Adaptive Diversity Maintenance and Convergence Guarantee in Multiobjective Evolutionary Algorithms

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Abstract- The trade-off between obtaining a well-converged and well-distributed set of Pareto optimal solutions, and obtaining them efficiently and automatically is an important issue in multi-objective evolutionary algorithms (MOEAs). Many studies have depicted different approaches that evolutionary algorithms can progress towards the Pareto optimal set with a wide-spread distribution of solutions. However, most mathematically convergent MOEAs desire certain prior knowledge of the solution space in order to efficiently maintain widespread solutions. In this paper, we propose, based on the E-dominance concept, an Adaptive Rectangle Archiving (ARA) strategy that overcomes this practically crucial problem. The MOEAs with this archiving technique provably converge to well-distributed Pareto sets without *a priori*. ARA complements the existing archiving techniques, and is useful to both researchers and practitioners.

1 Introduction

Most real-life optimization problems or decision-making problems are multi-objective in nature, since they normally have several (possibly conflicting) objectives that must be satisfied at the same time. Multi-Objective Evolutionary Algorithms (MOEAs) have been gaining an increasing attention among researchers and practitioners mainly because of the fact that they can be suitably applied to find multiple Pareto-optimal solutions in a single run [2]. This fact alone enables a user to have a less-subjective search in the first phase of finding a set of well-distributed solutions. Because of inherent cooperation among evolutionary search procedure, MOEAs are computationally promising for simultaneous discovery of multiple trade-off solutions. The features has attracted numerous researchers to develop different MOEAs — from MOGA [7], NPGA [9], and NSGA [15] with skillful fitness assignment and nondominated sorting, to SPEA [20], PESA [1], NASA–II [4], SPEA2 [19], IMOEA [16], and DMOEA [18] with elitism, diversity estimation and maintenance; to PAES [11] (based on AGA [10]) and ϵ-MOEA [3] (based on t-dominance [12]) with sound diversity and convergence guarantee.

Despite the great success of these MOEAs, there has been little successful attempt of *convergence-guaranteed* and *computationally efficient* maintenance a *well-distributed Pareto-optimal set* with *little prior knowledge of search space*. Most MOEAs may get widespread solutions using different diversity exploitation mechanisms [1, 4, 9, 17, 18, 20, 19], but little of them have convergence guarantee. Some early theoretical work has pointed out some approaches to enable MOEAs converge to Pareto optimal sets [8, 14], but little consider the distribution of the Pareto solutions obtained [12]. Several recent studies have made a big pace to generate diversified and Pareto optimal solutions [3, 10, 12]. The archiving techniques in [12], as well as [3], desire the distribution knowledge of the Pareto front beforehand. If the parameters are not set appropriately, in some extreme cases, only a solution is archived because it t-dominates all the others [10]. The adaptive grid archiving (AGA) strategy has been proved to converge to a Pareto optimal set of bounded size under certain condition [10]. Unfortunately, this condition is not easily satisfied. The solution oscillation problem has happened in practical applications [10] or been demonstrated empirically [6].

The basic idea of these efficient and successful diversity preserving mechanisms is to partition the whole objective space into mutually excluded regions, and then consider the diversity and Pareto optimality locally in these regions. Each region is of limited volume while the objective space is unknown in advance. This collision makes it difficult to explore the whole solution space, and results in the unexceptional difficulties in the recent work [12, 10]. Our proposal in the work is to introduce some "open" (hyper) rectangles in the space partitioning, such that the coverage of the "infinite" search space only required limited ones. We introduce an extended Pareto dominance (E-dominance) to achieve this point. In addition, the search space partition is adjusted adaptively based on the solutions found so far. The "crucial" regions will maintain more Pareto optimal solutions, while the "open" rectangles have choices to keep some accidental solutions. Therefore, our Adaptive Rectangle Archiving (ARA) technique can explore the whole search space and maintain the representative solutions automatically without any prior knowledge.

In the rest of the paper, we give the general MOEAs with archiving techniques. Then, we shortly review the existing MOEAs and discuss why they, if no prior knowledge available, do not have sound convergence and diversity guarantee. In section 4, E-dominance concept and E-Pareto set are introduced, and their corresponding archiving strategy, ARA, is established. The theoretical analysis of ARA, both iteration-based and infinite treads, is given. In section 6, conclusive comments and possible future research are discussed.

Procedure 1. ***MOEA with Archiving***

1. $t := 0$, $A^{(0)} := \emptyset$;
2. ***Repeat***:
3. $\quad t := t + 1$;
4. $\quad \mathbf{y}^{(t)} :=$ EVOLUTION(); /* *Generates a new solution* */
5. $\quad A^{(t)} :=$ ARCHIVE$(A^{(t-1)}, \mathbf{y}^{(t)})$; /* ***Updates_archive*** */
6. ***Termination***: *Until stopping criterion fulfilled;*
7. ***Output***: $A^{(t)}$, *t*.

