Nonlinear Observer Design for One-Sided Lipschitz Systems

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Abstract— Control and state estimation of nonlinear systems satisfying a Lipschitz continuity condition have been important topics in nonlinear system theory for over three decades, resulting in a substantial amount of literature. The main criticism behind this approach, however, has been the restrictive nature of the Lipschitz continuity condition and the conservativeness of the related results. This work deals with an extension to this problem by introducing a more general family of nonlinear functions, namely one-sided Lipschitz functions. The corresponding class of systems is a superset of its well-known Lipschitz counterpart and possesses inherent advantages with respect to conservativeness. In this paper, first the problem of state observer design for this class of systems is established, the challenges are discussed and some analysis-oriented tools are provided. Then, a solution to the observer design problem is proposed in terms of nonlinear matrix inequalities which in turn are converted into numerically efficiently solvable linear matrix inequalities.

Keywords: one-sided Lipschitz systems, quadratic inner-boundedness, nonlinear observers, nonlinear matrix inequalities, linear matrix inequalities

I. INTRODUCTION

THE observer design problem for nonlinear systems satisfying a Lipschitz continuity condition has been a topic of a constant research for the last three decades. Observers for Lipschitz systems were first considered by Thau [1] where he obtained a sufficient condition to ensure asymptotic stability of the observer error. Inspired by Thau's work, several authors have studied observer design for Lipschitz systems using various approaches [2]–[7]. Lipschitz systems constitute an important class of nonlinear systems for which observer design can be carried out using pseudo-linear techniques.

The Lipschitz constant of such functions is usually region-based and often dramatically increases as the operating region is enlarged. On the other hand, even if the nonlinear system is Lipschitz in the region of interest, it is generally the case that the available observer design techniques can only stabilize the error dynamics for dynamical systems with small Lipschitz constants but, as discussed later, fails to provide a solution when the Lipschitz constant becomes large. The problem becomes worse when dealing with *stiff* systems.

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Stiffness means that the ordinary differential equation (ODE) admits a smooth solution with moderate derivatives, together with nonsmooth ("transient") solutions rapidly converging towards the smooth one [8, p. 71]. This problem has been recognized in the mathematical literature and specially in the field of numerical analysis for some time and a powerful tool has developed to overcome this problem. This tool is a generalization of the Lipschitz continuity to a less restrictive condition known as *one-sided Lipschitz* continuity which has become one of the building blocks in numerical analysis and has been extensively applied to the stability analysis of ODE solvers [9]–[11].

Inspired by the these advances in the mathematical literature, in this paper, we extend this concept to the nonlinear observer design problem and consider stabilization of the observer error dynamics based on the one-sided Lipschitz condition. The advantages gained through this approach are two-fold: i) *Generalization:* we will show that the one-sided Lipschitz continuity covers a broad family on nonlinear systems which includes the well-known Lipschitz systems as a special case. ii) *Reduced conservativism:* Observer design techniques based on Lipschitz functions can guarantee stability only for small values of Lipschitz constants which directly translates into small stability regions. All available results on Lipschitz systems, however, provide only sufficient conditions for stability and the actual observer might still work with larger Lipschitz constants, even though the tool used in the analysis and design are unable to provide theoretical evidence. The implication is that there is a significant degree of conservativeness in the Lipschitz formulation, a critique that has often been reported by researchers, but that has been difficult to correct and has produced no valuable alternative.

In this work we provide this valuable alternative in the form of the one-sided Lipschitz condition. We will show that the one-sided Lipschitz condition generalizes the classical Lipschitz theory in the following sense: any dynamical system satisfying a Lipschitz condition satisfies also a one-sided Lipschitz. However, the one-sided Lipschitz constant is always smaller than its Lipschitz counterpart, a difference that can be significant even for very simple nonlinear functions [9]–[11]. Examples are presented illustrating this property as well as showing cases where a dynamical system satisfies a one-sided Lipschitz condition even-though it is not Lipschitz in the classical sense. Specially, when a dynamical system is *stiff*, the conventional Lipschitz constant inevitably becomes very large while one-sided Lipschitz constant is

still moderate [9]–[11]. As a result, more efficient and less conservative observers can be developed in this context.

