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Extensions of Partially Defined Boolean Functions with Missing Data

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Abstract. As a form of knowledge acquisition from data, we consider the problem which, given a partially defined Boolean function with missing data (pBmd) $(\tilde{T}, \tilde{F})$, where $\tilde{T} \subseteq \{0,1,*\}^n$ and $\tilde{F} \subseteq \{0,1,*\}^n$, respectively, represent "positive examples" and "negative examples" and "$*$" represents missing bits in the data, establishes a Boolean function (extension) f such that f is true (resp., false) in every given true (resp., false) vector. In particular, we study extensively three types of extensions called consistent, robust and most robust extensions for various classes of Boolean functions such as general, positive, regular, k-DNF, h-term DNF, Horn, self-dual, threshold, read-once, and decomposable. For certain classes we shall provide polynomial algorithms, and for other cases we prove their NP-hardness.

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Contents

1 Introduction

Knowledge acquisition in the form of Boolean logic has been intensively studied in the recent research (e.g., [3, 5, 8, 12, 18]): given a set of data, represented as a set T of binary "true n-vectors" (or "positive examples") and a set F of "false n-vectors" (or "negative examples"), establish a Boolean function (extension) f in a specified class $\mathcal{C}$, such that f is true (resp., false) in every given true (resp., false) vector; i.e., $T \subseteq T(f)$ and $F \subseteq F(f)$, where $T(f)$ (resp., $F(f)$) denotes the set of true vectors (resp., the set of false vectors) of f. A pair of sets (T, F) is called a *partially defined Boolean function* (*pdBf*) throughout this paper.

For instance, data x represent the symptoms to diagnose a disease, e.g., x_1 denotes whether temperature is high ($x_1 = 1$) or not ($x_1 = 0$), and x_2 denotes whether blood presure is high ($x_2 = 1$) or not ($x_2 = 0$), etc. Establishing an extension f, which is consistent with the given data, amounts to finding a logical diagnostic explanation of the given data. Therefore, this may be considered as a form of knowledge acquisition from given examples.

In this process, some knowledge or hypothesis about the extension f is usually available beforehand. Such knowledge may be obtained from experience or from the analysis of mechanisms that may or may not cause the phenomena under consideration. In the above example of diagnosing diseases, it would be natural to assume that we somehow know the direction of each variable that tends to cause the disease to appear. By changing the polarities of variables if necessary, therefore, the extension $f(x)$ can be assumed to be positive in all variables. In this paper, we discuss several classes of functions such as positive (also called monotone), regular, k-DNF, Horn and dual-comparable, which respectively arise in different context of applications.

Unfortunately, the real-world data might not be complete. As for the above examples, for some data x, temperature might not be measured, that is, it is not known whether $x_1 = 0$ or 1. For another instance, we have a battery of 45 biochemical tests for carcinogenicity. However, we do not usually apply all tests, since all tests cannot be checked in a laboratory or some tests are very expensive. When a test is not applied, we say that the test result is *missing*. A set of data $(\tilde{T}, \tilde{F})$, which includes the missing results, is called a *partially defined Boolean function with missing data* (*pBmd*), where $\tilde{T}$ (resp., $\tilde{F}$) denotes the set of "positive examples" (resp., "negative examples") of such vectors. To cope with such situations, we introduce in this paper three types of complete Boolean functions called *consistent*, *robust* and *most robust* extensions, respectively. More precisely, given a pBmd $(\tilde{T}, \tilde{F})$ and a class of Boolean functions $\mathcal{C}$, (i) a consistent extension is a Boolean function f in $\mathcal{C}$ such that, for every $\tilde{a} \in \tilde{T}$ (resp., $\tilde{F}$), there is a 0-1 vector a obtained from $\tilde{a}$ by fixing missing data appropriately, for which $f(a) = 1$ (resp., $f(a) = 0$) holds, (ii) a robust extension is a Boolean function f in $\mathcal{C}$ such that, for every $\tilde{a} \in \tilde{T}$ (resp., $\tilde{F}$), any 0-1 vector a obtained from $\tilde{a}$ by fixing missing data arbitrarily satisfies $f(a) = 1$ (resp., $f(a) = 0$), and (iii) a most robust extension is a Boolean function f in $\mathcal{C}$ which is a robust extension of a pBmd (T', F'), where (T', F') is obtained from $(\tilde{T}, \tilde{F})$ by fixing a smallest set of missing data appropriately (the remaining missing data in $T' \cup F'$ are assumed to take arbitrary values). All of these extensions provide logical explanations of a given pBmd $(\tilde{T}, \tilde{F})$ with varied freedom given to the missing data in $\tilde{T}$ and $\tilde{F}$. By definition, if $(\tilde{T}, \tilde{F})$ has a robust extension, it is also a most robust extension and is a consistent extension, and if $(\tilde{T}, \tilde{F})$ has a most robust extension, it is a consistent extension. In case of most robust and consistent extensions, they also provide information such that some missing data must take certain values if $(\tilde{T}, \tilde{F})$ can have a consistent extension in class $\mathcal{C}$. This type of information is also useful in analyzing incomplete data sets.

In this paper, we study the problems of deciding the existence of (and constructing) these extensions for a given pBmd $(\tilde{T}, \tilde{F})$ and class $\mathcal{C}$, mainly from the view point of their computational complexity. We obtain computationally efficient algorithms in some cases, and prove NP-completeness in some other cases. For a summary of the results obtained, see Tables 1–5.

2 Preliminaries

2.1 Boolean functions

A *Boolean function*, or a *function* in short, is a mapping $f : \mathbb{B}^n \mapsto \mathbb{B}$, where $\mathbb{B} = \{0, 1\}$, and $x \in \mathbb{B}^n$ is called a *Boolean vector* (or a *vector* in short). If $f(x) = 1$ (resp., 0), then x is called a *true* (resp., *false*) vector of f. The set of all true vectors (resp., false vectors) is denoted by $T(f)$ (resp., $F(f)$). Two special functions with $T(f) = \emptyset$ and $F(f) = \emptyset$ are respectively denoted by $f = \bot$ and $f = \top$. For two functions f and g on the same set of variables, we write $f \leq g$ if $f(x) = 1$ implies $g(x) = 1$ for all $x \in \mathbb{B}^n$, and we write $f < g$ if $f \leq g$ and $f \neq g$.

A function f is *positive* if $x \leq y$ (i.e., $x_i \leq y_i$ for all $i \in \{1, 2, \ldots, n\}$) always implies $f(x) \leq f(y)$. A positive function is also called *monotone*. For a subset $R \subseteq \{1, 2, \ldots, n\}$, let $x|R$ denote the vector obtained from x by switching the values 0 and 1 of all x_j, $j \in R$. Then f is called *unate* if there is a subset R such that $x|R \leq y|R$ always implies $f(x) \leq f(y)$. The variables $x_1, x_2, \ldots, x_n$ and their complements $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ are called *literals*. A *term* is a conjunction of literals such that at most one of x_i and $\bar{x}_i$ appears for each i. The constant 1 (viewed as the conjunction of an empty set of literals) is also considered as a term. A *disjunctive normal form* (*DNF*) is a disjunction of terms. Clearly, a DNF defines a function, and it is well-known that every function can be represented by a DNF (however, such a representation may not be unique). Throughout this paper, unless otherwise stated, we usually do not distinguish a DNF φ from the function it represents.

It is also well-known that a Boolean function f is positive if and only if f can be represented by a DNF, in which all the literals of each term are uncomplemented. A function is called a *k-DNF* if it has a DNF with at most k literals in each term, *h-term DNF* if it has a DNF with at most h terms, and *Horn* if it has a DNF with at most one complemented literal in each term. Furthermore, a function f is called *renamable Horn* (or sometimes *disguised Horn*) if there exists a subset $R \subseteq \{1, 2, \ldots, n\}$ such that function $f(x|R)$ is Horn. Obviously, a positive function is a special case of a Horn function, and a unate function is a special case of a renamable Horn function.

A Boolean expression is called *read-once* [1] if it contains at most one occurrence of each variable, where an expression can be given by using conjunctions, disjunctions and complementations. For instance, $\bar{x}_1 \vee x_2(x_3 \vee x_4\bar{x}_5)$ is a read-once expression. Read-once expressions are also called *μ-formulas* [20] or *Boolean trees*. A function is called *read-once* if it has a read-once expression.

The *dual* of a function f, denoted f^d, is defined by

$$f^d(x) = \bar{f}(\bar{x}),$$

where $\bar{f}$ and $\bar{x}$ denote the complement of f and x, respectively. As is well-known, a Boolean expression defining f^d can be obtained from an expression representing f by exchanging $\vee$ (or) and $\cdot$ (and), as well as the constants 0 and 1. It is easy to see that $(f \vee g)^d = f^d g^d$, and so on. A function f is called *dual-minor* if $f \le f^d$, *dual-major* if $f \ge f^d$, *dual-comparable* if $f \le f^d$ or $f \ge f^d$, and *self-dual* if $f^d = f$. It is known [5] that a function f is dual-minor (resp., dual-major, self-dual) if and only if at most (resp., at least, exactly) one of $f(a) = 1$ and $f(\bar{a}) = 1$ holds for every $a \in \mathbb{B}^n$.

An assignment A of binary values 0 or 1 to k variables $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ is called a *k-assignment*, and is denoted by

$$A = (x_{i_1} \leftarrow a_1, x_{i_2} \leftarrow a_2, \ldots, x_{i_k} \leftarrow a_k),$$

where each of $a_1, a_2, \ldots a_k$ is either 1 or 0. Let the complement of A, denoted by $\bar{A}$, represent the assignment obtained from A by complementing all the 1's and 0's in A. When a function f of n variables and a k-assignment A are given,

$$f_A = f_{(x_{i_1} \leftarrow a_1, x_{i_2} \leftarrow a_2, \ldots, x_{i_k} \leftarrow a_k)}$$

denotes the function of $(n \perp k)$ variables obtained by fixing variables $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ as specified by A. Let f be a function of n variables. If either $f_A \le f_{\bar{A}}$ or $f_A \ge f_{\bar{A}}$ holds for every k-assignment A, then f is said to be *k-comparable*. If f is k-comparable for every k such that $1 \le k \le m$, then f is said to be *m-monotonic*. (For more detailed discussion on these topics, see [16] for example.) In particular, f is 1-monotonic if $f_{(x_i \leftarrow 1)} \ge f_{(x_i \leftarrow 0)}$ or $f_{(x_i \leftarrow 1)} \le f_{(x_i \leftarrow 0)}$ holds for every $i \in \{1, 2, \ldots, n\}$. A function f is positive if and only if f is 1-monotonic and $f_{(x_i \leftarrow 1)} \ge f_{(x_i \leftarrow 0)}$ holds for all i.

Now consider a 2-assignment $A = (x_i \leftarrow 1, x_j \leftarrow 0)$. If

$$f_A \ge f_{\bar{A}} \;\; (\text{resp., } \; f_A > f_{\bar{A}})$$

holds, this is denoted $x_i \succeq_f x_j$ (resp., $x_i \succ_f x_j$). Variables x_i and x_j are said to be *comparable* if either $x_i \succeq_f x_j$ or $x_i \preceq_f x_j$ holds. When $x_i \succeq_f x_j$ and $x_i \preceq_f x_j$ hold simultaneously, it is denoted as $x_i \approx_f x_j$. If f is 2-monotonic, this binary relation $\succeq_f$ over the set of variables is known to be a total preorder [16]. A 2-monotonic positive function f of n variables is called *regular* if

$$x_1 \succeq_f x_2 \succeq_f \cdots \succeq_f x_n. \tag{1}$$

Any 2-monotonic positive function becomes regular by permuting variables. It is known that f is regular if and only if $f(x) \geq f(y)$ holds for all $x, y \in \mathbb{B}^n$ with $\sum_{j \leq k} x_j \geq \sum_{j \leq k} y_j$, $k = 1, 2, \ldots n$. The 2-monotonicity and related concepts have been studied under various names in the fields such as threshold logic [16] and hypergraph theory [6]. It was originally introduced in conjunction with threshold functions (e.g., [16]), where a positive function f is *threshold* if there exist $n+1$ nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and t such that:

$$f(x) = \begin{cases} 1, & \text{if } \sum \alpha_i x_i \geq t \\ 0, & \text{if } \sum \alpha_i x_i < t. \end{cases}$$

As $\alpha_i \geq \alpha_j$ implies $x_i \succeq_f x_j$ and $\alpha_i = \alpha_j$ implies $x_i \approx_f x_j$, a threshold function is always 2-monotonic, although the converse is not true [16].

Let $V = \{1, 2, \ldots, n\}$ denote the index set of variables. For a vector $x \in \mathbb{B}^n$ and $S \subseteq V$, $x[S]$ denotes the projection of x on S. To simplify notation, for a Boolean function h depending only on variables of $S \subseteq V$, we write $h(S)$ instead of $h(x[S])$. A function f is called $g(S_0, h_1(S_1), h_2(S_2), ..., h_k(S_k))$-*decomposable* [3, 14] if it satisfies the following three conditions:

(i) h_i depends only on variables in S_i, $i = 1, ..., k$,
(ii) g depends on the variables in S_0 and on the binary values $h_i(S_i)$ for $i = 1, ..., k$, (i.e., $g : \{0,1\}^{|S_0|+k} \to \{0,1\}$),
(iii) $f = g(S_0, h_1(S_1), h_2(S_2), ..., h_k(S_k))$.

Let us note that $S_0, S_1, \ldots, S_k$ are not necessarily assumed to be disjoint. Also, a function f is called *positive* $g(S_0, h_1(S_1), ..., h_k(S_k))$-*decomposable* if f is $g(S_0, h_1(S_1), ..., h_k(S_k))$-decomposable and functions g, h_i, $i = 1, 2, \ldots, k$ are all positive.

2.2 Partially defined Boolean functions (with missing data) and their extensions

A *partially defined Boolean function* (*pdBf*) is defined by a pair of sets (T, F) such that $T, F \subseteq \mathbb{B}^n$. A function f is called an *extension* (or *theory*) of the pdBf (T, F) if $T \subseteq T(f)$ and $F \subseteq F(f)$. We shall also say in this case that the function f *correctly classifies* all the vectors $a \in T$ and $b \in F$. Evidently, the disjointness of the sets T and F is a necessary and sufficient condition for the existence of an extension. It may not be evident, however, how to find out whether a given pdBf has a extension in $\mathcal{C}$, where $\mathcal{C}$ denotes a subclass of Boolean functions, such as the class of positive functions, the class of k-DNF's, etc. Therefore, we have considered in [5] the following problems.

Problem EXTENSION($\mathcal{C}$)
Input: a pdBf (T, F), where $T, F \subseteq \mathbb{B}^n$.
Question: Is there an extension $f \in \mathcal{C}$ of (T, F)?

Problem BEST-FIT($\mathcal{C}$)
Input: a pdBf (T, F), where $T, F \subseteq \mathbb{B}^n$, and a positive weight function $w : T \cup F \mapsto \mathbb{R}_+$.
Output: Subsets T^* and F^* such that $T^* \cap F^* = \emptyset$ and $T^* \cup F^* = T \cup F$, for which the pdBf (T^*, F^*) has an extension in $\mathcal{C}$, and $w(T \cap F^*) + w(F \cap T^*)$ is minimum.

Let us add that, in case the answer is YES in problem EXTENSION($\mathcal{C}$), we expect to be able to specify an extension $f \in \mathcal{C}$ as a justification. Hence, if EXTENSION($\mathcal{C}$) is solvable in polynomial time, we also provided such an extension $f \in \mathcal{C}$, either by a direct algebraic form, or by a polynomial time membership oracle [5]. Similarly, for BEST-FIT($\mathcal{C}$), an extension $f \in \mathcal{C}$ of the pBmd (T^*, F^*) has also to be specified [5]. Note that $w(T \cap F^*) + w(F \cap T^*)$ in the problem statement denotes the minimum weight sum of the vectors in $T \cup F$ which are erroneously classified by the obtained extension.

As a pdBf does not allow missing data, we then introduce set

$$\mathbb{M} = \{0, 1, *\},$$

and interpret the asterisk components $*$ of $v \in \mathbb{M}^n$ as missing bits. For a vector $v \in \mathbb{M}^n$, let $ON(v) = \{j \mid v_j = 1,\ j = 1, 2, \ldots, n\}$ and $OFF(v) = \{j \mid v_j = 0,\ j = 1, 2, \ldots, n\}$. For a subset $\tilde{S} \subseteq \mathbb{M}^n$, let $AS(\tilde{S}) = \{(v, j) | v \in \tilde{S},\ j \in V \setminus (ON(v) \cup OFF(v))\}$ be the collection of all missing bits of the vectors in $\tilde{S}$. If $\tilde{S}$ is a singleton $\{v\}$, we also denote $AS(\{v\})$ as $AS(v)$. Clearly, $\mathbb{B}^n \subseteq \mathbb{M}^n$, and $v \in \mathbb{B}^n$ if and only if $AS(v) = \emptyset$. Let us consider binary assignments $\alpha \in \mathbb{B}^Q$ to subsets $Q \subseteq AS(\tilde{S})$ of the missing bits. For a vector $v \in \tilde{S}$ and an assignment $\alpha \in \mathbb{B}^Q$, v^α denotes the vector obtained from v by replacing the $*$ components which belong to Q by the binary values assigned to them by α, i.e.,

$$v_j^\alpha = \begin{cases} v_j & \text{if } (v, j) \notin Q \\ \alpha(v, j) & \text{if } (v, j) \in Q. \end{cases}$$

For vectors $v, w \in \mathbb{M}^n$, we shall write $v \gtrapprox w$ (resp., $v \lessapprox w$) if there exists an assignment $\alpha \in \mathbb{B}^{AS(\{v,w\})}$ for which $v^\alpha \geq w^\alpha$ (resp., $v^\alpha \leq w^\alpha$) holds, and we say that v is *potentially greater* (resp., *smaller*) than w. If both $v \gtrapprox w$ and $v \lessapprox w$ hold then we write $v \approx w$, and say that v is *potentially identical* with w. Note that $v \approx w$ holds if and only if there is an assignment $\alpha \in AS(\{v, w\})$ such that $v^\alpha = w^\alpha$.

A *pdBf with missing data* (or in short *pBmd*) is a pair $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$. To a pBmd $(\tilde{T}, \tilde{F})$ we always associate the set $AS = AS(\tilde{T} \cup \tilde{F})$ of all missing bits. A function f is called a *robust extension* of the pBmd $(\tilde{T}, \tilde{F})$ if

$$f(a^\alpha) = 1 \text{ and } f(b^\alpha) = 0 \text{ for all } a \in \tilde{T},\ b \in \tilde{F} \text{ and for all } \alpha \in \mathbb{B}^{AS}.$$

We first consider the problem of deciding the existence of a robust extension of a given pBmd $(\tilde{T}, \tilde{F})$ in a specified class $\mathcal{C}$.

Problem RE($\mathcal{C}$)
Input: A pBmd $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$.
Question: Does $(\tilde{T}, \tilde{F})$ have a robust extension in class $\mathcal{C}$?

In case of a YES answer, we normally assume that a robust extension $f \in \mathcal{C}$ can also be provided, either by a direct algebraic form, or by a polynomial time membership oracle. It may happen that a pBmd $(\tilde{T}, \tilde{F})$ has no robust extension in $\mathcal{C}$, but it has an extension if we change some (or all) $*$ bits to appropriate binary values. A function f is called a *consistent extension* of pBmd $(\tilde{T}, \tilde{F})$, if there exists an assignment $\alpha \in \mathbb{B}^{AS}$ for which $f(a^\alpha) = 1$ and $f(b^\alpha) = 0$ for all $a \in \tilde{T}$ and $b \in \tilde{F}$. In other words, a pBmd $(\tilde{T}, \tilde{F})$ is said to have a consistent extension in $\mathcal{C}$ if, for some assignment $\alpha \in \mathbb{B}^{AS}$, the pdBf $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ defined by $\tilde{T}^\alpha = \{a^\alpha \mid a \in \tilde{T}\}$ and $\tilde{F}^\alpha = \{b^\alpha \mid b \in \tilde{F}\}$ has an extension in $\mathcal{C}$. This leads us to the problem of deciding the existence of a consistent extension of a given pBmd $(\tilde{T}, \tilde{F})$ in a specified class $\mathcal{C}$.

Problem CE($\mathcal{C}$)
Input: A pBmd $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$.
Question: Does $(\tilde{T}, \tilde{F})$ have a consistent extension in class $\mathcal{C}$?

Again in case of a YES answer, we normally assume that an assignment $\alpha \in \mathbb{B}^{AS}$, for which the pdBf $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has an extension in $\mathcal{C}$, is also specified together with such an extension $f \in \mathcal{C}$ (f is described either by a direct algebraic form, or by a polynomial time membership oracle).

It may also happen that not all missing bits should be specified in order to have a robust extension. In this case, call an assignment $\alpha \in \mathbb{B}^Q$ for a subset $Q \subseteq AS$ as a *robust assignment* if the resulting pBmd $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a robust extension in class $\mathcal{C}$. We are interested in finding a robust assignment with the smallest size $|Q|$. Such an extension is called a *most robust extension* of the given pBmd $(\tilde{T}, \tilde{F})$ in the specified class $\mathcal{C}$.

Problem MRE($\mathcal{C}$)
Input: A pBmd $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$.
Output: NO if $(\tilde{T}, \tilde{F})$ does ot have a consistent extension; otherwise a robust assignment $\alpha \in \mathbb{B}^Q$ for a subset $Q \subseteq AS$, which minimizes $|Q|$.

Similarly to the previous problems, a robust extension $f \in \mathcal{C}$ of pBmd $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ is also normally required to output. Let us define

$$\rho(\mathcal{C}; (\tilde{T}, \tilde{F})) = \min_{\substack{Q \subseteq AS \\ \exists \alpha \in B^Q \text{ s.t. } (\tilde{T}^\alpha, \tilde{F}^\alpha) \\ \text{has a robust extension in } \mathcal{C}}} |Q|, \tag{2}$$

where $\rho(\mathcal{C}; (\tilde{T}, \tilde{F})) = +\infty$ if there is no Q satisfying the stated condition. To simplify notation, we sometimes use $\rho(\tilde{T}, \tilde{F})$ in place of $\rho(\mathcal{C}; (\tilde{T}, \tilde{F}))$, unless confusion arises. Observe that a pBmd $(\tilde{T}, \tilde{F})$ has a robust extension if and only if $\rho(\tilde{T}, \tilde{F}) = 0$, and it has a consistent extension if and only if $\rho(\tilde{T}, \tilde{F}) \leq |AS|$.

It follows therefore that if RE($\mathcal{C}$) or CE($\mathcal{C}$) are NP-complete, then MRE($\mathcal{C}$) is NP-hard, and conversely, if MRE($\mathcal{C}$) is solvable in polynomial time, then both RE($\mathcal{C}$) and CE($\mathcal{C}$) are polynomially solvable. It seems also that RE($\mathcal{C}$) is, in general, easier than CE($\mathcal{C}$). Indeed, this is the case for many classes. NP-complete, Let us also note that, if $AS = \emptyset$ (i.e., $(\tilde{T}, \tilde{F})$ is a pdBf), then the notions of extension, robust extension and consistent extension all coincide. Thus, RE($\mathcal{C}$) and CE($\mathcal{C}$) are both at least as difficult as EXTENSION($\mathcal{C}$).

Let us add that we shall also consider various restricted variants of the above problems, in which the input pBmd $(\tilde{T}, \tilde{F})$ is restricted to satisfy certain conditions such as

(a) $|AS| \leq k$, where $k = O(\log(n + |\tilde{T}| + |\tilde{F}|))$.

(b) $|AS(a)| \leq k$ for every $a \in \tilde{T} \cup \tilde{F}$, where k is a given constant, or $k = O(\log(n + |\tilde{T}| + |\tilde{F}|))$.

3 Relations to EXTENSION and BEST-FIT

In this section, we examine more carefully the relation between our problems CE($\mathcal{C}$), RE($\mathcal{C}$), MRE($\mathcal{C}$) and those problems EXTENSION($\mathcal{C}$) and BEST-FIT($\mathcal{C}$) studied in [5]. As a result, we see that many complexity results follow from the results in [5].

3.1 Implications of EXTENSION

First of all, as we mentioned earlier, EXTENSION($\mathcal{C}$) is a special case of problems RE($\mathcal{C}$) and CE($\mathcal{C}$), since all these problems coincide if a pBmd $(\tilde{T}, \tilde{F})$ satisfies $AS = \emptyset$. Hence we have the following theorem.

Theorem 1 *If problem* EXTENSION($\mathcal{C}$) *is NP-complete, then problem* CE($\mathcal{C}$) *is NP-complete, and problem* RE($\mathcal{C}$) *is NP-hard. Furthermore, if the considered pBmds* $(\tilde{T}, \tilde{F})$ *are restricted to those satisfying* $|AS(a)| = O(\log(n + |\tilde{T}| + |\tilde{F}|)$ *for all* $a \in \tilde{T} \cup \tilde{F}$*, then* RE($\mathcal{C}$) *is NP-complete.* □

The slight difference between the conclusions for CE($\mathcal{C}$) and RE($\mathcal{C}$) comes from the fact that, although it is easy to see that CE($\mathcal{C}$) is in class NP, problem RE($\mathcal{C}$) may not belong to class NP, since the condition "for all $\alpha \in \mathbb{B}^{AS}$" is involved in the definition of RE($\mathcal{C}$). For instance, given a pBmd $(\tilde{T}, \tilde{F})$ and a cubic DNF $f \in \mathcal{C}_{3-DNF}$, the problem of deciding if f is a robust extension of $(\tilde{T}, \tilde{F})$, or not, can be shown to be co-NP-complete.

The second half of the theorem statement can be shown as follows. For a given pBmd $(\tilde{T}, \tilde{F})$, define

$$\begin{aligned} T^+ &= \{a^\alpha \mid a \in \tilde{T}, \alpha \in \mathbb{B}^{AS(a)}\} \\ F^+ &= \{b^\alpha \mid b \in \tilde{F}, \alpha \in \mathbb{B}^{AS(b)}\}. \end{aligned} \tag{3}$$

Then by the definition of a robust extension, a pBmd $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}$ if and only if a pdBf (T^+, F^+) has an extension in $\mathcal{C}$. Furthermore,

$$|T^+| + |F^+| = O((|\tilde{T}| + |\tilde{F}|) \times 2^{O(\log(n+|\tilde{T}|+|\tilde{F}|))}),$$

which is polynomial in $|\tilde{T}|$, $|\tilde{F}|$ and n. Thus, RE($\mathcal{C}$) is obviously in NP in this case.

Therefore, we immediately have the following corollary from the results in [5].

Corollary 1 *Problem* CE($\mathcal{C}$) *is NP-complete and problem* RE($\mathcal{C}$) *is NP-hard for the following classes of functions* $\mathcal{C}$*:* (1) *(positive) k-DNF,* (2) *(positive) h-term-DNF,* (3) *(positive) h-term-DNF with fixed* $h \geq 2$, (4) *(positive) h-term-k-DNF,* (5) *(positive) h-term-k-DNF with fixed* $h \geq 1$, (6) *(positive) h-term-k-DNF with fixed* $k \geq 1$, (7) *renamable Horn,* (8) *2-monotonic positive,* (9) *(positive) read-once, and* (10) *unate. Furthermore, problem* RE($\mathcal{C}$) *for these classes is NP-complete if pBmds* $(\tilde{T}, \tilde{F})$ *are restricted to those satisfying* $|AS(a)| = O(\log(n + |\tilde{T}| + |\tilde{F}|)$ *for all* $a \in \tilde{T} \cup \tilde{F}$. □

However, we also have the following positive results.

Theorem 2 *Let a pBmd* $(\tilde{T}, \tilde{F})$ *satisfy* $|AS(a)| = O(\log(n + |\tilde{T}| + |\tilde{F}|))$ *for all* $a \in \tilde{T} \cup \tilde{F}$, *where* $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$. *If problem* EXTENSION($\mathcal{C}$) *can be solved in polynomial time, then problem* RE($\mathcal{C}$) *is also polynomially solvable.*

Proof. As noted after Theorem 1, a pBmd $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}$ if and only if pdBf (T^+, F^+) has an extension in $\mathcal{C}$, where T^+ and F^+ are defined by (3). This and the polynomiality of EXTENSION($\mathcal{C}$) imply the polynomiality of RE($\mathcal{C}$). □

Corollary 2 *Let a pBmd* $(\tilde{T}, \tilde{F})$ *satisfy* $|AS(a)| = O(\log(n + |\tilde{T}| + |\tilde{F}|))$ *for all* $a \in \tilde{T} \cup \tilde{F}$, *where* $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$. *Then problem* RE($\mathcal{C}$) *can be solved in polynomial time for the following classes of functions* $\mathcal{C}$*:* (1) *general,* (2) *positive,* (3) *regular,* (4) *(positive) k-DNF with fixed k,* (5) *(positive)* 1*-term-DNF,* (6) *(positive) h-term-k-DNF with fixed h and k,* (7) *(positive) self-dual,* (8) *(positive) dual-minor,* (9) *(positive) dual-major,* (10) *(positive)* $g(S_0, h_1(S_1))$*-decomposable,* (11) *Horn, and* (12) *threshold.*

Proof. Combine Theorem 2 and the results in [5]. □

Theorem 3 *Let a pBmd* $(\tilde{T}, \tilde{F})$ *satisfy* $|AS| = O(\log(n + |\tilde{T}| + |\tilde{F}|))$, *where* $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$. *If problem* EXTENSION($\mathcal{C}$) *can be solved in polynomial time, then problem* MRE($\mathcal{C}$) *is also polynomially solvable.*

Proof. Let $(\tilde{T}, \tilde{F})$ be such a pBmd. Then, for each assignment $\alpha \in \mathbb{M}^{AS}$ (note that this may not be a binary assignment), check if the pBmd $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a robust extension. In this case, since $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ satisfies $|AS| = O(\log(n+|\tilde{T}|+|\tilde{F}|))$, RE($\mathcal{C}$) can be solved in polynomial time by Theorem 2. Therefore we find an assignment $\alpha^* \in \mathbb{M}^{AS}$ that maximizes $|AS(\alpha^*)|$ among those having robust extensions. Then set $Q = AS \setminus AS(\alpha^*)$ and binary assignment

$\beta \in \mathbb{B}^Q$ such that $\beta_j = \alpha_j^*$ for all $j \in Q$ provide a solution of MRE($\mathcal{C}$). As the number of assignments $\alpha \in \mathbb{M}^{AS}$ is $3^{|AS|}$, that is, polynomial in $|\tilde{T}|$, $|\tilde{F}|$ and n, we can find such an assignment $\alpha^* \in \mathbb{M}^{AS}$ in polynomial time. □

Corollary 3 *Let a pBmd $(\tilde{T}, \tilde{F})$ satisfy $|AS| = O(\log(n + |\tilde{T}| + |\tilde{F}|))$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$. Then problem* MRE($\mathcal{C}$) *can be solved in polynomial time for the following classes of functions $\mathcal{C}$:* (1) *general,* (2) *positive,* (3) *regular,* (4) (*positive*) *k-DNF with fixed k,* (5) (*positive*) 1-*term-DNF,* (6) (*positive*) *h-term-k-DNF with fixed h and k,* (7) (*positive*) *self-dual,* (8) (*positive*) *dual-minor,* (9) (*positive*) *dual-major,* (10) (*positive*) *$g(S_0, h_1(S_1))$-decomposable,* (11) *Horn, and* (12) *threshold.*

Proof. Combine Theorem 3 and the results in [5]. □

3.2 Implications of BEST-FIT

Let us recall problem BEST-FIT($\mathcal{C}$) described in Subsection 2.2, and denote by $\varepsilon(T, F)$ the required minimum weight sum of error vectors:

$$\varepsilon(T,F) = \min_{\substack{T^* \cap F^* = \emptyset, T^* \cup F^* = T \cup F \\ (T^*, F^*) \text{ has an extension in } \mathcal{C}}} w(T \cap F^*) + w(F \cap T^*).$$

For subsets $A, B \subseteq \mathbb{M}^n$, let $A \times B$ denote the set of vectors obtained as concatenations of vectors from A and B, i.e.,

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

We introduce the following properties for binary vectors $p, q, r \in \mathbb{B}^k$ for some k.

