

Numerical computation of the Tau approximation for the Volterra-Hammerstein integral equations

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Abstract In this work, we propose an extension of the algebraic formulation of the Tau method for the numerical solution of the nonlinear Volterra-Hammerstein integral equations. This extension is based on the operational Tau method with arbitrary polynomial basis functions for constructing the algebraic equivalent representation of the problem. This representation is an special semi lower triangular system whose solution gives the components of the vector solution. We will show that the operational Tau matrix representation for the integration of the nonlinear function can be represented by an upper triangular Toeplitz matrix. Finally, numerical results are included to demonstrate the validity and applicability of the method and some comparisons are made with existing results. Our numerical experiments show that the accuracy of the Tau approximate solution is independent of the selection of the basis functions.

Keywords Operational Tau method · Polynomial approximation · Volterra-Hammerstein integral equations · Algebraic nonlinearity · Numerical treatments

Mathematics Subject Classifications (2000) Primary 45G05 · Secondary 65R20

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1 Introduction

Nonlinear Volterra integral equations of the second kind often occur in Hammerstein form:

$$u(t) = f(t) + \int_0^t k(t, s)G(s, u(s))ds, \quad t \in [0, T] \quad (1.1)$$

where f , k and G are smooth given functions with $G(s, u)$ nonlinear in u , and $u(t)$ is a solution to be determined. These types of equations arise as a reformulation of some initial value problems for ordinary differential equations where the kernel is of convolution type and the free term is a polynomial whose coefficients are essentially given by the initial values [1]. Also, multi dimensional analogous and mixed type of these equations appear as various reformulation of nonlinear Volterra integro-differential equations and the dynamic model of a chemical reactor. Detailed description and analysis of these models may be found in [2–4].

The numerical solvability of the (1.1) and other related equations has been pursued by several authors. Brunner [1] applied the implicitly linear collocation method for (1.1) and discussed its connection with the iterated collocation method. Guoqiang [5] introduced and discussed the asymptotic error expansion of a collocation type method for Volterra-Hammerstein integral equations. Elnegar et al. in [6] were concerned with the Chebyshev spectral solution of (1.1). In [7], the rationalized Haar functions are developed to approximate solution of (1.1). Yalcinbas [12] has been concerned with the Taylor polynomials of certain nonlinear Volterra-Fredholm integral equations with algebraic nonlinearity. Recently, Shahmorad et al. [8–11] have been concerned with the numerical solution of integro-differential equations and some linear classes of (1.1) via the Tau method with the arbitrary bases especially Chebyshev and Legendre bases.

Spectral methods have been studied intensively in the last two decades because of their good approximation properties. The Tau method, through which the spectral methods can be described as special case, as shown in El-Daou and Ortiz [18, 22] and [17], has found extensive application for numerical solution of many operator equations in recent years. There has been considerable interest in solving integral equations using Tau methods (see e.g. [8–11]). Here, we are interested in the numerical Tau approximation of the nonlinear Volterra-Hammerstein integral equations. Our discussion based on the operational Tau method which proposed by Ortiz and Samara in [13–15]. We give a new approach of the operational Tau approximation for the numerical solution of the nonlinear integral equations. Detailed description, analysis and applications of the operational Tau method may be found in [13–15] and references therein.

To clarify the concepts of operational approach of the Tau method, we need assert some preliminaries and notations in Section 2 and then we introduce the new Tau algorithm and matrix representation of the method for linear integral equations as an auxiliary result. In Section 3, using an upper triangular

Toeplitz matrix we obtain the operational Tau matrix representation for the Volterra-Hammerstein integral equations as a semi lower triangular nonlinear system, which can be solved using forward substitution. Finally in Section 4, some numerical experiments are reported to clarify the method and some comparisons are made with existing methods in the literature.

2 Preliminaries

For any integrable functions $u(t)$ and $v(t)$ on $[0, T]$, we define the scalar product $\langle \cdot, \cdot \rangle_w$ by

$$\langle u(t), v(t) \rangle_w = \int_0^T u(t)v(t)w(t)dt,$$

$$\|u\|_w^2 = \langle u(t), u(t) \rangle_w,$$

where $w(t)$ is a positive weight function. Let $\phi_t = \{\phi_k(t)\}_{k=0}^\infty$ be a given set of arbitrary polynomial basis which are orthogonal with respect to the weight function $w(t)$ on $[0, T]$ and $L_w^2[0, T]$ is the space of all functions $f : [0, T] \rightarrow \mathbb{R}$, with $\|f\|_w^2 < \infty$.

Consider the operator equation $Iu = f$, where $f(t), u(t) \in L_w^2[0, T]$, and I is a linear integral operator. From the classical theory of well known Banach fixed point theorem, it follows that the equation has a unique solution. Existence and uniqueness results for solution of (1.1) may be found in [5, 6, 23]. We are seeking the exact polynomial solution to a perturbed projection of this operator equation onto a finite dimensional space. Orthogonal series expansion of the exact solution $u(t)$, using defined bases can be considered as:

$$u(t) = \sum_{i=0}^\infty a_i \phi_i(t),$$

where a_i 's are constant coefficients.

Let $\mathbf{\Pi}$ denote the infinite matrix of the Tau method generated by the linear operator I with respect to the orthogonal polynomial basis ϕ_t , and consider the linear Volterra integral equation of the second kind:

$$(Iu)(t) = u(t) - \int_0^t k(t, s)u(s)ds = f(t), \quad t \in [0, T] \quad (2.1)$$

where k is a bivariate given continuous function and $u(t)$ is a solution to be determined.

We define $u_n(t)$ as an approximation function of the exact solution $u(t)$ as follows:

$$u_n(t) = \sum_{i=0}^n a_i \phi_i(t) = \mathbf{a} \phi_t = \mathbf{a} \Phi \mathbf{X}_t,$$

where Φ is a non-singular lower triangular coefficient matrix given by $\phi_t = \Phi \mathbf{X}_t$ with a standard basis vector $\mathbf{X}_t = [1, t, t^2, \dots]^T$ and $\mathbf{a} = [a_0, a_1, \dots, a_n, 0, 0, \dots]$.

