Stability of N Interacting Queues in Random-Access Systems

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***Abstract*—We revisit the stability problem of systems consisting of N buffered terminals accessing a common receiver over the collision channel by means of the standard ALOHA protocol. We find that in the slotted ALOHA system queues have "instability rank" based on their individual average arrival rates and transmission probabilities. If a queue is stable, then the queue with lower instability rank is stable as well. The instability rank is used to intelligently set up the dominant systems. And the stability inner and outer bounds can be found by bounding the idle probability of some queues in the dominant system. Through analyzing those dominant systems one by one, we are able to obtain inner and outer bounds for stability. These bounds are tighter than the known ones although they still fail to identify the exact stability region for cases of $N > 2$. The methodology used is new and holds promise for successfully addressing other similar stability problems.**

***Index Terms*— Interacting queues, multiple access, slotted ALOHA, stability analysis.**

I. Introduction

The stability problem for bufferless terminals in the ALOHA systems has been extensively studied and is well understood [1]–[3]. In the buffered case, the problem becomes complicated because it involves interacting queues. Previous analyses have yielded only various bounds to the regions of arrival rate values for which the queues are stable [4]–[8]. Exact region identification has been achieved only for the case $N = 2$ [4], [7] and $N = 3$ [6].

Tsybakov and Mikailov provided a rigorous treatment on the problem in [7]. In the same paper, they implicitly used the concept of dominant system and stated rigorously some properties of the dominant system. In [4], Rao and Ephremides explicitly introduced the technique of dominant system, and pointed out that the dominant system, if properly set up, is indistinguishable from the original ALOHA system at saturation. In [6], Szpankowski treated the dominant system more rigorously and obtained a necessary and sufficient condition for stability by using Loynes' stability criteria [9]. Although the concept of dominant system is powerful in deriving the stability bound, how to set up a dominant system intelligently was not addressed before. In this paper, we identify an important property of the ALOHA system: queues have "stability ranks." By using this property, we intelligently set up the dominant system and obtain the improved bounds.

The system we consider is a discrete-time slotted ALOHA system with N terminals. Each terminal has a buffer of infinite capacity to store the incoming packets. Time is slotted. Transmission time of a packet is one slot. The packet arrival process at each terminal is Bernoulli,[1] and arrival processes at different terminals are independent. In each slot, the terminal i attempts to transmit the packet with probability p_i, if its buffer is not empty. If two or more terminals transmit in the same slot, a collision occurs. The packets involved in the collision wait to be retransmitted in the next slot with the same respective probabilities.

In Section II, we briefly set up the stability problem and describe the mathematical foundations which our later discussions are based on. In Section III, we investigate the stability problem. We use the concept of dominance to derive a lower bound in Section III-A. In Section III-B, we identify the relative "rank" of stability of the individual queues, and we obtain an upper bound. In Section III-C, we proceed to obtain the inner and outer stability regions by using the "ranking" technique, and we obtain an improved lower bound. Finally, in Section III-D, numerical results show the improvement of our bound over the previously obtained ones.

II. Background

Consider a slotted ALOHA system with N terminals. The packet arrival rate for the ith $(1 \le i \le N)$ terminal is λ_i. For an N-terminal system, we define an *arrival vector* $\mathbf{\Lambda}$ and a *vector of probabilities* $\boldsymbol{p}$ as follows:

$$\mathbf{\Lambda} \triangleq (\lambda_1, \lambda_2, \cdots, \lambda_N)^T \tag{1}$$

$$\boldsymbol{p} \triangleq (p_1, p_2, \cdots, p_N)^T \tag{2}$$

where $0 \le \lambda_i \le 1$, and $0 \le p_i \le 1$, for $i = 1, 2, \cdots, N$, and where "T" represents vector transposition.

We adopt the definition of stability used in [6].

Definition: Queue i of the system is *stable*, if

$$\lim_{t\to\infty} \Pr\{q_i(t) < x\} = F(x) \quad \text{and} \quad \lim_{x\to\infty} F(x) = 1 \tag{3}$$

where $q_i(t)$ is the size of the queue at time t. If

$$\lim_{x\to\infty} \lim_{t\to\infty} \inf \Pr\{q_i(t) < x\} = 1 \tag{4}$$

[1] In fact, the arrival process at each terminal may be arbitrary as long as it is stationary and ergodic. All theorems, except Theorem 5, are true for such arbitrary processes.

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the queue is *substable*. Obviously, if a queue is stable, then it is also substable. If a queue is not substable, then we say it is unstable.

We say $(\mathbf{\Lambda}, \boldsymbol{p})$ is *stable* or the ALOHA system with $(\mathbf{\Lambda}, \boldsymbol{p})$ is stable, if all the queues in the system are stable. If at least one queue in the system is unstable, the system is unstable.

Let $\boldsymbol{q(t)} \triangleq [q_1(t), q_2(t), \cdots, q_N(t)]$. The vector process $\boldsymbol{q(t)}$ is an irreducible and aperiodic Markov chain with countable number of states [7]. It is either positive recurrent, or null recurrent, or transient [10]. The stability as defined above is equivalent to positive recurrence of the Markov chain. Thus the system is stable if and only if the Markov chain $\boldsymbol{q}(t)$ has positive probability distribution when t approaches to infinity. This criterion is equivalent to the following: The ALOHA system is stable if and only if every queue i $(i = 1, 2, \cdots, N)$ has positive probability of being empty, i.e.,

$$\lim_{t\to\infty} \Pr\{q_i(t) = 0\} > 0.$$

We define the notion a *dominant system* corresponding to the original N-terminal system as in [4]. The *only* difference between the dominant system and the original system is that the ith terminal in the dominant system continues to transmit "dummy" packets with probability p_i even when its queue is empty, if queue i belongs to a designated set V. We denote the original system as Q, and the dominant system with persistent set V as Q^V. The dummy packets can result in collision, but the successful reception of the dummy packet does not reduce the queue size. Thus the dominant system always has larger queue size than the original system if both start from the same initial condition. If the dominant system is stable, then the original system is stable. Also we can see that any changes in the arrival rate or in the queue length of any queue in V have no effect on the other queues in the dominant system. Thus we can get a partially decoupled queuing system when we consider this kind of dominance.