PA(p): (T, F) has an extension in $\mathcal{C}$ if and only if $(T \times \{p\}, F \times \{p\})$ has an extension in $\mathcal{C}$.

PB(p, q, r): (T, F) has an extension in $\mathcal{C}$ if and only if $((T \times \{p\}) \cup (\mathbb{B}^n \times \{q\}), (F \times \{p\}) \cup (\mathbb{B}^n \times \{r\}))$ has an extension in $\mathcal{C}$.

Lemma 1 *Let (T, F) be a pdBf with $T, F \subseteq \mathbb{B}^n$ and $w(a) = 1$ for all $a \in T \cup F$ (i.e., $w(T \cap F^*) + w(F \cap T^*) = |T \cap F^*| + |F \cap T^*|$), and let us assume that class $\mathcal{C}$ satisfies properties* PA($(1,0)$) *and* PB($(1,0), (1,1), (0,0)$). *Then*

$$\varepsilon(T, F) = \rho(T \times \{(1, *)\}, F \times \{(*, 0)\}),$$

where ρ is defined by (2).

Proof. Let $\tilde{T} = T \times \{(1,*)\}$ and $\tilde{F} = F \times \{(*,0)\}$. Thus $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^{n+2}$. Let (T^*, F^*) be the solution of BEST-FIT($\mathcal{C}$) with input (T, F), i.e., $\varepsilon(T,F) = |T \cap F^*| + |F \cap T^*|$. Define $T' = T \setminus F^*$ and $F' = F \setminus T^*$. Obviously, the pdBf (T', F') has an extension in $\mathcal{C}$. Then by property PA$((1,0))$ the pdBf $(T' \times \{(1,0)\}, F' \times \{(1,0)\})$ has an extension in $\mathcal{C}$, and furthermore by property PB$((1,0),(1,1),(0,0))$, the pdBf (T'', F''), where

$$\begin{aligned} T'' &= (T' \times \{(1,0)\}) \cup (T \times \{(1,1)\}) \\ F'' &= (F' \times \{(1,0)\}) \cup (F \times \{(0,0)\}), \end{aligned}$$

also has an extension in $\mathcal{C}$, since $T \subseteq \mathbb{B}^n$ and $F \subseteq \mathbb{B}^n$. Now, define the sets

$$\begin{aligned} Q_1 &= AS\left((T \cap F^*) \times \{(1,*)\}\right) \\ Q_2 &= AS\left((F \cap T^*) \times \{(*,0)\}\right), \end{aligned}$$

and the assignment α on $Q = Q_1 \cup Q_2$ by $\alpha(q) = 1$ for all $q \in Q_1$ and $\alpha(q) = 0$ for all $q \in Q_2$. Then we conclude that the pBmd $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a robust extension in $\mathcal{C}$, since any vector $a \in \mathbb{B}^{n+2}$ obtainable from $\tilde{a} \in \tilde{T}^\alpha$ (resp., $\tilde{F}^\alpha$) by an assignment satisfies $a \in T''$ (resp., F''). This implies that

$$\varepsilon(T,F) = |T \cap F^*| + |F \cap T^*| = |Q_1| + |Q_2| \geq \rho(\tilde{T}, \tilde{F}).$$

For the converse inequality, let us assume next that a subset $Q \subseteq AS$ of the above pBmd $(\tilde{T}, \tilde{F})$ and an assignment $\alpha \in \mathbb{B}^Q$ satisfy $|Q| = \rho(\tilde{T}, \tilde{F})$, and that the resulting pBmd $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a robust extension f in $\mathcal{C}$. Define

$$\begin{aligned} T^\sharp &= \{a \in T \mid ((a,1,*), n+2) \notin Q\} \\ F^\sharp &= \{b \in F \mid ((b,*,0), n+1) \notin Q\}. \end{aligned}$$

Then $f(a,1,0) = 1$ and $f(b,1,0) = 0$ hold for all $a \in T^\sharp$ and $b \in F^\sharp$, by the definition of a robust extension. Thus, the pdBf $(T^\sharp \times \{(1,0)\}, F^\sharp \times \{(1,0)\})$ has an extension in $\mathcal{C}$. This, by property PA$((1,0))$, implies that $(T^\sharp, F^\sharp)$ has also an extension f' in $\mathcal{C}$. Then applying this extension f' to (T, F), f' can misclassify only vectors in $T \setminus T^\sharp$ and in $F \setminus F^\sharp$. Therefore,

$$\varepsilon(T,F) \leq |T \setminus T^\sharp| + |F \setminus F^\sharp| = |Q| = \rho(\tilde{T}, \tilde{F}),$$

since $|AS(a)| = 1$ holds for every $a \in \tilde{T} \cup \tilde{F}$. □

This lemma implies immediately the following theorem.

Theorem 4 *If a class of functions $\mathcal{C}$ satisfies properties* PA$((1,0))$ *and* PB$((1,0), (1,1), (0,0))$, *and problem* BEST-FIT($\mathcal{C}$) *is NP-hard when all $a \in T \cup F$ satisfy $w(a) = 1$, then problem* MRE($\mathcal{C}$) *is NP-hard, even if $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$.* □

We shall show later in Theorem 31 that Theorem 4 implies the NP-hardness of MRE($\mathcal{C}$) for the class $\mathcal{C}^+_{g(S_0,h_1(S_1))}$.

3.3 Positive extensions

Let us consider problems with subclasses of positive functions $\mathcal{C} \subseteq \mathcal{C}^+$. Given a vector $v \in \mathbb{M}^n$, let us denote by $\mathbf{1} \in \mathbb{B}^{AS(v)}$ (resp., by $\mathbf{0} \in \mathbb{B}^{AS(v)}$) the assignment of ones (resp., zeros) to all $(v,i) \in AS(v)$.

Lemma 2 *Consider a class of functions $\mathcal{C} \subseteq \mathcal{C}^+$. For a pBmd $(\tilde{T}, \tilde{F})$, let us associate a pdBf (T^-, F^+) by defining*

$$T^- = \{a^{\mathbf{0}} \mid a \in \tilde{T}\}$$
$$F^+ = \{b^{\mathbf{1}} \mid b \in \tilde{F}\}.$$

Then, the pBmd $(\tilde{T}, \tilde{F})$ has a robust extension in the class $\mathcal{C}$ if and only if the pdBf (T^-, F^+) has an extension in class $\mathcal{C}$.

Proof. Let us assume first that the pBmd $(\tilde{T}, \tilde{F})$ has a robust extension $f \in \mathcal{C}$. Then, by definition, f is an extension of the pdBf (T^-, F^+). For the converse direction, let us assume that the pdBf (T^-, F^+) has an extension g in class $\mathcal{C}$. For any assignment $\beta \in \mathbb{B}^{AS}$ and $a \in \tilde{T}$, the vector $a^{\mathbf{0}} \in T^-$ satisfies $a^{\mathbf{0}} \leq a^\beta$, and hence $g(a^\beta) = 1$ is implied by $g(a^\beta) \geq g(a^{\mathbf{0}}) = 1$. Similarly, for any assignment $\beta \in \mathbb{B}^{AS}$ and $b \in \tilde{F}$, the vector $b^{\mathbf{1}} \in F^+$ satisfies $b^{\mathbf{1}} \geq b^\beta$, and hence $g(b^\beta) = 0$ follows analogously. Therefore, g is a robust extension of the pBmd $(\tilde{T}, \tilde{F})$ in the class $\mathcal{C}$. □

Lemma 3 *Consider a class of functions $\mathcal{C} \subseteq \mathcal{C}^+$. For a pBmd $(\tilde{T}, \tilde{F})$, let us associate the pdBf (T^+, F^-) defined by*

$$T^+ = \{a^{\mathbf{1}} \mid a \in \tilde{T}\}$$
$$F^- = \{b^{\mathbf{0}} \mid b \in \tilde{F}\}.$$

Then, the pBmd $(\tilde{T}, \tilde{F})$ has a consistent extension in the class $\mathcal{C}$ if and only if the pdBf (T^+, F^-) has an extension in the same class.

Proof. Let us assume first that the pBmd $(\tilde{T}, \tilde{F})$ has a consistent extension $f \in \mathcal{C}$, i.e. that there exists an assignment $\beta \in \mathbb{B}^{AS}$ such that f is an extension of the pdBf $(\tilde{T}^\beta, \tilde{F}^\beta)$. Since $\mathcal{C} \subseteq \mathcal{C}^+$, for any $a \in \tilde{T}$ (resp., $b \in \tilde{F}$), $f(a^\beta) = 1$ (resp., $f(b^\beta) = 0$) implies $f(a^{\mathbf{1}}) = 1$ (resp., $f(b^{\mathbf{0}}) = 0$) by $a^\beta \leq a^{\mathbf{1}}$ (resp., $b^\beta \geq b^{\mathbf{0}}$). This implies that f is also an extension of the pdBf (T^+, F^-). The converse direction is immediate, since $(T^+, F^-) \equiv (\tilde{T}^\alpha, \tilde{F}^\alpha)$ for the assignment $\alpha \in \mathbb{B}^{AS}$ defined by $\alpha(v,i) = 1$ if $v \in \tilde{T}$, $(v,i) \in AS(v)$ and $\alpha(u,j) = 0$ for all $u \in \tilde{F}$, $(u,j) \in AS(u)$. □

The following theorem and its corollary are immediate by Lemmas 2 and 3, and by the results of [5].

Theorem 5 *If $\mathcal{C} \subseteq \mathcal{C}^+$ and problem* EXTENSION($\mathcal{C}$) *can be solved in polynomial time, then problems* RE($\mathcal{C}$) *and* CE($\mathcal{C}$) *can also be solved in polynomial time.* □

Corollary 4 *Problems* RE($\mathcal{C}$) *and* CE($\mathcal{C}$) *can be solved in polynomial time for the following classes of functions* $\mathcal{C}$*:* (1) *positive,* (2) *regular,* (3) *positive* k*-DNF with fixed* k*,* (4) *positive* 1*-term-DNF* (5) *positive* h*-term-*k*-DNF with fixed* h *and* k*,* (6) *positive self-dual,* (7) *positive dual-minor,* (8) *positive dual-major, and* (9) *positive* $g(S_0, h_1(S_1))$*-decomposable.* □

Furthermore, we have the following result.

Theorem 6 *If* $\mathcal{C} \subseteq \mathcal{C}^+$ *and problem* BEST-FIT($\mathcal{C}$) *can be solved in polynomial time, then* MRE($\mathcal{C}$) *can be solved in polynomial time for any pBmd* $(\tilde{T}, \tilde{F})$ *that satisfies* $|AS(a)| \leq 1$ *for all* $a \in \tilde{T} \cup \tilde{F}$*.*

Proof. Let us consider a pBmd $(\tilde{T}, \tilde{F})$ with $|AS(a)| \leq 1$ for all $a \in \tilde{T} \cup \tilde{F}$. Define a pdBf (T', F') by

$$\begin{aligned} T' &= \{a^{\mathbf{1}}, a^{\mathbf{0}} \mid a \in \tilde{T}\} \\ F' &= \{b^{\mathbf{1}}, b^{\mathbf{0}} \mid b \in \tilde{F}\}. \end{aligned}$$

Let us define the weights of the above vectors by

$$\begin{array}{ll} w(a^{\mathbf{1}}) = +\infty & \text{if } a \in \tilde{T} \\ w(b^{\mathbf{0}}) = +\infty & \text{if } b \in \tilde{F} \\ w(a^{\mathbf{0}}) = 1 & \text{if } a \in \tilde{T} \text{ and } AS(a) \neq \emptyset \\ w(b^{\mathbf{1}}) = 1 & \text{if } b \in \tilde{F} \text{ and } AS(a) \neq \emptyset. \end{array}$$

We claim that

$$\rho(\tilde{T}, \tilde{F}) = \varepsilon(T', F')$$

holds, which will prove the theorem.

First, if $\varepsilon(T', F') < +\infty$, then clearly, there is a consistent extension of $(\tilde{T}, \tilde{F})$ by the definition of w. Conversely, if there is a consistent extension f of $(\tilde{T}, \tilde{F})$, then $f(a^{\mathbf{1}}) = 1$ holds for all $a \in \tilde{T}$ and $f(b^{\mathbf{0}}) = 0$ holds all $b \in \tilde{F}$ by the positivity of f, which implies $\varepsilon(T', F') < +\infty$.

Let us assume next that there is a solution of MRE($\mathcal{C}$) for $(\tilde{T}, \tilde{F})$; i.e., a subset $Q \subseteq AS$ with $|Q| = \rho(\tilde{T}, \tilde{F})$ and an assignment $\beta \in \mathbb{B}^Q$ for which $(\tilde{T}^{\beta}, \tilde{F}^{\beta})$ has a robust extension f in $\mathcal{C}$. Then f correctly classifies all vectors in $T' \cup F'$, except for $a^{\bar{\beta}} \in T' \cup F'$ with $AS(a) \neq \emptyset$ (where $\bar{\beta}$ denotes the complement of β). Hence

$$\rho(\tilde{T}, \tilde{F}) = |Q| = \sum_{a \in T' \text{ s.t. } f(a)=0} w(a) \;\; + \sum_{b \in F' \text{ s.t. } f(b)=1} w(b) \;\; \geq \;\; \varepsilon(T', F').$$

For the converse inequality, consider a solution (T^*, F^*) to BEST-FIT($\mathcal{C}$) for the pdBf (T', F'), i.e., $T^* \cap F^* = \emptyset$, $T^* \cup F^* = T' \cup F'$, the pdBf (T^*, F^*) has an extension f in

$\mathcal{C}$, and $\varepsilon(T', F') = w(T' \cap F^*) + w(F' \cap T^*) < +\infty$. Then, by the positivity of f, we have $a^{\mathbf{1}} \in T^*$ for all $a \in \tilde{T}$ and $b^{\mathbf{0}} \in F^*$ for all $b \in \tilde{F}$. Thus define $Q = Q_1 \cup Q_2$, where

$$\begin{aligned} Q_1 &= \{(a,j) \mid a \in T', (a,j) \in AS(a) \text{ and } a^{\mathbf{0}} \in F^*\} \\ Q_2 &= \{(b,j) \mid b \in F', (b,j) \in AS(b) \text{ and } b^{\mathbf{1}} \in T^*\}, \end{aligned}$$

and an assignment $\beta \in \mathbb{B}^Q$ by $\beta(a,j) = 1$ for $(a,j) \in Q_1$, and $\beta(b,j) = 0$ for $(b,j) \in Q_2$. The resulting $(\tilde{T}^\beta, \tilde{F}^\beta)$ has a robust extension $f \in \mathcal{C}$. Consequently,

$$\varepsilon(T', F') = |Q_1| + |Q_2| = |Q| \geq \rho(\tilde{T}, \tilde{F}).$$

□

Corollary 5 *Let a pBmd $(\tilde{T}, \tilde{F})$ satisfy $|AS(a)| \leq 1$ for all $a \in \tilde{T} \cup \tilde{F}$. Then problem* MRE($\mathcal{C}$) *is polynomially solvable for the following classes of functions $\mathcal{C}$:* (1) *positive,* (2) *regular, and* (3) *positive h-term-k-DNF with fixed h and k.* □

In the rest of paper, we discuss complexity results of CE($\mathcal{C}$), RE($\mathcal{C}$) and MRE($\mathcal{C}$), which are not immediately derivable from the results for EXTENSION($\mathcal{C}$) and BEST-FIT($\mathcal{C}$) discussed in [5].

4 General functions

4.1 Problems RE and CE

Let $\mathcal{C}_{all}$ denote the class of all functions We first consider Problem RE($\mathcal{C}_{all}$).

Theorem 7 *Problem* RE($\mathcal{C}_{all}$) *can be solved in polynomial time.*

Proof. A pBmd $(\tilde{T}, \tilde{F})$ has a robust extension if and only if there exists an index j such that $a_j \neq b_j$ and $\{a_j, b_j\} = \{0, 1\}$ (i.e., either $a_j = 0$ and $b_j = 1$, or $a_j = 1$ and $b_j = 0$) for every pair of $a \in \tilde{T}$ and $b \in \tilde{F}$. Obviously, this can be checked in $O(n|T||F|)$ time. □

Let us now turn to problem CE($\mathcal{C}_{all}$). First, we note that CE($\mathcal{C}_{all}$) can be trivially solved if $|AS(a)| > 0$ holds for every $a \in \tilde{T} \cup \tilde{F}$. This is so, because $(\tilde{T}, \tilde{F})$ always has a consistent extension f. Indeed, let us consider an assignment $\alpha \in \mathbb{B}^{AS}$ such that $|ON(a^\alpha)|$ is odd for all $a \in \tilde{T}$, and $|ON(b^\alpha)|$ is even for all $b \in \tilde{F}$, and let f be the parity function such that $f(v) = 1$ if $|ON(v)|$ is odd, and $f(v) = 0$ if $|ON(v)|$ is even. Furthermore, CE($\mathcal{C}_{all}$) can be solved in polynomial time by Corollary 3 if $|AS| = O(\log(n + |\tilde{T}| + |\tilde{F}|))$. Problem CE($\mathcal{C}_{all}$) becomes more difficult when not all input vector has missing bits, but the number of missing bits in total is large, although, it remains polynomially solvable if no input vector contains more than one missing bit.

Theorem 8 *Problem* CE($\mathcal{C}_{all}$) *can be solved in polynomial time for a pBmd* $(\tilde{T}, \tilde{F})$ *for which every* $a \in \tilde{T} \cup \tilde{F}$ *satisfies* $|AS(a)| \leq 1$.

Proof. Let j_a be the index of the $*$ in each vector $a \in \tilde{T} \cup \tilde{F}$ (i.e., $AS(a) = \{(a, j_a)\}$), if any. Then $(\tilde{T}, \tilde{F})$ has a consistent extension if and only if (i) there is no pair of $a \in \tilde{T}$ and $b \in \tilde{F}$ such that $a, b \in \mathbb{B}^n$ and $a = b$, and (ii) there is an assignment $\alpha \in \mathbb{B}^{AS}$ satisfying the conditions

$$\begin{aligned} \alpha(a, j_a) \neq b_{j_a} &\qquad \text{if } a \notin \mathbb{B}^n \text{ and } b \in \mathbb{B}^n & (4)\\ \alpha(b, j_b) \neq a_{j_b} &\qquad \text{if } a \in \mathbb{B}^n \text{ and } b \notin \mathbb{B}^n & (5)\\ \alpha(a, j_a) \neq b_{j_a} \text{ or } \alpha(b, j_b) \neq a_{j_b} &\qquad \text{if } a, b \notin \mathbb{B}^n \text{ and } j_a \neq j_b & (6)\\ \alpha(a, j_a) \neq \alpha(b, j_b) &\qquad \text{if } a, b \notin \mathbb{B}^n \text{ and } j_a = j_b. & (7) \end{aligned}$$

for every pair of $a \in \tilde{T}$ and $b \in \tilde{F}$ with $a \approx b$. Obviously, condition (i) can be checked in $O(n|\tilde{T}||\tilde{F}|)$ time. To check (ii), let us observe that each of the conditions (4)–(7) can equivalently be expressed as clauses in the variables $\alpha(v, j)$ for $(v, j) \in AS$. Namely, conditions (4) and (5) are equivalent with linear clauses, (6) can be represented by a clause containing two variables, and condition (7) can be represented by the conjunction of two clauses, each of which contains two variables. E.g. (7) is equivalent with the condition

$$(\alpha(a, j_a) \vee \alpha(b, j_b))(\overline{\alpha(a, j_a)} \vee \overline{\alpha(b, j_b)}) = 1.$$

In total, we have a 2-SAT problem containing at most $2|\tilde{T}||\tilde{F}|$ clauses, which is solvable in time linear in its input size (see e.g., [2]). This shows that problem CE($\mathcal{C}_{all}$) can be solved in $O(n|\tilde{T}||\tilde{F}|)$ time. □

Example 1. Let us define $\tilde{T}, \tilde{F} \subseteq \{0, 1\}^3$ by

$$\tilde{T} = \left\{ \begin{array}{lcl} a^{(1)} & = & (1,1,*) \\ a^{(2)} & = & (0,0,1) \\ a^{(3)} & = & (0,1,*) \\ a^{(4)} & = & (*,0,0) \end{array} \right\}, \quad \tilde{F} = \left\{ \begin{array}{lcl} b^{(1)} & = & (1,1,1) \\ b^{(2)} & = & (0,*,1) \\ b^{(3)} & = & (*,0,0) \end{array} \right\}.$$

Then we have the following 2-SAT:

$$\overline{\alpha(a^{(1)},3)}\,\alpha(b^{(2)},2)(\overline{\alpha(a^{(3)},3)} \vee \overline{\alpha(b^{(2)},2)})(\alpha(a^{(4)},1) \vee \alpha(b^{(3)},1))(\overline{\alpha(a^{(4)},1)} \vee \overline{\alpha(b^{(3)},1)}) = 1.$$

For this, the assignment $\alpha \in \mathbb{B}^{AS}$ given by $\alpha(a^{(1)}, 3) = \alpha(a^{(3)}, 3) = \alpha(a^{(4)}, 1) = 0$ and $\alpha(b^{(2)}, 2) = \alpha(b^{(3)}, 1) = 1$, is a satisfying solution. □

In general, however, we have the following negative result.

Theorem 9 *Problem* CE($\mathcal{C}_{all}$) *is NP-complete, even if* $|AS(a)| \leq 2$ *holds for all* $a \in \tilde{T} \cup \tilde{F}$.

Proof. Given an assignment $\alpha \in \mathbb{B}^{AS}$, we can check in polynomial time if $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has an extension in $\mathcal{C}_{all}$, [5]. Hence problem CE($\mathcal{C}_{all}$) is in NP. Hence problem CE($\mathcal{C}_{all}$) belongs to NP.

Let us now consider a cubic CNF

$$\begin{aligned} \Phi &= \bigwedge_{k=1}^{m} C_k \\ C_k &= (u_k \vee v_k \vee w_k), \end{aligned}$$

where u_k, v_k and w_k for $k = 1, 2, ..., m$ are literals from set $L = \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$. The 3-SAT problem, i.e., deciding the existence of a binary vector $y \in \{0,1\}^n$ for which $\Phi(y) = 1$, is one of the well-known NP-complete problems (see [11]). We shall associate to Φ a pBmd $(\tilde{T}, \tilde{F})$, as follows, which has a consistent extension in $\mathcal{C}_{all}$ if and only if the 3-SAT $\Phi = 1$ has a solution.

Let us introduce subsets $A_z = \{p_{z1}, p_{z2}\}$, $z \in L$ and $B_k = \{q_{k1}, q_{k2}, q_{k3}\}$, $k = 1, 2, \ldots, m$ such that $A_z \cap L = B_k \cap L = A_z \cap B_k = \emptyset$, $A_z \cap A_{z'} = \emptyset$ for $z \neq z'$, and $B_i \cap B_j = \emptyset$ for $i \neq j$. Let

$$V = L \cup (\bigcup_{z \in L} A_z) \cup (\bigcup_{k=1}^{m} B_k).$$

Let us denote by $(R; S)$ the vector $v \in \mathbb{M}^V$ for which $ON(v) = R$ and $AS(v) = \{(v, j) \mid j \in S\}$. (Then $OFF(v) = V \setminus (R \cup S)$, i.e. if $S = \emptyset$, then v denotes a binary vector.)

Let us construct $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^V$ by setting $\tilde{T} = \tilde{T}_1 \cup \tilde{T}_2$ and $\tilde{F} = \tilde{F}_1 \cup \tilde{F}_2$, where

$$\tilde{T}_1 = \{(L \setminus \{x_i, \bar{x}_i\}; \{x_i, \bar{x}_i\}) \mid x_i \in L\} \cup \{((L \setminus \{z\}) \cup \{p_{zj}\}; \emptyset) \mid z \in L, j = 1, 2\}$$

$$\tilde{T}_2 = \left\{ \begin{array}{l|l} a^k = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{q_{k1}\}; \{q_{k2}, q_{k3}\}) & \\ a^{u_k 1} = ((L \setminus \{u_k\}) \cup A_{u_k}; \{q_{k1}\}) & \\ a^{u_k 2} = ((L \setminus \{u_k, \bar{u}_k, v_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \{v_k\}) & \\ a^{u_k 3} = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \{w_k\}) & \\ a^{u_k 4} = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{p_{u_k 2}\} \cup \{q_{k1}\}; \{p_{u_k 1}\}) & \\ a^{v_k 1} = ((L \setminus \{v_k\}) \cup \{q_{k2}\}; \{q_{k1}\}) & k = 1, 2, \ldots, m \\ a^{v_k 2} = ((L \setminus \{v_k, \bar{v}_k, w_k\}) \cup \{q_{k1}, q_{k2}\}; \{w_k\}) & \\ a^{v_k 3} = ((L \setminus \{v_k, \bar{v}_k, w_k, \bar{w}_k, u_k\}) \cup \{q_{k1}, q_{k2}\}; \{u_k\}) & \\ a^{w_k 1} = ((L \setminus \{w_k\}) \cup \{q_{k3}\}; \{q_{k1}\}) & \\ a^{w_k 2} = ((L \setminus \{w_k, \bar{w}_k, u_k\}) \cup \{q_{k1}, q_{k3}\}; \{u_k\}) & \\ a^{w_k 3} = ((L \setminus \{w_k, \bar{w}_k, u_k, \bar{u}_k, v_k\}) \cup \{q_{k1}, q_{k3}\}; \{v_k\}) & \end{array} \right\}$$

$$\tilde{F}_1 = \{(L;\emptyset)\} \cup \{(L\setminus\{x_i,\bar{x}_i\};\emptyset) \mid x_i \in L\} \cup \{(L\setminus\{z\};A_z) \mid z \in L\}$$

$$\tilde{F}_2 = \left\{ \begin{array}{l|l} b^k = ((L\setminus\{u_k,\bar{u}_k,v_k,\bar{v}_k,w_k,\bar{w}_k\}) \cup B_k;\emptyset) & \\ b^{u_k1} = ((L\setminus\{u_k,\bar{u}_k\}) \cup A_{u_k} \cup \{q_{k1}\};\{\bar{u}_k\}) & \\ b^{u_k2} = ((L\setminus\{u_k,\bar{u}_k,v_k,\bar{v}_k\}) \cup A_{u_k} \cup \{q_{k1}\};\{\bar{v}_k\}) & \\ b^{u_k3} = ((L\setminus\{u_k,\bar{u}_k,v_k,\bar{v}_k,w_k,\bar{w}_k\}) \cup A_{u_k} \cup \{q_{k1}\};\{\bar{w}_k\}) & \\ b^{u_k4} = ((L\setminus\{u_k,\bar{u}_k,v_k,\bar{v}_k,w_k,\bar{w}_k\}) \cup \{q_{k1}\};\{p_{u_k2}\}) & \\ b^{v_k1} = ((L\setminus\{v_k\});\{q_{k2}\}) & \\ b^{v_k2} = ((L\setminus\{v_k,\bar{v}_k\}) \cup \{q_{k1},q_{k2}\};\{\bar{v}_k\}) & k=1,2,\ldots,m \\ b^{v_k3} = ((L\setminus\{v_k,\bar{v}_k,w_k,\bar{w}_k\}) \cup \{q_{k1},q_{k2}\};\{\bar{w}_k\}) & \\ b^{v_k4} = ((L\setminus\{v_k,\bar{v}_k,w_k,\bar{w}_k,u_k,\bar{u}_k\}) \cup \{q_{k1},q_{k2}\};\{\bar{u}_k\}) & \\ b^{w_k1} = ((L\setminus\{w_k\});\{q_{k3}\}) & \\ b^{w_k2} = ((L\setminus\{w_k,\bar{w}_k\}) \cup \{q_{k1},q_{k3}\};\{\bar{w}_k\}) & \\ b^{w_k3} = ((L\setminus\{w_k,\bar{w}_k,u_k,\bar{u}_k\}) \cup \{q_{k1},q_{k3}\};\{\bar{u}_k\}) & \\ b^{w_k4} = ((L\setminus\{w_k,\bar{w}_k,u_k,\bar{u}_k,v_k,\bar{v}_k\}) \cup \{q_{k1},q_{k3}\};\{\bar{v}_k\}) & \end{array} \right\} \quad (8)$$

It is easy to see that $|AS(a)| \le 2$ holds for all $a \in \tilde{T} \cup \tilde{F}$.