For obtaining the matrix vector multiplication representation for the integral term, we assume that

$$k(t, s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k_{i,j} \phi_i(t) \phi_j(s),$$

such that by rearranging it we can write

$$k(t, s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \tilde{k}_{i,j} t^i s^j.$$

So, the integral term of (2.1) can be written as

$$\int_0^t k(t, s) u(s) ds = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \tilde{k}_{i,j} t^i \mathbf{a} \Phi \int_0^t s^j \mathbf{X}_s ds = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \tilde{k}_{i,j} t^i \mathbf{a} \Phi \mathbf{X}'_t(i, j),$$

where $\mathbf{X}'_t(i, j) = \left[\frac{t^{j+1+l}}{i+1+l} \right]_{l=0}^{\infty}$ ($i, j = 0, 1, 2, \dots$), and using simple computations we derive

$$\int_0^t k(t, s) u(s) ds = \mathbf{a} \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \begin{bmatrix} \frac{\tilde{k}_{i,j} t^{i+1+j}}{i+1} & 0 & 0 & 0 & \dots \\ 0 & \frac{\tilde{k}_{i,j} t^{i+1+j}}{i+2} & 0 & 0 & \dots \\ 0 & 0 & \frac{\tilde{k}_{i,j} t^{i+1+j}}{i+3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \mathbf{X}_t. \tag{2.2}$$

Let $\mathbf{K} = [K_{i,j}]_{i,j=0}^{\infty}$ be an infinite matrix such that all its elements are zero except the elements $(K)_{l,l+r+1} = \sum_{i=0}^r \frac{\tilde{k}_{i,r-i}}{r+i+l+1}$, ($l, r = 0, 1, 2, \dots$), the formula (2.2) can be written in compact form:

$$\int_0^t k(t, s) u(s) ds = \mathbf{a} \Phi \mathbf{K} \mathbf{X}_t.$$

Similarly, for the free term $f(t)$ of the (2.1) we set:

$$f(t) = \sum_{i=0}^{\infty} f_i \phi_i(t) = \mathbf{f} \phi_t,$$

where $\mathbf{f} = [f_0, f_1, \dots]$.

Thus, the matrix vector multiplication representation for the (2.1) is as follows:

$$\mathbf{a} \phi_t - \mathbf{a} \Phi \mathbf{K} \mathbf{X}_t = \mathbf{a} \phi_t - \mathbf{a} \Phi \mathbf{K} \Phi^{-1} \phi_t = \mathbf{f} \phi_t.$$

Let us introduce the matrices $\mathbf{K}_\Phi = \Phi \mathbf{K} \Phi^{-1}$ and $\mathbf{\Pi} = \mathbf{I} - \mathbf{K}_\Phi$, where \mathbf{I} represents the identity matrix. With these notations, this equation can be symbolically expressed as

$$\mathbf{a}\mathbf{\Pi}\phi_t = \mathbf{f} \phi_t. \tag{2.3}$$

Due to orthogonality of $\{\phi_i(t)\}_{i=0}^\infty$, projection of (2.3) on the basis functions yields

$$\left\langle \sum_{i=0}^\infty \mathbf{a} \mathbf{\Pi}_i \phi_i(t), \phi_j(t) \right\rangle_w = \left\langle \sum_{i=0}^\infty f_i \phi_i(t), \phi_j(t) \right\rangle_w, \quad j = 0, 1, 2, \dots$$

where $\mathbf{\Pi}_i$ is the i th column of $\mathbf{\Pi}$. So, orthogonality assumption of $\{\phi_i(t)\}_{i=0}^\infty$ leads to the following approximate system of equations

$$\mathbf{a} \mathbf{\Pi}_j = f_j, \quad j = 0, 1, 2, \dots, n$$

Using a finite truncated series of the above equations including the first $(n + 1)$ terms and solving the resulting system of equations, we can obtain the vector solution \mathbf{a} and so the approximate solution $u_n(t)$ will be determined.

3 Outline of the method for nonlinear integral equations

As a consequence of the previous section, in this section we derive formulas for numerical solvability of nonlinear integral equation (1.1) based on arbitrary polynomial basis functions of the operational Tau method.

3.1 Nonlinear function approximation

Let the nonlinear analytic function $G(t, u(t))$ defined on $[0, T] \times \mathbb{R}$, be approximated as:

$$G(t, u(t)) \simeq \sum_{i=0}^n \gamma_i(t) u^i(t).$$

This relation shows that the use of the Tau method requires that $u^i(t)$ must be written as the product of a matrix and a vector. The following result is concerned with approximation of the nonlinear function:

Lemma 1 *Let $v(t) = \sum_{i=0}^\infty v_i \phi_i(t) = \mathbf{v} \Phi \mathbf{X}_t$ be a polynomial with*

$$\mathbf{v} = [v_0, v_1, v_2, \dots], \quad \Phi = [\varphi_{i,j}]_{i,j=0}^\infty, \quad \mathbf{X}_t = [1, t, t^2, \dots]^T,$$

then for any natural number $p \in \mathbb{N}$, we have

$$v^p(t) = \mathbf{v} \Phi \mathbf{B}^{p-1} \mathbf{X}_t,$$

where \mathbf{B} is an infinite upper triangular Toeplitz matrix having the following structure

$$\mathbf{B} = \begin{bmatrix} \mathbf{v}\Phi_0 & \mathbf{v}\Phi_1 & \mathbf{v}\Phi_2 & \dots \\ \mathbf{0} & \mathbf{v}\Phi_0 & \mathbf{v}\Phi_1 & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{v}\Phi_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with $\Phi_i = [\varphi_{i,j}]_{j=0}^\infty$.