It is clear that if V includes all the queues, and if such a dominant system is still stable, then the original system is also stable. The following lemma states this fact.

Lemma 1: Given $(\mathbf{\Lambda}, \boldsymbol{p})$, if

$$\lambda_i < p_i \prod_{\substack{j\neq i\\ j=1}}^{N} (1 - p_i) \tag{5}$$

for all i $(i = 1, \cdots, N)$, then $(\mathbf{\Lambda}, \boldsymbol{p})$ is stable.

If *all* the queues in V of the dominant system Q^V are unstable, then the original system is also unstable, because in this case, all the queues in V tend to grow to infinity, so the probability of ever sending a dummy packet by a terminal of the dominant system is zero, hence the dominant system is indistinguishable from the original system under saturation. Thus we have the next lemma.

Lemma 2: Given $(\mathbf{\Lambda}, \boldsymbol{p})$, if

$$\lambda_i > p_i \prod_{\substack{j\neq i\\ j=1}}^{N} (1 - p_i)$$

for all i $(i = 1, \cdots, N)$, then $(\mathbf{\Lambda}, \boldsymbol{p})$ is unstable.

Loynes theorem is used throughout the paper. It states that if the arrival process and service process of a queue are all stationary, and the average arrival rate is less than the average service rate, then the queue is stable; if the average arrival rate is greater than the average service rate, the queue is unstable; if they are equal, which only occurs on the boundary of stability region, the queue can be either stable, or substable but not stable, or unstable [9]. In this paper, we are only concerned with the inner and outer stability region. We do not attempt to analyze the stability on the boundary points. To apply Loynes theorem, stationarities of arrival process and service process are required. Stationarity of arrival process is already given. But in the system with interacting queues, stationarity of the service process is not obvious. However, in our approach, this is not a hindrance. The reason is that in a dominant system Q^V, if all the queues outside of V are stable, then they yield a stationary slot availability (i.e., service) process. In addition, the queues in V also yield stationary slot availability process by definition of dominant system. Then the overall service process should also be stationary for any individual queue. This argument is due to Szpankowski who established stationarity and ergodicity of service process in the dominant system in [6].

III. Stability Problem

A. Sufficient Conditions for Stability

In this section we will consider the problem of whether the system is stable for given $(\mathbf{\Lambda}, \boldsymbol{p})$. Rao and Ephremides in [4] solved the case of $N = 2$ and provided approximate results for higher dimensional cases. Szpankowski provided a strict bound for $N = 3$ (see [6]). No exact identification of the stability region is known for the general case of $N > 3$.

The major difficulty, of course, is that the queues are interacting with each other so that we cannot decouple them and track the problem with single-dimensional Markovian analysis. Given the arrival vector $\mathbf{\Lambda}$, there is an optimal vector $\boldsymbol{p}$ so that the average delay is minimized. However, it is difficult to determine this $\boldsymbol{p}$. Even given an arbitrary $(\mathbf{\Lambda}, \boldsymbol{p})$, it is difficult to determine whether it is advantageous to increase any one of $\boldsymbol{p}$'s components. Conversely, to obtain the stable region of arrival rate vector $\mathbf{\Lambda}$ for a given $\boldsymbol{p}$ is equally intractable. So we take the approach of intelligently choosing dominant systems that we can track and that can tightly bound the given system.

So, consider a given pair $(\mathbf{\Lambda}, \boldsymbol{p})$. According to Lemma 1, if for every i, $1 \leq i \leq N$, we have that

$$\lambda_i < p_i \prod_{j\neq i} (1 - p_j) \tag{6}$$

then the system is stable. According to Lemma 2, if for all i, $1 \leq i \leq N$, we have

$$\lambda_i > p_i \prod_{j\neq i} (1 - p_j) \tag{7}$$

then the system is unstable. What interests us is whether the system is stable if for some pairs (λ_k, p_k), say, $1 \leq k < K$, we have

$$\lambda_k < p_k \prod_{j\neq k}(1-p_j) \tag{8}$$

and for other pairs $(K \geq k \leq N)$

$$\lambda_k \geq p_k \prod_{j\neq k}(1-p_j). \tag{9}$$

Lemma 3: In an ALOHA system with $(\mathbf{\Lambda}, \boldsymbol{p})$, if

$$\lambda_i < p_i \prod_{j\neq i}(1-p_j)$$

then queue i is stable.

Proof: Consider a dominant system in which all the queues except queue i transmit dummy packet. Then queue i in this dominant system is a $G/M/1$ Markov chain with service rate $p_i \prod_{j\neq i}(1-p_j)$. Because

$$\lambda_i < p_i \prod_{j\neq i}(1-p_j)$$

queue i is stable in the dominant system, thus it is also stable in the original ALOHA system. □

Now we proceed to obtain the sufficient conditions for the stability of the queues k, for which

$$\lambda_k \geq p_k \prod_{j\neq k}(1-p_j).$$

We set up the dominant system Q^V, for which $V = \{k : K \leq k \leq N\}$; we denote this system as Q^K, when no ambiguity arises. It is clear that if the original system Q is unstable, then the dominant system Q^K is unstable. Furthermore, if the dominant system Q^K is stable, then the original system is also stable. Our goal is to establish a sufficient condition for the stability of the dominant system, which will also be sufficient for the stability of the original system. Consider the first $K-1$ terminals which satisfy (8) in the dominant system Q^K. We define as average service rate μ_i of the ith terminal, the successful transmission probability when terminal i is not empty.