Let us first assume that there is a consistent extension f of $(\tilde{T},\tilde{F})$, and show that Φ is satisfiable. Now $(L\setminus\{x_i,\bar{x}_i\};\{x_i,\bar{x}_i\}) \in \tilde{T}_1$ and $(L;\emptyset)$, $(L\setminus\{x_i,\bar{x}_i\};\emptyset) \in \tilde{F}_1$ imply that either $f(L\setminus\{x_i\};\emptyset)=1$ or $f(L\setminus\{\bar{x}_i\};\emptyset)=1$ (or both) holds for each of $i=1,2,\ldots n$. Let us define a binary vector $y \in \mathbb{B}^n$ by

$$y_i = \begin{cases} 1 & \text{if } f(L\setminus\{x_i\};\emptyset)=0 \\ 0 & \text{otherwise,} \end{cases}$$

and show that this y satisfies $\Phi(y)=1$. By the definition of y, $y_i=1$ (resp., $y_i=0$) implies $f(L\setminus\{\bar{x}_i\};\emptyset)=1$ (resp., $f(L\setminus\{x_i\};\emptyset)=1$). Assuming that there exists a clause $C_k=(u_k \vee v_k \vee w_k)$, which is 0, we derive a contradiction.

(i) If $u_k=0$, then $f(L\setminus\{u_k\};\emptyset)=1$. Therefore $((L\setminus\{u_k\});A_{u_k}) \in \tilde{F}_1$ and $((L\setminus\{u_k\}) \cup \{p_{u_kj}\};\emptyset) \in \tilde{T}_1$ for $j=1,2$ implying

$$f((L\setminus\{u_k\}) \cup A_{u_k};\emptyset)=0. \quad (9)$$

Let us consider the sequence

$$a^{u_k1}\,(\in \tilde{T}_2), b^{u_k1}\,(\in \tilde{F}_2), \ldots, a^{u_k4}\,(\in \tilde{T}_2), b^{u_k4}\,(\in \tilde{F}_2).$$

The equation (9) and $a^{u_k1} \in \tilde{T}_2$ imply $f((L\setminus\{u_k\}) \cup A_{u_k} \cup \{q_{k1}\};\emptyset)=1$, which also yields $f((L\setminus\{u_k,\bar{u}_k\}) \cup A_{u_k} \cup \{q_{k1}\};\emptyset)=0$ by $b^{u_k1} \in \tilde{F}_2$. By applying a similar argument, we have

$$f((L\setminus\{u_k,\bar{u}_k,v_k,\bar{v}_k,w_k,\bar{w}_k\}) \cup \{q_{k1}\};\emptyset)=0. \quad (10)$$

(ii) If $v_k=0$, then $f(L\setminus\{v_k\};\emptyset)=1$ must hold. Let us consider the sequence

$$b^{v_k1}\,(\in \tilde{F}_2), a^{v_k1}\,(\in \tilde{T}_2), \ldots, a^{v_k3}\,(\in \tilde{T}_2), b^{v_k4}\,(\in \tilde{F}_2).$$

Then $f(L \setminus \{v_k\}; \emptyset) = 1$ and $b^{v_k 1} \in \tilde{F}_2$ imply $f((L \setminus \{v_k\}) \cup \{q_{k2}\}; \emptyset) = 0$, from which $f((L \setminus \{v_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) = 1$ follows by $a^{v_k 1} \in \tilde{T}_2$. By applying a similar argument, we have

$$f((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) = 0. \tag{11}$$

(iii) If $w_k = 0$, then similarly to (ii), we have

$$f((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) = 0. \tag{12}$$

The three equations (10), (11) and (12), and the fact that $b^k \in \tilde{F}_2$ together imply that no binary assignment to the missing bits of $a^k \in \tilde{T}_2$ can make it a true vector of f, contradicting the fact that f is a consistent extension of $(\tilde{T}, \tilde{F})$.

For the converse direction, let $y^* \in \mathbb{B}^n$ be a satisfying solution to Φ, and let

$$\begin{aligned} P_0 = \{(L \setminus \{x_i\}; \emptyset\}) \mid y_i^* = 0, i = 1, 2, \ldots, n\} \cup \{(L \setminus \{\bar{x}_i\}; \emptyset\}) \mid y_i^* = 1, i = 1, 2, \ldots, n\} \\ \cup \{((L \setminus \{z\}) \cup \{p_{zj}\}; \emptyset) \mid z \in L, j = 1, 2\}. \end{aligned}$$

For each clause $C_k = (u_k \vee v_k \vee w_k)$, let us define sets P_{k1}, P_{k2} and P_{k3} as follows. If $u_k = 1$ holds for the assignment y^*, then

$$P_{k1} = \left\{ \begin{array}{l} (a^k)' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{q_{k1}\}; \emptyset) \\ (a^{u_k 1})' = ((L \setminus \{u_k\}) \cup A_{u_k}; \emptyset) \\ (a^{u_k 2})' = ((L \setminus \{u_k, \bar{u}_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (a^{u_k 3})' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (a^{u_k 4})' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \end{array} \right| \left. k = 1, 2, \ldots, m \right\};$$

otherwise let

$$P_{k1} = \left\{ \begin{array}{l} (a^{u_k 1})' = ((L \setminus \{u_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (a^{u_k 2})' = ((L \setminus \{u_k, \bar{u}_k, v_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (a^{u_k 3})' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (a^{u_k 4})' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{p_{u_k 2}\} \cup \{q_{k1}\}; \emptyset) \end{array} \right| \left. k = 1, 2, \ldots, m \right\}.$$

If $v_k = 1$ holds for the assignment y^*, then

$$P_{k2} = \left\{ \begin{array}{l} (a^k)'' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \\ (a^{v_k 1})' = ((L \setminus \{v_k\}) \cup \{q_{k2}\}; \emptyset) \\ (a^{v_k 2})' = ((L \setminus \{v_k, \bar{v}_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \\ (a^{v_k 3})' = ((L \setminus \{v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \end{array} \right| \left. k = 1, 2, \ldots, m \right\};$$

otherwise,

$$P_{k2} = \left\{ \begin{array}{l} (a^{v_k 1})' = ((L \setminus \{v_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \\ (a^{v_k 2})' = ((L \setminus \{v_k, \bar{v}_k, w_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \\ (a^{v_k 3})' = ((L \setminus \{v_k, \bar{v}_k, w_k, \bar{w}_k, u_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \end{array} \right| \left. k = 1, 2, \ldots, m \right\}.$$

Finally, if $w_k = 1$ holds for the assignment y^*, then let

$$P_{k3} = \left\{ \begin{array}{l} (a^k)''' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \\ (a^{w_k1})' = ((L \setminus \{w_k\}) \cup \{q_{k3}\}; \emptyset) \\ (a^{w_k2})' = ((L \setminus \{w_k, \bar{w}_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \\ (a^{w_k3})' = ((L \setminus \{w_k, \bar{w}_k, u_k, \bar{u}_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \end{array} \middle| \; k = 1, 2, \ldots, m \right\};$$

otherwise set

$$P_{k3} = \left\{ \begin{array}{l} (a^{w_k1})' = ((L \setminus \{w_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \\ (a^{w_k2})' = ((L \setminus \{w_k, \bar{w}_k, u_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \\ (a^{w_k3})' = ((L \setminus \{w_k, \bar{w}_k, u_k, \bar{u}_k, v_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \end{array} \middle| \; k = 1, 2, \ldots, m \right\}.$$

Let us define a function f by

$$f(a) = \begin{cases} 1 & \text{if } a \in P \\ 0 & \text{otherwise,} \end{cases}$$

where $P = P_0 \cup (\bigcup_{k=1}^{m} (P_{k1} \cup P_{k2} \cup P_{k3}))$. We claim that this function f is a consistent extension of $(\tilde{T}, \tilde{F})$.

It is easy to see that for every $a \in \tilde{T}_1$, there exists an assignment $\alpha \in \mathbb{B}^{AS(a)}$ such that $a^\alpha \in P_0$, and for every $a \in \tilde{T}_2 \setminus \{a^k \mid k = 1, 2, \ldots m\}$, there exists an assignment $\alpha \in \mathbb{B}^{AS(a)}$ such that $a^\alpha = (a)'$. Finally, since y^* satisfies $C_k = 1$ for each $a^k \in \tilde{T}_2$, at least one of $(a^k)'$, $(a^k)''$ or $(a^k)'''$ belongs to P, and hence f is a consistent extension of pBmd $(\tilde{T}, \emptyset)$.

Let us show next that f is a consistent extension of $(\emptyset; \tilde{F})$. Let

$$\begin{array}{rl} Q_0 = \{(L; \emptyset)\} & \cup \{(L \setminus \{x_i, \bar{x}_i\}; \emptyset) \mid x_i \in L\} \\ & \cup \{((L \setminus \{x_i\}) \cup A_{x_i}; \emptyset), (L \setminus \{\bar{x}_i\}; \emptyset) \mid y_i^* = 0, i = 1, 2, \ldots, n\} \\ & \cup \; \{(L \setminus \{x_i\}; \emptyset), ((L \setminus \{\bar{x}_i\}) \cup A_{\bar{x}_i}; \emptyset) \mid y_i^* = 1, i = 1, 2, \ldots, n\} \\ & \cup \{b^k \in \tilde{F}_2 \mid k = 1, 2, \ldots, m\}. \end{array}$$

For each clause $C_k = (u_k \vee v_k \vee w_k)$, let us define sets Q_{k1}, Q_{k2} and Q_{k3} as follows. If $u_k = 1$ holds for the assignment y^*, then

$$Q_{k1} = \left\{ \begin{array}{l} (b^{u_k1})' = ((L \setminus \{u_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (b^{u_k2})' = ((L \setminus \{u_k, \bar{u}_k, v_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (b^{u_k3})' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (b^{u_k4})' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{p_{u_k2}\} \cup \{q_{k1}\}; \emptyset) \end{array} \middle| \; k = 1, 2, \ldots, m \right\};$$

otherwise let

$$Q_{k1} = \left\{ \begin{array}{l} (b^{u_k1})' = ((L \setminus \{u_k, \bar{u}_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (b^{u_k2})' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (b^{u_k3})' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup A_{u_k} \cup \{q_{k1}\}; \emptyset) \\ (b^{u_k4})' = ((L \setminus \{u_k, \bar{u}_k, v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{q_{k1}\}; \emptyset) \end{array} \middle| \; k = 1, 2, \ldots, m \right\}.$$

If $v_k = 1$ holds for the assignment y^*, then

$$Q_{k2} = \left\{ \begin{array}{l} (b^{v_k 1})' = ((L \setminus \{v_k\}); \emptyset) \\ (b^{v_k 2})' = ((L \setminus \{v_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \\ (b^{v_k 3})' = ((L \setminus \{v_k, \bar{v}_k, w_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \\ (b^{v_k 4})' = ((L \setminus \{v_k, \bar{v}_k, w_k, \bar{w}_k, u_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \end{array} \right| k = 1, 2, \ldots, m \Bigg\};$$

otherwise,

$$Q_{k2} = \left\{ \begin{array}{l} (b^{v_k 1})' = ((L \setminus \{v_k\}) \cup \{q_{k2}\}; \emptyset) \\ (b^{v_k 2})' = ((L \setminus \{v_k, \bar{v}_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \\ (b^{v_k 3})' = ((L \setminus \{v_k, \bar{v}_k, w_k, \bar{w}_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \\ (b^{v_k 4})' = ((L \setminus \{v_k, \bar{v}_k, w_k, \bar{w}_k, u_k, \bar{u}_k\}) \cup \{q_{k1}, q_{k2}\}; \emptyset) \end{array} \right| k = 1, 2, \ldots, m \Bigg\}.$$

Finally, if $w_k = 1$ holds for the assignment y^*, then let

$$Q_{k3} = \left\{ \begin{array}{l} (b^{w_k 1})' = ((L \setminus \{w_k\}); \emptyset) \\ (b^{w_k 2})' = ((L \setminus \{w_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \\ (b^{w_k 3})' = ((L \setminus \{w_k, \bar{w}_k, u_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \\ (b^{w_k 4})' = ((L \setminus \{w_k, \bar{w}_k, u_k, \bar{u}_k, v_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \end{array} \right| k = 1, 2, \ldots, m \Bigg\};$$

otherwise set

$$Q_{k3} = \left\{ \begin{array}{l} (b^{w_k 1})' = ((L \setminus \{w_k\}) \cup \{q_{k3}\}; \emptyset) \\ (b^{w_k 2})' = ((L \setminus \{w_k, \bar{w}_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \\ (b^{w_k 3})' = ((L \setminus \{w_k, \bar{w}_k, u_k, \bar{u}_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \\ (b^{w_k 4})' = ((L \setminus \{w_k, \bar{w}_k, u_k, \bar{u}_k, v_k, \bar{v}_k\}) \cup \{q_{k1}, q_{k3}\}; \emptyset) \end{array} \right| k = 1, 2, \ldots, m \Bigg\}.$$

It is easy to see that for every $b \in \tilde{F}_1 \cup \{b^k \mid k = 1, 2, \ldots, m\}$, there exists an assignment $\alpha \in \mathbb{B}^{AS(a)}$ such that $a^\alpha \in Q_0$, and for every $a \in \tilde{F}_2 \setminus \{b^k \mid k = 1, 2, \ldots m\}$, there exists an assignment $\alpha \in \mathbb{B}^{AS(a)}$ such that $a^\alpha = (a)'$. Hence f is a consistent extension of the pBmd $(\emptyset, \tilde{F})$.

Finally, let $Q = Q_0 \cup (\bigcup_{k=1}^m (Q_{k1} \cup Q_{k2} \cup Q_{k3}))$. It is easy to check that $P \cap Q = \emptyset$ holds. Therefore, by combining the above two results, we can conclude that f is a consistent extension of the pBmd $(\tilde{T}, \tilde{F})$. □

4.2 Problem MRE

We consider problem MRE($\mathcal{C}_{all}$) in this subsection. By Corollary 3 and Theorem 9, this problem is NP-hard, even if the number of missing bits is not more than 2 in each input vector. Therefore, we only consider the case in which

$$|AS(a)| \leq 1 \text{ for all } a \in \tilde{T} \cup \tilde{F},$$

and show that MRE($\mathcal{C}_{all}$) can be solved in polynomial time in such a case.

Let us remark first that any assignment $\alpha \in AS$ for which $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has an extension must satisfy the conditions (i) and (ii) in the proof of Theorem 8. Hence, some components of such an α may be forced to take a uniqe binary value by conditions (4) and (5) of (ii). Let us assume that we fix all such missing bits in advance, and let us consider only conditions (6) and (7) in the sequel.

Let us define a bipartite graph $G_{AS} = (V, E)$ by

$$\begin{aligned} V &= AS(\tilde{T}) \cup AS(\tilde{F}), \\ E &= \{(q, r; \alpha) \mid q = (a, i) \in AS(\tilde{T}), r = (b, j) \in AS(\tilde{F}), \\ &\qquad \text{and there exists an assignment } \alpha \in \mathbb{B}^{\{q,r\}} \text{ such that } a^\alpha = b^\alpha\}, \end{aligned} \tag{13}$$

where the label $c(e)$ of each edge $e = (q, r; c(e))$, as defined in (13), is called the *configuration* of e. If there are more than one assignments $\alpha \in \mathbb{B}^{\{q,r\}}$ for some $q \in AS(\tilde{T})$ and $r \in AS(\tilde{F})$, for which $a^\alpha = b^\alpha$ (this occurs if $q = (a, i)$ and $r = (b, j)$ satisfy $i = j$), then the graph G_{AS} has parallel edges corresponding to such different configurations. Let us note that, since $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$, every pair of $q = (a, i) \in AS(\tilde{T})$ and $r = (b, j) \in AS(\tilde{F})$ has at most two assignments $\alpha \in \mathbb{B}^{\{q,r\}}$ such that $a^\alpha = b^\alpha$.

Example 2. Let us define $\tilde{T}, \tilde{F} \subseteq \{0, 1\}^6$ by

$$\tilde{T} = \left\{ \begin{array}{lcl} a^{(1)} &=& (*, 1, 1, 1, 1) \\ a^{(2)} &=& (1, 1, 1, 1, *) \\ a^{(3)} &=& (1, 1, 1, *, 1) \\ a^{(4)} &=& (1, 1, *, 1, 1) \\ a^{(5)} &=& (1, *, 0, 1, 0) \end{array} \right\}, \quad \tilde{F} = \left\{ \begin{array}{lcl} b^{(1)} &=& (1, *, 1, 1, 1) \\ b^{(2)} &=& (1, 1, 1, 1, *) \\ b^{(3)} &=& (1, 1, *, 1, 0) \\ b^{(4)} &=& (1, 1, 0, 1, *) \end{array} \right\}.$$

Then graph G_{AS} is given in Figure 1, Although the configurations of the edges are not indicated, they are easy to find out. For example, edge $e = (a^{(1)}, b^{(1)})$ has $c(e) = (a_1^{(1)} = 1, b_2^{(1)} = 1)$, and double edges $e' = (a^{(2)}, b^{(2)})$ and $e'' = (a^{(2)}, b^{(2)})$ have $c(e') = (a_5^{(2)} = 0, b_5^{(2)} = 0)$ and $c(e'') = (a_5^{(2)} = 1, b_5^{(2)} = 1)$, respectively. □

Lemma 4 *Given a pBmd $(\tilde{T}, \tilde{F})$, an assignment $\beta \in \mathbb{B}^Q$ for a subset $Q \subseteq AS$ is a robust assignment of $(\tilde{T}, \tilde{F})$ (i.e., $(\tilde{T}^\beta, \tilde{F}^\beta)$ has a robust extension) if and only if, for every edge $e = (q, r; \alpha)$ of G_{AS}, we have either $q = (a, i) \in Q$ and $a^\beta \neq a^\alpha$, or $r = (b, j) \in Q$ and $b^\beta \neq b^\alpha$, or both.*

Proof. Let us first show the only-if-part. Let f be a robust extension of $(\tilde{T}^\beta, \tilde{F}^\beta)$, and let $e = (q, r; \alpha)$ be an edge of G_{AS}. We can assume without loss of generality, that $q = (a, i) \in AS(\tilde{T})$.

Let us assume that either $q \notin Q$ or $a^\beta = a^\alpha$. Let us show first that $f(a^\alpha) = 1$ is implied then. Indeed, if $q = (a, i) \notin Q$, then $(a^\beta)^\alpha = a^\alpha$, and since $\beta \in \mathbb{B}^Q$ is a robust assignment, $f(a^\alpha) = 1$ must hold. On the other hand, if $a^\beta = a^\alpha$, then obviously $f(a^\alpha) = f(a^\beta) = 1$ must hold, since $a \in \tilde{T}$.

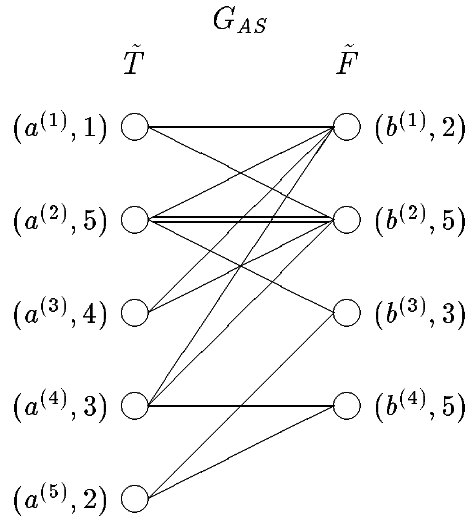

Figure 1: The graph G_{AS} of the pBmd $(\tilde{T}, \tilde{F})$ in Example 2.

We then show that $f(a^\alpha) = 1$ implies $r = (b, j) \in Q$ and $b^\beta \neq b^\alpha$. If $r \notin Q$, then $(b^\beta)^\alpha = b^\alpha = a^\alpha$, and hence $f(a^\alpha) = f(b^\alpha) = 0$ by $b \in \tilde{F}$, which is a contradiction. Similarly, $b^\beta = b^\alpha$ leads to the same contradiction. Hence $r \in Q$ and $b^\beta \neq b^\alpha$ must hold.

To prove the if-part, assume that $\beta \in \mathbb{B}^Q$ for a subset $Q \subseteq AS$ is not a robust assignment of $(\tilde{T}, \tilde{F})$. Then, by the definition of robustness, we have a pair of vectors $a \in \tilde{T}$ and $b \in \tilde{F}$ such that $a^\beta \approx b^\beta$. Then the edge $e = (q, r; \alpha)$ with $q = (a, i)$ and $r = (b, j)$ does not satisfy the statement of the lemma. □

For a vector $d \in \mathbb{B}^n$, let $E(d)$ denote the set of edges $e = (q, r; \alpha) \in E$ with $a^\alpha = b^\alpha = d$, where $q = (a, i)$ and $r = (b, j)$, and let $E = \cup_d E(d)$. Let us define a *coherent domain* $D(d)$ as the set of vertices incident to some edges of $E(d)$, and let D_0 denote the set of isolated vertices (i.e., incident to no edge $e \in E$). (Vertices in D_0 do not belong to any coherent domain.) In the following discussion, we only consider nonempty coherent domains $D(d)$. Figure 2 shows all nonempty coherent domains of the graph G_{AS} of $(\tilde{T}, \tilde{F})$ in Example 2.

Lemma 5 *Every coherent domain $D(d) \subseteq V$ of G_{AS} induces a complete bipartite subgraph of G_{AS}.*

Proof. Take any pair $q = (a, i) \in AS(\tilde{T})$ and $r = (b, j) \in AS(\tilde{F})$ that satisfy $q, r \in D(d)$. Then there exist assignments $\alpha \in \mathbb{B}^{\{q\}}$ and $\beta \in \mathbb{B}^{\{r\}}$ such that $d = a^\alpha = b^\beta$. We concatenate these assignments to have an assignment $\gamma = (\alpha, \beta) \in \mathbb{B}^{\{q,r\}}$ for which $a^\gamma = b^\gamma = d$, implying that there is an edge $(q, r) \in E(d)$. □

Lemma 6 *Let $D(d)$ and $D(d')$ be two coherent domains of G_{AS}, where $d, d' \in \mathbb{B}^n$ and $d \neq d'$. If $D(d) \cap D(d') \neq \emptyset$, then $\| d \perp d' \| = 1$ holds, where $\| x \| = \sum_{i=1}^n |x_i|$.*

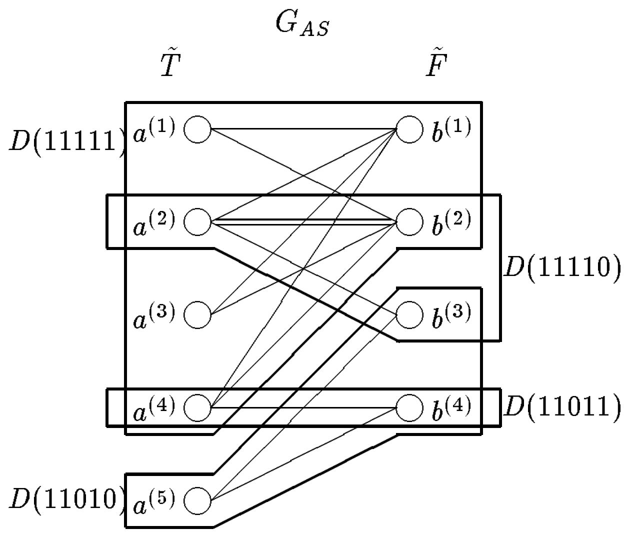

Figure 2: Coherent domains of the graph G_{AS} of $(\tilde{T}, \tilde{F})$ in Example 2.

Proof. Let $q = (a, i) \in D(d) \cap D(d')$. Then there exist two assignments $\alpha, \beta \in \mathbb{B}^{\{q\}}$ $(= \{0, 1\})$ such that $a^\alpha = d$ and $a^\beta = d'$. Since $|AS(a)| \leq 1$ is assumed, $\| d \perp d' \| = 1$ is implied. □

Lemma 7 *Let $D(d)$ and $D(d')$ be two coherent domains of G_{AS}, where $d, d' \in \mathbb{B}^n$ and $d \neq d'$. Then $|D(d) \cap D(d')| \leq 2$ holds. Furthermore, if $D(d) \cap D(d') = \{q, r\}$, then the graph G_{AS} has two parallel edges between q and r.*

Proof. If $q = (a, i), r = (b, j) \in D(d) \cap D(d')$, then by assigning 0 and 1 to q and r, each of a and b can become both d and d'. Since $\| d \perp d' \| = 1$ by Lemma 6, this can only happen if the vectors a and b are identical, missing the same component $i = j$. Therefore $|D(d) \cap D(d') \cap AS(\tilde{T})| \leq 1$ and $|D(d) \cap D(d') \cap AS(\tilde{F})| \leq 1$, and hence $|D(d) \cap D(d')| \leq 2$. Finally, if $D(d) \cap D(d') = \{q, r\}$, where $q = (a, i) \in AS(\tilde{T})$ and $r = (b, j) \in AS(\tilde{F})$, then $q = r$ implies that there are two assignments $\alpha, \beta \in \mathbb{B}^{\{q,r\}}$ such that $a^\alpha = b^\alpha = d$ and $a^\beta = b^\beta = d'$, i.e. the graph G_{AS} has two parallel edges between q and r. □

Let us now color the edges of G_{AS} by "yellow" and "blue", so that all edges of a set $E(d)$ get the same color, and for every pair $E(d)$ and $E(d')$ with $D(d) \cap D(d') \neq \emptyset$ the edges in $E(d)$ get different colors from the edges of $E(d')$. We call such a two coloring *alternating*. The following lemma shows that an alternating coloring is always possible. Furthermore, it can be uniquely completed after fixing a color of a set $E(d)$ in each connected component of G_{AS}.

Lemma 8 *Let $D(d^{(0)}), D(d^{(1)}), \ldots, D(d^{(l)})$ denote a cycle of coherent domains such that $d^{(i-1)} \neq d^{(i)}$ and $D(d^{(i-1)}) \cap D(d^{(i)}) \neq \emptyset$ hold for all $i = 1, 2, \ldots, l \perp 1$, and $D(d^{(l)}) = D(d^{(0)})$. Then l is even.*

Proof. Lemma 6 tells that $\| d^{(i-1)} \perp d^{(i)} \| = 1$ holds for all $i = 1, 2, \ldots, l \perp 1$. Since $\| d^{(0)} \perp d^{(l)} \| = 0$ is even, l must be even. □

Finally let us orient the edges of G_{AS} according to a given alternating coloring, as follows. Every yellow edge (q, r) is oriented from $q \in AS(\tilde{T})$ to $r \in AS(\tilde{F})$, and every blue edge (q, r) is oriented from $r \in AS(\tilde{F})$ to $q \in AS(\tilde{T})$. Let G'_{AS} denote the resulting directed graph. For example, Figure 3 shows the directed graph G'_{AS} for the pBmd $(\tilde{T}, \tilde{F})$ of Example 2. Let us observe that every directed path of this graph is alternating in colors, and every alternating undirected path is either forward directed or backward directed.

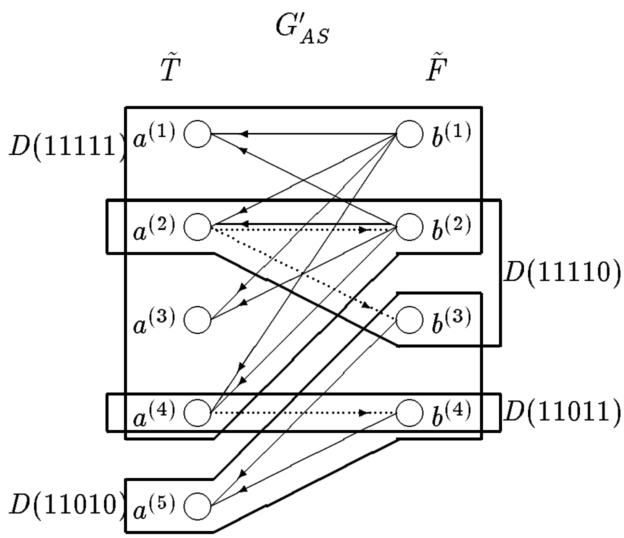

Figure 3: The directed graph G'_{AS} of $(\tilde{T}, \tilde{F})$ in Example 2.

The next lemma characterizes a robust assignment by a directed path of G'_{AS}.

Lemma 9 *Let $(\tilde{T}, \tilde{F})$ be a pBmd, and let $q^{(0)} \underset{e_1}{\mapsto} q^{(1)} \underset{e_2}{\mapsto} q^{(2)} \ldots q^{(l-1)} \underset{e_l}{\mapsto} q^{(l)}$ be a directed path in G'_{AS}. Then $\beta \in \mathbb{B}^Q$ for $Q \subseteq AS$ is a robust assignment if and only if the following properties hold, where $q^{(i)} = (a^{(i)}, j_i)$ and $\alpha_i = c(e_i)$ for all i.*

- **(i)** *If $q^{(0)} \notin Q$ or $(a^{(0)})^\beta = (a^{(0)})^{\alpha_1}$ holds, then $q^{(i)} \in Q$ and $(a^{(i)})^\beta \neq (a^{(i)})^{\alpha_i}$ hold for all $i = 1, 2, \ldots, l$.*
- **(ii)** *If $q^{(l)} \notin Q$ or $(a^{(l)})^\beta = (a^{(l)})^{\alpha_l}$ for some $l > 0$, then $q^{(i)} \in Q$ and $(a^{(i)})^\beta \neq (a^{(i)})^{\alpha_{i+1}}$ hold for all $i = 0, 1, \ldots, l \perp 1$.*

Proof. We first prove the only-if-part. For condition (i), we first consider $e_1 = (q^{(0)}, q^{(1)})$. By Lemma 4, $q^{(0)} \notin Q$ or $(a^{(0)})^\beta = (a^{(0)})^{\alpha_1}$ implies that $q^{(1)} \in Q$ and $(a^{(1)})^\beta \neq (a^{(1)})^{\alpha_1}$. Now, since $e_1 = (q^{(0)}, q^{(1)}) \in E(d)$ and $e_2 = (q^{(1)}, q^{(2)}) \in E(d')$ have different colors, we must have $d \neq d'$ and $q^{(1)} \in D(d) \cap D(d')$, and hence $\| d \perp d' \| = 1$ by Lemma 6. Therefore, $(a^{(1)})^\beta \neq (a^{(1)})^{\alpha_1} (= d)$ implies $(a^{(1)})^\beta = (a^{(2)})^{\alpha_2} (= d')$, and hence $q^{(2)}$ satisfies $(a^2)^\beta \neq (a^{(2)})^{\alpha_2}$ by

Lemma 4. This assignment can proceed in a similar manner to $q^{(i)}$, $i = 2, 3, \ldots, l$. Case (ii) is similar to (i).