These major advantages come along with a greater degree of difficulty encountered dealing with one-sided Lipschitz systems. Unlike Lipschitz functions, which lead to an inequality in a rather simple quadratic form, the one-sided Lipschitz formulation leads to a weighted bilinear form which imposes significant challenges in manipulating the Lyapunov derivative. Very recently, the nonlinear observers for one-sided Lipschitz systems are studied in [12], [13]. In these references, not only the one-sided Lipschitz constant is assumed to be known but also the Lyapunov candidate that stabilizes the observer error dynamics is assumed to be known apriori. Therefore the results in [12], [13] can only be used for analysis of a given observer and the design problem is left as an open problem. Our goal in this paper is to acknowledge the advantages of the one-sided Lipschitz formulation over the conventional Lipschitz assumption in the control and observation theory, and in particular to formulate the observer design problem based on that. In this respect, not only do we provide basic analysis tools but also we address the *design* problem and present a complete solution. The remainder of the paper is organized as follows: Section II introduces the one-sided Lipschitz condition and study its basis properties. In Section III we consider the observer problem based on this property and addressed observer stability. Section IV, which contains the main results, addresses observer design in the form of nonlinear matrix inequalities (NMIs). In the cycle, in order to use the efficient readily available numerical solvers, we convert the proposed NMI problem into linear matrix inequalities (LMIs). Section V presents an illustrative example.

II. MATHEMATICAL PRELIMINARIES AND PROBLEM STATEMENT

Throughout the paper $\mathbb{R}$ represents the field of real numbers, $\mathbb{R}^n$ the set of n-tuples of real numbers and $\mathbb{R}^{n \times p}$ the set of real matrices of order n by p. $<,>$ is the (often called "natural") inner product in the space $\mathbb{R}^n$, *i.e.* given $x, y \in \mathbb{R}^n$, then $< x, y >= x^T y$, where x^T is the transpose of the (column vector) $x \in \mathbb{R}^n$. $\|.\|$ is the vector 2-norm (the Euclidian norm) in $\mathbb{R}^n$ defined by $\|x\| = \sqrt{< x, x >}$.

Consider now the following continuous-time nonlinear dynamical system

$$\dot{x}(t) = Ax(t) + \Phi(x, u) \qquad A \in \mathbb{R}^{n \times n} \qquad (1)$$
$$y(t) = Cx(t) \qquad C \in \mathbb{R}^{n \times p}, \qquad (2)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ and $\Phi(x, u)$ represents a nonlinear function that is continuous with respect to both x and u. The system (1)-(2) is said to be *locally Lipschitz* in a region $\mathcal{D}$ including the origin with respect to x, uniformly in u, if there exist a constant $l > 0$ satisfying:

$$\|\Phi(x_1, u^*) - \Phi(x_2, u^*)\| \leqslant l\|x_1 - x_2\| \quad \forall x_1, x_2 \in \mathcal{D}, \quad (3)$$

where u^* is any admissible control signal. The smallest constant $l > 0$ satisfying (3) is known as the *Lipschitz*

constant. The region $\mathcal{D}$ is the *operational region* or our *region of interest*. If the condition (3) is valid everywhere in $\mathbb{R}^n$, then the function is said to be globally Lipschitz.

The following definition introduces one-sided Lipschitz functions.

Definition 1. [9] The nonlinear function $\Phi(x, u)$ is said to be *one-sided Lipschitz* if there exist $\rho \in \mathbb{R}$ such that $\forall x_1, x_2 \in \mathcal{D}$

$$\langle \Phi(x_1, u^*) - \Phi(x_2, u^*), x_1 - x_2 \rangle \leqslant \rho\|x_1 - x_2\|^2, \quad (4)$$

where $\rho \in \mathbb{R}$ is called the *one-sided Lipschitz constant*. As in the case of Lipschitz functions, the smallest ρ satisfying (4) is called the one-sided Lipschitz constant. ∎

Similarly to the Lipschitz property, the one-sided Lipschitz property might be local or global. Note that while the Lipschitz constant must be positive, the one-sided Lipschitz constant can be positive, zero or even negative. For any function $\Phi(x, u)$, we have:

$$| \langle \Phi(x_1, u^*) - \Phi(x_2, u^*), x_1 - x_2 \rangle |$$
$$\leqslant \|\Phi(x_1, u^*) - \Phi(x_2, u^*)\|\|x_1 - x_2\|$$
and if $\Phi(x, u)$ is Lipschitz, then: $\leqslant l\|x_1 - x_2\|^2.$

