fine–Kirk–Klassen [14].

Most proofs will be left to the reader; these are generally tedious but straightforward unwindings of the definitions. Usually a good picture contains the necessary ideas. Many choices must be made in the subsequent discussion, starting with a choice of left versus right principal bundles. It is quite easy to make incompatible choices, especially because these may cancel out later in the argument. We have carefully made consistent and correct choices.

All manifolds and maps in this appendix will be smooth. Let M be a connected manifold, G a Lie group, and $\pi\colon P \to M$ a principal G–bundle on M. Our principal bundles will always be equipped with a *right* action of the structure group G. A connection on P is a G–equivariant splitting of the map natural map $TP \to \pi^*TM$. The gauge group $\mathcal{G}(P)$ is the group of all equivariant maps $P \xrightarrow{\phi} P$ such that $\pi\circ\phi = \pi$; the gauge group acts on the left of $\mathcal{A}(P)$ via pushforward: $\phi_* A = D\phi \circ A \circ \tilde{\phi}^{-1}$.

Given a smooth curve $\gamma\colon [0,1] \to M$ we may define a parallel transport operator $T_\gamma\colon P_{\gamma(0)} \to P_{\gamma(1)}$ by following A–horizontal lifts of the path γ. An A–horizontal lift of γ is a curve $\tilde{\gamma}\colon [0,1] \to P$ satisfying

$$\pi \circ \tilde{\gamma} = \gamma \quad \text{and} \quad \tilde{\gamma}_p'(t) = A\left(\gamma'(t), \tilde{\gamma}_p(t)\right),$$

and is uniquely determined by its starting point $\tilde{\gamma}(0)$; we denote the lift starting at $p \in P_{\gamma(0)}$ by $\tilde{\gamma}_p$. Parallel transport is now defined by $T_\gamma(p) = \tilde{\gamma}_p(1)$.

Parallel transport is easily checked to be G–equivariant, and behaves appropriately with respect to composition and reversal of paths. Any *flat* connection A is locally trivial (see Spivak [41, p.349]), and a standard compactness argument shows that parallel transport is homotopy invariant for such connections.

Definition 8.1 *Let P be a principal G–bundle over M, and choose basepoints $m_0 \in M$, $p_0 \in P_{m_0}$. Associated to any flat connection A on P, the* holonomy representation

$$\rho_A\colon \pi_1(M, m_0) \to G$$

is defined by setting $\rho_A([\gamma])$ to be the unique element of G satisfying

$$p_0 = T^A_\gamma(p_0) \cdot \rho_A([\gamma]).$$

Here $\gamma\colon I \to M$ is a smooth loop based at m_0 and $[\gamma]$ is its class in $\pi_1(M, m_0)$.

From here on we assume that M is equipped with a basepoint $m_0 \in M$, and we assume all principal bundles P are equipped with basepoints $p_0 \in P_{m_0}$.

We denote the set of all (smooth) flat connections on a principal bundle P by $\mathcal{A}_{\text{flat}}(P)$. We now record the effect of the gauge group on holonomy.

Proposition 8.2 *For any $A \in \mathcal{A}_{\text{flat}}(P)$ and any $\phi \in \mathcal{G}(P)$ we have*

$$\rho_{\phi_* A} = \phi_{m_0} \rho_A \phi_{m_0}^{-1},$$

where $\phi_{m_0} \in G$ is the unique element such that $p_0 \cdot \phi_{m_0} = \phi(p_0)$. (Note that with this definition, the map $\phi \mapsto \phi_{m_0}$ is a homomorphism $\mathcal{G}(P) \to G$.)

Definition 8.3 *The based gauge group $\mathcal{G}_0(P) \subset \mathcal{G}(P)$ is the kernel of the restriction homomorphism $r\colon \mathcal{G}(P) \to G$, $r(\phi) = \phi_{m_0}$. Equivalently, $\mathcal{G}_0(P)$ is the subgroup of gauge transformations which are the identity on P_{m_0}.*

An immediate consequence of Proposition 6 is that the based gauge group $\mathcal{G}_0(P)$ acts trivially on holonomy; that is, for all $\phi \in \mathcal{G}_0(P)$ and all $A \in \mathcal{A}_{\text{flat}}(P)$ we have $\rho_{\phi_* A} = \rho_A$.