Conversely, if conditions (i) and (ii) hold, then, by Lemma 4, $\beta \in \mathbb{B}^Q$ is a robust assignment. $\square$

Let C_i, $i = 1, 2, \ldots, s$, denote all the strongly connected components of this directed graph G'_{AS}. Furthermore, let G^*_{AS} denote the transitive closure of G'_{AS} (i.e., (s, t) is an arc in G^*_{AS} if there is a s-t directed path in G'_{AS}), and let G_0 denote the directed subgraph of G^*_{AS} induced by

$$W = \cup_{i \text{ s.t. } |C_i|=1} C_i. \tag{14}$$

It is easy to see that the set of isolated vertices D_0 in G_{AS} satisfies $D_0 \subseteq W$. Figure 4 contains the graph G_0 of $(\tilde{T}, \tilde{F})$ in Example 2, where, for simplicity, arcs (u, v), for which there is a directed path of length at least 2 from u to v, are not indicated.

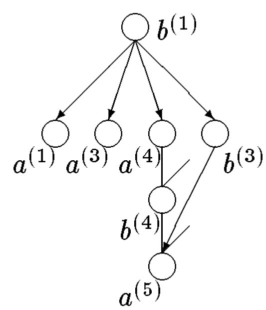

Figure 4: The graph G_0 corresponding to G'_{AS} of $(\tilde{T}, \tilde{F})$ in Example 2.

Lemma 10 *Let $(\tilde{T}, \tilde{F})$ be a pBmd, and let $\alpha \in \mathbb{B}^Q$ for some $Q \subseteq AS$ be a robust assignment, and let C_i and W be defined as above. Then the following two conditions hold:*

- **(i)** *$C_i \subseteq Q$ for all C_i with $|C_i| > 1$, and*
- **(ii)** *$W \setminus Q$ is an antichain in G_0 (i.e., for any pair of $q, r \in W \setminus Q$, there is no directed path from q and r in G_0, and vice versa).*

Proof. Consider a robust assignment $\alpha \in \mathbb{B}^Q$. Assume $q \in C_i \setminus Q$ for some C_i with $|C_i| > 1$. Then there is a directed cycle $q^{(0)} (= q), q^{(1)}, q^{(2)}, \ldots, q^{(l)} (= q)$ of length $l > 1$ in G'_{AS}, and $q \notin Q$ implies $q \in Q$ by Lemma 9, which is a contradiction. Hence condition (i) holds. To prove condition (ii), let us assume that for some pair of $q, r \in W \setminus Q$, there exists a directed path from q and r in G'_{AS}. This is again a contradiction since $q \notin Q$ implies $r \in Q$ by Lemma 9. $\square$

Lemma 11 *Let $(\tilde{T}, \tilde{F})$ be a pBmd, and let $S \subseteq W$ be any maximal antichain in G_0. Then for $Q = AS \setminus S$, there is a robust assignment $\alpha \in \mathbb{B}^Q$ of $(\tilde{T}, \tilde{F})$.*

Proof. For the above $Q = AS \setminus S$, we shall construct a robust assignment $\alpha \in \mathbb{B}^Q$. In the following, we shall consider the directed graph G'_{AS}, and lat us note that, by definition, S is also an antichain in G'_{AS}. Lemma 9 tells that, starting from a vertex $q \in S$ (i.e., $q \notin Q$), a robust assignment β for all vertices t which are either reachable from q or reachable to q is uniquely determined, unless the following cases of conflicts are encountered.

(i) For $q, r \in S$, there is a vertex t for which there are two directed paths $P_1 = q^{(0)} (= q) \to q^{(1)} \to \ldots \to q^{(k)} (= t)$ and $P_2 = r^{(0)} (= r) \to r^{(1)} \to \ldots \to r^{(l)} (= t)$ such that $t^\alpha \neq t^{\alpha'}$, where $\alpha = c(q^{(k-1)}, t)$ and $\alpha' = c(r^{(l-1)}, t)$.

(ii) For $q, r \in S$, there is a vertex t for which there are two directed paths $P_1 = q^{(0)} (= t) \to q^{(1)} \to \ldots \to q^{(k)} (= q)$ and $P_2 = r^{(0)} (= t) \to r^{(1)} \to \ldots \to r^{(l)} (= r)$ such that $t^\alpha \neq t^{\alpha'}$, where $\alpha = c(t, q^{(1)})$ and $\alpha' = c(t, r^{(1)})$.

If one of these conflicts occurs, Lemma 9 tells us that t must be assigned in different ways, and hence we cannot construct an appropriate robust assignment β.

However, we now show that none of these conflicts can occur. Let us consider case (i) only, since case (ii) can be analogously treated. Now $t^\alpha \neq t^{\alpha'}$ implies $(q^{(k-1)}, s) \in E(d)$ and $(r^{(l-1)}, s) \in E(d')$ for some $d \neq d'$. Thus $(q^{(k-1)}, t)$ and $(r^{(l-1)}, t)$ have different colors, since $D(d) \cap D(d') \neq \emptyset$. By the rule of orienting edges (yellow edges are oriented from $AS(\tilde{T})$ to $AS(\tilde{F})$, and blue edges are oriented from $AS(\tilde{F})$ to $AS(\tilde{T})$), this means that one of $(q^{(k-1)}, t)$ and $(r^{(l-1)}, t)$ is oriented towards t, and the other is away from t, a contradiction to the assumption in (i).

Let us denote by R the set of all vertices $t \notin S$ such that either t is reachable from some $q \in S$ or some $q \in S$ is reachable from t. The above argument says that a robust assignment β for R is uniquely determined by Lemma 9. Finally, we consider an assignment $\gamma \in \mathbb{B}^{AS \setminus (S \cup R)}$. By the maximality of S, every vertex $t \in AS \setminus (S \cup R)$ has an incoming arc $e = (r, t) \in E(d)$. Therefore, determine the robust assignment β of this t so that $t^\beta = d$ holds. This is well-defined because all incoming arcs to t belong to the same $E(d)$ by the definition of G'_{AS}. It is easy to see that the resulting β over AS is in fact a robust assignment. □

Lemmas 10 and 11 tell that problem MRE($\mathcal{C}_{all}$) is equivalent to the problem of finding a maximum antichain of G_0. Since G_0 is acyclic, we can find such an antichain in polynomial time by Dilworth's theorem (see e.g. [10]). Finally, we have the following theorem.

Theorem 10 *Problem* MRE($\mathcal{C}_{all}$) *can be solved in polynomial time for a pBmd* $(\tilde{T}, \tilde{F})$ *in which all* $a \in \tilde{T} \cup \tilde{F}$ *satisfy* $|AS(a)| \leq 1$. □

5 Positive and regular functions

Let $\mathcal{C}^+$ and $\mathcal{C}_{\models}$ denote the classes of positive functions and regular functions, respectively. Corollary 4 tells that problems RE($\mathcal{C}^+$), RE($\mathcal{C}_{\models}$), CE($\mathcal{C}^+$) and CE($\mathcal{C}_{\models}$) can be solved in polynomial time. Also by Corollaries 3 and 5, problems MRE($\mathcal{C}^+$) and MRE($\mathcal{C}_\models$) can be solved in polynomial time for the restricted instances.

Let us show first that problem MRE($\mathcal{C}^+$) is, in general, NP-hard.

Theorem 11 *Problem* MRE($\mathcal{C}^+$) *is NP-hard, even if* $|AS(a)| \le 2$ *holds for all* $a \in \tilde{T} \cup \tilde{F}$.

Proof. Let $G = (V, E)$ be a graph, where $V = \{1, 2, \ldots, n\}$, and let us define $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^V$ as follows.

$$\begin{array}{rcl} \tilde{T} &=& \{a^{(i,j)} = (\emptyset; \{i,j\}) \mid (i,j) \in E\} \\ \tilde{F} &=& \{b^{(0)} = (\emptyset; \emptyset)\} \cup \{b^{(i)} = (\emptyset; \{i\}) \mid i \in V\}, \end{array}$$

where $(R; S)$ denotes the vector $v \in \mathbb{M}^V$ such that $ON(v) = R$ and $AS(v) = \{(v, j) \mid j \in S\}$. It is easy to see that $|AS(a)| \le 2$ holds for all $a \in \tilde{T} \cup \tilde{F}$. We claim that

$$\rho(\tilde{T}, \tilde{F}) = |E| + \tau(G)$$

holds, where $\rho(\tilde{T}, \tilde{F})$ is defined by (2), and $\tau(G)$ denotes the cardinality of a minimum vertex cover of G. This will complete the proof of the theorem, since finding $\tau(G)$ is known to be NP-hard [11].

Let us first observe that, if $(\tilde{T}^\beta, \tilde{F}^\beta)$ has a robust positive extension for some $\beta \in \mathbb{B}^Q$, $Q \subseteq AS$, then either $\beta(a^{(i,j)}, i) = 1$ or $\beta(a^{(i,j)}, j) = 1$ (or both) holds for every $(i,j) \in E$, since otherwise we have $b^{(0)} = (a^{(i,j)})^\beta \in \tilde{F}$, which is a contradiction. Let

$$\begin{array}{rcl} E_1 &=& \{(i,j) \in E \mid \text{exactly one of } \beta(a^{(i,j)}, i) = 1 \text{ and } \beta(a^{(i,j)}, j) = 1 \text{ holds }\} \\ E_2 &=& \{(i,j) \in E \mid \beta(a^{(i,j)}, i) = \beta(a^{(i,j)}, j) = 1\}. \end{array}$$

If $(i,j) \in E_1$ and $\beta(a^{(i,j)}, i) = 1$ (resp., $\beta(a^{(i,j)}, j) = 1$), then $\beta(b^{(i)}, i) = 0$ (resp., $\beta(b^{(i)}, j) = 0$) (otherwise $(a^{(i,j)})^\beta \approx (b^{(i)})^\beta$ (resp., $(a^{(i,j)})^\beta \approx (b^{(j)})^\beta$) and β is not a robust assignment). This implies that $C = \{i \mid \beta(b^{(i)}, i) = 0\} \cup \{i \mid i < j, (i,j) \in E_2\}$ is a vertex cover of G. Hence

$$\begin{array}{rcl} |Q| &\ge& |E_1| + 2|E_2| + |\{i \mid \beta(b^{(i)}, i) = 0\}| \\ &=& (|E_1| + |E_2|) + (|E_2| + |\{i \mid \beta(b^{(i)}, i) = 0\}|) \\ &=& |E| + |C| \ge |E| + \tau(G). \end{array}$$

For the converse direction, let $C \subseteq V$ be a minimum vertex cover, and let us define a set $Q \subseteq AS$ and an assignment $\beta \in \mathbb{B}^Q$ by

$$Q = \{(a^{(i,j)}, i) \mid \text{ either } (i \in C, j \notin C) \text{ or } (i, j \in C, i < j)\} \cup \{(b^{(i)}, i) \mid i \in C\},$$

$\beta(a^{(i,j)}, i) = 1$ for $(a^{(i,j)}, i) \in Q$ and $\beta(b^{(i)}, i) = 0$ for $(b^{(i)}, i) \in Q$. It is easy to see that β is a robust assignment, and $|Q| = |E| + \tau(G)$ holds. □

Let us next consider problem MRE($\mathcal{C}_\models$). For a vector $v \in \mathbb{M}^n$, an assignment $\alpha \in \mathbb{B}^{AS(v)}$ is called (j-)*left-shifted* if there is an index j such that $\alpha(v, i) = 1$ for all $i \le j$, and $\alpha(v, i) = 0$

for all $i > j$. Analogously, α is called (j-)*right-shifted* if there is an index j for which $\alpha(v,i) = 0$ for all $i \leq j$, and $\alpha(v,i) = 1$ for all $i > j$. For example, if $v = (10*0**1)$, then an assignment $\alpha \in \mathbb{B}^{AS(v)}$ with $\alpha(v,3) = 1$ and $\alpha(v,5) = \alpha(v,6) = 0$ is left-shifted, and an assignment $\beta \in \mathbb{B}^{AS(v)}$ with $\beta(v,3) = \beta(v,5) = 0$ and $\beta(v,6) = 1$ is right-shifted. It is easy to see that if α is n-left-shifted (or 0-right-shifted), then $\alpha = \mathbf{1}$, while if β if 0-left-shifted (or n-right-shifted), then $\beta = \mathbf{0}$. (see the definition at the beginning of Subsection 3.3).

We can show the following result.

Theorem 12 *Problem* MRE($\mathcal{C}_{\models}$) *can be solved in polynomial time.*

Proof. For a pBmd $(\tilde{T}, \tilde{F})$, let us define a pdBf (T^*, F^*) by

$$\begin{aligned} T^* &= \{a^\alpha \mid a \in \tilde{T},\ \text{a left-shifted assignment } \alpha \in \mathbb{B}^{AS(a)}\} \\ F^* &= \{b^\alpha \mid b \in \tilde{F},\ \text{a right-shifted assignment } \alpha \in \mathbb{B}^{AS(b)}\}, \end{aligned}$$

and define the weights of the above vectors by

$$\begin{array}{ll} w(a^\alpha) = +\infty & \text{if either } a \in \tilde{T} \cap \mathbb{B}^n \text{ or } \alpha = \mathbf{1} \in \mathbb{B}^{AS(a)}, \\ w(b^\alpha) = +\infty & \text{if either } b \in \tilde{F} \cap \mathbb{B}^n \text{ or } \alpha = \mathbf{0} \in \mathbb{B}^{AS(b)}, \\ w(a^\alpha) = 1 & \text{if } a \in \tilde{T} \setminus \mathbb{B}^n \text{ and } \alpha \in \mathbb{B}^{AS(a)} \text{ is left-shifted, } \alpha \neq \mathbf{1}, \\ w(b^\alpha) = 1 & \text{if } b \in \tilde{F} \setminus \mathbb{B}^n \text{ and } \alpha \in \mathbb{B}^{AS(b)} \text{ is right-shifted, } \alpha \neq \mathbf{0}. \end{array}$$

We claim that

$$\rho(\tilde{T}, \tilde{F}) = \varepsilon(T^*, F^*) \tag{15}$$

holds, where $\rho(\tilde{T}, \tilde{F}) = +\infty$ means that there is no consistent extension of $(\tilde{T}, \tilde{F})$. This will prove the theorem, since BEST-FIT($\mathcal{C}_{\models}$) can be solved in polynomial time [5].

If $\rho(\tilde{T}, \tilde{F}) = +\infty$ holds, then by Lemma 3, for any function $f \in \mathcal{C}_{\models}$, there exists either a vector $a \in \tilde{T}$ with $f(a^{\mathbf{1}}) = 0$, or a vector $b \in \tilde{F}$ with $f(b^{\mathbf{0}}) = 1$ (or both). By the definition of w, this means that $\varepsilon(T^*, F^*) = +\infty$. Similarly, we can show that $\varepsilon(T^*, F^*) = +\infty$ implies $\rho(\tilde{T}, \tilde{F}) = +\infty$.

Let us continue with the case in which $\varepsilon(T^*, F^*), \rho(\tilde{T}, \tilde{F}) < +\infty$, and Let us assume that $\beta \in \mathbb{B}^Q$ for a subset $Q \subseteq AS$ is a robust assignment, an optimal solution of MRE($\mathcal{C}_{\models}$). Let $Q_1 = Q \cap AS(\tilde{T})$ and $Q_2 = Q \cap AS(\tilde{F})$. Then,

(i) $\beta(a,i) = 1$ holds for all $(a,i) \in \mathbb{B}^{Q_1}$, and $\beta(b,i) = 0$ holds for all $(b,i) \in \mathbb{B}^{Q_2}$.

This is because, if $\beta(a,i) = 0$ holds for some $(a,i) \in \mathbb{B}^{Q_1}$, the positivity of f implies that the restriction of β to the subset $Q' = (Q_1 \setminus \{(a,i)\}) \cup Q_2$ (i.e., a_i keeps $*$)would also be a robust extension, contradicting the optimality of β. . Similarly, $\beta(b,i) = 1$ for some $(b,i) \in \mathbb{B}^{Q_2}$ would also lead to a contradiction.

Furthermore, the regularity of f implies that Q satisfies the following conditions:

(ii) for every $a \in \tilde{T}$, there exists an index $j_a \in \{0, 1, \ldots, n\}$ such that all $(a,i) \in AS(a)$ with $i \leq j_a$ (resp., $i > j_a$) satisfy $(a,i) \in Q$ (resp., $(a,i) \notin Q$), and

(iii) for every $b \in \tilde{F}$, there exists an index $j_b \in \{0, 1, \ldots, n\}$ such that all $(b, i) \in AS(b)$ with $i \le j_b$ (resp., $i > j_b$) satisfy $(b, i) \in Q$ (resp., $(b, i) \notin Q$).

For example, assume that (ii) does not hold; i.e., some $a^* \in \tilde{T}$ and $i < j$ satisfy $(a^*, i) \in AS(a^*) \setminus Q$ and $(a^*, j) \in AS(a^*) \cap Q$. Then let $Q' = Q_1' \cup Q_2$, where $Q_1' = (Q_1 \setminus \{(a^*, j)\}) \cup \{(a^*, i)\}$, and let us define an assignment $\gamma \in \mathbb{B}^{Q'}$ by $\gamma(a, i) = 1$ for all $(a, i) \in Q_1'$, and $\gamma(b, i) = 0$ for all $(b, i) \in Q_2$. Then it follows from the regularity of f that γ is also a robust assignment. Condition (iii) can be similarly treaded. Thus by repeating this procedure, we have a $Q \in \mathbb{B}^{AS}$ with $|Q| = \rho(\tilde{T}, \tilde{F})$ that satisfies (ii) and (iii).

Now, we prove claim (15). Let a subset $Q \in \mathbb{B}^{AS}$ with $|Q| = \rho(\tilde{T}, \tilde{F})$ and an assignment $\beta \in \mathbb{B}^Q$ satisfy (i),(ii) and (iii), and let f be a robust extension of $(\tilde{T}^\beta, \tilde{F}^\beta)$ in $\mathcal{C}_\models$ (i.e., these give a solution to MRE($\mathcal{C}_\models$)). We shall show that

(iv) $|F(f) \cap \{a^\alpha \mid \text{ a left-shifted assignment } \alpha \in \mathbb{B}^{AS(a)}\}| = |Q \cap AS(a)|$ for every $a \in \tilde{T}$

(v) $|T(f) \cap \{b^\alpha \mid \text{ a right-shifted assignment } \alpha \in \mathbb{B}^{AS(b)}\}| = |Q \cap AS(b)|$ for every $b \in \tilde{F}$,

which will imply that

$$\begin{aligned}
\rho(\tilde{T}, \tilde{F}) &= |Q| = \sum_{a \in \tilde{T}} |Q \cap AS(a)| + \sum_{b \in \tilde{F}} |Q \cap AS(b)| \\
&= \sum_{a \in T^*} |F(f) \cap \{a^\alpha \mid \text{ a left-shifted assignment } \alpha \in \mathbb{B}^{AS(a)}\}| \\
&\quad + \sum_{b \in F^*} |T(f) \cap \{b^\alpha \mid \text{ a right-shifted assignment } \alpha \in \mathbb{B}^{AS(b)}\}| \\
&\ge \varepsilon(T^*, F^*).
\end{aligned}$$

To see (iv) let us observe that if $a \in \tilde{T}$ satisfies $Q \cap AS(a) = \emptyset$, then obviously $F(f) \cap \{a^\alpha \mid$ a left-shifted assignment $\alpha \in \mathbb{B}^{AS(a)}\} = \emptyset$. Otherwise, let j_a be the index of the above condition (ii). Then the j_a-left-shifted assignment $\gamma \in \mathbb{B}^{AS(a)}$ satisfies $f(a^\gamma) = 1$ by the definition of f. However, an l-left-shifted assignment $\gamma' \in \mathbb{B}^{AS(a)}$ satisfies $f(a^{\gamma'}) = 0$ if $l < j_a$, since otherwise the regularity of f implies that $Q' = Q \setminus \{(a, j_a)\}$ is also a solution to MRE($\mathcal{C}_\models$) (which is a contradiction to the minimality of Q), and $f(a^{\gamma'}) = 1$ if $l \ge j_a$ by the positivity of f. Hence (iv) holds since there are $|Q \cap AS(a)|$ such left-shifted assignments γ' with $l < j_a$.

Equality (v) can be treated similarly.

For the converse inequality, consider a best-fit extension $f \in \mathcal{C}_\models$ of pdBf (T^*, F^*). For a vector $v \in \mathbb{M}^n$, let $AS(v) = \{(v, l_1(v)), (v, l_2(v)), \ldots, (v, l_p(v))\}$ with $l_i(v) < l_j(v)$ for $i < j$. Define $Q = Q_1 \cup Q_2 \subseteq AS$ by

$$\begin{aligned}
Q_1 &= \{(a, l_{i+1}(a)) \in AS(a) \mid a \in \tilde{T},\ f(a^{l_i(a)}) = 0\} \\
Q_2 &= \{(b, l_{i+1}(b)) \in AS(b) \mid b \in \tilde{F},\ f(b^{l_i(b)}) = 1\},
\end{aligned}$$

where, for $a \in \tilde{T}$, $a^{l_i(a)}$ is the vector obtained from a by the $l_i(a)$-left-shifted assignment, and for $b \in \tilde{F}$, $b^{l_i(b)}$ is the vector obtained from b by the $l_i(b)$-right-shifted assignment. Note that

$f(a^{l_i(a)}) = 0$ (resp., $f(b^{l_i(b)}) = 1$) implies $f(a^{l_j(a)}) = 0$ (resp., $f(b^{l_j(b)}) = 1$) for all $j < i$ by the regularity of f. Let us define $\beta \in \mathbb{B}^Q$ by $\beta(a, i) = 1$ for all $(a, i) \in Q_1$, and $\beta(b, i) = 0$ for all $(b, i) \in Q_2$. Then f is a robust extension of $(\tilde{T}^\beta, \tilde{F}^\beta)$ in $\mathcal{C}_\models$, implying

$$\varepsilon(T^*, F^*) = |Q_1| + |Q_2| = |Q| \geq \rho(\tilde{T}, \tilde{F}).$$

□

6 Hereditary classes

A family $\mathcal{S}$ of DNF expressions is called *hereditary* if $\bigvee_{i \in I} t_i \in \mathcal{S}$ implies $\bigvee_{i \in I'} t_i \in \mathcal{S}$ for any $I' \subseteq I$, where t_i denote a term, i.e., an elementary conjunction. It is easy to observe that families of k-DNFs, h-term DNFs and Horn DNFs are all hereditary. For a family of expressions $\mathcal{S}$, let us define the corresponding class of functions by $\mathcal{C}_\mathcal{S} = \{f \mid f$ has a DNF expression in $\mathcal{S}\}$. A class $\mathcal{C}_\mathcal{S}$ of functions is then called *hereditary* if $\mathcal{S}$ is hereditary. In the following subsections we shall consider hereditary families, such as k-DNFs, h-term DNFs and Horn DNFs.

6.1 k-DNF functions

A DNF

$$\varphi = \bigvee_{i=1}^{m} \prod_{j \in P_i} x_j \prod_{j \in N_i} \bar{x}_j$$

is called a *k-DNF* if $|N_i \cup P_i| \leq k$ for $i = 1, .., m$. It is a *positive k-DNF* if, in addition, $N_i = \emptyset$ for $i = 1, ..., m$. Let $\mathcal{C}_{k\text{-}DNF}$ and $\mathcal{C}^+_{k\text{-}DNF}$, respectively, denote the corresponding classes of Boolean functions. In this section, we sometimes do not distinguish a DNF φ from the function it represents.

Let us first consider robust and consistent extensions in the classes $\mathcal{C}_{k\text{-}DNF}$ and $\mathcal{C}^+_{k\text{-}DNF}$. For a general k, Corollary 1 tells that problems RE($\mathcal{C}_{k\text{-}DNF}$), RE($\mathcal{C}^+_{k\text{-}DNF}$), CE($\mathcal{C}_{k\text{-}DNF}$) and CE($\mathcal{C}^+_{k\text{-}DNF}$) are all NP-complete. However, for a fixed k, by Corollary 4, problems RE($\mathcal{C}^+_{k\text{-}DNF}$) and CE($\mathcal{C}^+_{k\text{-}DNF}$) can be solved in polynomial time.

Among the remaining problems, we start with problem RE($\mathcal{C}_{k\text{-}DNF}$) for a fixed k. For a vector $v \in \mathbb{M}^n$, let $A(v)$ denote the assignment to the variables x_i defined by

$$A(v) = (x_i \leftarrow v_i \mid v_i \neq *), \tag{16}$$

e.g., $v = (1, *, 0, 0, *)$, then $A(v) = (x_1 \leftarrow 1, x_3 \leftarrow 0, x_4 \leftarrow 0)$. Recall that $f_{A(v)}$ (resp., $\varphi_{A(v)}$) denotes the function (resp., DNF) obtained by fixing the variables x_i as specified by $A(v)$.

Lemma 12 *Consider a vector $v \in \mathbb{M}^n$ and a term $t = \prod_{j \in P} x_j \prod_{j \in N} \bar{x}_j$. Then $t(v^\alpha) = 0$ holds for all assignments $\alpha \in \mathbb{B}^{AS(v)}$ if and only if $ON(v) \cap N \neq \emptyset$ or $OFF(v) \cap P \neq \emptyset$.*

Proof. It is easy to see that the if-part holds. For the only-if-part, assume $ON(v) \cap N = OFF(v) \cap P = \emptyset$, and define an assignment $\alpha \in \mathbb{B}^{AS(v)}$ by

$$\alpha(v,i) = \begin{cases} 1, & \text{if } i \in P, (v,i) \in AS(v) \\ 0, & \text{if } i \in N, (v,i) \in AS(v). \end{cases}$$

This assignment $\alpha \in \mathbb{B}^{AS(v)}$ obviously satisfies $t(v^{\alpha}) = 1$. □

Lemma 13 *Let φ be a DNF of n variables, and let $v \in \mathbb{M}^n$. For a subset $Q \subseteq AS(v)$ and an assignment $\alpha \in \mathbb{B}^Q$,*

(i) *$\varphi(v^{(\alpha,\beta)}) = 1$ holds for all assignments $\beta \in \mathbb{B}^{AS(v)\setminus Q}$ if and only if $\varphi_{A(v^{\alpha})} = \top$, and*

(ii) *$\varphi(v^{(\alpha,\beta)}) = 0$ holds for all assignments $\beta \in \mathbb{B}^{AS(v)\setminus Q}$ if and only if $\varphi_{A(v^{\alpha})} = \bot$.*

Proof. (i) We claim that a subset $Q \subseteq AS(v)$ and an assignment $\alpha \in \mathbb{B}^Q$ satisfy that $\varphi(v^{(\alpha,\beta)}) = 1$ for all assignments $\beta \in \mathbb{B}^{AS(v)\setminus Q}$ if and only if $\prod_{i \in ON(v^{\alpha})} x_i \prod_{i \in OFF(v^{\alpha})} \bar{x}_i \leq \varphi$ holds. It is clear that the if-part holds. To show the only-if-part, let us assume that $\prod_{i \in ON(v^{\alpha})} x_i \prod_{i \in OFF(v^{\alpha})} \bar{x}_i \not\leq \varphi$ holds. Then $\varphi(v^{(\alpha,\beta)}) = 0$ for some assignment $\beta \in \mathbb{B}^{AS(v)\setminus Q}$. The condition $\prod_{i \in ON(v^{\alpha})} x_i \prod_{i \in OFF(v^{\alpha})} \bar{x}_i \leq \varphi$ is equivalent to $\varphi_{A(v^{\alpha})} = \top$.

(ii) is similar to (i). □

For a k-DNF φ, the problem of checking if $\varphi \neq \top$ is called k-NONTAUTOLOGY [11]. It is known that its complexity is the same as of k-SAT. For $k \leq 2$, k-SAT can be solved in polynomial time, but for $k \geq 3$, k-SAT is NP-complete [11]. The problem of checking $\varphi = \top$ is called k-TAUTOLOGY. It follows from the result about k-SAT that k-TAUTOLOGY is co-NP-complete for $k \geq 3$.

Theorem 13 *If $k \leq 2$, then problem* RE($\mathcal{C}_{k\text{-}DNF}$) *can be solved in polynomial time.*

Proof. The following algorithm solves problem RE($\mathcal{C}_{k\text{-}DNF}$).

Algorithm CHECK-RE($\mathcal{C}_{k\text{-}DNF}$)
Input: a pBmd $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$.
Output: If a pBmd $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{k\text{-}DNF}$, then output such a DNF φ; otherwise, NO.

Step 1. Generate all possible terms t with at most k literals. Let φ be the disjunction of all those terms for which $t(b^{\alpha}) = 0$ holds for all $b \in \tilde{F}$ and $\alpha \in \mathbb{B}^{AS(b)}$.

Step 2. If $\varphi_{A(a)} = \top$ for all $a \in \tilde{T}$, then output φ; otherwise, output NO.

It is easy to see that the φ obtained in Step 1 is a k-DNF, and furthermore it is the maximum k-DNF (with respect to $T(\varphi)$) such that $\varphi(b^\alpha) = 0$ for all $b \in \tilde{F}$ and $\alpha \in \mathbb{B}^{AS(b)}$. By Lemma 13 (the case of $Q = \emptyset$), if φ passes the test of Step 2, then $\varphi(a^\alpha) = 1$ must hold for any $a \in \tilde{T}$ and $\alpha \in \mathbb{B}^{AS(a)}$. Hence this φ represents a robust extension of $(\tilde{T}, \tilde{F})$; otherwise there is no robust extension.

Let us next consider its time complexity. In Step 1, by Lemma 12, checking of each term t can be done in $O(n|\tilde{F}|)$ time. Since there are at most $M = \sum_{j=0}^{k} \binom{2n}{j} = O(n^k)$ such terms, Step 1 can be done in $O(n^{k+1}|\tilde{F}|)$ time. In Step 2, we solve a k-SAT for each $a \in \tilde{T}$ to check whether $\varphi_{A(a)} = \top$ holds. Hence if $k \leq 2$, this can be solved in $O(|\varphi_{A(a)}|)$ time [2], where $|\varphi|$ denotes the number of literals in φ. Since $\varphi_{A(a)}$ can be constructed in $O(|\varphi|)$ time and $|\varphi_{A(a)}| \leq |\varphi| = O(kn^k)$ holds, Step 2 can be done in $O(kn^k|\tilde{T}|)$ time. Totally, CHECK-RE($\mathcal{C}_{k\text{-}DNF}$) can be executed in $O(n^k(k|\tilde{T}| + n|\tilde{F}|))$ time. □

For $k \geq 3$, however, CHECK-RE($\mathcal{C}_{k\text{-}DNF}$) does not run in polynomial time since it must check if $\varphi_{A(v)} = \top$, which is co-NP-complete. In fact, RE($\mathcal{C}_{k\text{-}DNF}$) for $k \geq 3$ can be shown to be co-NP-complete.