The average service rate μ_i of the ith terminal satisfies the following inequality:

$$\mu_i \geq p_i \prod_{j\neq i}(1-p_j). \tag{10}$$

That this is so follows from the fact that since the first K terminals are stable, $(1-p_j)$ is the "nontransmission" probability of queue j when it is not empty. So $(1-p_j)$ is the underestimation of the overall (unconditional) "nontransmission" probability of queue j.

Assume that the "nontransmission" probability for the first $K-1$ terminals is $P_E^{(K)}$; then

$$\begin{aligned} P_E^{(K)} = {} & P(\text{queue 1 is empty}) \\ & \times P(\text{queues from 2 to } K-1 \text{ do not} \\ & \qquad \text{transmit/queue 1 is empty}) \\ & + P(\text{queue 1 is not empty}) \\ & \times P(\text{queues from 1 to } K-1 \text{ do not} \\ & \qquad \text{transmit/queue 1 is not empty}). \end{aligned} \tag{11}$$

We know that

$$\begin{aligned} & P(\text{all the queues from 2 to } K-1 \text{ do not} \\ & \qquad \text{transmit/queue 1 is empty}) \geq \prod_{j=2}^{K-1}(1-p_j) \end{aligned} \tag{12}$$

and that

$$\begin{aligned} & P(\text{all the queues from 1 to } K-1 \text{ do not} \\ & \qquad \text{transmit/queue 1 is not empty}) \geq \prod_{j=1}^{K-1}(1-p_j) \end{aligned} \tag{13}$$

Furthermore,

$$P(\text{queue 1 is not empty}) = 1 - P(\text{queue 1 is empty}). \tag{14}$$

Substituting from (12)–(14) into (11), we obtain

$$\begin{aligned} P_E^{(K)} \geq {} & P(\text{queue 1 is empty}) \prod_{j=2}^{K-1}(1-p_j) \\ & + (1 - P(\text{queue 1 is empty})) \prod_{j=1}^{K-1}(1-p_j). \end{aligned} \tag{15}$$

We know that queue 1 is stable by Lemma 3. And by Little's theorem

$$P(\text{queue 1 is empty}) = 1 - \frac{\lambda_1}{\mu_1}. \tag{16}$$

Substituting from (10) in (16), we obtain

$$P(\text{queue 1 is empty}) \geq 1 - \frac{\lambda_1}{p_1 \prod_{j=2}^{N}(1-p_j)}. \tag{17}$$

Using (17), we can rewrite (15) as

$$P_E^{(K)} \geq \prod_{j=1}^{K-1}(1-p_j) + p_1 \prod_{j=2}^{K-1}(1-p_j) - \frac{\lambda_1}{\prod_{j=K}^{N}(1-p_j)}. \tag{18}$$

The mth terminal, $m \geq K$, is stable in the dominant system Q^K if

$$\lambda_m < p_m P_E^{(K)} \prod_{\substack{j\neq m \\ K\leq j\leq N}}(1-p_j). \tag{19}$$

By using (18), a sufficient condition for queue m to be stable is

$$\begin{aligned} \lambda_m < {} & p_m \left\{ \prod_{j=1}^{K-1}(1-p_j) + p_1 \prod_{j=2}^{K-1}(1-p_j) - \frac{\lambda_1}{\prod_{j=K}^{N}(1-p_j)} \right\} \\ & \times \prod_{\substack{j\neq m \\ K<j\leq N}}(1-p_j) \\ = {} & p_m \prod_{j\neq m}(1-p_m) + \frac{p_m}{1-p_m}\left[p_1 \prod_{j\neq 1}(1-p_j) - \lambda_1\right]. \end{aligned} \tag{20}$$

Without loss of generality, we assume that

$$p_1 \prod_{j\neq 1}(1-p_j) - \lambda_1 = \max_{1\leq k\leq N}\left[p_k \prod_{j\neq k}(1-p_j) - \lambda_k\right]. \tag{21}$$

Then we obtain the following lemma.

Lemma 4: Given $(\boldsymbol{\Lambda}, \boldsymbol{p})$, if there is an integer K $(2 \le K \le N)$ such that $\lambda_i < p_i \prod_{j \ne i} (1 - p_j)$, for $1 \le i < K$, and

$$\lambda_i \ge p_i \prod_{j \ne i} (1 - p_j)$$

for $K \le i \le N$, and if

$$\lambda_i < p_i \prod_{j \ne i} (1 - p_j) + \frac{p_i}{1 - p_i} \max_{1 \le k \le N} \left[p_k \prod_{j \ne k} (1 - p_j) - \lambda_k \right] \tag{22}$$

for $K \le i \le N$, then the system is stable.

The condition above in the lemma can be written in a concise way. First, we can let $K = 2$; secondly, we note that $\lambda_i \ge p_i \prod_{j \ne i} (1 - p_j)$ for $K \le i \le N$ is not necessary because the violation of this condition simply means smaller arrival rates. Note that the system with smaller arrival rates will be stable if the original one is. So the conditions in Lemma 4 can be summarized as follows:

1)

$$p_1 \prod_{j \ne 1} (1 - p_j) - \lambda_1 = \max_{1 \le k \le N} \left[p_k \prod_{j \ne k} (1 - p_j) - \lambda_k \right] > 0;$$

2) for queues i $(2 \le i \le N)$, (22) is satisfied.

Actually, even for $i = 1$, (22) is satisfied; and if (22) is satisfied for $i = 1$

$$p_1 \prod_{j \ne 1} (1 - p_j) - \lambda_1 > 0.$$

Therefore, we have the following theorem.