Proof The validity of the Lemma for $p = 1$ is obvious. Let $v^2(t) = (\mathbf{v}\Phi\mathbf{X}_t) \times (\mathbf{v}\Phi\mathbf{X}_t)$. Simple manipulation we conclude

$$v^2(t) = \mathbf{v}\Phi(\mathbf{X}_t \times (\mathbf{v}\Phi\mathbf{X}_t)).$$

Now, we will show that

$$\mathbf{X}_t \times (\mathbf{v}\Phi\mathbf{X}_t) = \mathbf{B}\mathbf{X}_t.$$

We may set

$$\mathbf{X}_t \times (\mathbf{v}\Phi\mathbf{X}_t) = \mathbf{X}_t \times \left(\sum_{s=0}^\infty \sum_{r=0}^\infty v_r \varphi_{r,s} t^s \right) = \left[\sum_{s=0}^\infty \sum_{r=0}^\infty v_r \varphi_{r,s} t^{s+i} \right]_{i=0}^\infty,$$

and

$$\mathbf{B}\mathbf{X}_t = \left[\sum_{j=0}^\infty B_{ij} t^j \right]_{i=0}^\infty = \left[\sum_{j=0}^\infty \left(\sum_{r=0}^\infty v_r \varphi_{r,j-i} \right) t^j \right]_{i=0}^\infty.$$

Concerning $B_{ij} = 0$, for $i > j$, it follows that

$$\mathbf{B}\mathbf{X}_t = \left[\sum_{j=i}^\infty \left(\sum_{r=0}^\infty v_r \varphi_{r,j-i} \right) t^j \right]_{i=0}^\infty,$$

and by rearranging the indices we get

$$\mathbf{B}\mathbf{X}_t = \left[\sum_{j=0}^\infty \sum_{r=0}^\infty v_r \varphi_{r,j} t^{j+i} \right]_{i=0}^\infty,$$

which states the Lemma holds for $p = 2$. Now, we proceed by induction. So, we assume the validity of the Lemma for k and transit to $k + 1$ as follows:

$$\begin{aligned} v^{k+1}(t) &= v^k(t)v(t) = (\mathbf{v}\Phi\mathbf{B}^{k-1}\mathbf{X}_t) \times (\mathbf{v}\Phi\mathbf{X}_t) = \mathbf{v}\Phi\mathbf{B}^{k-1}(\mathbf{X}_t \times (\mathbf{v}\Phi\mathbf{X}_t)) \\ &= \mathbf{v}\Phi\mathbf{B}^{k-1}(\mathbf{B}\mathbf{X}_t) = \mathbf{v}\Phi\mathbf{B}^k\mathbf{X}_t. \end{aligned}$$

□

3.2 Operational Tau matrix representation for the integration term

In this section we present the operational Tau representation of the integration term for a class of Volterra-Hammerstein integral equations. Actually, we will show that the effect of integrating $k(t, s)u^p(s)$ will be represented as the product of a matrix and vector. We give the following theorem which its proof is based mainly on Lemma 1.

Theorem 1 *Suppose that the analytic functions $u(s)$ and $k(t, s)$ can be expressed as:*

$$u(s) = \sum_{i=0}^{\infty} a_i \phi_i(s) = \mathbf{a} \Phi \mathbf{X}_s, \quad k(t, s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k_{i,j} \phi_i(t) \phi_j(s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \tilde{k}_{i,j} t^i s^j,$$

where $\mathbf{a} = [a_0, a_1, \dots]$, Φ is a non-singular lower triangular matrix and $\mathbf{X}_s = [1, s, s^2, \dots]^T$, then we have

$$\int_0^t k(t, s)u^p(s)ds = \mathbf{a} \Phi \mathbf{B}^{p-1} \mathbf{M} \mathbf{X}_t,$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & \tilde{k}_{0,0} & \tilde{k}_{0,1} + \frac{1}{2}\tilde{k}_{1,0} & \tilde{k}_{0,2} + \frac{1}{2}\tilde{k}_{1,1} + \frac{1}{3}\tilde{k}_{2,0} & \dots \\ 0 & 0 & \frac{1}{2}\tilde{k}_{0,0} & \frac{1}{2}\tilde{k}_{0,1} + \frac{1}{3}\tilde{k}_{1,0} & \dots \\ 0 & 0 & 0 & \frac{1}{3}\tilde{k}_{0,0} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & \dots & 0 & \frac{1}{n}\tilde{k}_{0,0} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

and \mathbf{B} has been given in Lemma 1.

Proof Using Lemma 1, we have:

$$u^p(s) = \mathbf{a} \Phi \mathbf{B}^{p-1} \mathbf{X}_s,$$

also

$$k(t, s)u^p(s) = \mathbf{a} \Phi \mathbf{B}^{p-1} [k(t, s), sk(t, s), \dots]^T.$$

Noting that for any $m \geq 0$, we can write:

$$k(t, s)s^m = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \tilde{k}_{i,j} t^i s^{m+j},$$

so, the desired integration term can be written as:

$$\begin{aligned} \int_0^t k(t, s)u^p(s)ds &= \mathbf{a}\Phi\mathbf{B}^{p-1} \left[\int_0^t k(t, s)s^m ds \right]_{m=0}^\infty \\ &= \mathbf{a}\Phi\mathbf{B}^{p-1} \left[\sum_{i=0}^\infty \sum_{j=0}^\infty \tilde{k}_{i,j}t^i \frac{t^{m+j+1}}{m+j+1} \right]_{m=0}^\infty. \end{aligned}$$

On the other hand, we can show

$$\sum_{i=0}^\infty \sum_{j=0}^\infty \tilde{k}_{i,j} \frac{t^{m+j+1+i}}{m+j+1} = \left[\frac{1}{m+j+1} \right]_{j=0}^\infty \tilde{\mathbf{k}}(m)\mathbf{X}_t,$$

such that $\tilde{\mathbf{k}}(m)$ is a matrix having the following entries

$$\tilde{k}_{i,j}(m) = \begin{cases} \tilde{k}_{i,j-i-1-m}, & j > m+i, \\ 0, & j \leq m+i. \end{cases}$$