Theorem 14 *For a fixed $k \geq 3$, problem* RE($\mathcal{C}_{k\text{-}DNF}$) *is co-NP-complete.*

Proof. Apply algorithm CHECK-RE($\mathcal{C}_{k\text{-}DNF}$) given in the proof of Theorem 13. Step 1 is carried out in polynomial time as noted therein. Step 2 consists of checking if $\varphi_{A(a)} = \top$ for polynomially many a, each of which is obviously a computation in co-NP. Therefore, RE($\mathcal{C}_{k\text{-}DNF}$) for $k \geq 3$ belongs to co-NP.

To prove its co-NP-hardness, let $\mathcal{H} = (V, E)$ be a 3-uniform hypergraph, where $V = \{1, 2, \ldots, n\}$ and each $H \in E$ satisfies $H \subseteq V$ and $|H| = 3$. We may assume $n \geq 4$ without loss of generality. Let $V_1 = \{n+1, n+2, \ldots, 2n\}$, $V_2 = \{2n+1, 2n+2, \ldots, 3n\}$, $V_3 = \{3n+1, 3n+2, \ldots, 3n+(k \perp 3)\}$ and $V' = V \cup V_1 \cup V_2 \cup V_3$. Define $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^{V'}$ as follows.

$$\begin{aligned} \tilde{T} &= \{a = (V \cup V_3; V_1 \cup V_2)\} \\ \tilde{F} &= \tilde{F}_0 \cup \tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{F}_3, \end{aligned}$$

where

$$\begin{aligned}
\tilde{F}_0 &= \{(V_3; \emptyset)\} \\
\tilde{F}_1 &= \{(V \cup I; V_1 \cup V_2) \mid I \subseteq V_3, I \neq V_3\} \\
\tilde{F}_2 &= \left\{ \begin{array}{l|c}
(\{i_1, i_2, i_3\} \cup \{2n+i_1, n+i_2, n+i_3\} \cup V_3; \emptyset) & \\
(\{i_1, i_2, i_3\} \cup \{n+i_1, 2n+i_2, n+i_3\} \cup V_3; \emptyset) & \\
(\{i_1, i_2, i_3\} \cup \{n+i_1, n+i_2, 2n+i_3\} \cup V_3; \emptyset) & i_1, i_2, i_3 \in V, \\
(\{i_1, i_2, i_3\} \cup \{2n+i_1, 2n+i_2, n+i_3\} \cup V_3; \emptyset) & i_1 \neq i_2, i_2 \neq i_3, i_3 \neq i_1 \\
(\{i_1, i_2, i_3\} \cup \{2n+i_1, n+i_2, 2n+i_3\} \cup V_3; \emptyset) & \\
(\{i_1, i_2, i_3\} \cup \{n+i_1, 2n+i_2, 2n+i_3\} \cup V_3; \emptyset) &
\end{array} \right\} \\
\tilde{F}_3 &= \{(I \cup V_3; \emptyset) \mid I \subseteq V_1 \cup V_2, |I| = 3, \{n+i, 2n+i\} \notin I \text{ for all } i \in V, \text{ and} \\
&\qquad I \neq \{n+i_1, n+i_2, n+i_3\}, \{2n+i_1, 2n+i_2, 2n+i_3\} \text{ for all } \{i_1, i_2, i_3\} \in E\},
\end{aligned}$$

and $(R;S)$ denotes again the vector $y \in \mathbb{M}^{V'}$ for which $ON(y) = R$ and $AS(y) = \{(y,j) \mid j \in S\}$. We claim that $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{k\text{-}DNF}$ if and only if $\mathcal{H}$ is not 2-colorable, which will complete the proof because deciding if $\mathcal{H}$ is 2-colorable is NP-complete [11].

To prove the claim, we first show that a k-DNF

$$\begin{aligned}\varphi = \bigvee_{H\in E}\Big(\prod_{j\in H} x_{n+j}(\prod_{l\in V_3} x_l) \ \vee\ \prod_{j\in H} x_{2n+j}(\prod_{l\in V_3} x_l)\Big) \ \vee \\ \bigvee_{i=1}^{n}\Big(x_{n+i}x_{2n+i}(\prod_{l\in V_3} x_l) \ \vee\ x_i\bar{x}_{n+i}\bar{x}_{2n+i}(\prod_{l\in V_3} x_l)\Big)\end{aligned} \tag{17}$$

satisfies the following conditions:

(i) Every term t in φ has an assignment $\alpha \in \mathbb{B}^{AS}$ such that $t(a^\alpha) = 1$ for $a \in \tilde{T}$.

(ii) The equation $\varphi(b^\alpha) = 0$ holds for all $b \in \tilde{F}$ and $\alpha \in \mathbb{B}^{AS}$.

(iii) The cardinality $|T(\varphi)|$ is maximum among all the k-DNFs satisfying (i) and (ii).

Conditions (i), (ii) and (iii) imply that if $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{k\text{-}DNF}$, then φ is such an extension.

Let us consider conditions (i), (ii) and (iii). It is easy to see that (i) holds. For (ii), every term $t = \prod_{j\in P} x_j \prod_{j\in N} \bar{x}_j$ satisfies

$$P \cap (V \cup V_1 \cup V_2) \neq \emptyset \tag{18}$$

$$P \supseteq V_3 \tag{19}$$

Since $OFF(b) \supseteq V \cup V_1 \cup V_2$ holds for $b \in \tilde{F}_0$, (18) implies that $\varphi(b^\alpha) = 0$ for all $b \in \tilde{F}_0$ and $\alpha \in \mathbb{B}^{AS}$. Since $OFF(b) \cap V_3 \neq \emptyset$ holds for all $b \in \tilde{F}_1$, (19) implies that $\varphi(b^\alpha) = 0$ for all $b \in \tilde{F}_1$ and $\alpha \in \mathbb{B}^{AS}$. For $b \in \tilde{F}_2$, we can see that $b \in \mathbb{B}^{V'}$ and

$$|ON(b) \cap V_1|,\ |ON(b) \cap V_2| \leq 2. \tag{20}$$

$$n+i \in ON(b) \cap V_1 \implies 2n+i \in OFF(b) \tag{21}$$

$$i \in ON(b) \cap V \implies \{n+i, 2n+i\} \not\subseteq OFF(b). \tag{22}$$

Then (20), (21) and (22), respectively, imply

$$\begin{aligned}\bigvee_{H\in E}(\prod_{j\in H} x_{n+j}(\prod_{l\in V_3} x_l) \vee \prod_{j\in H} x_{2n+j}(\prod_{l\in V_3} x_l))(b),= 0 \\ \bigvee_{i=1}^{n}(x_{n+i}x_{2n+i}(\prod_{l\in V_3} x_l))(b)= 0 \text{ and} \\ \bigvee_{i=1}^{n}(x_i\bar{x}_{n+i}\bar{x}_{2n+i}(\prod_{l\in V_3} x_l))(b),= 0\end{aligned}$$

for $b \in \tilde{F}_2$. Hence $\varphi(b^\alpha) = 0$ for all $b \in \tilde{F}_2$ and $\alpha \in \mathbb{B}^{AS}$. Similarly, $b \in \tilde{F}_3$ satisfies $b \in \mathbb{B}^{V'}$, (20), (21) and (22), and hence $\varphi(b^\alpha) = 0$ for all $b \in \tilde{F}_3$ and $\alpha \in \mathbb{B}^{AS}$. Therefore, (ii) holds.

For (iii), let us consider a term $t = \prod_{j\in P} x_j \prod_{j\in N} \bar{x}_j$ with $|P \cup N| \leq k$ satisfying (i) and (ii). We show that such a term t satisfies $t \leq \varphi$, which implies (iii). For this, we prove first the following relations.

(a) $P \supseteq V_3$ holds. Otherwise, $t(b^\alpha) = 1$ for some $b \in \tilde{F}_1$ and $\alpha \in \mathbb{B}^{AS}$, since $t(a^\alpha) = 1$ holds for some $\alpha \in \mathbb{B}^{AS}$ by (i). This means $|(P \cup N) \setminus V_3| \leq 3$ by $|t| \leq k$ and $|V_3| = k \perp 3$.

(b) $|N| = 0$ or 2 holds. Otherwise, if $|N| = 1$, then $t(b) = 1$ holds for some $b \in \tilde{F}_2$, which is a contradiction. Furthermore, if $|N| = 3$, then $t(b) = 1$ holds for $b \in \tilde{F}_0$.

(c) If $|N| = 2$, then $P = V_3 \cup \{i\}$ and $N = \{n+i, 2n+i\}$ hold for some $i \in \{1, 2, \dots n\}$. Otherwise, $t(b) = 1$ holds for some $b \in \tilde{F}_2$. This means that such a term t is in φ, i.e., $t \leq \varphi$ holds.

(d) If $N = \emptyset$, then $|P \cap V| \leq 1$ holds. Otherwise, either $|P \cap V| = 3$ and $|P \cap (V_1 \cup V_2)| = 0$, or $|P \cap V| = 2$ and $|P \cap (V_1 \cup V_2)| \leq 1$. In either case, $t(b) = 1$ holds for some $b \in \tilde{F}_2$.

(e) If $N = \emptyset$ and $P \cap V = \{j\}$, then $t = x_j x_{n+i} x_{2n+i} (\prod_{l \in V_3} x_l)$ holds for some $i \in V$. Otherwise, $t(b) = 1$ holds for some $b \in \tilde{F}_2$. This means that such a term t satisfies $t \leq \varphi$ because φ has terms $(x_{n+i} x_{2n+i} (\prod_{l \in V_3} x_l)$ for all $i \in V$.

(f) If $N = P \cap V = \emptyset$, then by the definition of $\tilde{F}_3$, t satisfies $t = \prod_{j \in H} x_{n+j} (\prod_{l \in V_3} x_l)$ for some $H \in E$, $t = \prod_{j \in H} x_{2n+j} (\prod_{l \in V_3} x_l)$ for some $H \in E$, or $t = x_{n+i} x_{2n+i} (\prod_{l \in V_3} x_l)$ for some $i \in V$. Hence, such a term t is in φ, i.e., $t \leq \varphi$ holds.

By (a) $\sim$ (f), we have (iii).

Now we show the claim. If $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{k\text{-}DNF}$, then by the above argument, φ of (17) is such an extension. Assume that $\mathcal{H}$ is 2-colorable, i.e., there is a subset $C \subseteq V$ such that $H \cap C \neq \emptyset$ and $H \cap (V \setminus C) \neq \emptyset$ hold for all $H \in E$. Define an assignment $\alpha \in \mathbb{B}^{AS}$ for $a \in \tilde{T}$ by

$$\begin{array}{ll} \alpha(a, n+i) = 1 & \text{if } i \in C \text{ and } i \in V \\ \alpha(a, n+i) = 0 & \text{if } i \notin C \text{ and } i \in V \\ \alpha(a, 2n+i) = 1 & \text{if } i \notin C \text{ and } i \in V \\ \alpha(a, 2n+i) = 0 & \text{if } i \in C \text{ and } i \in V \end{array}$$

Then $\varphi(a^\alpha) = 0$ holds, which is a contradiction. Hence $\mathcal{H}$ is not 2-colorable.

Conversely, assume that $\mathcal{H}$ is not 2-colorable, and take any assignment $\alpha \in \mathbb{B}^{AS}$. If $\alpha(a, n+i) = \alpha(a, 2n+i)$ holds for some $i \in V$, then

$$\bigvee_{i=1}^{n} (x_{n+i} x_{2n+i} (\prod_{l \in V_3} x_l) \vee x_i \bar{x}_{n+i} \bar{x}_{2n+i} (\prod_{l \in V_3} x_l))(a^\alpha) = 1$$

holds. Otherwise, since $\mathcal{H}$ is not 2-colorable,

$$\{i \mid \alpha(a, n+i) = 1, i \in V\} \supseteq H \text{ or } \{i \mid \alpha(a, 2n+i) = 1, i \in V\} \supseteq H$$

holds for some H. This means that such an assignment α satisfies

$$\bigvee_{H \in E} (\prod_{j \in H} x_{n+j} (\prod_{l \in V_3} x_l) \vee \prod_{j \in H} x_{2n+j} (\prod_{l \in V_3} x_l))(a^\alpha) = 1.$$

Hence combining (ii), we see that φ of (17) is a robust extension of $(\tilde{T}, \tilde{F})$. $\square$

We now turn to problem CE($\mathcal{C}_{k\text{-}DNF}$) for a fixed k.

Let us first consider problem CE($\mathcal{C}_{1\text{-}DNF}$).

Let $V = \{1, 2, \ldots, n\}$, and let us consider a pBmd $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^V$. For a vector $v \in \mathbb{M}^V$ and a subset $I \subseteq V$, let $v[I]$ denote the projection of v on I; e.g., if $v = (1, 0, 1, 1, *, 0, *)$, $w = (1, *, *, 0, 0, *, 1)$ and $I = \{2, 3, 5\}$, then $v[I] = (0, 1, *)$, and $w[I] = (*, *, 0)$. Furthermore, for a set $\tilde{S} \subseteq \mathbb{M}^V$ and a subset $I \subseteq V$, let $\tilde{S}[I]$ denote the projection of $\tilde{S}$ on I (we assume that this projection keeps its multiplicity), and if I is a singleton, say $I = \{j\}$, we write simply $\tilde{S}[j]$ instead of $\tilde{S}[\{j\}]$.

We shall show that the following algorithm can solve problem CE($\mathcal{C}_{1\text{-}DNF}$) in polynomial time.

Algorithm FIND-CE($\mathcal{C}_{1\text{-}DNF}$)

Input: A pBmd $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^V$ and $V = \{1, 2, \ldots, n\}$.

Output: If the pBmd $(\tilde{T}, \tilde{F})$ has a consistent extension in $\mathcal{C}_{1\text{-}DNF}$, then output an assignment $\alpha \in \mathbb{B}^{AS}$ such that $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has an extension in $\mathcal{C}_{1\text{-}DNF}$ and its 1-DNF expression φ; otherwise, NO.

Step 1. Let $I_0 := \{j \in V \mid 0 \in \tilde{F}[j], 1 \notin \tilde{F}[j]\}$, $I_1 := \{j \in V \mid 1 \in \tilde{F}[j], 0 \notin \tilde{F}[j]\}$, $I_{01} := \{j \in V \mid 0, 1 \in \tilde{F}[j]\}$, and $I := V \setminus (I_0 \cup I_1 \cup I_{01})$ (i.e.,$\tilde{F}[j]$ for $j \in I$ contains only $*$). Define an assignment α by

$$\alpha(a, j) := \begin{cases} 1 & \text{if either (i) } j \in I_{01}, \text{ or (ii) } a \in \tilde{T} \text{ and } j \in I_0, \text{ or (iii) } a \in \tilde{F} \text{ and } j \in I_1 \\ 0 & \text{if either (iv) } a \in \tilde{F} \text{ and } j \in I_0, \text{ or (v) } a \in \tilde{T} \text{ and } j \in I_1, \end{cases} \tag{23}$$

and 1-DNF

$$\varphi := \bigvee_{i \in I_0} x_i \vee \bigvee_{i \in I_1} \bar{x}_i. \tag{24}$$

Step 2. Define a pBmd $(\tilde{T}', \tilde{F}')$ with $\tilde{T}', \tilde{F}' \subseteq \mathbb{M}^I$ by

$$\begin{aligned} \tilde{T}' &:= (\tilde{T} \setminus \tilde{S}_1)[I] \\ \tilde{F}' &:= \tilde{F}[I], \end{aligned}$$

where I was defined in Step 1, and $\tilde{S}_1 = \{a \in \tilde{T} \mid a_j \in \{1, *\} \text{ for some } j \in I_0\} \cup \{a \in \tilde{T} \mid a_j \in \{0, *\} \text{ for some } j \in I_1\}$.

Step 3. For each $j \in I$, introduce a binary variable y_j (these variables define an assignment $\beta \in \mathbb{B}^{AS(\tilde{T}' \cup \tilde{F}')}$ such that $\beta(a, j) = y_j$ for all $(a, j) \in AS(\tilde{T}')$ and $\beta(b, j) = \bar{y}_j$ for all $(b, j) \in AS(\tilde{F}')$). Let $\tilde{T}'' := \tilde{T}' \cap \mathbb{B}^I$, and construct a CNF (conjunctive normal form)

$$\begin{aligned} \Phi(y) &= \textstyle\bigwedge_{a \in \tilde{T}''} C_a \\ C_a &= \textstyle\bigvee_{j \in ON(a)} y_j \vee \bigvee_{j \in OFF(a)} \bar{y}_j. \end{aligned}$$

Find a solution satisfying $\Phi(y) = 1$ (i.e., solve problem SAT). If there exists a solution y^*, then let $\varphi' = \bigvee_{j \in ON(y^*)} x_j \vee \bigvee_{j \in OFF(y^*)} \bar{x}_j$, and output $\varphi := \varphi \vee \varphi'$ and the concatenated assignment (α, β), where β is obtained by substituting $y_j = y_j^*$ in the way as shown above; otherwise, output NO.

To see the correctness of algorithm FIND-CE($\mathcal{C}_{1\text{-}DNF}$), let us show the following lemma.

Lemma 14 *A pBmd $(\tilde{T}, \tilde{F})$ has a consistent extension in $\mathcal{C}_{1\text{-}DNF}$ if and only if $(\tilde{T}', \tilde{F}')$ obtained in Step* **2** *of* FIND-CE($\mathcal{C}_{1\text{-}DNF}$) *has a consistent extension in $\mathcal{C}_{1\text{-}DNF}$.*

Proof. Let φ be the 1-DNF of (24), and let φ' be a 1-DNF consistent extension of $(\tilde{T}', \tilde{F}')$. Then we claim that the 1-DNF $\varphi \vee \varphi'$ defines a consistent extension of $(\tilde{T}, \tilde{F})$, which will prove the if-part. By the assignment α of (23), $\varphi(a^\alpha) = 1$ holds for all $a \in \tilde{S}_1$, and $\varphi(b^\alpha) = 0$ holds for all $b \in \tilde{F}$. Furthermore, since φ' is a consistent extension of $(\tilde{T}', \tilde{F}')$, some assignment $\beta \in \mathbb{B}^{AS(\tilde{T}' \cup \tilde{F}')}$ satisfies that $\varphi(a^\beta) = 1$ holds for all $a \in \tilde{T}'$, and $\varphi(b^\beta) = 0$ holds for all $b \in \tilde{F}'$. Hence, by the definition of $\tilde{F}'$, $\varphi(b^{(\alpha,\beta)}) = 0$ holds for all $b \in \tilde{F}$, where (α, β) is the concatenation of α and β. This implies that $\varphi \vee \varphi'$ is a 1-DNF extension of $(\tilde{T}^{(\alpha,\beta)}, \tilde{F}^{(\alpha,\beta)})$, that is, $\varphi \vee \varphi'$ is a 1-DNF consistent extension of $(\tilde{T}, \tilde{F})$.

Conversely, let $\gamma \in \mathbb{B}^{AS}$ be an assignment such that $(\tilde{T}^\gamma, \tilde{F}^\gamma)$ has a 1-DNF extension

$$\varphi^* = \bigvee_{i \in P} x_i \vee \bigvee_{i \in N} \bar{x}_i.$$

Then the following properties hold:

(i) $I_{01} \cap (P \cup N) = \emptyset$

(ii) $I_0 \cap N = \emptyset$

(iii) $I_1 \cap P = \emptyset$,

since otherwise some vector $b \in \tilde{F}$ would satisfy $f(b^\gamma) = 1$, a contradiction. Let $\varphi' = \bigvee_{i \in P \setminus I_0} x_i \vee \bigvee_{i \in N \setminus I_1} \bar{x}_i$, and let $\beta = \gamma[AS(\tilde{T}' \cup \tilde{F}')]$, i.e., $\beta \in \mathbb{B}^{AS(\tilde{T}' \cup \tilde{F}')}$ be the projection of γ on $AS(\tilde{T}' \cup \tilde{F}')$. By (i), (ii) and (iii), φ' is defined on I. We now show that φ' is an extension of $((\tilde{T}')^\beta, (\tilde{F}')^\beta)$, which will prove the only-if-part. By the definition of φ', all $b \in \tilde{F}'$ satisfy $\varphi'(b^\beta) = 0$. Assume that $a[I] \in \tilde{T}'$ of some $a \in \tilde{T}$ satisfies $\varphi'(a^\beta) = 0$. Then $(\bigvee_{i \in I_0} x_i \vee \bigvee_{i \in I_1} \bar{x}_i)(a^\beta) = 1$ holds. However, by the definition of $\tilde{T}' = (\tilde{T} \setminus \tilde{S}_1)[I]$, $a_j = 0$ must hold for $j \in I_0$, and $a_j = 1$ must hold for $j \in I_1$, which is a contradiction. Hence φ' is an extension of $((\tilde{T}')^\beta, (\tilde{F}')^\beta)$. □

Let us now consider a consistent extension of the pBmd $(\tilde{T}', \tilde{F}')$, i.e. an assignment $\beta \in \mathbb{B}^{AS(\tilde{T}' \cup \tilde{F}')}$ for which $((\tilde{T}')^\beta, (\tilde{F}')^\beta)$ has a 1-DNF extension. Note that $AS(b) = I$ holds for all $b \in \tilde{F}'$, i.e., all vectors in $\tilde{F}'$ are $\{(*, *, \ldots, *)\}$. Furthermore, if $\beta(b, j) = 1$ (resp., 0) holds for some $b \in \tilde{F}'$, then any 1-DNF extension φ' of $((\tilde{T}')^\beta, (\tilde{F}')^\beta)$ has no term x_j (resp., $\bar{x}_j$). Since $\varphi'(a^\beta) = 1$ must holds for all $a \in \tilde{T}'$, we would like to make $|T(\varphi')|$ as larger as possible, under the condition that $\varphi'(b^\beta) = 0$ holds for all $b \in \tilde{F}'$. This means that we only need to consider an assignment $\beta \in \mathbb{B}^{AS(\tilde{T}' \cup \tilde{F}')}$ such that $\beta(a, j) = y_j$ for all $a \in \tilde{T}'$ and $\beta(b, j) = \bar{y}_j$ for all $b \in \tilde{F}'$, where $y \in \mathbb{B}^I$, and a 1-DNF

$$\varphi' = \bigvee_{j \in ON(y)} x_j \vee \bigvee_{j \in OFF(y)} \bar{x}_j \tag{25}$$

as an extension. Then, it is easy to see that all $a \in \tilde{T}' \setminus \mathbb{B}^I$ satisfy $\varphi'(a^\beta) = 1$, and all $b \in \tilde{F}'$ satisfy $\varphi'(b^\beta) = 0$. Hence we must choose a $y \in \mathbb{B}^I$ such that $\varphi'(a^\beta) = 1$ for all $a \in \tilde{T}' \cap \mathbb{B}^I$. This condition can be written as $\Phi(y) = 1$ in Step 3. Therefore, $(\tilde{T}', \tilde{F}')$ has a 1-DNF consistent extension φ' of (25) if and only if $\Phi(y) = 1$ holds.

Theorem 15 *Problem* CE($\mathcal{C}_{1\text{-}DNF}$) *can be solved in polynomial time.*

Proof. The above discussion shows the correctness of algorithm FIND-CE($\mathcal{C}_{1\text{-}DNF}$). Let us consider its time complexity. Obviously, we can execute Steps 1 and 2 in $O(n(|\tilde{T}|+|\tilde{F}|))$ time. In Step 3, we must find a solution of $\Phi(y) = \bigwedge_{a \in \tilde{T}''} C_a = 1$ (i.e., solve a exact $|I|$-SAT, where exact k-SAT is a SAT satisfying that each of clauses has exact k literals). Exact k-SAT is in general NP-complete, but in this case, $k = |I|$, that is, k is equal to the dimension of SAT. Hence, this can be solved in $O(n|\tilde{T}''|)$ time by checking if the number of different vectors in $\tilde{T}''$ is equal to $2^{|I|}$ (in this case, $\Phi(y)$ is not satisfiable), and finding a vector $y^* \in \mathbb{B}^I$ such that $\overline{y^*} \notin \tilde{T}''$ if not so (by using a binary tree as a data structure [17]) (in this case, y^* is a solution). In total, we need $O(n(|\tilde{T}| + |\tilde{F}|))$ time. □

For $k \geq 2$, however, we have the following negative result.

Theorem 16 *For a fixed $k \geq 2$, problem* CE($\mathcal{C}_{k\text{-}DNF}$) *is NP-complete, even if $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$.*

Proof. Given an assignment $\alpha \in \mathbb{B}^{AS}$, we can check in polynomial time if $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a k-DNF extension, since EXTENSION($\mathcal{C}_{k\text{-}DNF}$) can be solved in polynomial time [5]. Hence this problem is in NP. To show its NP-hardness, let

$$\Phi = \bigwedge_{i=1}^{m} C_i, \;\; C_i = (u_i \vee v_i \vee w_i),$$

be a cubic CNF, where u_i, v_i and w_i for $i = 1, 2, \ldots, m$ are literals from the set $L = \{x_1, \bar{x}_1, ..., x_n, \bar{x}_n\}$. We write $x_j \in C_i$ (resp., $\bar{x}_j \in C_i$) if either $u_i = x_j$ or $v_i = x_j$, or $w_i = x_j$ holds (resp., either $u_i = \bar{x}_j$ or $v_i = \bar{x}_j$, or $w_i = \bar{x}_j$ holds). Let $V_1 = \{1, 2, \ldots, n\}$, $V_2 = \{n+1, n+2, \ldots, n+m\}$, $V_3 = \{n+m+1, n+m+2, \ldots, n+m+k \perp 2\}$ and $V = V_1 \cup V_2 \cup V_3$. We construct $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^V$ as follows.

$$\begin{aligned}
\tilde{T} &= \{a^{(i)} = (W_i \cup \{n+i\}; \emptyset) \mid i = 1, 2, \ldots, m\} \\
\tilde{F} &= \{(\emptyset; \emptyset), (V_1; \emptyset)\} \cup \{(\{j\}; \emptyset) \mid j \in V_1\} \\
&\qquad \cup \{b^{(i)} = (W_i \cup \{n+i\} \cup \{l\}; \emptyset) \mid i = 1, 2, \ldots, m, l \in V_3\} \\
&\qquad \cup \{c^{(i)} = (W_{\bar{i}} \cup \{n+i\}; \emptyset) \mid i = 1, 2, \ldots, m\} \cup \{d^{(j)} = (U_j; \{j\}) \mid j \in V_1\},
\end{aligned}$$

where $(R; S)$ denotes the vector $v \in \mathbb{M}^V$ such that $ON(v) = R$ and $AS(v) = \{(v, j) \mid j \in S\}$, and $W_i = \{j \mid x_j \in C_i\}$, $W_{\bar{i}} = \{j \mid \bar{x}_j \in C_i\}$ and $U_j = \{n + i \mid x_j \in C_i \text{ or } \bar{x}_j \in C_i\}$. It is easy to see that $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$. We claim that this pBmd $(\tilde{T}, \tilde{F})$ has a consistent k-DNF extension if and only if the 3-SAT problem for Φ has a solution (i.e., if there is a binary vector $y \in \{0,1\}^n$ for which $\Phi(y) = 1$). This will complete the proof, because 3-SAT is NP-complete [11].

To prove the claim, let $\alpha \in \mathbb{B}^{AS}$ be an assignment such that $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a k-DNF extension φ, and let $t_i = \prod_{j \in P_i} x_j \prod_{j \in N_i} \bar{x}_j$, where $P_i \cap N_i = \emptyset$ and $|P_i \cup N_i| \leq k$, be a term in φ such that $t_i(a^{(i)}) = 1$ for $a^{(i)} \in \tilde{T}$. Then such terms t_i, $i = 1, 2, \ldots, m$, satisfy the following properties:

(a) $N_i \supseteq V_3$ holds. Otherwise, $t_i(b^{(i)}) = 1$, which is a contradiction. Since $|V_3| = k \perp 2$, this means $|(P_i \cup N_i) \setminus V_3| \leq 2$.

(b) $|(P_i \cup N_i) \cap V_2| = 1$ holds. Otherwise, we have $|(P_i \cup N_i) \cap V_2| = 0$ or 2 by (a). If $|(P_i \cup N_i) \cap V_2| = 0$, then at least one vector b in $\{(\emptyset; \emptyset), (V_1; \emptyset)\} \cup \{(\{j\}; \emptyset) \mid j \in V_1\}$ $(\subseteq \tilde{F})$ satisfies $t_i(b) = 1$, which is a contradiction. Furthermore, if $|(P_i \cup N_i) \cap V_2| = 2$, then $t_i(a^{(i)}) = 1$ implies that $c^{(i)} \in \tilde{F}$ satisfies $t_i(c^{(i)}) = 1$, which is again a contradiction.

(c) $P_i \cap V_2 = \{n + i\}$ holds. Otherwise, (b) implies $N_i \cap V_2 = \{n + h\}$, where $h \in \{1, 2, \ldots, m\}$ and $h \neq i$, and then at least one vector b in $\{(\emptyset; \emptyset), (V_1; \emptyset)\} \cup \{(\{j\}; \emptyset) \mid j \in V_1\}$ $(\subseteq \tilde{F})$ satisfies $t_i(b) = 1$, which is a contradiction. Therefore $t_i = x_{n+i}(\prod_{l \in V_3} \bar{x}_l)$ or $t_i = z_j x_{n+i}(\prod_{l \in V_3} \bar{x}_l)$ with $z_j \in L$.