Theorem 1: Given $(\boldsymbol{\Lambda}, \boldsymbol{p})$, if for every i, $i = 1, 2, \cdots, N$

$$\lambda_i < p_i \prod_{j \ne i} (1 - p_j) + \frac{p_i}{1 - p_i} \max_{1 \le k \le N} \left[p_k \prod_{j \ne k} (1 - p_j) - \lambda_k \right] \tag{23}$$

then the system is stable.

A concise proof of Theorem 1 consists of setting up a dominant system in which all the queues except queue 1 send dummy packets; and we can then prove that the service rate for queue i $(2 \le i \le N)$, for the dominant system, is given by the expression on the right-hand side of (22). If (22) is satisfied, then queue i is stable in the dominant system, and hence stable in the original system.

Consider the following example. Let $N = 2$; if

$$\lambda_1 < p_1 (1 - p_2) \tag{24}$$

the stable condition for λ_2, according to Theorem 1, is given by

$$\lambda_2 < p_2 (1 - p_1) + \frac{p_2}{1 - p_2} [p_1 (1 - p_2) - \lambda_1] \tag{25}$$

or, equivalently,

$$\lambda_2 < p_2 \left(1 - \frac{\lambda_1}{1 - p_2} \right). \tag{26}$$

Symmetrically, we consider the case where

$$\lambda_2 < p_2 (1 - p_1) \tag{27}$$

and we obtain that

$$\lambda_1 < p_1 \left(1 - \frac{\lambda_2}{1 - p_1} \right). \tag{28}$$

Interestingly, this is the *necessary and sufficient* condition for stability as in [4].

Similar results were also derived in [5]. Later in the paper, we will provide bounds that improve further the one given here.

B. Ranking of the Queues

The previous section discussed a sufficient condition for stability. Although we are not able to obtain a condition that is both sufficient and necessary, we do derive here a separate necessary condition. When $N = 2$, the two conditions coincide. The following theorem is the basis for our discussion thereafter.

Theorem 2: Given $(\boldsymbol{\Lambda}, \boldsymbol{p})$, we order the indices of the terminals $1, 2, \cdots, N$, so that $\lambda_j (1 - p_j)/p_j \le \lambda_i (1 - p_i)/p_i$, if $j < i$. If queue i is stable, then queue j $(j < i)$ is also stable.

Proof: If the system is stable then the theorem is trivial. If the system is unstable and among the unstable queues there is a queue j, for which $\lambda_j (1 - p_j)/p_j$ is less than $\lambda_i (1 - p_i)/p_i$, and queue i is stable, we will arrive at a contradiction. Define U as the set which contains only those queues that are unstable. Hence, if $\bar{U}$ is the complement of U, $\bar{U}$ contains only those queues that are not unstable. Now let us consider the dominant system Q^U; we note that all the queues in U are unstable in Q^U, so Q^U is indistinguishable from Q.

Because queue i is stable, the successful transmission probability is equal to its arrival rate, that is,

$$P(\text{queue } i \text{ transmits and no other queue transmits}) = \lambda_i. \tag{29}$$

Queue i independently decides whether to transmit when nonempty. So

$$\begin{aligned} \lambda_i &= P(\text{queue } i \text{ nonempty and no other queue transmits}) p_i \\ &= P(\text{queue } i \text{ transmits and no other queue transmits}) \end{aligned} \tag{30}$$

and

$$\begin{aligned} &P(\text{queue } i \text{ nonempty, no queue transmits}) \\ &\quad = P(\text{queue } i \text{ nonempty, no other queue transmits})(1 - p_i). \end{aligned} \tag{31}$$

Let us rewrite (30) as

$$P(\text{queue } i \text{ nonempty, no \textit{other} queue transmits}) = \frac{\lambda_i}{p_i} \tag{32}$$

and substitute it into (31); we obtain

$$P(\text{queue } i \text{ nonempty and no queue transmits}) = \frac{\lambda_i(1-p_i)}{p_i}. \tag{33}$$

Furthermore, the probability that no queue in U transmits is equal to $\prod_{j\in U}(1-p_j)$, which is independent of the status of the queues in $\bar{U}$. Thus we have

$$\begin{aligned} &P(\text{queue } i \text{ nonempty, no queue in } \bar{U} \text{ transmits}) \prod_{k\in U}(1-p_k) \\ &\quad = P(\text{queue } i \text{ nonempty, no queue transmits}) \\ &\quad = \frac{\lambda_i(1-p_i)}{p_i}. \end{aligned} \tag{34}$$

It then follows that

$$P(\text{queue } i \text{ nonempty, no queue in } \bar{U} \text{ transmits}) = \frac{\lambda_i(1-p_i)}{p_i\prod_{k\in U}(1-p_k)}. \tag{35}$$

Now we know that queue j is unstable. Therefore,

$$\begin{aligned} \lambda_j &> \mu_j \\ &= \frac{p_j}{1-p_j} P(\text{no queue in } \bar{U} \text{ transmits}) \prod_{k\in U}(1-p_k) \\ &> \frac{p_j}{1-p_j} P(\text{queue } i \text{ nonempty, no queue in } \bar{U} \text{ transmits}) \\ &\quad \times \prod_{k\in U}(1-p_k) \\ &= \frac{p_j}{(1-p_j)} \frac{\lambda_i(1-p_i)}{p_i\prod_{k\in U}(1-p_k)} \prod_{k\in U}(1-p_k) \\ &= \frac{p_j}{(1-p_j)} \frac{(1-p_i)}{p_i}\lambda_i. \end{aligned} \tag{36}$$

However, the last inequality violates the assumption that

$$\lambda_i(1-p_i)/p_i \geq \lambda_j(1-p_j)/p_j$$

and thus it leads to a contradiction. Hence, the statement in Theorem 2 is proven. □

The above theorem offers an illuminating insight. It points out that if a system is unstable then the unstable queues are those that have larger values of $\lambda_i(1-p_i)/p_i$. Furthermore, the theorem holds true for general packet arrival pattern.