Therefore, we can write

$$\begin{aligned} \int_0^t k(t, s)u^p(s)ds &= \mathbf{a}\Phi\mathbf{B}^{p-1} \left[\left[\frac{1}{m+i+1} \right]_{i=0}^\infty \tilde{\mathbf{k}}(m)\mathbf{X}_t \right]_{m=0}^\infty, \\ &= \mathbf{a}\Phi\mathbf{B}^{p-1} \left[\left[\frac{1}{m+i+1} \right]_{i=0}^\infty \tilde{\mathbf{k}}(m) \right]_{m=0}^\infty \mathbf{X}_t, \\ &= \mathbf{a}\Phi\mathbf{B}^{p-1}\mathbf{M}\mathbf{X}_t. \end{aligned}$$

□

3.3 Numerical Tau approximation of the Volterra-Hammerstein equations

Here we apply the previous results for constructing the Tau approximate solution of the Volterra-Hammerstein integral equation (1.1). Two typical applied nonlinearities (e.g. algebraic and exponential nonlinearity) arise naturally in the modeling of n^{th} order isothermal, irreversible reaction in a planner geometry or in steady two-dimensional heat transfer in a fin placed in a vacuum environment and in the study of magneto hydrodynamics and biological models [19–21].

Without loss of generality and owing to the large variety of kernels and nonlinearities that occur in practice, we can assume that in (1.1), the nonlinear analytic function may be expanded as

$$G(s, u(s)) \simeq \sum_{p=0}^n \gamma_p(s)u^p(s).$$

Now, we consider the Tau approximation of the (1.1), as follows:

$$u(t) = f(t) + \sum_{p=0}^n \int_0^t k(t, s) \gamma_p(s) u^p(s) ds, \quad t \in [0, T]$$

in which it can be written as

$$u(t) = f(t) + \int_0^t k(t, s) \gamma_0(s) ds + \sum_{p=1}^n \int_0^t k(t, s) \gamma_p(s) u^p(s) ds, \quad t \in [0, T]. \quad (3.1)$$

Let us set $\bar{k}_{(p)}(t, s) = k(t, s) \gamma_p(s)$, where $\gamma_p(s)$, ($p = 0, \dots, n$) are continuous functions. Using the given notations in Theorem 1, (3.1) can be transformed to the following matrix form:

$$\mathbf{a} \Phi \mathbf{X}_t = \mathbf{f} \Phi \mathbf{X}_t + \mathbf{M}_0 \mathbf{X}_t + \sum_{p=1}^n \mathbf{a} \Phi \mathbf{B}^{p-1} \mathbf{M}_p \mathbf{X}_t, \quad (3.2)$$

where for $p = 0, \dots, n$

$$\mathbf{M}_p = \begin{bmatrix} 0 & \bar{k}_{(p)0,0} & \bar{k}_{(p)0,1} + \frac{1}{2} \bar{k}_{(p)1,0} & \bar{k}_{(p)0,2} + \frac{1}{2} \bar{k}_{(p)1,1} + \frac{1}{3} \bar{k}_{(p)2,0} & \dots \\ 0 & 0 & \frac{1}{2} \bar{k}_{(p)0,0} & \frac{1}{2} \bar{k}_{(p)0,1} + \frac{1}{3} \bar{k}_{(p)1,0} & \dots \\ 0 & 0 & 0 & \frac{1}{3} \bar{k}_{(p)0,0} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & \dots & 0 & \frac{1}{n} \bar{k}_{(p)0,0} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Equation (3.2) can be written as:

$$\mathbf{a} \Phi \mathbf{X}_t = \mathbf{f} \Phi \mathbf{X}_t + \mathbf{M}_0 \Phi^{-1} \Phi \mathbf{X}_t + \sum_{p=1}^n \mathbf{a} \Phi \mathbf{B}^{p-1} \mathbf{M}_p \Phi^{-1} \Phi \mathbf{X}_t,$$

and so

$$\mathbf{a} \phi_t = \mathbf{f} \phi_t + \mathbf{M}_0 \Phi^{-1} \phi_t + \mathbf{a} \Phi \sum_{p=1}^n \mathbf{B}^{p-1} \mathbf{M}_p \Phi^{-1} \phi_t.$$

As we pointed out in the previous section, the orthogonality of $\{\phi_j(t)\}_{j=0}^\infty$, enables us to project the above equation on the $\{\phi_i(t)\}_{i=0}^n$, so we have

$$\mathbf{a} = \mathbf{f} + \mathbf{M}_0 \Phi^{-1} + \mathbf{a} \Phi \sum_{p=1}^n \mathbf{B}^{p-1} \mathbf{M}_p \Phi^{-1}. \quad (3.3)$$

According to structure of the matrices Φ , \mathbf{M}_p and upper triangular Toeplitz matrix \mathbf{B} with the same diagonal entries, an special upper triangular system of $(n + 1) \times (n + 1)$ nonlinear equations is obtained whose solution gives the unknown components of the vector $\mathbf{a} = [a_0, a_1, \dots, a_n]$. In what follows, we give

more details regarding the structure of the above mentioned nonlinear system and the analysis of obtaining the solution procedure.

Remark 1 Clearly, for $G(s, u(s)) = u^p(s)$, the method described above remain applicable and system (3.3) can be represented as a simple form

$$\mathbf{a} = \mathbf{f} + \mathbf{a}\Phi\mathbf{B}^{p-1}\mathbf{M}\Phi^{-1}. \tag{3.4}$$

3.4 The construction of Tau approximation system with complexity analysis

In this subsection, we show how to compute the matrices \mathbf{B} , \mathbf{B}^{p-1} and \mathbf{M} , which are required in the construction of nonlinear system (3.4) and then we discuss the analysis of obtaining its solution. In other words, our objectives are to explain the principles underlying the basic direct technique, to illustrate this on the nonlinear system of (3.4) that arise in the Tau numerical solution of Volterra-Hammerstein integral equations with algebraic nonlinearity. The solution of nonlinear system of equations is an important component of many spectral algorithms.