(d) $t_i = u_i x_{n+i}(\prod_{l \in V_3} \bar{x}_l)$, $v_i x_{n+i}(\prod_{l \in V_3} \bar{x}_l)$ or $w_i x_{n+i}(\prod_{l \in V_3} \bar{x}_l)$ holds. If $t_i = x_{n+i}(\prod_{l \in V_3} \bar{x}_l)$, then $c^{(i)} \in \tilde{F}$ satisfies $t_i(c^{(i)}) = 1$, which is a contradiction. On the other hand, if $z_j \in L \setminus \{u_i, v_i, w_i\}$, then z_j must be a negative literal x_k, and furthermore $k \notin W_{\bar{i}}$. This means $t_i(c^{(i)}) = 1$, which is again a contradiction.

(e) There is no pair of terms t_k and t_h such that $t_k = x_j x_{n+k}(\prod_{l \in V_3} \bar{x}_l)$ and $t_h = \bar{x}_j x_{n+h}(\prod_{l \in V_3} \bar{x}_l)$. Otherwise, let t_k and t_h be such terms. Then $(t_k \vee t_h)((d^{(j)})^\beta) = 1$ holds for all assignments $\beta \in \mathbb{B}^{AS}$, which is a contradiction.

Let us define a binary vector $y \in \mathbb{B}^n$ by

$$y_j = \begin{cases} 1 & \text{if } t_i = x_j x_{n+i}(\prod_{l \in V_3} \bar{x}_l) \text{ for some } i \in \{1, 2, \ldots, m\} \\ 0 & \text{otherwise.} \end{cases}$$

Then properties (d) and (e) show that this y satisfies $\Phi(y) = 1$.

Let us next consider the converse direction. For a binary vector $y \in \mathbb{B}^n$ satisfying $\Phi(y) = 1$, define an assignment $\alpha \in \mathbb{B}^{AS}$ by

$$\alpha(d^{(j)}, j) = \begin{cases} 1 & \text{if } y_j = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and a k-DNF function φ^* by

$$\varphi^* = \bigvee_{i=1}^{m} t_i^*,$$
$$t_i^* = z_j x_{n+i} (\prod_{l \in V_3} \bar{x}_l), \tag{26}$$

where $z_j \in \{u_i, v_i, w_i\} = C_i$ and $z_j = 1$ is implied by y. Then we can see that φ^* is an extension of $(\tilde{T}^\alpha, \tilde{F}^\alpha)$, that is, φ^* is a consistent extension of $(\tilde{T}, \tilde{F})$. □

Finally we consider the most robust extensions. By Corollary 3, in the restricted case of $|AS| = O(\log(n + |\tilde{T}| + |\tilde{F}|))$, MRE($\mathcal{C}_{k\text{-}DNF}$) and MRE($\mathcal{C}^+_{k\text{-}DNF}$) are polynomially solvable for a fixed k. However, if the number of missing bits is not limited, we have the following theorem.

Theorem 17 *For a fixed k, problems* MRE($\mathcal{C}_{k\text{-}DNF}$) *and* MRE($\mathcal{C}^+_{k\text{-}DNF}$) *are NP-hard, even if $|AS(a)| \le 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$.*

Proof. Let $G = (V, E)$ be a graph, where $V = \{1, 2, \ldots, n\}$, and let $W = \{n+1, n+2, \ldots, n+k \perp 1\}$. Let us define $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^{V \cup W}$ as follows.

$$\begin{aligned} \tilde{T} &= \{a^{(i,j)} = (\{i,j\} \cup W; \emptyset) \,|\, (i,j) \in E\} \\ \tilde{F} &= \{b^{(0)} = (W; \emptyset)\} \cup \{b^{(i)} = (W; \{i\}) \,|\, i \in V\} \\ &\quad \cup \{b^{(i,j)} = (\{i,j\} \cup (W \setminus \{l\}); \emptyset) \,|\, (i,j) \in E, l \in W\}, \end{aligned}$$

where $(R; S)$ denotes the vector $v \in \mathbb{M}^{V \cup W}$ such that $ON(v) = R$ and $AS(v) = \{(v, j) \,|\, j \in S\}$. It is easy to see that $|AS(a)| \le 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$. We claim that

$$\rho(\mathcal{C}_{k\text{-}DNF}; (\tilde{T}, \tilde{F})) = \rho(\mathcal{C}^+_{k\text{-}DNF}; (\tilde{T}, \tilde{F})) = \tau(G) \tag{27}$$

holds, where $\tau(G)$ denotes the cardinality of a minimum vertex cover of graph G. This will complete the proof because finding $\tau(G)$ is known to be NP-hard [11].

To prove the claim, we show first that

$$\rho(\mathcal{C}_{k\text{-}DNF}; (\tilde{T}, \tilde{F})) \le \rho(\mathcal{C}^+_{k\text{-}DNF}; (\tilde{T}, \tilde{F})) \le \tau(G) \tag{28}$$

The first inequality follows from $\mathcal{C}_{k\text{-}DNF} \supseteq \mathcal{C}^+_{k\text{-}DNF}$. For the second one, let us associate a k-DNF φ_C to any subset $C \subseteq V$ by defining

$$\varphi_C = \bigvee_{i \in C} x_i x_{n+1} x_{n+2} \ldots x_{n+k-1},$$

and let us consider φ_{C^*}, where $C^* \subseteq V$ is a minimum vertex cover of G. Define $Q \subseteq AS$ and $\alpha \in \mathbb{B}^Q$ by $Q = \{(b^{(i)}, i) \,|\, i \in C^*\}$ and $\alpha((b^{(i)}, i)) = 0$ for all $(b^{(i)}, i) \in Q$, respectively. Then φ_{C^*} is a robust extension of $(\tilde{T}^\alpha, \tilde{F}^\alpha)$, i.e., $\rho(\mathcal{C}^+_{k\text{-}DNF}; (\tilde{T}, \tilde{F})) \le |C^*| = \tau(G)$.

Next, we show that

$$\rho(\mathcal{C}_{k\text{-}DNF}; (\tilde{T}, \tilde{F})) \geq \tau(G), \tag{29}$$

which together with (28) will imply (27). For this end, let $\alpha \in \mathbb{B}^Q$ for $Q \subseteq AS$ be an assignment such that $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a robust k-DNF extension, and let

$$\varphi = \bigvee_{i \in I} t_i,$$

be such a k-DNF with a minimal I, where $t_i = \prod_{j \in P_i} x_j \prod_{j \in N_i} \bar{x}_j$, $P_i \cap N_i = \emptyset$ and $|P_i \cup N_i| \leq k$ for all $i \in I$. Then the minimality of I implies that for every term t_i, there is $a^{(h_i,l_i)} \in \tilde{T}^\alpha$ such that $t_i(a^{(h_i,l_i)}) = 1$. Thus $P_i \supseteq W$ holds for every $i \in I$, since otherwise the vector $b^{(h_i,l_i)} \in \tilde{F}^\alpha$ also satisfies $t_i(b^{(h_i,l_i)}) = 1$, which is a contradiction. This implies $|(P_i \cup N_i) \cap V| \leq 1$ by $|P_i \cup N_i| \leq k$. However, $|P_i \cap V| = 1$ holds for every $i \in I$; otherwise (i.e., $P_i \cap V = \emptyset$), $t_i(b^{(0)}) = 1$ holds for $b^{(0)} \in \tilde{F}^\alpha$, which is again a contradiction. Let us now define

$$C = \{j \mid \{j\} = P_i \cap V, i \in I\} \ (\subseteq V).$$

Then this set C is a vertex cover, since for every $a^{(h,l)} \in \tilde{T}^\alpha$, there exists a term t_i such that $P_i \cap V = \{h\}$ or $\{l\}$. Hence $\varphi \equiv \varphi_C$ holds for some vertex cover $C \subseteq V$, which implies (29) by applying a discussion similar to that of (28). □

6.2 h-term DNF functions

A DNF

$$\varphi = \bigvee_{i=1}^{m} \prod_{j \in P_i} x_j \prod_{j \in N_i} \bar{x}_j$$

is called an *h-term DNF*, if $m \leq h$. It is a *positive h-term DNF* if, in addition, $N_i = \emptyset$ for $i = 1, ..., m$. Let $\mathcal{C}_{h\text{-}term}$ and $\mathcal{C}^+_{h\text{-}term}$, respectively, denote the corresponding classes of Boolean functions.

For a general h and a fixed $h \geq 2$, Corollary 1 tells that problems RE($\mathcal{C}_{h\text{-}term}$), RE($\mathcal{C}^+_{h\text{-}term}$), CE($\mathcal{C}_{h\text{-}term}$) and CE($\mathcal{C}^+_{h\text{-}term}$) are NP-complete. On the other hand, problems RE($\mathcal{C}^+_{1\text{-}term}$) and CE($\mathcal{C}^+_{1\text{-}term}$) can be solved in polynomial time, by Corollary 4. Here we consider the remaining cases.

Theorem 18 *Problems* RE($\mathcal{C}_{1\text{-}term}$) *and* CE($\mathcal{C}_{1\text{-}term}$) *are polynomially solvable. Problems* MRE($\mathcal{C}_{1\text{-}term}$) *and* MRE($\mathcal{C}^+_{1\text{-}term}$) *are however NP-hard, even if $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$.*

Proof. The problems RE($\mathcal{C}_{1\text{-}term}$), CE($\mathcal{C}_{1\text{-}term}$), MRE($\mathcal{C}_{1\text{-}term}$) and MRE($\mathcal{C}^+_{1\text{-}term}$) are dual to RE($\mathcal{C}_{1\text{-}DNF}$), CE($\mathcal{C}_{1\text{-}DNF}$), MRE($\mathcal{C}_{1\text{-}DNF}$) and MRE($\mathcal{C}^+_{1\text{-}DNF}$), respectively. In other words, defining $\tilde{T}^d = \{\bar{b} \mid b \in \tilde{F}\}$ and $\tilde{F}^d = \{\bar{a} \mid a \in \tilde{T}\}$, where $\bar{a}$ denotes the vector such that $\bar{a}_i = 1$ (resp., 0) if $a_i = 0$ (resp., 1), and $\bar{a}_i = *$ if $a_i = *$, the pBmd $(\tilde{T}, \tilde{F})$ has a robust (resp., consistent) extension in $\mathcal{C}^{(+)}_{1\text{-}term}$ if and only if $(\tilde{T}^d, \tilde{F}^d)$ has a robust (resp., consistent) extension in $\mathcal{C}^{(+)}_{1\text{-}DNF}$. Thus Theorems 13, 15 and 17 imply this theorem. □

6.3 h-term k-DNF functions

A DNF

$$\varphi = \bigvee_{i=1}^{m} \prod_{j \in P_i} x_j \prod_{j \in N_i} \bar{x}_j$$

is an *h-term k-DNF* if $m \leq h$ and $|P_i \cup N_i| \leq k$ for all $i = 1, ..., m$. If, in addition, $N_i = \emptyset$ for $i = 1, .., m$, then φ is called a *positive h-term k-DNF*. Let $\mathcal{C}_{h,k\text{-}DNF}$ and $\mathcal{C}^+_{h,k\text{-}DNF}$ denote the corresponding classes of h-term k-DNF functions and positive h-term k-DNF functions, respectively.

By Corollary 1, problems RE($\mathcal{C}_{h,k\text{-}DNF}$), RE($\mathcal{C}^+_{h,k\text{-}DNF}$), CE($\mathcal{C}_{h,k\text{-}DNF}$) and CE($\mathcal{C}^+_{h,k\text{-}DNF}$) are all NP-hard, if at least one of h and k is not fixed. We therefore consider only problem MRE for the classes $\mathcal{C}_{h,k\text{-}DNF}$ and $\mathcal{C}^+_{h,k\text{-}DNF}$ with fixed h and k, and show that these can be solved in polynomial time. This also tells that problems CE and RE are polynomially solvable in these cases.

Theorem 19 *Problems* MRE($\mathcal{C}_{h,k\text{-}DNF}$) *and* MRE($\mathcal{C}^+_{h,k\text{-}DNF}$) *can be solved in polynomial time, if both h and k are fixed constants.*

Proof. (i) MRE($\mathcal{C}_{h,k\text{-}DNF}$): The following algorithm solves MRE($\mathcal{C}_{h,k\text{-}DNF}$).

Algorithm FIND-MRE($\mathcal{C}_{h,k\text{-}DNF}$)
Input: a pBmd $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$.
Output: If $(\tilde{T}, \tilde{F})$ has a consistent extension in $\mathcal{C}_{h,k\text{-}DNF}$, then output a subset $Q \subseteq AS$, an assignment $\alpha \in \mathbb{B}^Q$ and an h term k-DNF φ such that φ represents a robust extension of $(\tilde{T}^\alpha, \tilde{F}^\alpha)$, for which $|Q|$ is minimum; otherwise output NO.

Step 1. For each h-term k-DNF φ, we construct a subset $Q_\varphi \subseteq AS$ and an assignment $\alpha_\varphi \in \mathbb{B}^{Q_\varphi}$ as follows.

- For every $a \in \tilde{T}$, find a subset $Q_a \subseteq AS(a)$ and an assignment $\alpha \in \mathbb{B}^{Q_a}$ such that $\varphi_{A(a^\alpha)} = \top$ (where $A(a^\alpha)$ is defined in (16)) and $|Q_a|$ is minimum. If there is no such Q_a for some $a \in \tilde{T}$, abandon φ; otherwise, let $Q_\varphi := \bigcup_{a \in \tilde{T}} Q_a$, and let $\alpha_\varphi(a, j) := \alpha(a, j)$ for $(a, j) \in Q_a$, $a \in \tilde{T}$.
- For every $b \in \tilde{F}$, find a subset $Q_b \subseteq AS(b)$ and an assignment $\alpha \in \mathbb{B}^{Q_b}$ such that $\varphi_{A(b^\alpha)} = \bot$ and $|Q_b|$ is minimum. If there is no such Q_b for some $b \in \tilde{F}$, abandon φ; otherwise, let $Q_\varphi := Q_\varphi \cup \bigcup_{b \in \tilde{F}} Q_b$, and let $\alpha_\varphi(b, j) := \alpha(b, j)$ for $(b, j) \in Q_b$, $b \in \tilde{F}$.

Step 2. Among those h-term k-DNFs φ, which are not abandoned in Step 1, find $\varphi = \varphi^*$ with minimum $|Q_\varphi|$. If such φ^* exists, then output Q_{φ^*}, $\alpha_{\varphi^*} \in \mathbb{B}^{Q_{\varphi^*}}$ and the DNF φ^*; otherwise, output NO. STOP.

The correctness of algorithm FIND-MRE($\mathcal{C}_{h,k\text{-}DNF}$) is immediate from Lemma 13. Let us consider its time complexity. Let M be the number of terms with at most k literals, i.e., $M \leq \sum_{j=0}^{k} \binom{2n}{j} = O(n^k)$. Then there are $\sum_{m=0}^{h} \binom{M}{m} = O(n^{kh})$ h-term k-DNF expressions.

In Step 1, for each h-term k-DNF φ, we check if there exist Q_a, $a \in \tilde{T}$ and Q_b, $b \in \tilde{F}$, which satisfy the stated conditions. If there exist $Q_a \subseteq AS(a)$ and $\alpha \in \mathbb{B}^{Q_a}$ such that $\varphi_{A(a^\alpha)} = \top$, then $\varphi_{A(a)} \neq \bot$, implying that $\varphi_{A(a)}$ has a term $t = \prod_{j \in P} x_j \prod_{j \in N} \bar{x}_j \leq \varphi_{A(a)}$. If we set $Q_a = \{(a,j) \mid j \in P \cup N\}$, and $\alpha(a,j) = 1$ (resp., 0) for $j \in P$ (resp., N), then we have $\varphi_{A(a^\alpha)} = \top$. This Q_a satisfies $|Q_a| = |P| + |N| \leq k$, since $\varphi_{A(a)}$ is a k-DNF. Hence the subset $Q_a \subseteq AS(a)$ that minimizes $|Q_a|$ also satisfies $|Q_a| \leq k$, and there are $\sum_{j=0}^{k} \binom{2n}{j} = O(n^k)$ such Q_as. Therefore, we apply the above check to all $Q_a \subseteq AS(a)$. Checking of $\varphi_{A(a^\alpha)} = \top$ (i.e., problem TAUTOLOGY) can be done by dualizing $\varphi_{A(a^\alpha)}$ and checking if $\varphi^d_{A(a^\alpha)} = \bot$ holds (recall that checking $\varphi = \bot$ for a DNF is trivial). This can be done in $O(k^h)$ time, since $\varphi_{A(a^\alpha)}$ can be obtained from φ in $O(|\varphi|) = O(kh)$ time, and $\varphi^d_{A(a^\alpha)}$ can be obtained from $\varphi_{A(a^\alpha)}$ in $O(k^h)$ time, since $\varphi_{A(a^\alpha)}$ is a h-term k-DNF. Thus this computation requires $O(k^h M) = O(k^h n^k)$ time for each h-term k-DNF φ and $a \in \tilde{T}$. Similarly, we require $O(khM) = O(khn^k)$ time for each h-term k-DNF φ and $b \in \tilde{F}$ to check $\varphi_{A(b^\alpha)} = \bot$ for all subsets $Q_b \subseteq AS(b)$ and assignments $\alpha \in \mathbb{B}^{Q_b}$ with $|Q_b| \leq k$. Therefore, Step 1 can be carried out in $O(n^{k(h+1)}(k^h|\tilde{T}| + kh|\tilde{F}|))$ time, since there are $O(n^{kh})$ h-term k-DNF expressions. Step 2 can be carried out in $O(n^{kh})$ time. Totally, algorithm FIND-MRE($\mathcal{C}_{h,k\text{-}DNF}$) requires $O(n^{k(h+1)}(k^h|\tilde{T}| + kh|\tilde{F}|))$ time, which is polynomial if k and h are constants.

(ii) MRE($\mathcal{C}^+_{h,k\text{-}DNF}$): We can solve MRE($\mathcal{C}^+_{h,k\text{-}DNF}$) in polynomial time by modifying algorithm FIND-MRE($\mathcal{C}_{h,k\text{-}DNF}$) as follows. In this case, h-term k-DNFs are restricted to be positive, and in Step 1, for each $a \in \tilde{T}$ (resp., $b \in \tilde{F}$), all assignments $\alpha \in \mathbb{B}^{Q_a}$ (resp., $\alpha \in \mathbb{B}^{Q_b}$) with $Q_a \subseteq AS(a)$ (resp., $Q_b \subseteq AS(b)$) are restricted to be positive (resp., negative). Hence for each positive h-term k-DNF φ and $a \in \tilde{T}$, we can check in $O(kh)$ time if a subset $Q_a \subseteq AS(a)$ and assignment $\alpha \in \mathbb{B}^{Q_a}$ satisfy $f_{A(a^\alpha)} = \top$, and similarly for each positive h-term k-DNF φ and $b \in \tilde{F}$, we can check in $O(kh)$ time if a subset $Q_b \subseteq AS(b)$ and assignment $\alpha \in \mathbb{B}^{Q_b}$ satisfy $f_{A(a^\alpha)} = \bot$. Therefore, MRE($\mathcal{C}^+_{h,k\text{-}DNF}$) can be solved in $O(khn^{k(h+1)}(|\tilde{T}| + |\tilde{F}|))$ total time. □

6.4 Horn functions

A DNF

$$\varphi = \bigvee_{i=1}^{m} \prod_{j \in P_i} x_j \prod_{j \in N_i} \bar{x}_j$$

is called *Horn* if $|N_i| \leq 1$ for all terms $i = 1, ..., m$. Let us denote by $\mathcal{C}_{Horn}$ the class of Horn functions.

Theorem 20 *Problem* RE($\mathcal{C}_{Horn}$) *can be solved in polynomial time.*

Proof. Let $(\tilde{T}, \tilde{F})$ be a pBmd. For each $a \in \tilde{T}$, let us define $B(a) = \{b \in \tilde{F} \mid b \gtrapprox a\}$. We claim that $(\tilde{T}, \tilde{F})$ has a robust Horn extension if and only if for every $a \in \tilde{T}$, there exists an index j such that $a_j = 0$ and $b_j = 1$ for all $b \in B(a)$. The latter condition can be easily checked in $O(n|\tilde{T}||\tilde{F}|)$ time.

To prove the claim, let us assume first that for every $a \in \tilde{T}$, there exists an index j such that $a_j = 0$ and $b_j = 1$ for all $b \in B(a)$. Then for any $\alpha \in \mathbb{B}^{AS}$, all $b^\alpha \in B(a)^\alpha$ satisfies $b^\alpha_j = 1$. Thus, for the Horn term

$$t_a = \left(\prod_{i \in ON(a)} x_i\right) \bar{x}_j,$$

we have $t_a(a^\alpha) = 1$ and $t_a(b^\alpha) = 0$ for all $\alpha \in \mathbb{B}^{AS}$ and $b \in \tilde{F}$. Hence, the Horn DNF

$$\varphi = \bigvee_{a \in \tilde{T}} t_a$$

provides a Horn extension of $(\tilde{T}, \tilde{F})$.

For the converse direction, let us assume that for some $a \in \tilde{T}$, every index j with $a_j = 0$ has a vector $b \in B(a)$ with $b_j \in \{0, *\}$. For such a vector a, consider the assignments $\alpha \in \mathbb{B}^{AS(a)} \cup \mathbb{B}^{AS(B(a))}$ defined by

$$\alpha(a, i) = \begin{cases} \prod_{b \in B(a)\,\text{s.t.}\, b_i \neq *} b_i & \text{if there is a vector } b \in B(a) \text{ with } b_i \in \{0, 1\} \\ 1 & \text{otherwise} \end{cases}$$

for $(a, i) \in AS(a)$, and $\alpha(b, i) = a^\alpha_i$ for $(b, i) \in AS(B(a))$. Then $\{b^\alpha \in \tilde{F}^\alpha \mid b^\alpha \geq a^\alpha\} = B(a)^\alpha$ satisfies

$$a^\alpha = \bigwedge_{\{b^\alpha \in \tilde{F}^\alpha \mid b^\alpha \geq a^\alpha\}} b^\alpha,$$

by the above assumption on a and $B(a)$, where $\bigwedge$ denotes the componentwise AND operation, e.g., $(010111) \bigwedge (100101) = (000101)$. However, it is known [5, 13] that a pdBf (T, F) has an extension in $\mathcal{C}_{Horn}$ if and only if

$$\bigwedge_{b \in F\ \text{s.t.}\ b \geq a} b \neq a \tag{30}$$

holds for every $a \in T$. Hence, $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has no extension in $\mathcal{C}_{Horn}$. □

Now, we turn to problem CE($\mathcal{C}_{Horn}$).

Theorem 21 *Problem* CE($\mathcal{C}_{Horn}$) *is NP-complete, even if* $|AS(a)| \leq 1$ *holds for all* $a \in \tilde{T} \cup \tilde{F}$.

Proof. Similarly to the proof of Theorem 9, CE($\mathcal{C}_{Horn}$) is in NP. To show its NP-hardness, let $\mathcal{H} = (V, E)$ be a 3-uniform hypergraph, i.e., E is a collection of 3 element subsets of $V = \{1, 2, \ldots, n\}$. Let $\mathcal{H}' = (V', E')$ be a copy of $\mathcal{H}$, i.e., $V' = \{1', 2', ..., n'\}$ and $E' = \{H' = \{i', j', k'\} \mid H = \{i, j, k\} \in E\}$. Let us define $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^{V \cup V'}$ as follows.

$$\begin{aligned} \tilde{T} &= \{a^{1H} = ((V \setminus H) \cup (V' \setminus H'); \emptyset), a^{2H} = (V \cup (V' \setminus H'); \emptyset) \mid H \in E\} \\ \tilde{F} &= \{b^{(i)} = ((V \setminus \{i\}) \cup (V' \setminus \{i'\}); \{i\}) \mid i \in V\}, \end{aligned}$$

where $(R; S)$ denotes the vector $v \in \mathbb{M}^V$ such that $ON(v) = R$ and $AS(v) = \{(v, j) \mid j \in S\}$. It is easy to see that $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$. We claim that this $(\tilde{T}, \tilde{F})$ has a consistent Horn extension if and only if $\mathcal{H}$ is 2-colorable, i.e., if there is a partition $(C, V \setminus C)$, for which $C \cap H \neq \emptyset$ and $(V \setminus C) \cap H \neq \emptyset$ hold for all $H \in E$. This will complete the proof because deciding the existence of a 2-coloring of a 3-uniform hypergraph is NP-complete [11].

Let us first consider an assignment $\alpha \in \mathbb{B}^{AS}$ such that $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a consistent Horn extension. Let

$$C = \{i \in V \mid \alpha(b^{(i)}, i) = 1\}.$$

Then we shall show that $(C, V \setminus C)$ is a good 2-coloring. For this, let us assume otherwise; i.e., there is an edge $H \in E$ such that either $H \cap C = \emptyset$ or $H \cap (V \setminus C) = \emptyset$. If $H \cap C = \emptyset$ holds, then $a^{1H} \in \tilde{T}^\alpha$ does not satisfy condition (30) for the existence of a Horn extension, since $\{b \in \tilde{F}^\alpha \mid b \geq a^{1H}\} = \{(b^{(i)})^\alpha \mid i \in H\}$ and $\bigwedge_{i \in H} (b^{(i)})^\alpha = a^{1H}$. On the other hand, if $H \cap (V \setminus C) = \emptyset$, then $a^{2H} \in \tilde{T}^\alpha$ does not satisfy (30), since $\{b \in \tilde{F}^\alpha \mid b \geq a^{2H}\} = \{(b^{(i)})^\alpha \mid i \in H\}$ and $\bigwedge_{i \in H} (b^{(i)})^\alpha = a^{2H}$.

For the converse direction, let $(C, V \setminus C)$ be a 2-coloring. Let us define an assignment $\alpha \in \mathbb{B}^{AS}$ by

$$\alpha(b^{(i)}, i) = \begin{cases} 1 & \text{if } i \in C \\ 0 & \text{otherwise.} \end{cases}$$

Then (30) holds for every $a \in \tilde{T}^\alpha$, that is, $(\tilde{T}, \tilde{F})$ has a consistent Horn extension. □

7 Dual-comparable functions

Let us recall that a Boolean function f is *dual-minor* (resp., *dual-major*, *self-dual*) if $f \leq f^d$ (resp., $f \geq f^d$, $f = f^d$), where the dual f^d of f is defined by $f^d(x) = \bar{f}(\bar{x})$. Let $\mathcal{C}_{DMI}$, $\mathcal{C}_{DMA}$ and $\mathcal{C}_{SD}$ denote the corresponding classes of dual-minor, dual-major and self-dual functions, respectively. Analogously, let $\mathcal{C}^+_{DMI}$, $\mathcal{C}^+_{DMA}$ and $\mathcal{C}^+_{SD}$ denote the classes of dual-minor, dual-major and self-dual positive functions, respectively. It is known [5] that a function f is dual-minor (resp., dual-major, self-dual) if and only if at most (resp., at least, exactly) one of $f(a) = 1$ and $f(\bar{a}) = 1$ holds for every $a \in \{0,1\}^n$.

Corollary 4 tells that the robust and consistent extensions can be found in polynomial time for classes of $\mathcal{C}^+_{DMI}$, $\mathcal{C}^+_{DMA}$ and $\mathcal{C}^+_{SD}$. Therefore, let us consider the robust and consistent extensions for classes of $\mathcal{C}_{DMI}$, $\mathcal{C}_{DMA}$ and $\mathcal{C}_{SD}$. We start with the next lemma.

Lemma 15 **(i)** *A pBmd $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{DMI}$ if and only if there exists an index j such that $a_j = b_j \in \mathbb{B}$ for each pair of $a, b \in \tilde{T}$.*

(ii) *A pBmd $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{DMA}$ if and only if there exists an index j such that $a_j = b_j \in \mathbb{B}$ for each pair of $a, b \in \tilde{F}$.*

(iii) *A pBmd $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{SD}$ if and only if there exists an index j such that $a_j = b_j \in \mathbb{B}$ for each pair of $a, b \in \tilde{T}$ and each pair of $a, b \in \tilde{F}$.*

Proof. (i) A pBmd $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{DMI}$ if and only if, for any assignment $\alpha \in \mathbb{B}^{AS}$, there is no pair of $a, b \in \tilde{T}$ such that $a^\alpha = \overline{b^\alpha}$. The latter condition is equivalent to that there exists an index j such that $a_j = b_j \in \mathbb{B}$ for every pair of $a, b \in \tilde{T}$.

(ii) is similar to (i), and (iii) is obtained by combining (i) and (ii). □

Theorem 22 *Problems* RE($\mathcal{C}_{DMI}$), RE($\mathcal{C}_{DMA}$) *and* RE($\mathcal{C}_{SD}$) *can be solved in polynomial time.*

Proof. By Lemma 15, it is easy to see that RE($\mathcal{C}_{DMI}$), RE($\mathcal{C}_{DMA}$) and RE($\mathcal{C}_{SD}$) can be solved in time $O(n|\tilde{T}|^2)$, $O(n|\tilde{F}|^2)$ and $O(n(|\tilde{T}|^2 + |\tilde{F}|^2)$, respectively. □

Theorem 23 *Problems* CE($\mathcal{C}_{DMI}$), CE($\mathcal{C}_{DMA}$) *and* CE($\mathcal{C}_{SD}$) *can be solved in polynomial time for a pBmd* $(\tilde{T}, \tilde{F})$ *such that* $|AS(a)| \leq 1$ *holds for all* $a \in \tilde{T} \cup \tilde{F}$.