By repeating the steps used in the proof of Theorem 2, we obtain the following corollary.

Corollary 1: Given $(\boldsymbol{\Lambda}, \boldsymbol{p})$, and a dominant system Q^k, in which the terminal indices are ordered so that

$$\lambda_1(1-p_1)/p_1 \leq \cdots \leq \lambda_N(1-p_N)/p_N$$

we have that, in system Q^k, for $i < j$, if queue j is stable then queue i is also stable.

Proof: There are three cases: 1) $i < j < k$, 2) $i < k \leq j$, and 3) $k \leq i < j$. For Case 1, the logic is identical with that used in the proof of Theorem 2. For Case 3, if queue j is stable, then it follows that

$$\lambda_j < \frac{p_j}{(1-p_j)} \prod_{l\geq k}(1-p_l)P_E^{(k)} \tag{37}$$

where $P_E^{(k)}$ is the probability that none of the queues from 1 to $k-1$ transmit. Because $\lambda_j(1-p_j)/p_j \geq \lambda_i(1-p_i)/p_i$, we have

$$\lambda_i < \frac{p_i}{(1-p_i)} \prod_{l\geq k}(1-p_l)P_E^{(k)}. \tag{38}$$

So queue i is also stable. Thus we prove the statement for Case 3. For Case 2, if queue j is stable but queue i is not, then from Case 1 we know that in system Q^k all the queues from i to $k-1$ are unstable. So system Q^k is indistinguishable from system Q^i. Then from Case 3, we see that queue j is unstable in the Q^i system because queue i is unstable in that system. So queue j is unstable in system Q^k which contradicts the hypothesis. Hence the corollary has to be true for Case 2 as well. □

Now, consider the order of the indices of the queues as before. For a stable system Q, in the corresponding dominant system Q^K those persistent queues cannot be all unstable. Otherwise, the probability of sending a dummy packet of the dominant system is zero, and hence there is no distinction between the dominant system and the original system. And then the instability of the dominant system implies the instability of the original system. By Corollary 1, the queues from 1 to K should be stable. Indeed, we only need to check whether queue K is stable. If it is stable then queues from 1 to $K-1$ are also stable; if not, the original system is unstable. Based on this idea, we obtain the necessary condition for the stability as follows.

Theorem 3: Given $(\boldsymbol{\Lambda}, \boldsymbol{p})$, and

$$\lambda_1(1-p_1)/p_1 \leq \cdots \leq \lambda_N(1-p_N)/p_N$$

a necessary condition for stability is that for every $k, 1 \leq k \leq N$, we have

$$\frac{\lambda_k(1-p_k)}{p_k} < \prod_{j=k}^{N}(1-p_j) - \sum_{i=1}^{k-1}\lambda_i. \tag{39}$$

Proof: Assume that we have a stable system with parameters $(\boldsymbol{\Lambda}, \boldsymbol{p})$, and that

$$\lambda_1(1-p_1)/p_1 \leq \cdots \leq \lambda_N(1-p_N)/p_N.$$

Given an arbitrary index k, $(1 \leq k \leq N)$, we consider the associated dominant system Q^k. In Q^k, queue k is stable. Hence it follows that

$$\lambda_k < \frac{p_k}{(1-p_k)} P_E^{(k)} \prod_{j=k}^{N}(1-p_j) \tag{40}$$

where $P_E^{(k)}$ is the probability that no terminal from the first to the $(k-1)$th transmits in system Q^k. We also know that the queues from 1 to $k-1$ should be stable, otherwise, the queues from k to N would all be unstable; and then the unstable system Q^k would be indistinguishable from the original system Q. The probability of successful transmission by one of the queues from 1 to $(k-1)$ is given by $\sum_{i=1}^{k-1} \lambda_i$. Because the queues from k to N are sending packets independently, the probability that only one of the queues from 1 to $(k-1)$ transmits a packet is given by

$$\sum_{i=1}^{k-1} \lambda_i \bigg/ \prod_{j=k}^{N} (1-p_j).$$

Therefore, we have

$$P_E^{(k)} < 1 - \frac{\sum_{i=1}^{k-1} \lambda_i}{\prod_{j=k}^{N} (1-p_j)} \tag{41}$$

and, hence, it follows that

$$\begin{aligned} \lambda_k &< \frac{p_k}{(1-p_k)} \left[1 - \frac{\sum_{i=1}^{k-1} \lambda_i}{\prod_{j=k}^{N} (1-p_j)} \right] \prod_{j=k}^{N} (1-p_j) \\ &= \frac{p_k}{(1-p_k)} \left[\prod_{j=k}^{N} (1-p_j) - \sum_{j=1}^{k-1} \lambda_j \right] \end{aligned} \tag{42}$$

which we can rewrite as

$$\frac{\lambda_k(1-p_k)}{p_k} < \prod_{j=k}^{N} (1-p_j) - \sum_{j=1}^{k-1} \lambda_j. \tag{43}$$

Note that k was arbitrarily chosen. Then the proof of the theorem is complete. □

C. Tighter Lower Bound

With the help of the concept of dominant system, we may improve our insight concerning the interaction of the queues in the ALOHA system. If an N-terminal ALOHA system is not stable, we can identify the unstable queues on the basis of the comparison of the values $\lambda_i(1-p_i)/p_i$. Let us assume that all queues j $(k \le j \le N)$ are unstable. Then for the corresponding system Q^k, the same queues j $(k \le j \le N)$ are also unstable. Theorem 2 implies a "rank" of the queues. Specifically, if we order the queues so that

$$\lambda_1(1-p_1)/p_1 \le \cdots \le \lambda_N(1-p_N)/p_N$$

then either all the queues are stable or there is an integer K, such that every queue j $(K \le j \le N)$ is unstable, and every queue i $(1 \le i < K)$ is stable. These observations can be summarized in the following form.