2 Preliminaries

We assume, without loss of generality, minimization multiobjective problems in this work. For a multiobjective function Γ from $X(\subset \Re^d)$ to a finite set $Y \subset \Re^m (m \geq 2)$, an objective vector $\mathbf{y}^{(1)} = [y_1^{(1)}, y_2^{(1)}, \cdots, y_m^{(1)}]^T \in Y$ *dominates* another one $\mathbf{y}^{(2)} = [y_1^{(2)}, \cdots, y_m^{(2)}]^T$, if and only if

$$\begin{cases} y_i^{(1)} \leq y_i^{(2)}, & \forall i \in \{1, \cdots, m\} \\ y_j^{(1)} < y_j^{(2)}, & \exists j \in \{1, \cdots, m\}. \end{cases} \tag{1}$$

We also denote it as $\mathbf{y}^{(1)} \prec \mathbf{y}^{(2)}$. For convenience, we also denote $\neg\left(\mathbf{y}^{(1)} \prec \mathbf{y}^{(2)}\right)$ as $\mathbf{y}^{(1)} \not\prec \mathbf{y}^{(2)}$. $\mathbf{y}^{(1)}$ is said to be *nondominated* (or *incomparable*) with $\mathbf{y}^{(2)}$ if $\neg(\mathbf{y}^{(1)} \prec \mathbf{y}^{(2)} \vee \mathbf{y}^{(2)} \prec \mathbf{y}^{(1)} \vee \mathbf{y}^{(1)} = \mathbf{y}^{(2)})$. It is denoted as $\mathbf{y}^{(1)} \sim \mathbf{y}^{(2)}$. Therefore, $\mathbf{y}^{(1)} \not\prec \mathbf{y}^{(2)}$ means $\mathbf{y}^{(1)} = \mathbf{y}^{(2)}$ or $\mathbf{y}^{(2)} \prec \mathbf{y}^{(1)}$ or $\mathbf{y}^{(1)} \sim \mathbf{y}^{(2)}$.

Likewise, the *dominates* and *nondominated* relations can be defined between a vector $\mathbf{y}(\in Y)$ and a set $A(\subseteq Y)$:

$$\mathbf{y} \prec A \iff \exists \mathbf{a} \in A, \quad \mathbf{y} \prec \mathbf{a} \tag{2}$$

$$A \prec \mathbf{y} \iff \exists \mathbf{a} \in A, \quad \mathbf{a} \prec \mathbf{y} \tag{3}$$

$$\mathbf{y} \sim A \iff \forall \mathbf{a} \in A, \quad \mathbf{y} \sim \mathbf{a} \tag{4}$$

$$\mathbf{y} \not\prec A \iff \forall \mathbf{a} \in A, \quad \mathbf{y} \not\prec \mathbf{a}. \tag{5}$$

Given the set of vectors Y, its *Pareto set* Y^* contains all vectors $\mathbf{y}^* \in Y$ that are not dominated by any vector $\mathbf{y} \in Y$. That is, $Y^* = \{\mathbf{y}^* \in Y | \nexists \mathbf{y} \in Y, \mathbf{y} \prec \mathbf{y}^*\}$, which is also known as the *Pareto front*. Each $\mathbf{y}^* \in Y^*$ is called *Pareto optimal*, or *a nondominated solution*. A Pareto optimal solution reaches a good tradeoff among these conflicting objectives: one objective cannot be improved without worsening any other objectives. In this paper, we assume, valid to almost all multiobjective problems, at least two different values for each objective in Y^*.

For many multiobjective optimization problems, the unique Pareto set Y^* is of substantial size. Thus, the determination of Y^* is computationally prohibitive, and Y^* as a result of an optimization is not easy to maintain and is questionable [6, 12]. Furthermore, the value of presenting such a large set of solutions to a decision maker is doubtful in the context of decision support, instead one should provide him with a set of the best representative solutions. Finally, in limiting the size of solution set, preference information could be used to steer the process to certain parts of the search space. Therefore, all practical implementations of MOEAs have maintained (off-line) an archive of best (nondominated) solutions found so far, and the archive is of bounded size [10].

In order to facilitate our analysis on archiving strategies, we separate the evolutionary procedure and the archiving procedure. Procedure 1 gives an abstract description of a general MOEA with archive. The integer t denotes the iteration count, the m-dimensional vector $\mathbf{y} \in Y$ is the solution generated at iteration t, and the set $A^{(t)}$ will be called the archive at iteration t and should contain a representative subset of the points in the objective space Y. The function EVOLUTION represents an evolutionary algorithm, where the evolutionary operator is associated with variation (recombination, mutation, and selection). It can generate a population of points, possibly using the contents of the old archive set $A^{(t-1)}$. However, for convenience, it only outputs a new solution in each iteration t. The function ARCHIVE gets the new solution $y^{(t)}$ and the old archive set $A^{(t-1)}$ and determines the updated one $A^{(t)}$. Its purpose is to gather useful information about the underlying search problem during the run. The use of archive is usually twofold: on one hand, it is used to store the best solutions found so far; on the other hand, the search operator exploits this information to steer the search to promising regions.