Therefore, any Lipschitz function is also one-sided Lipschitz. The converse, however, is not true. For Lipschitz functions,

$$-l\|x_1 - x_2\|^2 \leqslant \langle \Phi(x_1, u^*) - \Phi(x_2, u^*), x_1 - x_2 \rangle$$
$$\leqslant l\|x_1 - x_2\|^2,$$

which is a *two-sided* inequality v.s. the *one-sided* inequality in (4). If the nonlinear function $\Phi(x, u)$ satisfies the one-sided Lipschitz continuity condition globally in $\mathbb{R}^n$, then the results are valid globally. For continuously differentiable nonlinear functions it is well-known that the smallest possible constant satisfying (3) (*i.e.*, the Lipschitz constant) is the supremum of the norm of Jacobian of the function over the region $\mathcal{D}$, that is:

$$l = \limsup \left(\left\| \frac{\partial \Phi}{\partial x} \right\| \right), \qquad \forall x \in \mathcal{D}. \qquad (5)$$

Alternatively, the one-sided Lipschitz constant is associated with the *logarithmic matrix norm (matrix measure)* of the Jacobian [11]. The logarithmic matrix norm of a matrix A is defined as [11]:

$$\mu(A) = \lim_{\epsilon \to 0} \frac{|||I + \epsilon A||| - 1}{\epsilon}, \qquad (6)$$

where the symbol $|||.|||$ represents any matrix norm. Then, we have [11]

$$\rho = \limsup \left[\mu \left(\frac{\partial \Phi}{\partial x} \right) \right], \qquad \forall x \in \mathcal{D}. \qquad (7)$$

If the norm used in (6) is indeed the induced 2-norm (the spectral norm) then it can be shown that $\mu(A) = \lambda_{max} \left(\frac{A + A^T}{2} \right)$ [14]. On the other hand, from the Fan's theorem (see for example [15]) we know that for any matrix, $\lambda_{max} \left(\frac{A + A^T}{2} \right) \leq \sigma_{max}(A) = \|A\|$ [15]. Therefore $\rho \leq l$. Usually one-sided Lipschitz constant can be found to be

much smaller that than the Lipschitz constant [11]. It is well-known in numerical analysis that for stiff ODE systems, $\rho << l$ [10], [11]. See [16] for an example of a nonlinear function which satisfies one-sided Lipschitz continuity but not the Lipschitz continuity.

III. OBSERVER STRUCTURE

In this section we consider the observation problem; *i.e.* given the dynamical system (1)-(2) we assume that only the input u and output y are available and study the feasibility of reconstructing the state x. To this end we consider an observer of the following form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \Phi(\hat{x}, u) + L(y - C\hat{x}). \tag{8}$$

The observer error dynamics is given by $e(t) \triangleq x(t) - \hat{x}(t)$,

$$\dot{e}(t) = (A - LC)e(t) + \Phi - \hat{\Phi} \tag{9}$$

where $\Phi \triangleq \Phi(x, u)$ and $\hat{\Phi} \triangleq \Phi(\hat{x}, u)$. Using a Lyapunov function candidate of the form $V(t) = e^T P e$, the derivatives along the error trajectories are $\dot{V} = e^T[(A-LC)^T P + P(A-LC)]e + 2e^T P(\Phi - \hat{\Phi})$. Our goal is, assuming that Φ satisfies a one-sided Lipschitz condition, find an observer gain L such that the observer error dynamics is asymptotically stable.

A. Motivating problem

Given the form of the Lyapunov derivative, the standard approach used in the Lipschitz case is to find L such that $(A - LC)^T P + P(A - LC) = -Q$ is *negative definite* and $e^T Q e > 2e^T P(\Phi - \hat{\Phi})$. This approach, first proposed by Thau in 1973 [1], has dominated the literature on Lipschitz systems ever since. The implicit idea behind this approach is to use of the output injection term in the observer dynamics to ensure that the linear part of the observer error dominates the nonlinear terms. This, in turn, is facilitated by the strong square norm condition (3) satisfied by Lipschitz systems which leads to the conservative nature of the result. In other words, in the process of employing the Lipschitz property (3), the term $e^T P(\Phi - \hat{\Phi})$ in the Lyapunov derivative is replaced by a strictly positive term forcing the rest of the derivative to be sufficiently negative to compensate the remaining terms. It is important to note that the term $e^T[(A-LC)^T P + P(A-LC)]e < 0$ if and only if $A - LC$ has eigenvalues with negative real part. Unlike the Lipschitz constant l, which is positive by definition, the one-sided Lipschitz constant ρ can be any real number. Thus, the term $2e^T P(\Phi - \hat{\Phi})$ can be negative. Hence, a negative Lyapunov derivative may be guaranteed even with a positive definite $(A-LC)^T P + P(A-LC)$ and consequently the linear terms in $A - LC$ is not necessarily required to have eigenvalues with negative real part. This means that the linear terms not necessarily dominate the nonlinear function Φ, which in turn can lead to less conservative results. The mathematical description behind the need of $A - LC$ having eigenvalues in the left half plane can be traced back through a substantial body of literature for Lipschitz systems such as [1]–[5] while the freedom of the one-sided counterpart from such necessity is established in this article.