Theorem 4: Given an N-terminal ALOHA system with parameters $(\mathbf{\Lambda}, p)$, and if

$$\lambda_1(1-p_1)/p_1 \le \cdots \le \lambda_N(1-p_N)/p_N$$

the sufficient condition for stability is that

$$\lambda_k < \frac{p_k}{(1-p_k)} \prod_{j=k}^{N} (1-p_j) P_E^{(k)} \tag{44}$$

for all k, $(1 \le k \le N)$; the sufficient condition for instability is that there exists k, $1 \le k \le N$

$$\lambda_k > \frac{p_k}{(1-p_k)} \prod_{j=k}^{N} (1-p_j) P_E^{(k)} \tag{45}$$

where $P_E^{(k)}$ is the probability that none of the queues from 1 to $k-1$ transmit in the corresponding dominant system Q^k.

Proof: The right-hand side of (44) is actually the probability of the successful transmission for queue k in system Q^k. The inequality means that the arrival rate at the kth queue is less than the successful transmission probability, so that the kth queue is stable in system Q^k. Hence, it is stable in the original system Q. This is true for all k $(1 \le k \le N)$. So if (44) holds for all k, the original system Q is stable.

Now let us prove the second part. If there exists k, for which (45) holds, then queue k is unstable in the dominant system Q^k. By Corollary 1, all the queues with higher rank than that of queue k are unstable as well. So system Q^k is indistinguishable from the original system Q. Thus the original system is unstable. □

Theorem 4 does not tell us in case of equality in (44), whether the system is stable. Thus right on the boundary of the stability region it is not clear whether the system is stable or not.

From Theorem 4, it is clear that evaluating the bound for stability is equivalent to evaluating the quantity $P_E^{(k)}$ of the dominant systems Q^k. Unfortunately, the precise value of $P_E^{(k)}$ is not known or easy to evaluate. However, we can steer our efforts toward bounding its value and thus toward obtaining separate necessary and sufficient conditions from upper and lower bounds, respectively. Szpankowski obtained a similar theorem to Theorem 4 and applied the theorem to obtain a tight stability bound for three-terminal ALOHA system [6]. The difference of our theorem from his is that we observe the "rank" of the queues so that we do not need to evaluate the values of $P_E^{(k)}$ for every combination of the dominant systems. We can start from system Q^1, check whether queue 1 is stable or unstable in Q^1. If queue 1 is unstable, we stop there and conclude that system Q is unstable, and queues from 1 to N are all unstable; if queue 1 is stable we then continue with system Q^2 and check queue 2. The procedure continues until we find that in system Q^K, queue K is unstable; or ends with system Q^N in which queue N is stable. For the former case, we can conclude that in the original system Q, queues from K to N are all unstable; and for the latter case, we can conclude that Q is stable.

Now, we will explore in detail how to derive the bounds by using Theorem 4. Before we begin to bound the value of

$P_E^{(k)}$, we need to introduce some notation. We denote by $\mu_j^{(k)}$ the successful transmission probability of the jth queue for system Q^k when queue j is nonempty. In addition, we denote by U_k the set of queues from k to N, and by $\overline{U_k}$ the set of queues from 1 through $k-1$.

Following the procedure as described above, assume we continue with index k. Then we can say that in system Q^k, queues from 1 to $k-1$ are all stable.

In system Q^k, we know that

$$\mu_k^{(k)} = \frac{p_k}{(1-p_k)} \prod_{j=k}^{N} (1-p_j) P_E^{(k)} \tag{46}$$

where $P_E^{(k)}$ is the probability that no queue in $\overline{U_k}$ transmits in system Q^k. We also have that in system Q^k

$$P_E^{(k)} = 1 - P(\text{only one queue in } \overline{U_k} \text{ transmits}) - P(\text{more than one queues in } \overline{U_k} \text{ transmit}). \tag{47}$$

We may compute one of the terms in (47) exactly; that is, under stable operation the throughput of the first $(k-1)$ queues is given by

$$P(\text{only one queue in } \overline{U_k} \text{ transmits}) \cdot \prod_{j=k}^{N} (1-p_j) = \sum_{j=1}^{k-1} \lambda_j \tag{48}$$

or

$$P(\text{only one queue in } \overline{U_k} \text{ transmits}) = \frac{\sum_{j=1}^{k-1} \lambda_j}{\prod_{j=k}^{N} (1-p_j)} \tag{49}$$

The other term in (47) can only be bounded from above or below. Specifically, we have

$$\begin{aligned} &\max_{1\le j\le k-1} P(\text{a packet of queue } j \text{ collides with packets from other queues in } \overline{U_k}) \\ &\le P(\text{more than one queue in } \overline{U_k} \text{ transmits}) \\ &\le \frac{1}{2}\sum_{j=1}^{k-1} P(\text{a packet of queue } j \text{ collides with packets from other queues in } \overline{U_k}). \end{aligned} \tag{50}$$

The factor "$\frac{1}{2}$" in (50) accounts for the fact that every collision involves at least two queues. Next, we have

$$\begin{aligned} &P(\text{a packet of queue } j \text{ collides with packets from other queues in } \overline{U_k}) \\ &= P(\text{terminal } j \text{ attempts transmission}) - \frac{\lambda_j}{\prod_{i=k}^{N}(1-p_i)} \\ &= P(\text{queue } j \text{ nonempty}) p_j - \frac{\lambda_j}{\prod_{i=k}^{N}(1-p_i)} \\ &= p_j \frac{\lambda_j}{\mu_j^{(k)}} - \frac{\lambda_j}{\prod_{i=k}^{N}(1-p_i)}. \end{aligned} \tag{51}$$