Proof. For a given pBmd $(\tilde{T}, \tilde{F})$ with $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$, let us define a pdBf (T, F) by

$$\begin{aligned} T &= \{a^\alpha \mid a \in \tilde{T}, \alpha \in \mathbb{B}^{AS}\} \\ F &= \{b^\alpha \mid b \in \tilde{F}, \alpha \in \mathbb{B}^{AS}\}. \end{aligned}$$

Now, for each $a \in T \cup F$, let us introduce a new binary variable X_a, where X_a corresponds to the value $f(a)$ of a consistent extension f of $(\tilde{T}, \tilde{F})$. Then the following quadratic equations must hold.

$$\begin{array}{lll} \text{(i)} & \overline{X}_a = 0 & \text{if } a \in \tilde{T} \cap \mathbb{B}^n \\ \text{(ii)} & \overline{X}_{a^\alpha}\overline{X}_{a^{\bar{\alpha}}} = 0 & \text{if } a \in \tilde{T} \setminus \mathbb{B}^n \\ \text{(iii)} & X_b = 0 & \text{if } b \in \tilde{F} \cap \mathbb{B}^n \\ \text{(iv)} & X_{b^\alpha}X_{b^{\bar{\alpha}}} = 0 & \text{if } b \in \tilde{F} \setminus \mathbb{B}^n \\ \text{(v)} & X_a X_{\bar{a}} = 0 & \text{if } a, \bar{a} \in T \\ \text{(vi)} & X_b X_{\bar{b}} = 0 & \text{if } b, \bar{b} \in F. \end{array}$$

The equations (i), (ii), (iii) and (iv) express the conditions that $(\tilde{T}, \tilde{F})$ has a consistent extension in $\mathcal{C}_{all}$. The equations (v) and (vi) express that a consistent extension of $(\tilde{T}, \tilde{F})$ must be dual-minor and dual-major, respectively. Hence in order to solve CE($\mathcal{C}_{DMI}$) (resp., CE($\mathcal{C}_{DMA}$) and CE($\mathcal{C}_{SD}$)), we check if the quadratic systems consisting of (i), (ii), (iii), (iv) and (v) (resp., (i), (ii), (iii), (iv), (vi), and (i), (ii), (iii), (iv), (v), (vi)) has a solution. Checking if the quadratic systems has a solution (i.e., 2-SAT) can be done in time linear in its size [2]. Therefore, we can solve CE($\mathcal{C}_{DMI}$), CE($\mathcal{C}_{DMA}$) and CE($\mathcal{C}_{SD}$) in polynomial time, if $|AS(a)| \leq 1$ holds for every $a \in \tilde{T} \cup \tilde{F}$. □

In general, however, we have the following negative result.

Theorem 24 *All three problems* CE($\mathcal{C}_{DMI}$), CE($\mathcal{C}_{DMA}$) *and* CE($\mathcal{C}_{SD}$) *are NP-complete, even if* $|AS(a)| \leq 2$ *holds for all* $a \in \tilde{T} \cup \tilde{F}$.

Proof. These problems are all in NP, similarly to Theorem 9. To show the NP-hardness, for a given pBmd $(\tilde{T}, \tilde{F})$ with $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$, let us define a pBmd $(\tilde{T}', \tilde{F}')$ by

$$\begin{aligned} \tilde{T}' &= \{(a,1) \mid a \in \tilde{T}\} \\ \tilde{F}' &= \{(b,1) \mid b \in \tilde{F}\}, \end{aligned}$$

where $\tilde{T}', \tilde{F}' \subseteq \mathbb{M}^{n+1}$. We show that $(\tilde{T}, \tilde{F})$ has a consistent extension f in $\mathcal{C}_{all}$ if and only if $(\tilde{T}', \tilde{F}')$ has consistent extensions in $\mathcal{C}_{DMI}$, $\mathcal{C}_{DMA}$ and $\mathcal{C}_{SD}$, respectively.

First, if $(\tilde{T}', \tilde{F}')$ has a consistent extension in one of the classes $\mathcal{C}_{DMI}$, $\mathcal{C}_{DMA}$ and $\mathcal{C}_{SD}$, then obviously $(\tilde{T}, \tilde{F})$ has a consistent extension in $\mathcal{C}_{all}$. To prove the converse direction, let f be a consistent extension of $(\tilde{T}, \tilde{F})$ in $\mathcal{C}_{all}$. Then define f' of $n+1$ variables by

$$f'(d) = 1 \quad \text{iff either } f(a) = 1 \text{ and } d_{n+1} = 1, \text{ or } f(\bar{a}) = 0 \text{ and } d_{n+1} = 0,$$

where $d = (a, d_{n+1}) \in \mathbb{B}^{n+1}$. We claim that f' is a consistent extension in class $\mathcal{C}_{SD}$ (i.e., $\mathcal{C}_{DMI} \cap \mathcal{C}_{DMA}$). It is easy to see that f' is a consistent extension in $\mathcal{C}_{all}$. Let us show the self-duality (i.e., dual-minority and dual-majority) of f'.

(i) $f'(a, d_{n+1}) = 1$ implies either (i) $f(a) = 1$ and $d_{n+1} = 1$, or (ii) $f(\bar{a}) = 0$ and $d_{n+1} = 0$. If (i) holds, then $f'(\overline{a, d_{n+1}}) = 0$, since $f(\bar{\bar{a}}) = 1$ and $\bar{d}_{n+1} = 0$ hold. If (ii) holds, then $f'(\overline{a, d_{n+1}}) = 0$, since $f(\bar{a}) = 0$ and $\bar{d}_{n+1} = 1$ hold. Similarly, $f'(a, d_{n+1}) = 0$ implies $f'(\overline{a, d_{n+1}}) = 1$. Thus f' is in $\mathcal{C}_{SD}$.

Therefore, the theorem follows from Theorem 9. □

Let us finally consider the problem of most robust extensions for the classes of positive dual-comparable functions. Recall [5] that

(i) A pdBf (T, F) has an extension in $\mathcal{C}^+_{DMI}$ if and only if (T, F) has a positive extension, and for all $a \in T$, there is no $a' \in T$ such that $a' \leq \bar{a}$.

(ii) A pdBf (T, F) has an extension in $\mathcal{C}^+_{DMA}$ if and only if (T, F) has a positive extension, and for all $b \in F$, if there is no $b' \in F$ such that $b' \geq \bar{b}$.

(iii) A pdBf (T, F) has an extension in $\mathcal{C}^+_{SD}$ if and only if (T, F) has a positive extension, and for all $a \in T$, there is no $a' \in T$ such that $a' \leq \bar{a}$, and for all $b \in F$, if there is no $b' \in F$ such that $b' \geq \bar{b}$.

Theorem 25 *Problems* MRE($\mathcal{C}^+_{DMI}$), MRE($\mathcal{C}^+_{DMA}$) *and* MRE($\mathcal{C}^+_{SD}$) *are NP-hard, even if* $|AS(a)| \leq 1$ *holds for all* $a \in \tilde{T} \cup \tilde{F}$.

Proof. MRE($\mathcal{C}^+_{DMI}$) and MRE($\mathcal{C}^+_{SD}$): Let $G = (V, E)$ be a Δ-free graph, where $V = \{v_1, v_2, \ldots, v_n\}$, and G is called Δ-free if G has no clique of size 3, i.e., there is no set of vertices $v_1, v_2, v_3 \in V$ such that $(v_1, v_2), (v_2, v_3), (v_3, v_1) \in E$. Let $W = \{w_{ij} \mid (v_i, v_j) \notin E, i \neq j\}$, where $W \cap V = \emptyset$, and let V' be the base set with $V' = V \cup W$. Now define $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^{V'}$ as follows.

$$\begin{array}{rcl} \tilde{T} &=& \{a^{(i)} = (N_{v_i} \cup \{w_{ij} \in W \mid v_j \in V\}; \{v_i\}) \mid i = 1, 2, \ldots, n\} \\ \tilde{F} &=& \emptyset, \end{array} \tag{31}$$

where $N(v_i) = \{v_l \in V \mid (v_l, v_i) \in E\}$, and $(R; S)$ denotes the vector $y \in \mathbb{M}^{V'}$ such that $ON(y) = R$ and $AS(y) = \{(y, j) \mid j \in S\}$. It is easy to see that $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$.

We claim that

$$\rho(\mathcal{C}^+_{SD}; (\tilde{T}, \tilde{F})) = \rho(\mathcal{C}^+_{DMI}; (\tilde{T}, \tilde{F})) = \tau(G), \tag{32}$$

where $\tau(G)$ denotes the cardinality of a minimum vertex cover of graph G. This will prove the statement because finding $\tau(G)$ for a Δ-free graph G is known to be NP-hard [11]. To prove the claim, we first show that

$$\rho(\mathcal{C}^+_{SD}; (\tilde{T}, \tilde{F})) = \rho(\mathcal{C}^+_{DMI}; (\tilde{T}, \tilde{F})). \tag{33}$$

$\rho(\mathcal{C}^+_{SD}; (\tilde{T}, \tilde{F})) \geq \rho(\mathcal{C}^+_{DMI}; (\tilde{T}, \tilde{F}))$ follows from $\mathcal{C}^+_{SD} \subseteq \mathcal{C}^+_{DMI}$. For the converse inequality, let $\alpha \in Q$ for $Q \subseteq AS$ be a solution of MRE($\mathcal{C}^+_{DMI}$), and let (T', F') be a pdBf on V', where

$$\begin{array}{rcl} T' &=& \{a^{(\alpha,\beta)} \mid a \in \tilde{T}, \beta \in \mathbb{B}^{AS \setminus Q}\} \\ F' &=& \emptyset \ (= \tilde{F}). \end{array} \tag{34}$$

By the definition of most robust extension, $\alpha \in Q$ for $Q \subseteq AS$ is a solution of MRE($\mathcal{C}$) if and only if (T', F') has an extension in $\mathcal{C}$. $\mathcal{C} = \mathcal{C}^+_{DMI}$ implies that (T', F') satisfies the condition (i) (before this theorem). In case of $F' = \emptyset$, condition (i) is equivalent to (iii), that is, (T', F') has an extension in $\mathcal{C}^+_{SD}$. This means that $\alpha \in Q$ for $Q \subseteq AS$ is also a solution to MRE($\mathcal{C}^+_{SD}$). Hence $\rho(\mathcal{C}^+_{SD}; (\tilde{T}, \tilde{F})) \leq \rho(\mathcal{C}^+_{DMI}; (\tilde{T}, \tilde{F}))$, which will imply (33).

Next we show

$$\rho(\mathcal{C}^+_{DMI}; (\tilde{T}, \tilde{F})) = \tau(G), \tag{35}$$

which will complete the proof. Let $\alpha \in Q$ for $Q \subseteq AS$ be a solution to MRE($\mathcal{C}^+_{DMI}$). By the above argument, a pdBf (T', F') of (34) has an extension in $\mathcal{C}^+_{DMI}$. We show that $\{(a^{(i)}, v_i), (a^{(j)}, v_j)\} \cap Q \neq \emptyset$ holds for all pair of i and j with $(v_i, v_j) \in E$, which implies

$$\rho(\mathcal{C}^+_{DMI}; (\tilde{T}, \tilde{F})) \geq \tau(G). \tag{36}$$

Assume otherwise, i.e., $(a^{(i)}, v_i), (a^{(j)}, v_j) \notin Q$ holds for some i and j with $(v_i, v_j) \in E$. Let $a, a' \in T'$ be the vectors, which are respectively obtained from $a^{(i)}$ and $a^{(j)}$ by assigning $\beta(a^{(i)}, v_i) = 0$ and $\beta(a^{(j)}, v_j) = 0$. Then we can see that $a' \leq \bar{a}$ holds, since G is Δ-free. By (i), this implies that (T', F') has no extension in $\mathcal{C}^+_{DMI}$, which is a contradiction. Hence (36) holds.

Finally, consider the converse inequality, i.e.,

$$\rho(\mathcal{C}^+_{DMI}; (\tilde{T}, \tilde{F})) \leq \tau(G). \tag{37}$$

Let $C \subseteq V$ be a minimum vertex cover of G. Let $Q = \{(a^{(i)}, v_i) \mid i \in C\}$, and define $\alpha \in \mathbb{B}^Q$ by $\alpha(a^{(i)}, v_i) = 1$ for all $(a^{(i)}, v_i) \in Q$. For this α, we show that $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a robust extension in $\mathcal{C}^+_{DMI}$, that is, (T', F') of (34) has an extension in $\mathcal{C}^+_{DMI}$. This implies (37). By (i) together with $F' = \emptyset$, we only have to check if there is no $a' \in T'$ such that $a' \leq \bar{a}$ for all $a \in T'$.

For every pair of $a, a' \in T'$, which are respectively obtained from $a^{(i)}$ and $a^{(j)}$ with $(v_i, v_j) \in E$, $\alpha(a^{(i)}, v_i) = 1$ or $\alpha(a^{(j)}, v_j) = 1$ holds (i.e., $a_{v_i} = 1$ or $a'_{v_j} = 1$), by the definition of α. Since $a_{v_j} = a'_{v_i} = 1$, we have $a_{v_i} = a'_{v_i} = 1$ or $a_{v_j} = a'_{v_j} = 1$. Hence

$$a' \not\leq \bar{a} \ \text{ (i.e., } a \not\leq \bar{a}'). \tag{38}$$

For other pairs of $a, a' \in T'$ (i.e., if a and a' are respectively obtained from $a^{(i)}$ and $a^{(j)}$ with $(v_i, v_j) \notin E$), (38) also holds since $a_{w_{ij}} = a'_{w_{ij}} = 1$. Hence we have (37), which together with (36) will imply (35).

The case of MRE($\mathcal{C}^+_{DMA}$) is dual to MRE($\mathcal{C}^+_{DMI}$); it can be shown to be NP-hard by using the instance $(\tilde{T}^d = \{\bar{b} \mid b \in \tilde{F}\}, \tilde{F}^d = \{\bar{a} \mid a \in \tilde{T}\})$ constructed from the above instance $(\tilde{T}, \tilde{F})$, where $\bar{a}$ denotes the vector such that $\bar{a}_i = 1$ (resp., 0) if $a_i = 0$ (resp., 1), and $\bar{a}_i = *$ if $a_i = *$. □

8 Threshold functions

Let us denote by $\mathcal{C}_{TH}$ the class of threshold functions.

Theorem 26 *Problem* RE($\mathcal{C}_{TH}$) *can be solved in polynomial time.*

Proof. For a pBmd $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$, let us consider the following linear programming problem (LP):

$$\begin{array}{lll}
\max & \xi = \sum_{i=1}^n y_i \perp \sum_{i=1}^n z_i & \\
\text{subject to} & & \\
& \sum_{i \in ON(a)} w_i + \sum_{(a,i) \in AS(a)} y_i \geq t & \forall a \in \tilde{T} \\
& \sum_{i \in ON(b)} w_i + \sum_{(b,i) \in AS(b)} z_i \leq t \perp 1 & \forall b \in \tilde{F} \\
& y_i \leq w_i, \ \ y_i \leq 0 & i = 1, 2, \ldots, n \\
& z_i \geq w_i, \ \ z_i \geq 0 & i = 1, 2, \ldots, n.
\end{array} \tag{39}$$

We claim that the LP problem (39) has a feasible solution with a finite optimum value ξ if and only if $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{TH}$.

Let us assume first that $(\tilde{T}, \tilde{F})$ has a robust extension $f \in \mathcal{C}_{TH}$, and let w_i, $i = 1, 2, \ldots, n$, and t be the coefficients of f. Then by setting $y_i = \min\{0, w_i\}$ and $z_i = \max\{0, w_i\}$, we have a feasible solution of (39). Since $\sum_{i=1}^n y_i \perp \sum_{i=1}^n z_i \leq 0$, problem (39) has a feasible solution with a finite optimum.

Then assume conversely that w_i, y_i, z_i, $i = 1, 2, \ldots, n$, and t are an optimal solution of problem (39) (with a finite optimum). Then $y_i = \min\{0, w_i\}$ and $z_i = \max\{0, w_i\}$ hold since otherwise it would not be a optimum. This implies that w_i, $i = 1, 2, \ldots, n$, are finite, and hence w_i, $i = 1, 2, \ldots, n$, and t define a threshold function, which is a robust extension of $(\tilde{T}, \tilde{F})$. □

However, problem CE($\mathcal{C}_{TH}$) appears to be harder than RE($\mathcal{C}_{TH}$).

Theorem 27 *Problem* CE($\mathcal{C}_{TH}$) *is NP-complete, even if* $|AS(a)| \leq 1$ *holds for all* $a \in \tilde{T} \cup \tilde{F}$.

Proof. Similarly to the proof of Theorem 9, this problem is in NP. We now show that it is NP-hardness.

Let us consider a cubic CNF

$$\begin{aligned} \Phi &= \bigwedge_{k=1}^{m} C_k \\ C_k &= (u_k \vee v_k \vee w_k), \end{aligned}$$

where u_k, v_k and w_k for $k = 1, 2, \ldots, m$ are literals from set $L = \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$. Let $L' = \{x'_1, \bar{x}'_1, ..., x'_n, \bar{x}'_n\}$, and define $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^{L \cup L'}$ as follows.

$$\begin{aligned} \tilde{T} &= \{a^{x_i} = (\{x'_i\}; \{x_i\}), a^{\bar{x}_i} = (\{\bar{x}'_i\}; \{\bar{x}_i\}) \mid i = 1, 2, \ldots, n\} \\ \tilde{F} &= \{(b^{x_i} = \{x_i, \bar{x}_i\}; \emptyset), b^{x'_i} = (\{x'_i, \bar{x}'_i\}; \emptyset), \mid i = 1, 2, \ldots, n\} \\ &\quad \cup \{(b^{C_k} = (\{u_k, v_k, w_k, u'_k, v'_k, w'_k\}; \emptyset) \mid k = 1, 2, \ldots, m\}, \end{aligned} \tag{40}$$

where $(R; S)$ denotes the vector $v \in \mathbb{M}^{L \cup L'}$ such that $ON(v) = R$ and $AS(v) = \{(v, j) \mid j \in S\}$. It is easy to see that $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$. We claim that this pBmd $(\tilde{T}, \tilde{F})$ has a consistent threshold extension if and only if 3-SAT for Φ has a solution, which completes the proof.

Let us first assume that $\alpha \in \mathbb{B}^{AS}$ is an assignment such that $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a threshold extension:

$$f(d) = \begin{cases} 1 & \text{if } \sum_{z \in L \cup L'} w_z d_z \geq t \\ 0 & \text{otherwise,} \end{cases}$$

where $d \in \mathbb{B}^{L \cup L'}$. We shall first show that $\alpha(a^{x_i}, x_i) \neq \alpha(a^{\bar{x}_i}, \bar{x}_i)$ must hold for (a^{x_i}, x_i), $(a^{\bar{x}_i}, \bar{x}_i) \in AS$. If $\alpha(a^{x_i}, x_i) = \alpha(a^{\bar{x}_i}, \bar{x}_i) = 1$ holds, then $(a^{x_i})^\alpha \in \tilde{T}^\alpha$ and $(a^{\bar{x}_i})^\alpha \in \tilde{T}^\alpha$, respectively, imply $w_{x_i} + w_{x'_i} \geq t$ and $w_{\bar{x}_i} + w_{\bar{x}'_i} \geq t$, and hence

$$w_{x_i} + w_{x'_i} + w_{\bar{x}_i} + w_{\bar{x}'_i} \geq 2t. \tag{41}$$

However, $b^{x_i} \in \tilde{F}^\alpha$ and $b^{x'_i} \in \tilde{F}^\alpha$, respectively, implying $w_{x_i} + w_{\bar{x}_i} < t$ and $w_{x'_i} + w_{\bar{x}'_i} < t$, and hence $w_{x_i} + w_{x'_i} + w_{\bar{x}_i} + w_{\bar{x}'_i} < 2t$ follows, which is a contradiction to (41). Furthermore, if $\alpha(a^{x_i}, x_i) = \alpha(a^{\bar{x}_i}, \bar{x}_i) = 0$ holds, then $(a^{x_i})^\alpha \in \tilde{T}^\alpha$ and $(a^{\bar{x}_i})^\alpha \in \tilde{T}^\alpha$, respectively, implying

$w_{x'_i} \geq t$ and $w_{\bar{x}'_i} \geq t$, and hence $w_{x'_i} + w_{\bar{x}'_i} \geq 2t$ would follow, which is a contradiction to $f(b^{x'_i}) = 0$. Hence $\alpha(a^{x_i}, x_i) \neq \alpha(a^{\bar{x}_i}, \bar{x}_i)$ holds. Let us now define a binary vector $y \in \mathbb{B}^n$ by

$$y_i = \begin{cases} 1 & \text{if } \alpha(a^{x_i}, x_i) = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and show that this y satisfies $\Phi(y) = 1$. For this, assume otherwise that there is a clause C_k that satisfies $C_k(y) = 0$ (i.e., $u_k = v_k = w_k = 0$ holds by y), that is, $\alpha(a^{u_k}, u_k) = \alpha(a^{v_k}, v_k) = \alpha(a^{w_k}, w_k) = 1$. Then taking three vectors $(a^{u_k})^\alpha, (a^{v_k})^\alpha, (a^{w_k})^\alpha \in \tilde{T}^\alpha$, we have $w_{u_k} + w_{u'_k} \geq t$, $w_{v_k} + w_{v'_k} \geq t$ and $w_{w_k} + w_{w'_k} \geq t$, and hence $w_{u_k} + w_{v_k} + w_{w_k} + w_{u'_k} + w_{v'_k} + w_{w'_k} \geq 3t$, which is a contradiction to $f(b^{C_k}) = 0$.

For the converse direction, let us assume that $\Phi(y) = 1$ holds for some $y \in \mathbb{B}^n$. Let us define an assignment $\alpha \in \mathbb{B}^{AS}$ by $\alpha(a^{x_i}, x_i) = \bar{y}_i$ and $\alpha(a^{\bar{x}_i}, \bar{x}_i) = y_i$ for $i = 1, 2, \ldots, n$, and let

$$w_z = \begin{cases} \perp 3 & \text{if either } z = x_i \text{ and } y_i = 1, \text{ or } z = \bar{x}_i \text{ and } y_i = 0 \\ +2 & \text{if either } z = x_i \text{ and } y_i = 0, \text{ or } z = \bar{x}_i \text{ and } y_i = 1 \\ +1 & \text{if either } z = x'_i \text{ and } y_i = 1, \text{ or } z = \bar{x}'_i \text{ and } y_i = 0 \\ \perp 1 & \text{if either } z = x'_i \text{ and } y_i = 0, \text{ or } z = \bar{x}'_i \text{ and } y_i = 1, \end{cases}$$

and $t = 1$. Then $\sum_{z \in L \cup L'} w_z a_z \geq 1$ holds for all $a \in \tilde{T}^\alpha$, and $\sum_{z \in L \cup L'} w_z b_z \leq 0$ holds for all $b \in \tilde{F}^\alpha$. Hence $(\tilde{T}, \tilde{F})$ has a consistent threshold extension. □

9 Decomposable functions

The decomposability was defined at the end of Subsection 2.1. We only consider the following fundamental classes of decomposable functions.

$\mathcal{C}_{g(S_0, h_1(S_1))}$: class of $g(S_0, h_1(S_1))$-decomposable functions,
$\mathcal{C}^+_{g(S_0, h_1(S_1))}$: class of positive $g(S_0, h_1(S_1))$-decomposable functions,

where $S_1, S_2 \subseteq V$. It is known that the problems EXTENSION($\mathcal{C}_{g(S_0, h_1(S_1))}$) and EXTENSION($\mathcal{C}^+_{g(S_0, h_1(S_1))}$) can be solved in polynomial time [3], and that both of the problems BEST-FIT($\mathcal{C}_{g(S_0, h_1(S_1))}$) and BEST-FIT($\mathcal{C}^+_{g(S_0, h_1(S_1))}$) are NP-hard [5].

Let us first consider the class $\mathcal{C}_{g(S_0, h_1(S_1))}$.

Theorem 28 *Problem* CE($\mathcal{C}_{g(S_0, h_1(S_1))}$) *is NP-complete, even if* $|AS(a)| \leq 1$ *holds for all* $a \in \tilde{T} \cup \tilde{F}$.

Proof. Similarly to the proof of Theorem 9, this problem is in NP. We now show its NP-hardness. Let $\mathcal{H} = (V, E)$ be a 3-uniform hypergraph, where $V = \{1, 2, \ldots, n\}$ and $E = \{H_i = \{u_i, v_i, w_i\} \mid u_i < v_i < w_i, i = 1, 2, \ldots, m\}$. Let $S_0 = \{0, 1, 2, \ldots, p\}$ and $S_1 = \{p+1, p+2, \ldots, p+q\}$, and let the base set S be defined by $S = S_0 \cup S_1$. Let $W_i \subseteq S_0 \setminus \{0\}$, $i = 1, 2, \ldots, m$, be subsets such that $W_i \neq W_j$ for $i \neq j$, and let $U_l \subseteq S_1$, $l \in V$, be subsets such that $U_l \neq U_{l'}$ for $l \neq l'$. This is possible if we use S_0 and S_1 satisfying $p = O(\lceil \log m \rceil)$ and $q = O(\lceil \log n \rceil)$. Then we define $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^S$ as follows:

$$\begin{aligned} \tilde{T} &= \{a^{(i)} = (W_i \cup U_{u_i}; \{0\}) \mid H_i = \{u_i, v_i, w_i\} \in E\} \\ \tilde{F} &= \{b^{(i1)} = (\{0\} \cup W_i \cup U_{v_i}; \emptyset), b^{(i2)} = (W_i \cup U_{w_i}; \emptyset) \mid H_i \in E\}, \end{aligned}$$

where $(P; R)$ denotes the vector $v \in \mathbb{M}^S$ such that $ON(v) = P$ and $AS(v) = \{(v, i) \mid i \in R\}$. It is easy to see that $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$. We claim that this pBmd $(\tilde{T}, \tilde{F})$ has a consistent extension in $\mathcal{C}_{g(S_0, h_1(S_1))}$ if and only if $\mathcal{H}$ is 2-colorable, which completes the proof.

Let us first assume that $(C, V \setminus C)$ is a 2-coloring of $\mathcal{H}$, i.e., $C \cap H \neq \emptyset$ and $(V \setminus C) \cap H \neq \emptyset$ for all $H \in E$. Then define an assignment $\alpha \in \mathbb{B}^{AS}$ by

$$\alpha(a^{(i)}, 0) = \begin{cases} 1 & \text{if either } (u_i \in C \text{ and } v_i \notin C) \text{ or } (u_i \notin C \text{ and } v_i \in C) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h_1(U_l; \emptyset) = \begin{cases} 1 & \text{if } l \in C \\ 0 & \text{otherwise.} \end{cases}$$

Now we shall show that

$$(a^\alpha[S_0], h_1(a^\alpha[S_1])) \neq (b^\alpha[S_0], h_1(b^\alpha[S_1])) \qquad (42)$$

holds for all pair of vectors $a^\alpha \in \tilde{T}^\alpha$ and $b^\alpha \in \tilde{F}^\alpha$, which implies that $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has an extension in $\mathcal{C}_{g(S_0, h_1(S_1))}$. It is easy to see that (42) holds for every pair of $(a^{(i)})^\alpha \in \tilde{T}^\alpha$ and $(b^{(jk)})^\alpha \in \tilde{F}^\alpha$, where $i \neq j$ and $k = 1, 2$. Let us consider the pair of $(a^{(i)})^\alpha \in \tilde{T}^\alpha$ and $(b^{(ik)})^\alpha \in \tilde{F}^\alpha$ for $k = 1, 2$. Since $(C, V \setminus C)$ is a 2-coloring, $h_1(U_{u_i}; \emptyset) \neq h_1(U_{v_i}; \emptyset)$ or $h_1(U_{u_i}; \emptyset) \neq h_1(U_{w_i}; \emptyset)$ holds. If $h_1(U_{u_i}; \emptyset) \neq h_1(U_{v_i}; \emptyset)$ holds, then clearly (42) holds for the pair of $(a^{(i)})^\alpha \in \tilde{T}^\alpha$ and $(b^{(i1)})^\alpha \in \tilde{F}^\alpha$, and by the definition of $\alpha \in \mathbb{B}^{AS}$, $(a^{(i)})^\alpha[0] \neq (b^{(i2)})^\alpha[0]$, implying (42) for the pair of $(a^{(i)})^\alpha \in \tilde{T}^\alpha$ and $(b^{(i2)})^\alpha \in \tilde{F}^\alpha$. Hence (42) holds for all pair of $(a^{(i)})^\alpha \in \tilde{T}^\alpha$ and $(b^{(ik)})^\alpha \in \tilde{F}^\alpha$, where $k = 1, 2$. Also for the case of $h_1(U_{u_i}; \emptyset) = h_1(U_{v_i}; \emptyset) \neq h_1(U_{w_i}; \emptyset)$, we can show by a similar argument that (42) holds for all pair of $(a^{(i)})^\alpha \in \tilde{T}^\alpha$ and $(b^{(ik)})^\alpha \in \tilde{F}^\alpha$, where $k = 1, 2$.

For the converse direction, let $\alpha \in \mathbb{B}^{AS}$ is an assignment such that $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has an extension in $\mathcal{C}_{g(S_0, h_1(S_1))}$, and let us define

$$C = \{l \mid h_1(U_l; \emptyset) = 1\}. \qquad (43)$$

Then we claim that $(C, V \setminus C)$ is a 2-coloring of $\mathcal{H}$. For this, assume that some $H_i \in E$ is monochromatic, that is, either $H_i \subseteq C$ or $H_i \subseteq V \setminus C$ holds. Then by (43),

$$h_1((a^{(i)})^\alpha[S_1]) = h_1((b^{(i1)})^\alpha[S_1]) = h_1((b^{(i2)})^\alpha[S_1]) \qquad (44)$$

holds. Since $(\tilde{T}^\alpha, \tilde{F}^\alpha)$ has a $g(S_0, h_1(S_1))$-decomposable extension,

$$((a^{(i)})^\alpha[S_0], h_1((a^{(i)})^\alpha[S_1])) \neq ((b^{(i1)})^\alpha[S_0], h_1((b^{(i1)})^\alpha[S_1]))$$

and

$$((a^{(i)})^\alpha[S_0], h_1((a^{(i)})^\alpha[S_1])) \neq ((b^{(i2)})^\alpha[S_0], h_1((b^{(i2)})^\alpha[S_1]))$$

hold. However, by (44), this implies $(a^{(i)})^\alpha[S_0] \neq (b^{(i1)})^\alpha[S_0]$ and $(a^{(i)})^\alpha[S_0] \neq (b^{(i2)})^\alpha[S_0]$, which is impossible for any assignment $\alpha \in \mathbb{B}^{AS}$. □

Next let us consider the robustness of $\mathcal{C}_{g(S_0,h_1(S_1))}$, where it is emphasized that $S_0 \cap S_1 \neq \emptyset$ generally holds. For a subset $\tilde{S} \subseteq \mathbb{M}^V$, let $AS_k(\tilde{S}) = \{(v,j) \in AS(\tilde{S}) \mid j \in S_k\}$ for $k = 1, 2$, and $AS_0 = AS_0(\tilde{T} \cup \tilde{F})$. Let us define a graph $G_{(\tilde{T},\tilde{F})} = (W, E_1 \cup E_2)$, with $W \subseteq \mathbb{B}^{S_1 \cap S_0} \times \mathbb{M}^{S_1 \setminus S_0}$ by

$$\begin{aligned} W &= \{w, w' \mid \text{there exist } a \in \tilde{T}, b \in \tilde{F}, \alpha \in \mathbb{B}^{AS_0(a)} \text{ and } \beta \in \mathbb{B}^{AS_0(b)} \\ &\qquad \text{such that } w = a^\alpha[S_1], w' = b^\beta[S_1] \text{ and } a^\alpha[S_0] = b^\beta[S_0]\} \\ E_1 &= \{(w, w') \mid \text{there exist } a \in \tilde{T}, b \in \tilde{F}, \alpha \in \mathbb{B}^{AS_0(a)} \text{ and } \beta \in \mathbb{B}^{AS_0(b)} \\ &\qquad \text{such that } w = a^\alpha[S_1], w' = b^\beta[S_1] \text{ and } a^\alpha[S_0] = b^\beta[S_0]\} \\ E_2 &= \{(w, w') \mid \text{there exist } a, b \in \tilde{T} \cup \tilde{F}, \alpha \in \mathbb{B}^{AS_0(a)} \text{ and } \beta \in \mathbb{B}^{AS_0(b)} \\ &\qquad \text{such that } w = a^\alpha[S_1], w' = b^\beta[S_1] \text{ and } a^\alpha[S_1] \approx b^\beta[S_1]\}. \end{aligned}$$

Furthermore, denote by $G'_{(\tilde{T},\tilde{F})}$ the graph obtained from $G_{(\tilde{T},\tilde{F})}$ by contracting all edges in E_2.