B. Challenging Obstacle

We now return to our main objective and endeavor to find L that makes $\dot{V} < 0$. This problem is nontrivial. Unfortunately, to the best of the authors' knowledge, there is no result in mathematics relating the weighted bilinear form $2e^T P(\Phi - \hat{\Phi})$ to the bilinear form $2e^T(\Phi - \hat{\Phi})$ which is required to take full advantage of the properties offered by the one-sided Lipschitz formulation. This is the main obstacle to overcome in this work. A very simple first approach to this problem is to consider $P = I$. With this choice, we have:

$$\dot{V} = e^T[(A - LC)^T + (A - LC)]e + 2e^T(\Phi - \hat{\Phi})$$
$$\leq e^T[(A - LC)^T + (A - LC) + 2\rho I]e.$$

where we substituted $e^T(\Phi - \hat{\Phi}) \leq \rho e^T e$. Hence, in order to have $\dot{V} < 0$, we must have

$$(A - LC)^T + (A - LC) + 2\rho I < 0$$
$$\Rightarrow \mu(A - LC) < -\rho. \tag{10}$$

Inequality (10) is an LMI which can be efficiently solved using any available LMI solver to find the observer gain L. For the logarithmic matrix norm the following inequality can be used [14]:

$$-\mu(-A) \leq \Re\lambda_i(A) \leq \mu(A), \quad i = 1, \ldots, n. \tag{11}$$

Therefore, $-\mu(-(A - LC)) \leq \Re\lambda_i(A - LC)$. On the other hand, we want $\mu(A-LC) < -\rho$, so as a necessary condition, we must have

$$\max_i \Re\lambda_i(A - LC) < -\rho. \tag{12}$$

Furthermore, suppose $A - LC$ is not stable (not stabilizable). We can always find $\alpha > 0$ such that $(A - LC - \alpha I)$ is stable where $\alpha > \max_i \Re\lambda_i(A - LC)$. Then the observer error is asymptotically stable if

$$(A - LC)^T + (A - LC) + 2\rho I = (A - LC - \alpha I)^T$$
$$+ (A - LC - \alpha I) + 2\alpha I + 2\rho I < 0 \tag{13}$$

Now, a sufficient condition for (13) to be true is

$$\alpha + \rho < 0 \Rightarrow \rho < -\alpha < -\max_i \Re\lambda_i(A - LC).$$

The above discussion provides some analysis insight but does address the fundamental design problem in a satisfactory manner. In the next section we propose a complete solution to this rather involved design problem.

IV. MAIN RESULTS

In this section, we first introduce the concept of *quadratic inner-boundedness* for the function $\Phi(x, u)$. Our design solution will make extensive use of this concept.

Definition 2. The nonlinear function $\Phi(x, u)$ is called *quadratically inner-bounded* in the region $\tilde{\mathcal{D}}$ if $\forall x_1, x_2 \in \tilde{D}$ there exist $\beta, \gamma \in \mathbb{R}$ such that

$$(\Phi(x_1, u) - \Phi(x_2, u))^T (\Phi(x_1, u) - \Phi(x_2, u)) \leq$$
$$\beta\|x_1 - x_2\|^2 + \gamma \langle x_1 - x_2, \Phi(x_1, u) - \Phi(x_2, u) \rangle. \blacksquare \tag{14}$$

It is clear that any Lipschitz function is also quadratically inner-bounded (e.g. with $\gamma = 0$ and $\beta > 0$). Thus, Lipschitz continuity implies quadratic inner-boundedness. The converse is, however, not true. We emphasize that γ in (14) can be any real number and is not necessarily positive. In fact, if γ is restricted to be positive, then from the above definition, it can be shown that Φ must be Lipschitz which is only a special case of our proposed class of systems. From now on we assume that $\Phi(x, u)$ is one-sided Lipschitz in $\mathcal{D}$ and quadratically inner-bounded in $\widetilde{\mathcal{D}}$. All of our results will be valid in the intersection $\mathcal{D} \cap \widetilde{\mathcal{D}}$ (the operational region). With the above notation, the following inequality holds for the estimation error.

$$(\Phi - \hat{\Phi})^T (\Phi - \hat{\Phi}) \leq \beta \|e\|^2 + \gamma e^T (\Phi - \hat{\Phi}). \qquad (15)$$

In the following Theorem, we propose a method for observer design for one-sided Lipschitz systems.