TABLE I
ITERATION RECORD OF THEOREM 5, FOR $N = 3$ AND $p_1 = p_2 = p_3 = 0.5$, $\lambda_1 = \lambda_2 = 0.6$

k	1	2	3
B_k	0.125	0.190	0.341
C_k	0.125	0.190	0.341
D_k	0.125	0.190	0.276

TABLE II
ITERATIONRECORD OF THEOREM 5, FOR $N = 3$ AND $p_1 = p_2 = p_3 = 0.5$, $\lambda_1 = 0.12$, $\lambda_2 = 0.13$

k	1	2	3
B_k	0.125	0.130	0.130
C_k	0.125	0.130	0.130
D_k	0.125	0.130	0.130

TABLE III
ITERATION RECORD OF THEOREM 5, FOR $N = 5$ AND $p_1 = p_2 = p_3 = p_4 = p_5 = 0.5$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.03$

k	1	2	3	4	5
B_k	0.0312	0.0325	0.0362	0.0402	0.0482
C_k	0.0312	0.0325	0.0362	0.0354	-0.0049
D_k	0.0312	0.0325	0.0349	0.0392	0.04669

Note that $\lambda_j / \prod_{i=k}^{N}(1-p_i)$ is the probability that terminal j transmits a packet in a slot and no other terminal in $\overline{U_k}$ transmits a packet in the same slot.

From (47) and (49), we obtain

$$P_E^{(k)} = 1 - \frac{\sum_{j=1}^{k-1} \lambda_j}{\prod_{j=k}^{N}(1-p_j)} - P(\text{more than one terminal in } \overline{U_k} \text{ transmit}) \tag{52}$$

and, then, by using (50) and (51), we upper-bound $\mu_k^{(k)}$ as follows:

$$\mu_k^{(k)} \le \frac{p_k}{(1-p_k)} \prod_{j=k}^{N}(1-p_j) \times \left\{ 1 - \frac{\sum_{i=1}^{k-1}\lambda_i}{\prod_{j=k}^{N}(1-p_j)} - \max_{j<k}\left[p_j \frac{\lambda_j}{\mu_j^{(k)}} - \frac{\lambda_j}{\prod_{i=k}^{N}(1-p_i)} \right] \right\} \tag{53}$$

and we lower-bound $\mu_k^{(k)}$ by

$$\mu_k^{(k)} \ge \frac{p_k}{(1-p_k)} \left\{ \prod_{j=k}^{N}(1-p_j) - \sum_{j=1}^{k-1}\lambda_j - \frac{1}{2}\sum_{j=1}^{k-1}\left[\left(\frac{\lambda_j}{\mu_j^k}\right) p_j \prod_{i=k}^{N}(1-p_i) - \lambda_j \right] \right\}. \tag{54}$$

TABLE IV
ITERATION RECORD OF THEOREM 5, FOR $N = 10$ AND $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = p_9 = p_{10} = 0.5$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0.036$, $\lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = 0.039$

k	1	2	3	4	5	6	7	8	9	10
B_k	0.0387	0.0390	0.0394	0.0398	0.0402	0.0406	0.0408	0.0410	0.0412	0.0414
C_k	0.0387	0.0390	0.0394	0.0398	0.0402	0.0406	0.0406	0.0404	0.0400	0.0392
D_k	0.0387	0.0390	0.0393	0.0398	0.0401	0.0406	0.0408	0.0410	0.0412	0.0414

In [4], it was proved that $\mu_j^{(k)} \geq \mu_j^{(j)}$, for $j < k$. Thus (54) becomes

$$\mu_k^{(k)} \geq \frac{p_k}{(1-p_k)}\left\{\prod_{j=k}^{N}(1-p_j) - \sum_{j=1}^{k-1}\lambda_j - \frac{1}{2}\sum_{j=1}^{k-1}\left[\left(\frac{\lambda_j}{\mu_j^{(j)}}\right)p_j\prod_{i=k}^{N}(1-p_i) - \lambda_j\right]\right\} \tag{55}$$

which leads to a sufficient condition for stability. Furthermore, in [4] a lower bound for $\mu_k^{(k)}$ was obtained and is given by

$$\mu_k^{(k)} \geq \frac{p_k}{(1-p_k)}\prod_{j=1}^{N}\left[1 + \sum_{i=1}^{k-1}\left(1 - \frac{\lambda_i}{\mu_i^{(i)}}\right)\frac{p_i}{(1-p_i)}\right]. \tag{56}$$

To obtain an improved sufficient condition we simply combine (55) and (56) as stated in the following theorem. Let us define C_k and D_k as the terms on the right-hand sides of (55) and (56), respectively; that is, for $k \geq 2$

$$C_k = \frac{p_k}{(1-p_k)}\left\{\prod_{j=k}^{N}(1-p_j) - \sum_{j=1}^{k-1}\lambda_j - \frac{1}{2}\sum_{j=1}^{k-1}\left[\left(\frac{\lambda_j}{B_j}\right)p_j\prod_{i=k}^{N}(1-p_i) - \lambda_j\right]\right\} \tag{57}$$

and

$$D_k = \frac{p_k}{(1-p_k)}\prod_{j=1}^{N}(1-p_j)\left[1 + \sum_{i=1}^{k-1}\left(1 - \frac{\lambda_i}{B_i}\right)\frac{p_i}{(1-p_i)}\right] \tag{58}$$

where the quantity B_k is chosen as the maximum of C_k and D_k for $k \geq 2$, i.e.,

$$B_k = \max(C_k, D_k) \tag{59}$$

and B_1 is equal to $p_1\prod_{j=2}^{N}(1-p_j)$.

Theorem 5: Given an N-terminal ALOHA system with parameters $(\boldsymbol{\Lambda}, \boldsymbol{p})$, and such that

$$\lambda_1(1-p_1)/p_1 \leq \cdots \leq \lambda_N(1-p_N)/p_N$$

if for every k $(1 \leq k \leq N)$

$$\lambda_k < B_k \tag{60}$$

then the system is stable.