We describe the matrix structure produced by the Tau method owing to their tensor-product nature. In order to describe the key ideas without having to complex notations involving products of matrices and vectors, we will assume the nonlinearity in (1.1) be as an algebraic form (see e.g Remark 1), so the structure of the nonlinear system (3.4) will be discussed. Obviously, the following analysis can be easily extended to the computation of matrices \mathbf{M}_p and the system (3.3).

Multiplying both sides of (3.4) by Φ , we get

$$\mathbf{a}\Phi = \mathbf{f}\Phi + \mathbf{a}\Phi\mathbf{B}^{p-1}\mathbf{M}.$$

Let us introduce $\tilde{\mathbf{a}} = \mathbf{a}\Phi$ and $\tilde{\mathbf{f}} = \mathbf{f}\Phi$ with $\tilde{\mathbf{a}} = [\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n]$ and $\tilde{\mathbf{f}} = [\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_n]$, then the above equation take the form

$$\tilde{\mathbf{a}} = \tilde{\mathbf{f}} + \tilde{\mathbf{a}}\mathbf{B}^{p-1}\mathbf{M}. \tag{3.5}$$

We will show that (3.5) is a semi lower triangular nonlinear system, which can be solved using forward substitution algorithm.

Following the structure of matrix \mathbf{B} in Lemma 1, we may write

$$\mathbf{B} = \mathbf{a} \otimes \begin{bmatrix} \Phi_0 & \Phi_1 & \Phi_2 & \dots \\ \mathbf{0} & \Phi_0 & \Phi_1 & \dots \\ \mathbf{0} & \mathbf{0} & \Phi_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (\mathbf{a}\Phi) \otimes \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \dots \\ \mathbf{0} & \mathbf{e}_1 & \mathbf{e}_2 & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{e}_1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\Phi_i = [\varphi_{i,j}]_{j=0}^\infty$, \mathbf{e}_i and Φ are unit and lower triangular matrices, respectively and \otimes denotes the kronecker product.

We take $\tilde{\mathbf{a}} = \mathbf{a}\Phi$, therefore the matrix \mathbf{B} can be represented as an upper triangular Toeplitz form

$$\mathbf{B} = \begin{bmatrix} \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_n \\ 0 & \tilde{a}_0 & \tilde{a}_1 & \cdots & \tilde{a}_{n-1} \\ 0 & 0 & \tilde{a}_0 & \cdots & \tilde{a}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \tilde{a}_0 \end{bmatrix}.$$

In order to investigate the properties of the product matrix \mathbf{B}^{p-1} , we note that the product of upper triangular Toeplitz matrices is an upper triangular Toeplitz matrix whose principal diagonal entries are the product of the main diagonal entries of the individual matrices and the other minor diagonal entries are as a multiplication of the corresponding and preceding diagonal entries. This property enables us to compute the system solution directly from the preceding relations. Indeed, these are the properties that affect the direct solution of the system.

Through simple calculations the upper triangular Toeplitz matrix \mathbf{B}^{p-1} is as follows:

$$\mathbf{B}^{p-1} = \begin{bmatrix} (\tilde{a}_0)^{p-1} & c_0(\tilde{a}_0)^{p-2}\tilde{a}_1 & c_1(\tilde{a}_0)^{p-3}(\tilde{a}_1)^2 + c_2(\tilde{a}_0)^{p-2}\tilde{a}_2 & \cdots \\ 0 & (\tilde{a}_0)^{p-1} & c_0(\tilde{a}_0)^{p-2}\tilde{a}_1 & \cdots \\ 0 & 0 & (\tilde{a}_0)^{p-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & (\tilde{a}_0)^{p-1} \end{bmatrix},$$

where c_0, c_1, c_2, \dots are constants.

Lemma 2 Consider the Volterra-Hammerstien integral equation (1.1) with algebraic nonlinearity and assume that the functions $G(s, u(s)), k(t, s)$ and $f(t)$ are continuous on $[0, T]$. Then the nonlinear spectral Tau system (3.5) defines a unique Tau solution $u_n(t)$.

Proof Firstly, we show that the entries of vector $\tilde{\mathbf{a}}\mathbf{B}^{p-1}\mathbf{M}$ can be written as

$$\tilde{\mathbf{a}}\mathbf{B}^{p-1}\mathbf{M} = [0, g_1(\tilde{a}_0), g_2(\tilde{a}_0, \tilde{a}_1), g_3(\tilde{a}_0, \tilde{a}_1, \tilde{a}_2), \dots, g_n(\tilde{a}_0, \dots, \tilde{a}_{n-1})]^T, \quad (3.6)$$

where $g_i(\tilde{a}_0, \dots, \tilde{a}_{i-1})$ is a nonlinear function of elements $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_{i-1}$. Actually, we obtain the j th entry of the vector $\tilde{\mathbf{a}}\mathbf{B}^{p-1}\mathbf{M}$ as a function of elements $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_{j-1}$. Due to structure of the matrix \mathbf{M} , it is clear the first entry of

vector $\widetilde{\mathbf{a}}\mathbf{B}^{p-1}\mathbf{M}$, for $j = 0$ is zero and the other entries for $j = 1, \dots, n$, are as follows:

$$\begin{aligned}
 (\widetilde{\mathbf{a}}\mathbf{B}^{p-1}\mathbf{M})_j &= \widetilde{\mathbf{a}}\mathbf{B}^{p-1} [M_{i,j}]_{i=0}^n \\
 &= \widetilde{\mathbf{a}} \left[\overbrace{0, \dots, 0}^i, B_{i,i}^{p-1}(\widetilde{a}_0), B_{i,i+1}^{p-1}(\widetilde{a}_0, \widetilde{a}_1), \dots, B_{i,n}^{p-1}(\widetilde{a}_0, \dots, \widetilde{a}_{n-i}) \right]_{i=0}^n \begin{bmatrix} M_{0,j} \\ M_{1,j} \\ \vdots \\ M_{j-1,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
 &= \widetilde{\mathbf{a}} \begin{bmatrix} \sum_{l=0}^{j-1} B_{0,l}^{p-1}(\widetilde{a}_0, \dots, \widetilde{a}_l) M_{l,j} \\ \sum_{l=0}^{j-2} B_{1,1+l}^{p-1}(\widetilde{a}_0, \dots, \widetilde{a}_l) M_{1+l,j} \\ \vdots \\ \sum_{l=0}^{j-j} B_{j-1,l+j-1}^{p-1}(\widetilde{a}_0, \dots, \widetilde{a}_l) M_{l+j-1,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} \widetilde{a}_0, \widetilde{a}_1, \dots, \widetilde{a}_{j-1}, \widetilde{a}_j, \dots, \widetilde{a}_n \end{bmatrix} \begin{bmatrix} g'_{0,j}(\widetilde{a}_0, \dots, \widetilde{a}_{j-1}) \\ g'_{1,j}(\widetilde{a}_0, \dots, \widetilde{a}_{j-2}) \\ \vdots \\ g'_{j-1,j}(\widetilde{a}_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
 &= \sum_{s=0}^{j-1} \widetilde{a}_s g'_{s,j}(\widetilde{a}_0, \dots, \widetilde{a}_{j-1-s}) = g_j(\widetilde{a}_0, \dots, \widetilde{a}_{j-1}),
 \end{aligned}$$

where for the simplicity we have set $\mathbf{B}^{p-1} = [B_{i,j}^{p-1}]_{i,j=0}^n$ and

$$g'_{i,j}(\widetilde{a}_0, \dots, \widetilde{a}_{j-1-i}) = \sum_{l=0}^{j-1-i} B_{i,l+i}^{p-1}(\widetilde{a}_0, \dots, \widetilde{a}_l) M_{l+i,j}, \quad i = 0, 1, \dots, n.$$

Substituting (3.6) in (3.5) yields

$$\begin{bmatrix} \widetilde{a}_0 \\ \widetilde{a}_1 \\ \vdots \\ \widetilde{a}_n \end{bmatrix} = \begin{bmatrix} \widetilde{f}_0 \\ \widetilde{f}_1 \\ \vdots \\ \widetilde{f}_n \end{bmatrix} + \begin{bmatrix} 0 \\ g_1(\widetilde{a}_0) \\ \vdots \\ g_n(\widetilde{a}_0, \dots, \widetilde{a}_{n-1}) \end{bmatrix}.$$

Finally, we end up with an special system of algebraic equation for the unknown \tilde{a}_i , ($i = 0, 1, \dots, n$) corresponding to the (1.1) which can be solved uniquely using the forward substitution process. \square

From Lemma 2, the matrix (3.5) can be transformed to

$$\tilde{\mathbf{a}} = [\tilde{f}_0, \tilde{f}_1 + g_1(\tilde{a}_0), \tilde{f}_2 + g_2(\tilde{a}_0, \tilde{a}_1), \dots, \tilde{f}_n + g_n(\tilde{a}_0, \dots, \tilde{a}_{n-1})]^T,$$

or equivalently

$$\begin{aligned} \tilde{a}_0 &= \tilde{f}_0, \\ \tilde{a}_1 &= \tilde{f}_1 + g_1(\tilde{a}_0), \\ \tilde{a}_2 &= \tilde{f}_2 + g_2(\tilde{a}_0, \tilde{a}_1), \\ &\vdots \\ \tilde{a}_n &= \tilde{f}_n + g_n(\tilde{a}_0, \dots, \tilde{a}_{n-1}). \end{aligned}$$

Therefore the desired approximation to the solution $u_n(t)$ of (1.1) can be obtained from $\mathbf{a} = \tilde{\mathbf{a}}\Phi^{-1}$.

The following algorithm summarizes the proposed Tau method:

Algorithm 1 The construction of Tau approximation system

Step 1. Choose n , form the arbitrary orthogonal bases: $\phi_i(t), i = 0, 1, \dots, n$ and compute the nonsingular coefficient matrix Φ as $\{\phi_i(t)\}_{i=0}^n = \Phi[1, t, t^2, \dots]^T$.

Step 2. Compute $\mathbf{f} = \mathbf{f}\Phi$ by using orthogonality condition of $\{\phi_i(t)\}_{i=0}^n$ as

$$f(t) \simeq \sum_{i=0}^n f_i \phi_i(t),$$

where $f_i = \frac{1}{\|\phi_i(t)\|_w^2} \int_0^T f(t)\phi_i(t)w(t)dt$ $i = 0, 1, \dots, n$ and $w(t)$ is a positive weight function.

Step 3. Compute the matrices \mathbf{B} , \mathbf{B}^k and \mathbf{M} for $k = 1, \dots, p - 1$ from Lemma 1 and Theorem 1.

Step 4. Take $\tilde{\mathbf{a}} = [\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n]$ and obtain the entries of the vector solution $\tilde{\mathbf{a}}$ from the nonlinear system of equation: $\tilde{\mathbf{a}} = \tilde{\mathbf{a}}\mathbf{B}^{p-1}\mathbf{M} + \mathbf{f}$ as follows:

$$\begin{aligned} \tilde{a}_0 &= \tilde{f}_0, \\ \tilde{a}_1 &= \tilde{f}_1 + \tilde{a}_0^p k_{0,0}, \\ \tilde{a}_2 &= \tilde{f}_2 + \tilde{a}_0^p \left(\tilde{k}_{0,1} + \frac{1}{2}\tilde{k}_{1,0} \right) + \frac{1}{2}c_{k-1,0}\tilde{a}_1^p \tilde{a}_0^{p-2}\tilde{k}_{0,0}, \\ &\vdots \end{aligned}$$

Step 5. Using forward substitution, compute $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n$ from Step 4, and set $\mathbf{a} = \tilde{\mathbf{a}}\Phi^{-1}$, then $u_n(t) = \mathbf{a}[1, t, \dots, t^n]^T$.