Theorem 1. Consider a nonlinear system satisfying inequalities (4) and (14) with constants ρ, β and γ, along with the observer (8). The observer error dynamics is asymptotically stable if there exist positive definite matrix P, symmetric matrix Q, matrix L and a positive scalar $\alpha > 0$ such that the following matrix inequalities problem is feasible:

$$(A - LC)^T P + P(A - LC) \leq -Q, \qquad (16)$$

$$\xi \lambda_{max}(P) - \lambda_{min}(P) < \alpha \lambda_{min}(Q), \qquad (17)$$

$$\gamma + 2\alpha > 0, \qquad (18)$$

$$\frac{\lambda_{max}(P)}{\lambda_{min}(P)} \cdot (\alpha^2 - 1) < \alpha^2, \qquad (19)$$

where $\xi = (\beta + 1) + \rho(\gamma + 2\alpha)$.

Proof: For any nonzero $\alpha \in \mathbb{R}$ we can write

$$2e^T P(\Phi - \hat{\Phi}) = \frac{1}{\alpha} \left[e + \alpha(\Phi - \hat{\Phi}) \right]^T P \left[e + \alpha(\Phi - \hat{\Phi}) \right]$$
$$- \frac{1}{\alpha} e^T P e - \alpha(\Phi - \hat{\Phi})^T P(\Phi - \hat{\Phi}).$$

Assuming $\alpha > 0$, then

$$2e^T P(\Phi - \hat{\Phi}) \leq$$
$$\frac{1}{\alpha} \lambda_{max}(P) \left[e + \alpha(\Phi - \hat{\Phi}) \right]^T \left[e + \alpha(\Phi - \hat{\Phi}) \right]$$
$$- \frac{1}{\alpha} e^T P e - \alpha(\Phi - \hat{\Phi})^T P(\Phi - \hat{\Phi}). \qquad (20)$$

Using the quadratic boundedness property we have

$$\left[e + \alpha(\Phi - \hat{\Phi}) \right]^T \left[e + \alpha(\Phi - \hat{\Phi}) \right] \leq (\beta + 1)e^T e$$
$$+ (\gamma + 2\alpha)e^T (\Phi - \hat{\Phi}) + (\alpha^2 - 1)\|\Phi - \hat{\Phi}\|^2. \qquad (21)$$

Substituting (21) into (20) leads to

$$2e^T P(\Phi - \hat{\Phi}) \leq \frac{1}{\alpha} \lambda_{max}(P)[(\beta + 1)e^T e + \cdots$$
$$\cdots (\gamma + 2\alpha)e^T (\Phi - \hat{\Phi}) + (\alpha^2 - 1)\|\Phi - \hat{\Phi}\|^2]$$
$$- \frac{1}{\alpha} e^T P e - \alpha(\Phi - \hat{\Phi})^T P(\Phi - \hat{\Phi}). \qquad (22)$$

Based on the Rayleigh's inequality, for any $\alpha > 0$ we have

$$(\Phi - \hat{\Phi})^T P(\Phi - \hat{\Phi}) \geq \lambda_{min}(P)\|\Phi - \hat{\Phi}\|^2 \Rightarrow$$
$$- \alpha(\Phi - \hat{\Phi})^T P(\Phi - \hat{\Phi}) \leq -\alpha \lambda_{min}(P)\|\Phi - \hat{\Phi}\|^2.$$

Hence, from (22), using the one-sided Lipschitz inequality (4) and knowing that $\gamma + 2\alpha > 0$, we obtain

$$2e^T P(\Phi - \hat{\Phi}) \leq$$
$$\left[\frac{1}{\alpha} \lambda_{max}(P)(\alpha^2 - 1) - \alpha \lambda_{min}(P) \right] \|\Phi - \hat{\Phi}\|^2$$
$$+ \frac{1}{\alpha} \lambda_{max}(P) \left[(\beta + 1) + \rho(\gamma + 2\alpha) \right] e^T e - \frac{1}{\alpha} e^T P e.$$