Example 3. Let $S_0 = \{1,2,3,4\}$, $S_1 = \{4,5,6\}$ and $V = S_0 \cup S_1$ (i.e., $V = \{1,2,\ldots,6\}$), and define $\tilde{T}, \tilde{F} \subseteq \{0,1\}^V$ by

$$\tilde{T} = \left\{ \begin{array}{lcl} a^{(1)} &=& (1,1,1,1,*,0) \\ a^{(2)} &=& (0,*,1,*,1,*) \\ a^{(3)} &=& (0,0,0,0,1,0) \end{array} \right\}, \quad \tilde{F} = \left\{ \begin{array}{lcl} b^{(1)} &=& (0,1,*,*,0,0) \\ b^{(2)} &=& (0,0,1,0,0,*) \\ b^{(3)} &=& (0,0,0,0,0,1) \end{array} \right\}.$$

Graphs $G_{(\tilde{T},\tilde{F})}$ and $G'_{(\tilde{T},\tilde{F})}$ are given in Figure 5. Note that graph $G_{(\tilde{T},\tilde{F})}$ does not have vertex $(1,*,0) \in \mathbb{B}^{S_1 \cap S_0} \times \mathbb{M}^{S_1 \setminus S_0}$, which is obtained from $a^{(1)} \in \tilde{T}$, since there is no $b \in \tilde{F}$ such that $(a^{(1)})^\alpha[S_0] = b^\beta[S_0]$ holds for some $\alpha \in \mathbb{B}^{AS_0(a^{(1)})}$ and $\beta \in \mathbb{B}^{AS_0(b)}$. □

Lemma 16 *Let $(\tilde{T},\tilde{F})$ be a pBmd. Then $(\tilde{T},\tilde{F})$ has a robust $g(S_0,h_1(S_1))$-decomposable extension if and only if $G'_{(\tilde{T},\tilde{F})}$ is bipartite.*

Proof. Let us first show the only-if-part. Assume that $(\tilde{T},\tilde{F})$ has a robust $g(S_0,h_1(S_1))$-decomposable extension, but $G'_{(\tilde{T},\tilde{F})}$ is not bipartite. In other words, there is a cycle $w^{(0)} \to w^{(1)} \to \ldots \to w^{(l)} \,(= w^{(0)})$ in $G_{(\tilde{T},\tilde{F})} = (W, E_1 \cup E_2)$ such that

$$|E_1 \cap \{(w^{(i)}, w^{(i+1)}) \mid i = 0, 1, \ldots, l \perp 1\}| \text{ is odd.} \tag{45}$$

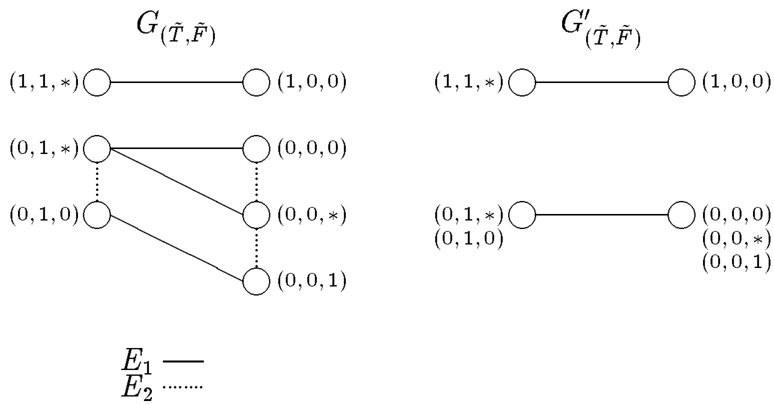

Figure 5: Graphs $G_{(\tilde{T},\tilde{F})}$ and $G'_{(\tilde{T},\tilde{F})}$ of $(\tilde{T},\tilde{F})$ in Example 3.

Let us consider the values of h_1 on $\{(w^{(i)})^\alpha \mid \alpha \in \mathbb{B}^{AS_1(w^{(i)})}\}$, $i = 0, 1, \ldots, l \perp 1$. For each $(w^{(i)}, w^{(i+1)}) \in E_1$, by the definition of E_1, we must have $h_1((w^{(i)})^\beta) \neq h_1((w^{(i+1)})^\gamma)$ for all assignments $\beta \in \mathbb{B}^{AS_1(w^{(i)})}$ and $\gamma \in \mathbb{B}^{AS_1(w^{(i+1)})}$. This means that

$$\begin{array}{ll} h_1((w^{(i)})^\beta) = p & \text{for all } \beta \in \mathbb{B}^{AS_1(w^{(i)})}, \text{ and} \\ h_1((w^{(i+1)})^\gamma) = \bar{p} & \text{for all } \gamma \in \mathbb{B}^{AS_1(w^{(i+1)})}, \end{array} \tag{46}$$

where $p \in \{0, 1\}$. On the other hand, if $(w^{(i)}, w^{(i+1)}) \in E_2$,

$$h_1((w^{(i)})^\beta) = h_1((w^{(i+1)})^\gamma) \tag{47}$$

holds for all $\beta \in \mathbb{B}^{AS_1(w^{(i)})}$ and $\gamma \in \mathbb{B}^{AS_1(w^{(i+1)})}$, because the definition of W and (46) imply that $h_1((w^{(i)})^\beta) = p$ for all $\beta \in \mathbb{B}^{AS_1(w^{(i)})}$, $h_1((w^{(i+1)})^\gamma) = q$ for all $\gamma \in \mathbb{B}^{AS_1(w^{(i+1)})}$, and $p = q$ by $(w^{(i)}, w^{(i+1)}) \in E_2$. Thus (46) and (47) contradict (45).

Conversely, if $G'_{(\tilde{T},\tilde{F})}$ is bipartite, then there is a partition $(Y, W \setminus Y)$ of W such that

$$E_1 \subseteq Y \times (W \setminus Y) \tag{48}$$

$$E_2 \subseteq (Y \times Y) \cup ((W \setminus Y) \times (W \setminus Y)). \tag{49}$$

By (49), we can define the value of h_1 for W by

$$h_1((w)^\beta) = \begin{cases} 1 & \text{if } w \in Y \text{ and } \beta \in \mathbb{B}^{AS_1(w)} \\ 0 & \text{if } w \in W \setminus Y \text{ and } \beta \in \mathbb{B}^{AS_1(w)}. \end{cases} \tag{50}$$

Furthermore, define g by

$$\begin{array}{ll} g(a^\alpha[S_0], h_1(a^\alpha[S_1])) = 1 & \text{for all } a \in \tilde{T} \text{ and } \alpha \in \mathbb{B}^{AS(a)} \\ g(b^\alpha[S_0], h_1(b^\alpha[S_1])) = 0 & \text{for all } b \in \tilde{F} \text{ and } \alpha \in \mathbb{B}^{AS(b)}. \end{array}$$

If $a^\alpha[S_0] = b^\alpha[S_0]$ holds for some $a \in \tilde{T}$, $b \in \tilde{F}$ and $\alpha \in \mathbb{B}^{AS(\{a,b\})}$, then $w_a, w_b \in W$ with $a^\alpha[S_1] \approx w_a$ and $b^\alpha[S_1] \approx w_b$ satisfy $(w_a, w_b) \in E_1$, and we have $h_1(a^\alpha[S_1])) \neq h_1(b^\alpha[S_1]))$ by (48) and (50). Hence g is also well-defined. Therefore, by extending this h_1 to $\mathbb{B}^{S_1}$, we see that $(\tilde{T}, \tilde{F})$ has a robust $g(S_0, h_1(S_1))$-decomposable extension . □

In general, however, the size of a graph $G'_{(\tilde{T},\tilde{F})}$ is exponential in $|S_0|$, and the above lemma does not directly lead to an efficient algorithm to $\mathrm{RE}(\mathcal{C}_{g(S_0,h_1(S_1))})$.

Theorem 29 *Problem* $\mathrm{RE}(\mathcal{C}_{g(S_0,h_1(S_1))})$ *is co-NP-complete.*

Proof. First we show that the problem is in co-NP. For a pBmd $(\tilde{T}, \tilde{F})$, we show that every simple cycle C in $G'_{(\tilde{T},\tilde{F})}$ satisfies $|C| \leq |\tilde{T}| + |\tilde{F}|$. By the definition of E_1, $w[S_0 \cap S_1] = w'[S_0 \cap S_1]$ holds for all edges $(w, w') \in E_1$. The same condition also holds for all edges $(w, w') \in E_2$ by the property $w[S_1] \approx w'[S_1]$. Thus

$$w[S_0 \cap S_1] = w'[S_0 \cap S_1] \tag{51}$$

holds if there is a path from w to w' in $G_{(\tilde{T},\tilde{F})}$. In particular, all vertices in a cycle C in $G'_{(\tilde{T},\tilde{F})}$ have this property, and, by the definition of $G'_{(\tilde{T},\tilde{F})}$, all vertices w in C have different $w[S_1 \setminus S_0]$, implying that they are generated from different vectors in $\tilde{T} \cup \tilde{F}$. This proves $|C| \leq |\tilde{T}| + |\tilde{F}|$. Since $(\tilde{T}, \tilde{F})$ has a robust $g(S_0, h_1(S_1))$-decomposable extension if and only if there is no cycle C of odd length in $G'_{(\tilde{T},\tilde{F})}$, we can then conclude that $\mathrm{RE}(\mathcal{C}_{g(S_0,h_1(S_1))})$ is in co-NP.

We next show its co-NP-hardness. Let $\mathcal{H} = (U, E)$ be a 3-uniform hypergraph, where $U = \{1, 2, \ldots, n\}$, $E = \{H_i \,|\, i = 1, 2, \ldots, m\}$ and m is odd. Let $S_0 = \{1, 2, \ldots, n + m\}$, $S_1 = \{1, 2, \ldots, n\} \cup \{n + m + 1, n + m + 2, \ldots n + 2m\}$ and the base set $V = S_0 \cup S_1$. Obviously, $S_0 \cap S_1 = U = \{1, 2, \ldots, n\}$ holds in this case. Then define $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^V$ as follows.

$$\begin{aligned}
\tilde{T} &= \{(A_i \cup \{n+i\} \cup \{n+m+i\}; U \setminus H_i) \,|\, A_i \subset H_i, A_i \neq \emptyset, 1 \leq i \leq m\}\} \\
\tilde{F} &= \{(A_i \cup \{n+i\} \cup \{n+m+(i \pmod m)+1\}; U \setminus H_i) \,|\, A_i \subset H_i, A_i \neq \emptyset, 1 \leq i \leq m\}\},
\end{aligned}$$

where $\subset$ denotes the proper inclusion, and $(R; S)$ denotes the vector $v \in \mathbb{M}^V$ such that $ON(v) = R$ and $AS(v) = \{(v, i) \,|\, i \in S\}$. We claim that this pBmd $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_{g(S_0,h_1(S_1))}$ if and only if $\mathcal{H}$ is not 2-colorable, which completes the proof because deciding if $\mathcal{H}$ is 2-colorable is NP-complete (even if $|E|$ is restricted to be odd). For this $(\tilde{T}, \tilde{F})$, we have $E_2 = \emptyset$, because any $(v, j) \in AS$ satisfies $j \in S_0 \cap S_1$, implying that $v \in \mathbb{B}^{S_1}$ holds for any vertex v in $G_{(\tilde{T},\tilde{F})}$. This means that $G'_{(\tilde{T},\tilde{F})}$ is bipartite if and only if so is $G_{(\tilde{T},\tilde{F})}$. Thus Lemma 16 tells that $(\tilde{T}, \tilde{F})$ has a robust $g(S_0, h_1(S_1))$-decomposable extension if and only if $G_{(\tilde{T},\tilde{F})} = (W, E_1 \cup E_2)$ is bipartite.

Let us first assume that $(C, U \setminus C)$ is a 2-coloring of $\mathcal{H}$, i.e., $C \cap H_i \neq \emptyset$ and $(U \setminus C) \cap H_i \neq \emptyset$ for all $H_i \in E$. Then C can be represented by

$$C = \bigcup_{i=1}^{m} A_i^* \tag{52}$$

for some $\emptyset \neq A_i^* \subset H_i$, and we have

$$((C \cup \{n+m+i\}; \emptyset), (C \cup \{n+m+(i \pmod{m})+1\}; \emptyset)) \in E_1 \quad \text{for } i = 1, 2, \ldots, m.$$

Hence, we have a cycle $w^{(1)} \to w^{(2)} \to \ldots \to w^{(m)} \to w^{(1)}$ in $G_{(\tilde{T},\tilde{F})}$, where

$$w^{(i)} = (C \cup \{n+m+i\}; \emptyset), \;\; i = 1, 2, \ldots, m.$$

Since m is supposed to be odd, this implies that $(\tilde{T}, \tilde{F})$ has no robust extension in $\mathcal{C}_{g(S_0,h_1(S_1))}$.

For the converse direction, let us assume that $G_{(\tilde{T},\tilde{F})}$ has a cycle. By property (51), we can only consider a cycle in $G_{(\tilde{T},\tilde{F})}[W_C]$ for some $C \in U$, where $W_C = \{w \in W \mid ON(w[U]) = C\}$ and $G_{(\tilde{T},\tilde{F})}[W_C]$ is the subgraph of $G_{(\tilde{T},\tilde{F})}$ induced by W_C. By the definition of the above $(\tilde{T}, \tilde{F})$, such a cycle is of the form

$$(C\cup\{n+m+1\}; \emptyset) \to (C\cup\{n+m+2\}; \emptyset) \to \ldots \to (C\cup\{n+2m)\}; \emptyset) \to (C\cup\{n+m+1\}; \emptyset).$$

Thus the length of this cycle is odd. This C obviously satisfies (52) and is a 2-coloring of $\mathcal{H}$. □

However, if $S_0 \cap S_1 = \emptyset$, then $\mathrm{RE}(\mathcal{C}_{g(S_0,h_1(S_1))})$ is polynomially solvable.

Theorem 30 *If $S_0 \cap S_1 = \emptyset$, problem $\mathrm{RE}(\mathcal{C}_{g(S_0,h_1(S_1))})$ can be solved in polynomial time.*

Proof. Since $S_0 \cap S_1 = \emptyset$, a graph $G_{(\tilde{T},\tilde{F})} = (W, E_1 \cup E_2\}$ can be represented by

$$\begin{aligned} W &= \{a[S_1], b[S_1] \mid a \in \tilde{T}, b \in \tilde{F} \text{ and } a[S_0] \approx b[S_0]\}, \\ E_1 &= \{(a[S_1], b[S_1]) \mid a \in \tilde{T}, b \in \tilde{F} \text{ and } a[S_0] \approx b[S_0]\}, \\ E_2 &= \{(a[S_1], b[S_1]) \mid a, b \in \tilde{T} \cup \tilde{F} \text{ and } a[S_1] \approx b[S_1]\}. \end{aligned}$$

It is easy to see that this graph $G'_{(\tilde{T},\tilde{F})}$ has polynomially many vertices and can be constructed in polynomial time. Then, by applying Lemma 16, $\mathrm{RE}(\mathcal{C}_{g(S_0,h_1(S_1))})$ can be solved in polynomial time. □

Let us finally consider class $\mathcal{C}^+_{g(S_0,h_1(S_1))}$. Problems $\mathrm{RE}(\mathcal{C}^+_{g(S_0,h_1(S_1))})$ and $\mathrm{CE}(\mathcal{C}^+_{g(S_0,h_1(S_1))})$ can be solved in polynomial time, by Corollary 4. For problem $\mathrm{MRE}(\mathcal{C}^+_{g(S_0,h_1(S_1))})$, however, we have the following negative result.

Theorem 31 *Problem $\mathrm{MRE}(\mathcal{C}^+_{g(S_0,h_1(S_1))})$ is NP-hard, even if $|AS(a)| \leq 1$ holds for all $a \in \tilde{T} \cup \tilde{F}$.*

Proof. We claim that class $\mathcal{C}^+_{g(S_0,h_1(S_1))}$ satisfies properties PA((1,0)) and PB((1,0), (1,1), (0,0)) of Subsection 3.2, where we consider that the two added variables of the vectors (1,0), (1,1) and (0,0) used in PA and PB are in S_0. Combining this with the NP-hardness of problem BEST-FIT($\mathcal{C}^+_{g(S_0,h_1(S_1))}$) [5], Theorem 4 will prove the above theorem statement, since the specification of the added components does not affect Theorem 4. Let us denote the two added variables by x_{n+1} and x_{n+2} in the following.

PA((1,0)): Let $f = g(S_0, h_1(S_1))$ be an extension of a pdBf (T, F), where $T, F \subseteq \mathbb{B}^n$. Then f is also an extension of $(T \times \{(1,0)\}, F \times \{(1,0)\})$ (in this case, just ignore x_{n+1} and x_{n+2}). Conversely, let $f' = g'(S_0, h_1(S_1))$ be an extension of $(T \times \{(1,0)\}, F \times \{(1,0)\})$. Then $f = f'_{\{x_{n+1} \leftarrow 1, x_{n+2} \leftarrow 0\}}$ is obviously a $g(S_0, h_1(S_1))$-decomposable extension of (T, F).

PB((1,0),(1,1),(0,0)): It is easy to see that $f x_{n+1} \bar{x}_{n+2} \vee x_{n+1} x_{n+2}$ is a $g(S_0, h_1(S_1))$ decomposable extension of $((T \times \{(1,0)\}) \cup (\mathbb{B}^n \times \{(1,1)\}), (F \times \{(1,0)\}) \cup (\mathbb{B}^n \times \{(0,0)\}))$ if f is an extension of (T, F). Conversely, let $f' = g'(S_0, h_1(S_1))$ be an extension of $((T \times \{(1,0)\}) \cup (\mathbb{B}^n \times \{(1,1)\}), (F \times \{(1,0)\}) \cup (\mathbb{B}^n \times \{(0,0)\}))$. Then $f'_{\{x_{n+1} \leftarrow 1, x_{n+2} \leftarrow 0\}}$ is a $g(S_0, h_1(S_1))$-decomposable extension of (T, F), since f' is also an extension of $(T \times \{(1,0)\}, F \times \{(1,0)\})$. □

10 Conclusion

In this paper, we have considered three types of extensions, consistent, robust and most robust, for partially defined Boolean functions that contain missing data. In the tables below, we summarize their complexity for various classes of functions $\mathcal{C}$, which are considered in this paper. In real-world applications, the sizes of $(\tilde{T}, \tilde{F})$ are usually large, and there are many missing data. Hence it would be important to develop fast heuritic algorithms for these NP-hard cases.

References

[1] D. Angluin, L. Hellerstein and M. Karpinski, Learning read-once formulas with queries, *J. ACM*, 40 (1993) 185-210.

[2] B. Apswall, M.F. Plass and R.E.Tarjan, A linear-time algorithm for testing the truth of certain quantified Boolean formulas, *Information Processing Letters* 8 (1979) 121-123.

[3] E. Boros, V. Gurvich, P.L. Hammer, T. Ibaraki, and A. Kogan, Decompositions of partially defined Boolean functions, *Discrete Applied Mathematics* 62 (1995) 51-75.

[4] E. Boros, P. L. Hammer, and J. N. Hooker, Predicting cause-effect relationships from incomplete discrete observations, *SIAM Journal on Discrete Mathematics* 7 (1994) 531-543.

[5] E. Boros, T. Ibaraki and K. Makino, Error-free and best-fit extensions of a partially defined Boolean function, RUTCOR Research Report RRR 14-95, Rutgers University, 1995.

[6] V. Chvátal and P.L. Hammer, Aggregation of inequalities in integer programming, *Annals of Discrete Mathematics* 1 (1977) 145-162.

[7] C. J. Colbourn, *The Combinatorics of Network Reliability*, Oxford University Press, 1987.

[8] Y. Crama, P. L. Hammer and T. Ibaraki, Cause-effect relationships and partially defined boolean functions, *Annals of Operations Research* 16 (1988) 299-326.

[9] R. Dechter and J. Pearl, Structure identification in relational data, *Artificial Intelligence* 58 (1992) 237-270.

[10] L. R. Ford and D. R. Fulkerson, *Flows in Networks*, Princeton University Press, 1962.

[11] M. R. Garey and D. S. Johnson, *Computers and Intractability*, Freeman, New York, 1979.

[12] D. Kavvadias, C. H. Papadimitriou and M. Sideri, On Horn envelopes and hypergraph transversals, *ISAAC'93* Algorithms and Computation, edited by K. W. Ng et al., Springer Lecture Notes in Computer Science 762 (1993) 399-405.

[13] K. Makino, K. Hatanaka and T. Ibaraki, Horn extensions of a partially defined Boolean function, RUTCOR Research Report RRR 27-95, Rutgers University, 1995.

[14] K. Makino, K. Yano and T. Ibaraki, Positive and Horn decomposability of partially defined Boolean functions, RUTCOR Research Report RRR 29-95, Rutgers University, 1995.

[15] D. Mundici, Functions computed by monotone Boolean formulas with no repeated variables, *Theoretical Computer Science* 66 (1989) 113-114.

[16] S. Muroga, *Threshold Logic and Its Applications*, Wiley-Interscience, 1971.

[17] J. S. Provan and M. O. Ball, Efficient recognition of matroids and 2-monotonic systems, *Applications of Discrete Mathematics*, edited by R. Ringeisen and F. Roberts, SIAM, Philadelphia (1988) 122-134.

[18] J. R. Quinlan, Induction of decision trees, *Machine Learning* 1 (1986) 81-106.

[19] K. G. Ramamurthy, *Coherent Structures and Simple Games*, Kluwer Academic Publishers, 1990.

[20] L. G. Valiant, A theory of the learnable, *Communications of the ACM* 27 (1984) 1134-1142.

Function classes	RE	CE	MRE
	$\lvert AS(a)\rvert \le 1$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 1$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 1$ for all $a \in \tilde{T} \cup \tilde{F}$
	$\lvert AS(a)\rvert \le 2$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 2$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 2$ for all $a \in \tilde{T} \cup \tilde{F}$
	$\lvert AS(a)\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$
	General case	General case	General case
General	P	P	P
	P	NPC	NPH
	P	NPC	P
	P	NPC	NPH
Positive	P	P	P
	P	P	NPH
	P	P	P
	P	P	NPH
Regular	P	P	P
	P	P	P
	P	P	P
	P	P	P

P: Polynomial, NPC: NP-complete, NPH: NP-hard

Table 1: Transitive classes.

Function classes	RE	CE	MRE
	$\lvert AS(a)\rvert \le 1$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 1$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 1$ for all $a \in \tilde{T} \cup \tilde{F}$
	$\lvert AS(a)\rvert \le 2$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 2$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 2$ for all $a \in \tilde{T} \cup \tilde{F}$
	$\lvert AS(a)\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$
	General case	General case	General case
(Positive) k-DNF	NPC	NPC	NPH
	NPC	NPC	NPH
	NPC	NPC	NPH
	NPH	NPC	NPH
(Positive) 1-DNF	P	P	NPH
	P	P	NPH
	P	P	P
	P	P	NPH
2-DNF	P	NPC	NPH
	P	NPC	NPH
	P	NPC	P
	P	NPC	NPH
Positive 2-DNF	P	P	NPH
	P	P	NPH
	P	P	P
	P	P	NPH
k-DNF with fixed $k \ge 3$	P	NPC	NPH
	P	NPC	NPH
	P	NPC	P
	co-NPC	NPC	NPH
Positive k-DNF with fixed $k \ge 3$	P	P	NPH
	P	P	NPH
	P	P	P
	P	P	NPH

P: Polynomial, NPC: NP-complete, NPH: NP-hard, co-NPC: co-NP-complete

Table 2: Hereditary classes (i).

Function classes	RE	CE	MRE
	$\|AS(a)\| \leq 1$ for all $a \in \tilde{T} \cup \tilde{F}$ $\|AS(a)\| \leq 2$ for all $a \in \tilde{T} \cup \tilde{F}$ $\|AS(a)\| = O(\log(n + \|\tilde{T}\| + \|\tilde{F}\|))$ for all $a \in \tilde{T} \cup \tilde{F}$ General case	$\|AS(a)\| \leq 1$ for all $a \in \tilde{T} \cup \tilde{F}$ $\|AS(a)\| \leq 2$ for all $a \in \tilde{T} \cup \tilde{F}$ $\|AS(a)\| = O(\log(n + \|\tilde{T}\| + \|\tilde{F}\|))$ for all $a \in \tilde{T} \cup \tilde{F}$ General case	$\|AS(a)\| \leq 1$ for all $a \in \tilde{T} \cup \tilde{F}$ $\|AS(a)\| \leq 2$ for all $a \in \tilde{T} \cup \tilde{F}$ $\|AS\| = O(\log(n + \|\tilde{T}\| + \|\tilde{F}\|))$ General case
(Positive) h-term-DNF	NPC NPC NPC NPH	NPC NPC NPC NPC	NPH NPH NPH NPH
(Positive) h-term-DNF with fixed $k \geq 2$	NPC NPC NPC NPH	NPC NPC NPC NPC	NPH NPH NPH NPH
(Positive) 1-term-DNF	P P P P	P P P P	NPH NPH P NPH
(Positive) h-term-k-DNF	NPC NPC NPC NPH	NPC NPC NPC NPC	NPH NPH NPH NPH
(Positive) h-term-k-DNF with fixed $h \geq 1$	NPC NPC NPC NPH	NPC NPC NPC NPC	NPH NPH NPH NPH
(Positive) h-term-k-DNF with fixed $k \geq 1$	NPC NPC NPC NPH	NPC NPC NPC NPC	NPH NPH NPH NPH
(Positive) h-term-k-DNF with fixed h, k	P P P P	P P P P	P P P P
Horn	P P P P	NPC NPC NPC NPC	NPH NPH P NPH

P: Polynomial, NPC: NP-complete, NPH: NP-hard

Table 3: Hereditary classes (ii).

Function classes	RE	CE	MRE
	$\lvert AS(a)\rvert \le 1$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 1$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 1$ for all $a \in \tilde{T} \cup \tilde{F}$
	$\lvert AS(a)\rvert \le 2$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 2$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert \le 2$ for all $a \in \tilde{T} \cup \tilde{F}$
	$\lvert AS(a)\rvert = O(\log(n + \lvert T\rvert + \lvert F\rvert))$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS(a)\rvert = O(\log(n + \lvert T\rvert + \lvert F\rvert))$ for all $a \in \tilde{T} \cup \tilde{F}$	$\lvert AS\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$
	General case	General case	General case
Self-dual	P	P	NPH
	P	NPC	NPH
	P	NPC	P
	P	NPC	NPH
Dual-minor	P	P	NPH
	P	NPC	NPH
	P	NPC	P
	P	NPC	NPH
Dual-major	P	P	NPH
	P	NPC	NPH
	P	NPC	P
	P	NPC	NPH
Positive self-dual	P	P	NPH
	P	P	NPH
	P	P	P
	P	P	NPH
Positive dual-minor	P	P	NPH
	P	P	NPH
	P	P	P
	P	P	NPH
Positive dual-major	P	P	NPH
	P	P	NPH
	P	P	P
	P	P	NPH

P: Polynomial, NPC: NP-complete, NPH: NP-hard

Table 4: Dual-comparable classes.

Function classes	RE	CE	MRE
	$\lvert AS(a)\rvert \le 1$ for all $a \in T \cup F$	$\lvert AS(a)\rvert \le 1$ for all $a \in T \cup F$	$\lvert AS(a)\rvert \le 1$ for all $a \in T \cup F$
	$\lvert AS(a)\rvert \le 2$ for all $a \in T \cup F$	$\lvert AS(a)\rvert \le 2$ for all $a \in T \cup F$	$\lvert AS(a)\rvert \le 2$ for all $a \in T \cup F$
	$\lvert AS\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$	$\lvert AS\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$	$\lvert AS\rvert = O(\log(n + \lvert\tilde{T}\rvert + \lvert\tilde{F}\rvert))$
	General case	General case	General case
$g(S_0, h_1(S_1))$-decomposable	P	NPC	NPH
	P	NPC	NPH
	P	NPC	P
	co-NPC	NPC	NPH
Positive $g(S_0, h_1(S_1))$-decomposable	P	P	NPH
	P	P	NPH
	P	P	P
	P	P	NPH
Renamable Horn	NPC	NPC	NPH
	NPC	NPC	NPH
	NPC	NPC	NPH
	NPH	NPC	NPH
Threshold	P	NPC	NPH
	P	NPC	NPH
	P	NPC	P
	P	NPC	NPH
2-monotonic positive	NPC	NPC	NPH
	NPC	NPC	NPH
	NPC	NPC	NPH
	NPH	NPC	NPH
(Positive) read-once	NPC	NPC	NPH
	NPC	NPC	NPH
	NPC	NPC	NPH
	NPH	NPC	NPH
Unate	NPC	NPC	NPH
	NPC	NPC	NPH
	NPC	NPC	NPH
	NPH	NPC	NPH

P: Polynomial, NPC: NP-complete, NPH: NP-hard, co-NPC: co-NP-complete

Table 5: Other classes.