Remark 2 For the computational complexity, we note that the cost for storage requirement and computing the matrices \mathbf{M} and \mathbf{B}^{p-1} is $\mathcal{O}(n^2)$ and $\mathcal{O}(p n^2)$ flops, respectively. Therefore, the nonlinear system $\tilde{\mathbf{a}} = \tilde{\mathbf{a}}\mathbf{B}^{p-1}\mathbf{M} + \tilde{\mathbf{f}}$, can be obtained efficiently in $\mathcal{O}(n^2 + n^2 + n^2 + p n^2) = \mathcal{O}((p + 3)n^2)$ flops and can be solved by using step 4, in $\mathcal{O}(\frac{n^2}{2})$ flops.

Remark 3 Clearly, the given analysis can be easily extended to the computation of matrices \mathbf{M}_p and construction of the nonlinear system of equation (3.3), related to the Volterra-Hammerstien integral equation (1.1) with any types of nonlinearities which we refrain from going into details.

4 Numerical experiments and some comments

In this section, four test problems were solved using proposed operational Tau algorithm based on the standard, Chebyshev and Legendre basis functions. All calculations were supported by the Mathematica®. The “Maximal Error” in Tables 1, 2, 3 refers to the maximal difference between approximation and exact solution at the equally spaced, Gauss-Lobatto-Chebyshev and Gauss-Lobatto-Legendre nodal points for the standard, Chebyshev and Legendre bases, respectively. In all cases any non-polynomial term was replaced by a suitable truncated series expansion. The error behaviors of the given method with three different basis functions with respect to other recent numerical methods have been shown in Figs. 1, 2, 3.

Example 1 (From Elnagar and Kazemi [6]) Consider the following Volterra-Hammerstein integral equation:

$$u(x) = f(x) + \int_0^x k(x, t)u^2(t)dt, \quad x \in [0, 1],$$

where

$$f(x) = \frac{-1}{4}x^5 - \frac{2}{3}x^4 - \frac{5}{6}x^3 - x^2 + 1, \quad k(x, t) = xt + 1,$$

with the exact solution: $u(x) = x + 1$. For computational details and numerical implementation of the proposed Tau algorithm, we take $p = 2$ and $n = 5$, so the following simple matrices in the case of Chebyshev basis functions will be obtained:

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 & 0 \\ 1 & 0 & -8 & 0 & 8 & 0 \\ 0 & 5 & 0 & -20 & 0 & 16 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Table 1 Numerical results of Example 2 using operational Tau method with different bases

n	Maximal error			
	Standard base	Chebyshev base	Legendre base	Method in [7]
4	6.60×10^{-3}	4.46×10^{-3}	2.52×10^{-3}	–
8	9.45×10^{-7}	6.66×10^{-7}	3.97×10^{-7}	–
12	8.41×10^{-11}	2.07×10^{-11}	1.28×10^{-11}	–
16	8.91×10^{-16}	6.50×10^{-16}	7.10×10^{-16}	2.00×10^{-4}

$$\mathbf{B} = \begin{bmatrix} \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 & \tilde{a}_4 & \tilde{a}_5 \\ 0 & \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 & \tilde{a}_4 \\ 0 & 0 & \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\ 0 & 0 & 0 & \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 \\ 0 & 0 & 0 & 0 & \tilde{a}_0 & \tilde{a}_1 \\ 0 & 0 & 0 & 0 & 0 & \tilde{a}_0 \end{bmatrix}, \quad \tilde{\mathbf{f}} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -\frac{5}{6} \\ -\frac{2}{3} \\ -\frac{1}{4} \end{bmatrix},$$

and using the given algorithm, we obtain the nonlinear system of equations

$$\begin{aligned}
 \tilde{a}_0 &= 1, \\
 \tilde{a}_1 &= \tilde{a}_0^2, \\
 \tilde{a}_2 &= -1 + \tilde{a}_0\tilde{a}_1, \\
 \tilde{a}_3 &= \frac{-5}{6} + \frac{\tilde{a}_0^2}{2} + \frac{\tilde{a}_1^2}{3} + \frac{2}{3}\tilde{a}_0\tilde{a}_2, \\
 \tilde{a}_4 &= \frac{-2}{3} + \frac{2}{3}\tilde{a}_0\tilde{a}_1 + \frac{1}{2}\tilde{a}_1\tilde{a}_2 + \frac{1}{2}\tilde{a}_0\tilde{a}_3, \\
 \tilde{a}_5 &= \frac{-1}{4} + \frac{\tilde{a}_1^2}{4} + \frac{1}{2}\tilde{a}_0\tilde{a}_2 + \frac{\tilde{a}_2^2}{5} + \frac{2}{5}\tilde{a}_1\tilde{a}_3 + \frac{2}{5}\tilde{a}_0\tilde{a}_4,
 \end{aligned}$$

with the exact solution: $\tilde{a}_0 = 1, \tilde{a}_1 = 1, \tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$. Thus $\tilde{\mathbf{a}} = [1, 1, 0, 0, 0, 0]$ and $u_n(x) = \tilde{\mathbf{a}}X_x$ which is the exact solution of the equation. Actually, following [13, 24], Tau approximation for equations with polynomial solution is exact, while degree of Tau approximation is at least equal to the degree of the polynomial solution. Numerical results of the Chebyshev spectral method proposed by Elnegar et al. in [6] and rationalized Haar functions proposed by Razzaghi et al. in [7] show that the given operational Tau method for equations with polynomial solution is very powerful and give the exact solution in comparison with numerical schemes [6] and [7]. (Under the conditions that the mentioned nonlinear system of equations which arise from the Tau method can be exactly solved.)