Of course, we also know that

$$\mu_j^{(k)} \leq p_j\prod_{i=k}^{N}(1-p_i)$$

TABLE V
COMPARISON OF BOUNDS FOR λ_N FOR $N = 3$ AND $p_1 = p_2 = p_3 = 0.5$

	λ_1	λ_2	Upp. bound	Low. bound	[4]'s bound
G1	0.0	0.0	0.500	0.500	0.375
	0.0	0.12	0.380	0.380	0.315
G2	0.06	0.06	0.380	0.341	0.276
G3	0.12	0.123	0.257	0.140	0.137
G4	0.12	0.13	0.250	0.130	0.130

TABLE VI
COMPARISON OF BOUNDS FOR λ_N FOR $N = 3$ AND $p_1 = 0.6$, $p_2 = 0.7$, $p_3 = 0.8$

	λ_1	λ_2	Upp. bound	Low. Bound	[4]'s bound
G1	0.0	0.0	0.800	0.800	0.464
	0.0	0.05	0.600	0.600	0.384
G2	0.018	0.028	0.616	0.508	0.328
G3	0.03	0.05	0.480	0.240	0.184
G4	0.035	0.0561	0.4356	0.1152	0.1086
	0.025	0.0563	0.4748	0.2777	0.2096

and thus (53) yields an upper bound for $\mu_k^{(k)}$, namely,

$$\mu_k^{(k)} < \frac{p_k}{(1-p_k)}\prod_{j=k}^{N}(1-p_j)\left[1 - \frac{\sum_{i=1}^{k-1}\lambda_i}{\prod_{j=k}^{N}(1-p_j)}\right] \tag{61}$$

which provides a necessary condition for stability that is identical to the one we obtained in Theorem 3.

D. Comparison of the Bounds

Tables I–IV provide the comparison of the bounds corresponding to the quantities C_k or D_k separately. In these tables, "B_k" represents the combined maximum value as given in (59).

Table I shows a case in which C_k is always greater than or equal to D_k. Note that C_1 is always equal to D_1 in Theorem 5. Table II shows a case in which C_k is always equal to D_k. Table III shows a case in which at first C_k is greater than D_k (i.e., for small values of k) but as the iteration of (57), (58) proceeds, C_k decreases rapidly compared to D_k, and finally C_k is less than D_k. We can see in this case that C_5 is less than zero which means the bound for λ_5 is meaningless. So using only C_k or D_k separately cannot give the best bound across all values of k. Table IV shows another example in which the combined use of C_k and D_k improves over the use of C_k or D_k alone. From our observations, the trend seems to be that C_k is usually better for the beginning (low) values of k and becomes worse when the iteration proceeds toward large value of k.

In Tables V–VIII, "Upp. bound" represents the necessary condition calculated from Theorem 3. "Low. bound" represents the sufficient condition calculated from Theorem 5.

TABLE VII
COMPARISON OF BOUNDS FOR λ_N FOR $N = 5$ AND $p_1 = p_2 = p_3 = p_4 = p_5 = 0.5$

	λ_1	λ_2	λ_3	λ_4	Upp. bound	Low. Bound	[4]'s bound
G1	0.0	0.0	0.0	0.0	0.500	0.500	0.156
	0.0	0.0	0.0	0.015	0.485	0.485	0.153
	0.0	0.0	0.015	0.015	0.470	0.462	0.147
	0.0	0.015	0.015	0.015	0.455	0.422	0.139
G2	0.015	0.015	0.015	0.015	0.440	0.337	0.120
G3	0.03	0.03	0.03	0.03	0.380	0.048	0.047
G4	0.03	0.03	0.03	0.033	0.377	0.0458	0.0443
	0.033	0.032	0.031	0.03	0.374	0.0393	0.0381
	0.0325	0.032	0.0315	0.03	0.374	0.0377	0.0369

TABLE VIII
COMPARISON OF BOUNDS FOR λ_N FOR $N = 10$ AND $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = p_9 = p_{10} = 0.1$

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	Upp. bound	Low. Bound	[4]'s bound
G1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.077
	0.0	0.019	0.019	0.019	0.019	0.019	0.019	0.019	0.019	0.083	0.077	0.065
G2	0.019	0.019	0.019	0.019	0.019	0.019	0.019	0.019	0.019	0.081	0.073	0.062
G3	0.036	0.036	0.036	0.036	0.036	0.036	0.036	0.036	0.036	0.064	0.04386	0.04284
G4	0.039	0.036	0.036	0.036	0.036	0.036	0.036	0.036	0.036	0.0636	0.04255	0.04253
	0.039	0.039	0.039	0.039	0.036	0.036	0.036	0.036	0.036	0.0627	0.04141	0.04139

The values of the arrival rates λ_i chosen for each table are organized into four groups, designated as G1, G2, G3, G4 as was done in [4]. In G1, one or more values of λ_i are zero. In G1, G2, and G3 every given value of λ_i is less than the corresponding value of $p_i \prod_{j \neq i} (1 - p_j)$, and G3 is close to the symmetric case [7]. In G4 one or more of the λ_i are more than the corresponding value of $p_i \prod_{j \neq i} (1 - p_j)$. We can see from the Tables V–VIII that our upper and lower bounds are very close to each other for the G1 group. In particular, they are equal to each other for $N = 2$ which shows why the case of $N = 2$ was amenable to a necessary and sufficient condition from the outset in [4]. In the G4 group, we find that the two lower bounds are very close and both are also close to the tight lower bound.

IV. CONCLUSION

We revisited the problem of stability of an N-terminal slotted ALOHA system. We identified the relative liability for instability of the queues by identifying the notion of the "rank" in the Theorem 2. And, based on Theorem 2, we improved the inner bound for stability. An outer bound was also separately obtained.

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