We know that $\kappa(P)(\alpha^2 - 1) < \alpha^2$, where $\kappa(P)$ is the condition number of P or $\frac{1}{\alpha} \lambda_{max}(P)(\alpha^2 - 1) - \alpha \lambda_{min}(P) < 0$. Then,

$$2e^T P(\Phi - \hat{\Phi}) < \frac{1}{\alpha} \lambda_{max}(P) \left[(\beta + 1) + \rho(\gamma + 2\alpha) \right] e^T e$$
$$- \frac{1}{\alpha} e^T P e. \qquad (23)$$

Now we substitute (23) into the Lyapunov derivative. We obtain

$$\dot{V} = e^T[(A - LC)^T P + P(A - LC)]e + 2e^T P(\Phi - \hat{\Phi})$$
$$< e^T \left[(A - LC)^T P + P(A - LC) \right] e$$
$$+ \frac{\xi}{\alpha} \lambda_{max}(P)e^T e - \frac{1}{\alpha} e^T P e, \qquad (24)$$

where $\xi \triangleq (\beta + 1) + \rho(\gamma + 2\alpha)$. Therefore, in order to have $\dot{V} < 0$ it suffices to have:

$$\frac{\xi}{\alpha} \lambda_{max}(P) +$$
$$\lambda_{max} \left[(A - LC)^T P + P(A - LC) - \frac{1}{\alpha} P \right] < 0. \qquad (25)$$

For any two symmetric matrices A and B, it can be shown that $\lambda_i \leq \lambda_i(A) + \lambda_i(B)$, where λ_is are the sorted eigenvalues [15]. Thus,

$$\lambda_{max} \left[(A - LC)^T P + P(A - LC) - \frac{1}{\alpha} P \right] \leq$$
$$\lambda_{max} \left[(A - LC)^T P + P(A - LC) \right] + \lambda_{max} \left(-\frac{P}{\alpha} \right) =$$
$$\lambda_{max} \left[(A - LC)^T P + P(A - LC) \right] - \frac{1}{\alpha} \lambda_{min}(P). \qquad (26)$$

Now without loss of generality we assume that there exists a symmetric matrix Q, such as $(A - LC)^T P + P(A - LC) \leq -Q$. Note that Q is not necessarily positive definite (meaning that $(A - LC)$ is not necessarily stable). Thus,

$$\lambda_{max} \left[(A - LC)^T P + P(A - LC) \right]$$
$$\leq \lambda_{max}(-Q) = -\lambda_{min}(Q). \qquad (27)$$

Substituting from (27), (26) and (25) into (24), we get

$$\frac{\xi}{\alpha} \lambda_{max}(P) - \frac{1}{\alpha} \lambda_{min}(P) - \lambda_{min}(Q) < 0. \qquad \blacksquare \qquad (28)$$

A. LMI formulation

Theorem 1, provides a design method for nonlinear observers for one-sided Lipschitz systems in the form of the nonlinear matrix inequalities (NMIs) (16)-(19). The difficulty, however, is that although Theorem 1 provides a legitimate solution to our problem, there is currently no efficient solution in the numerical analysis literature capable of solving NMIs. Unlike the nonlinear case, however, *linear* matrix inequalities (LMIs) can be efficiently solved using commercially available packages such as the Matlab LMI solver. We now show how to cast the proposed nonlinear matrix inequalities solution into the LMI framework to take advantage of the efficient numerical LMI solvers readily available. Using Fan's theorem [15], we can write

$$\lambda_{max}[(A - LC)^T P + P(A - LC)]$$
$$\leq 2\sigma_{max}[P(A - LC)] \leq 2\lambda_{max}(P)\sigma_{max}(A - LC).$$

Substituting this back to (25) yields to

$$\frac{1}{\alpha}\lambda_{max}(P)\xi - \frac{1}{\alpha}\lambda_{min}(P) + 2\lambda_{max}(P)\sigma_{max}(A - LC) < 0$$
$$\Rightarrow \sigma_{max}(A - LC) < \frac{1}{2\alpha}\left(\frac{1}{\kappa(P)} - \xi\right),$$

which by means of Schur's complement and change of variable $\lambda = \frac{1}{\kappa}$ is equivalent to the LMI (29). LMI (31) represents the condition $\kappa(\alpha^2 - 1) < \alpha^2$.

Based on the above discussion which also serves as the proof, the following corollary provides an LMI solution to our observer design problem.