Holonomy defines a map

$$\mathcal{H}\colon \coprod_{[P]} \mathcal{A}_{\text{flat}}(P) \to \text{Hom}(\pi_1(M, m), G) \tag{11}$$

via the formula $\mathcal{H}(A) = \rho_A$. The disjoint union ranges over some chosen set of representatives for the *unbased* isomorphism classes of (based) principal G–bundles. In other words, we choose a set of representatives for the unbased isomorphism classes, and then choose, arbitrarily, a basepoint in each representative (at which we compute holonomy).

Proposition 8.2 shows that we have a diagram

$$\begin{array}{ccc} \coprod_{[P]} \mathcal{A}_{\text{flat}}(P) & \xrightarrow{\quad\mathcal{H}\quad} & \text{Hom}(\pi_1(M, m_0), G) \\ & \searrow \qquad \nearrow_{\overline{\mathcal{H}}} & \\ & \coprod_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P), & \end{array}$$

and our next goal is to explain the equivariance properties of this diagram. When G is connected, Lemma 3.3 shows that G acts on the space

$$\coprod_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P).$$

Specifically, the action of $g \in G$ on an equivalence class $[A] \in \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P)$ is given by the formula

$$g \cdot [A] = [(\phi^g)_* A],$$

where $\phi^g \in \mathcal{G}(P)$ is any gauge transformation satisfying $(\phi^g)_{m_0} = g$. We can now state the main result of this appendix.

Proposition 8.4 *The holonomy map defines a (continuous) bijection*

$$\overline{\mathcal{H}}\colon \coprod_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P) \to \text{Hom}(\pi_1 M, G),$$

and if G is connected then this map is G–equivariant with respect to the G–action described above.

Proof We begin by noting that equivariance is immediate from Proposition 8.2, and continuity of the holonomy map is immediate from its definition in terms of integral curves of vector fields (here we are thinking of the C^∞–topology on $\mathcal{A}_{\text{flat}}(P)$).

In order to prove bijectivity of $\overline{\mathcal{H}}$, we will need to introduce the mixed bundles associated to representations $\rho\colon \pi_1 M \to G$. This will provide a proof of surjectivity. Injectivity requires a slightly tricky argument, based on the idea that maps between flat bundles can be described in terms of parallel transport.

Let $\rho\colon \pi_1 M \to G$ be a representation. We define the *mixed bundle* $E_\rho = \widetilde{M} \times_\rho G$ by

$$E_\rho = \left(\widetilde{M} \times G\right) \Big/ (x, g) \sim (x \cdot \gamma, \rho(\gamma)^{-1} g)$$

Here $\widetilde{M} \xrightarrow{\pi_{\widetilde{M}}} M$ is the universal cover of M, equipped with a basepoint $\widetilde{m}_0 \in \widetilde{M}$ lying over $m_0 \in M$.

It is easy to check that E_ρ is a principal G–bundle on M, with projection $[(\widetilde{m}, g)] \mapsto \pi_{\widetilde{M}}(\widetilde{m})$. We denote this map by $\pi_\rho\colon E_\rho \to M$. Note that since we have chosen basepoints $m_0 \in M$ and $\widetilde{m}_0 \in \widetilde{M}$, E_ρ acquires a canonical basepoint $[(\widetilde{m}_0, e)] \in E_\rho$ making E_ρ a based principal G–bundle. (Here $e \in G$ denotes the identity element.)