Table 2 Numerical results of Example 3 using operational Tau method with different bases

n	Maximal error			
	Standard base	Chebyshev base	Legendre base	Method in [12]
4	9.94×10^{-3}	9.94×10^{-3}	9.94×10^{-3}	2.42×10^{-3}
6	2.26×10^{-4}	2.24×10^{-4}	2.24×10^{-4}	2.30×10^{-4}
8	3.05×10^{-6}	1.22×10^{-6}	3.19×10^{-6}	1.00×10^{-5}
12	1.72×10^{-10}	1.00×10^{-10}	2.23×10^{-10}	–
16	7.41×10^{-15}	2.11×10^{-15}	5.01×10^{-15}	–

Table 3 Numerical results of Example 4 using operational Tau method with different bases

n	Maximal error			
	Standard base	Chebyshev base	Legendre base	Method in [16]
4	9.00×10^{-3}	8.10×10^{-3}	7.20×10^{-3}	8.14×10^{-3}
6	4.50×10^{-4}	1.90×10^{-4}	1.10×10^{-4}	1.90×10^{-4}
8	5.20×10^{-6}	2.70×10^{-6}	2.20×10^{-6}	4.00×10^{-5}
10	5.48×10^{-8}	3.08×10^{-8}	2.48×10^{-8}	–
12	3.59×10^{-10}	2.40×10^{-10}	1.14×10^{-10}	–

Fig. 1 The Tau approximation errors of degree 5 for Example 2. using three different classic bases w.r. to the method in [7]

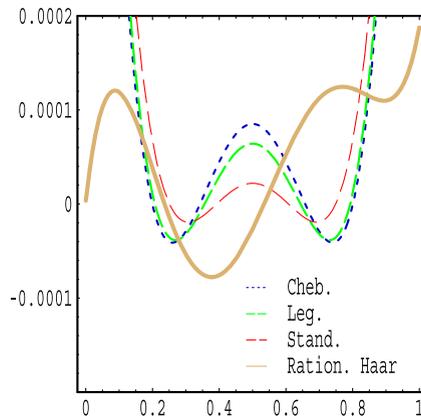


Fig. 2 The Tau approximation errors of degree 6 for Example 3 using three different classic bases w.r. to the method in [12]

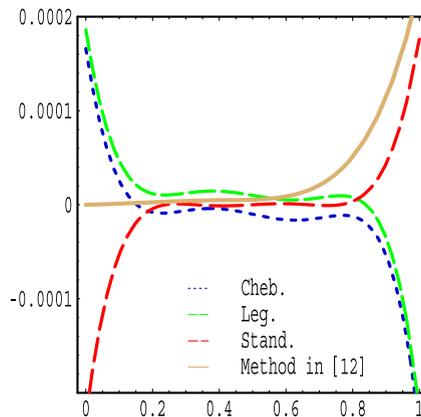
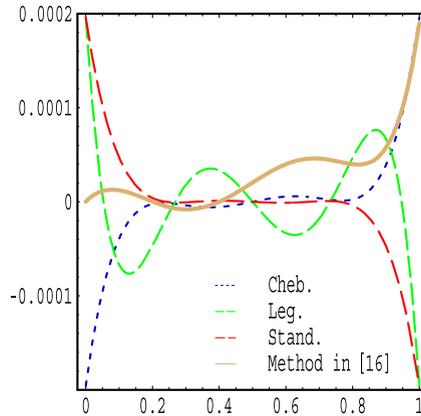


Fig. 3 The Tau approximation errors of degree 6 for Example 4 using three different classic bases w.r. to the method in [16]



Example 2 (From Razzaghi and Ordokhani [7])

$$u(x) = 1 + \sin^2(x) - \int_0^x 3 \sin(x - t)u^2(t)dt, \quad x \in [0, 1],$$

with the exact solution: $u(x) = \cos(x)$. As we expected, Tau approximation has produced highly numerical results with respect to rationalized Haar functions approximation of the problem proposed in [7]. Table 1 represents the error estimates of the method for different basis functions together with the issued maximum error of rationalized Haar functions [7]. The maximum errors listed, show that high accuracy are obtained for $n = 16$ in comparison to the numerical results in [7].

Example 3 (From Yalcinbas [12])

$$u(x) = e^x - \frac{1}{3}e^{3x} + \frac{1}{3} + \int_0^x u^3(t)dt, \quad x \in [0, 1],$$

with the exact solution: $u(x) = e^x$.

Example 4 (From Hadizadeh and Azizi [12])

$$u(x) = -x^3(-1 + e^{\sin x}) + \sin x + \int_0^x x^3 \cos t e^{u(t)} dt, \quad x \in [0, 1],$$

with the exact solution : $u(x) = \sin x$.

Examples 3 and 4 were solved in [12] and [16] by a method based on Taylor series expansion approach. The reported results of the proposed Tau method and the method of [12] and [16] for $n = 4, 6, 8$ show that both methods have produced nearly equivalent approximate solutions for small values of n . However, additional numerical experiments indicate that we can achieve good numerical results for $n \geq 8$. Also, due to some restrictions and more complexity of the Taylor series approximation, in contrast the low computational

complexity of the Tau approximation and its accurate solutions, the comparative effect of our proposed Tau method will become obvious.

It is observed that from our numerical experiments and Tables 1–3, the accuracy of the given Tau method is independent of the selection of basis functions.

5 Conclusion

In this research, a variation of operational Tau method has been used for the approximate solution of the nonlinear Volterra-Hammerstein integral equations. With the availability of this methodology, it will now be possible to investigate the approximate solution of other types of applied nonlinear integral equations. Comparing the given algorithm and other numerical methods [6, 7, 12, 16], we may conclude the low computational complexity and storage requirement with high accuracy of the proposed procedure. The difficulty which we will be faced with is the approximation of the kernels and input functions with appropriate polynomials, which is not very significant due to regular conditions.

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