Corollary 1. Consider a nonlinear system satisfying inequalities (4) and (14) with constants ρ, β and γ, along with the observer (8). The observer error dynamics is asymptotically stable if there exists a matrix L and positive scalars $\alpha > 0$ and $0 < \lambda < 1$ such that the following matrix inequalities problem is feasible:

$$\begin{bmatrix} \frac{1}{2\alpha}(\lambda - \xi)I & (A - LC)^T \\ (A - LC) & \frac{1}{2\alpha}(\lambda - \xi)I \end{bmatrix} > 0, \qquad (29)$$

$$\gamma + 2\alpha > 0, \qquad (30)$$

$$\lambda > 1 - \frac{1}{\alpha^2}, \qquad (31)$$

where $\xi = (\beta + 1) + \rho(\gamma + 2\alpha)$.

We now summarize the observer design procedure.

B. Observer Design Procedure

- **Step 1:** Pick an $\alpha > 0$ such that $2\alpha + \gamma > 0$.
- **Step 2:** Calculate $\xi = \beta + \rho(\gamma + 2\alpha) + 1$.
- **Step 3:** Check if the condition $\kappa(\alpha^2 - 1) < \alpha^2$ and $\frac{1}{\kappa} - \xi > 0$ are consistent (each condition provides a bound on κ). If Yes, go to Step 4 otherwise go to Step 1.
- **Step 4:** Solve the LMIs in Corollary 1 for L and λ.

Note that if Step 3 is passed, the LMIs in Step 4 are always feasible meaning that an observer gain L will always be found. The variable κ calculated in Corollary 1 is the condition number of the P matrix used in the Lyapunov function. Any P with such condition number would be acceptable. Although easy to do, as our goal of finding the observer gain L is already achieved, this step (finding P) is unnecessary except for analysis purposes of the results.

V. Illustrative Example

In this section, we illustrate the proposed observer design procedure through a numerical example.

Example. Suppose that the equations of motion of a moving object is given in the polar coordinates as follows:

$$\dot{r} = r - r^3,$$
$$\dot{\theta} = 1,$$

and the measurements are given as $r\sin(\theta)$. The goal is to design an observer to find r and θ. First we convert the equations into the Cartesian coordinates. We get:

$$\dot{x} = x - y - x(x^2 + y^2),$$
$$\dot{y} = x + y - y(x^2 + y^2),$$

and the measurements are y. We define the state vector as $\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}^T$ and $\mathbf{y}$ as the output, in which the variables are bolded to avoid ambiguity. We have:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -x(x^2 + y^2) \\ -y(x^2 + y^2) \end{bmatrix},$$
$$\mathbf{y} = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{x}.$$

Knowing that, $x_1 x_2 = \frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2}(x_1 - x_2)^2$, we get,

$$\langle \Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_2), \mathbf{x}_1 - \mathbf{x}_2 \rangle = -\frac{\|\mathbf{x}_1 - \mathbf{x}_2\|^2}{2}\left[\|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2\right]$$
$$-\frac{1}{2}\left[\|\mathbf{x}_1\|^2 - \|\mathbf{x}_2\|^2\right]^2 \leq 0. \qquad (32)$$

This means that the systems is globally one-sided Lipschitz with the one-sided Lipschitz constant $\rho = 0$. Now, lets verify the Lipschitz continuity property. Φ is continuously differentiable, so an estimate for the Lipschitz constant is the supremum of the norm of the Jacobian matrix, $\|J\| = 3r^2$. This means that the system is locally Lipschitz and on any set $\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq r\}$, the Lipschitz constant l is $3r^2$, i.e. the Lipschitz constant rapidly increases with the increase of r. We need to verify the quadratic inner-boundedness property of the system, as well. After some algebraic manipulations, the left hand side of the quadratic inner-boundedness (14) is:

$$\text{LHS} : [\Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_2)]^T [\Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_2)] =$$

$$\left[\|\mathbf{x}_1\|^2 - \|\mathbf{x}_2\|^2\right]^2 \cdot \left[\|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2\right] + \|\mathbf{x}_1 - \mathbf{x}_2\|^2\|\mathbf{x}_1\|^2\|\mathbf{x}_2\|^2$$

The right hand side of (14) is:

$$\text{RHS} : \gamma\langle \Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_2), \mathbf{x}_1 - \mathbf{x}_2 \rangle + \beta\|\mathbf{x}_1 - \mathbf{x}_2\|^2 =$$

$$\|\mathbf{x}_1 - \mathbf{x}_2\|^2\left[\beta - \frac{\gamma}{2}\left(\|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2\right)\right] - \frac{\gamma}{2}\left[\|\mathbf{x}_1\|^2 - \|\mathbf{x}_2\|^2\right]^2$$

in which (32) is used. We have to find values for β and γ and a region $\widetilde{\mathcal{D}}$ such that for all $\mathbf{x} \in \widetilde{\mathcal{D}}$, LHS $\leq$ RHS. Comparing

the two, it suffices to have:

$$\|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2 \leq -\frac{\gamma}{2},$$

$$\|\mathbf{x}_1\|^2.\|\mathbf{x}_2\|^2 \leq \beta - \frac{\gamma}{2}\left[\|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2\right]^2 \leq \beta + \frac{\gamma^2}{4}.$$