The trivial bundle $\widetilde{M} \times G$ has a natural horizontal connection, which descends to a canonical flat connection $\mathbb{A}_\rho$ on the bundle E_ρ. This connection is given by the formula

$$\mathbb{A}_\rho\left([\tilde{x}, g], \vec{\mathbf{v}}_x\right) = Dq\left(\left(D_{\tilde{x}} \pi_{\widetilde{M}}\right)^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_g\right).$$

On the left, $x \in M$, $\vec{\mathbf{v}}_x \in T_x M$, $\tilde{x} \in \pi_{\widetilde{M}}^{-1}(x) \subset \widetilde{M}$, and $g \in G$. On the right, $\vec{\mathbf{0}}_g \in T_g G$ denotes the zero vector, q denotes the quotient map $\widetilde{M} \times G \to \widetilde{M} \times_\rho G = E_\rho$, and

$D_{\tilde{x}}\pi_{\widetilde{M}}$ is invertible because $\pi_{\widetilde{M}}\colon \widetilde{M} \to M$ is a covering map. We leave it to the reader to check that the connection $\mathbb{A}_\rho$ is flat, with holonomy representation $\mathcal{H}(\mathbb{A}_\rho) = \rho$. This proves surjectivity of the holonomy map.

Injectivity will follow from:

Proposition 8.5 *Let (P, p_0) and (Q, q_0) be based principal G–bundles over M with flat connections A_P and A_Q, respectively. If $\mathcal{H}(A_P) = \mathcal{H}(A_Q)$, then there is a based isomorphism $\phi\colon P \to Q$ such that $\phi_* A_P = A_Q$.*

The map ϕ is defined by setting $\phi(p_0 \cdot g) = q_0 \cdot g$ and then extending via parallel transport:

$$\phi(p) = T_\gamma^{A_Q} \circ \phi \circ T_{\overline{\gamma}}^{A_P},$$

where $\gamma\colon [0,1] \to M$ is any path with $\gamma(0) = m_0$ and $\gamma(1) = \pi(p)$. Using the fact that $\mathcal{H}(A_P) = \mathcal{H}(A_Q)$, one may check that ϕ is well-defined.

To prove that $\phi_* A_P = A_Q$, we work locally. Given $p \in P$, choose a path $\gamma\colon [0,1] \to M$ with $\gamma(0) = m_0$, $\gamma(1) = \pi_P(p)$ and cover $\gamma([0,1]) \subset M$ by open sets $U_1, \ldots, U_k$ over which the connections A_P and A_Q are both trivial. We may now subdivide γ into subpaths $\gamma_i\colon [t_{i-1}, t_i] \to M$, where $i = 1, \ldots, k$, $t_0 = 0, t_1 = 1$ and (after renumbering the U_i if necessary) $\gamma_i([t_{i-1}, t_i]) \subset U_i$. Since A_P and A_Q are both trivial over U_i, we may choose isomorphisms

$$\psi_i\colon P|_{U_i} \to Q|_{U_i}$$

such that $(\psi_i)_* A_P = A_Q$. Moreover, we may choose the ψ_i in order and assume that $\psi_1(p_0) = q_0$, and then (inductively) we may assume that $\psi_i = \psi_{i+1}$ on the fiber over t_i (here we use the fact that the trivial connection on $U_i \times G$ is fixed by the constant gauge transformation $(u, g) \mapsto (u, hg)$ for any $h \in G$). It will now suffice to check that $\psi_i = \phi|_{U_i}$. This is easily proven by induction on k, using the following lemma.

Lemma 8.6 *Let $\phi\colon (P, p_0) \to (Q, q_0)$ be a map of principal G–bundles, and let A be a flat connection on P. Then for any $p \in P$, $\phi(p)$ is given by the formula*

$$\phi(p) = T_\gamma^{\phi_* A} \circ \phi_{m_0} \circ T_{\overline{\gamma}}^{A}(p),$$

where $\gamma\colon [0,1] \to M$ is any smooth path with $\gamma(0) = m_0$ and $\gamma(1) = \pi_P(p)$, and ϕ_{m_0} is the restriction of ϕ to the fiber over m_0 (so $\phi_{m_0}(p_0 \cdot g) = q_0 \cdot g$).

This completes the proof of Proposition 8.4. □

As an easy consequence of this result, one obtains the more well-known bijection between (unbased) isomorphism classes of flat connections and conjugacy classes of homomorphisms. A proof of the latter result is given by Morita [35, Theorem 2.9]; however, Morita does not prove an analogue of Proposition 8.5 and consequently his argument does not make the injectivity portions of these results clear.

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