Considering the set $\widetilde{\mathcal{D}} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq r\}$, it suffices to have: $2r^2 \leq -\frac{\gamma}{2} \to r \leq \sqrt{-\frac{\gamma}{4}}$ and $r^4 \leq \beta + \frac{\gamma^4}{4} \to r \leq \sqrt[4]{\beta + \frac{\gamma^2}{4}}$. Hence,

$$r = \min\left(\sqrt{-\frac{\gamma}{4}} \ , \ \sqrt[4]{\beta + \frac{\gamma^2}{4}}\right), \quad \gamma < 0, \ \beta + \frac{\gamma^2}{4} > 0.$$

Also, since $\rho = 0$, $\xi = \beta + 1$ and thus according the LMIs in Corollary 1, since $\lambda < 1$, in order to have $\lambda - \xi > 0$, β has to be negative. As the system is globally one-sided Lipschitz ($\mathcal{D} = \mathbb{R}^2$), $\mathcal{D} \cap \widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}$. It is clear that by choosing appropriate values for γ and β, the region $\widetilde{\mathcal{D}}$ can be made arbitrarily large. If we take $\beta = -200, (\xi = -199)$ and $\gamma = -141$, we get $r = 5.9372$. Then we take $\alpha = 70.6$ (to ensure $\gamma + 2\alpha > 0$) and solve the LMIs is Corollary 1. We get $\lambda = 0.999892$, $L = \begin{bmatrix} -1.000000 & 1.000000 \end{bmatrix}^T$. Figure 1 shows the system trajectories along with their estimates and the system phase plane. For comparison purposes we
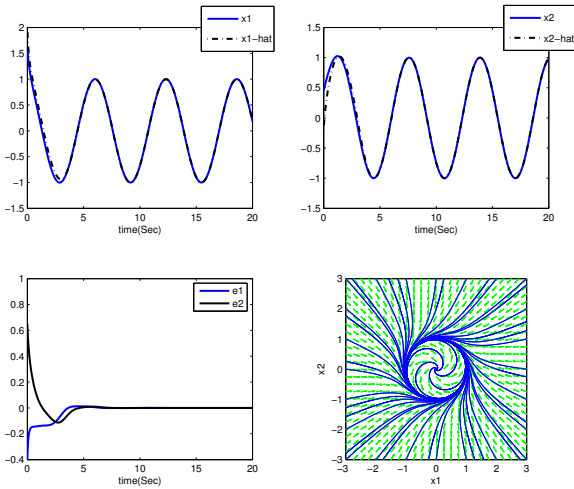

Fig. 1. The system and observer states and the phase plane

now consider the conventional Lipschitz formulation. With $r = 5.9372$, the corresponding Lipschitz constant is $l = 3r^2 = 105.75$ (compare it with the one-sided Lipschitz constant $\rho = 0$). It is highly unlikely that an observer designed based on the conventional Lipschitz approach can work with such a large Lipschitz constant. The *maximum* Lipschitz constant that those observers can handle are normally *at least* an order of magnitude less than this. For example, the maximum admissible Lipschitz constant for the observer designed using [7] in this case is $l = 1.0324$. The advantage of the new approach in this case, is evident. Note that $A - LC = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is indeed unstable, confirming

our finding of $A - LC$ not being necessarily stable. To verify our result lets calculate the Lyapunov derivative. Any positive definite matrix P with condition number $\kappa = \frac{1}{\lambda}$ is acceptable. Lets take $P = \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & 1 \end{bmatrix}$. From (25) we obtain $\dot{V} \leq -0.4187 < 0$.

VI. CONCLUSIONS

This article introduces a new class of nonlinear systems, namely the one-sided Lipschitz systems, as a generalization of the well-known class of Lipschitz systems. The observer design problem for this class of systems is established. The advantages of designing observers in this context are explained and the challenges discussed. A observer design procedure is given that can be easily applied to the considered class of system using the available numerically efficient LMI solvers. The efficiency of the approach is shown through an illustrative example.

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