This paper mainly deals with the function ARCHIVE, i.e., how to appropriately update the archive. For each $\mathbf{y}$, its additional information about the corresponding decision values could be associated to the archive but will be of no concern in this paper. We also refer to an objective vector as a solution. According to requirements of MOEAs, an ideal archive strategy should maintain solutions having the following properties:

Pareto optimal: Converge to Pareto optimal solutions in each run;

Well distributed: Solutions are uniformly distributed in the whole objective space;

Computational Efficiency: The time and memory complexity should be low;

Little Prior Knowledge: Little knowledge about the multiobjective problem is desired beforehand.

This last property may facilitate users greatly, since most of time, we have to make decisions on some conflicting problems with little prior knowledge. As mentioned above, some information about the objective space has been stored in the archive during search. Thus, we may adjust our archive adaptively without any prior information. We will construct such an algorithm in Section 4, after the discussion of the existing approaches in next section.

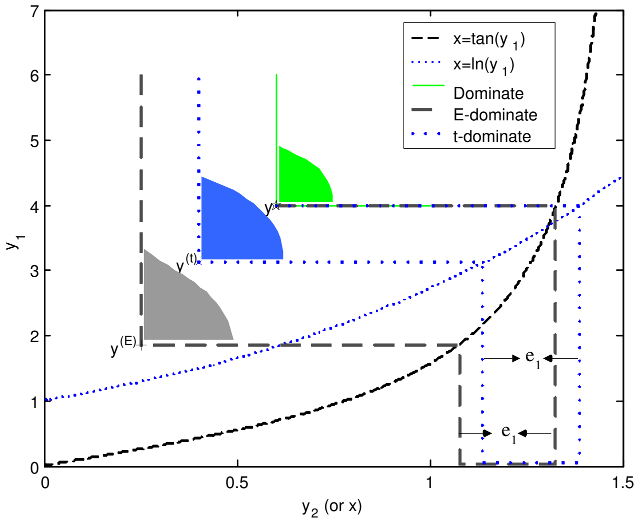

Figure 1: Illustration of E-dominance, t-dominance, and Pareto dominance. The regions dominated by $\mathbf{y}$ under three different dominance relations are visualized. The calculation of vectors $\mathbf{y}^E$ and $\mathbf{y}^t$ is illustrated in bottom right corner.

3 MOEAs and their Limitations

We discuss a number of archiving or elitism strategies that appeared in the literature of MOEAs.

Early theoretical work of MOEAs mainly concentrates on convergence. Hanne [8] gave a convergence proof for a $(\mu + \lambda)$-MOEA with Gaussian mutation distributions over a compact real search space by the application of a (negative) efficiency preservation selection scheme, which only accepting new solutions that dominate at least one of the archived solutions. There is no assumption on the distribution of solutions, and arbitrary regions may become unreachable with (negative) efficiency preservation [12]. Rudolph and Agapie [14], using stochastic process techniques, developed several sophisticated selection operators to precludes the problem of deterioration. Their algorithms with evolutionary operators having a positive transition probability matrix provably converge to the Pareto optimal set, but they do not guarantee a good distribution of the solutions archived.

A number of elitist MOEAs have been developed to especially address diversity of the archived solutions using different mechanisms. The diversity exploitation mechanisms include mating restriction, fitness sharing (NPGA [9]), clustering (SPEA [20], SPEA2 [19]), nearest neighbor distance (NAGA-II [4]), crowding count (DMOEA [18], PESA [1]), or some preselection operators [2]. Most of them have been more or less successful, but little of them have convergence guarantee.

Recently, Laumanns *et al.* [12] proposed several archiving strategies that guarantee both progress towards the Pareto front and covering the whole range of nondominated solutions. The algorithms maintain a finite-sized archive of nondominated solutions that is iteratively updated in the presence of a new solution based on the concept of t-dominance. However, the t value, which determines solution resolution, must either be pre-set or be determined adaptively. In the former case, the size of the archive is bounded only by some function of the objective space ranges, which usually unknown in advance. Whereas in the latter case, t may become arbitrarily large and so finally, only a poor representation of the sequence of solutions presented to the archive are stored. In some extreme cases, only one solution is archived since it t-dominates all other Pareto solutions [10].

More recently, Knowles and Corne [10] has analyzed a metric-based archiving and an adaptive grid archiving one. The metric-based strategy requires S-metric which assigns a scalar value to each possible approximation set reflecting its quality and fulfilling certain monotonicity conditions. Convergence is then defined as the achievement of a local optimum of the quality function. However, its computational overhead is prohibitively high for more than a few objectives. The adaptive grid archiving strategy implemented in PAES [10] provably maintains solutions in some "critical" hyperboxes of the Pareto set once they have been found. The strategy is provably convergent when the Pareto optimal set spans the feasible objective space in all objectives. This condition is not true for many optimization problems with more than two objectives. Thus, the oscillation problem of the archive has happened in practical applications [10] or been demonstrated empirically [6].

In order to diversify the solutions, the density estimation or diversity preservation has been locally made in some boxes for computational efficiency. However, the objective space is unknown in advance, and it is sometimes impractical to use restricted volumed boxes to envelop it appropriately. This point results in oscillation of AGA [10] and probably poor representation of Pareto set in [12], though they can generate widespread solutions.

4 Adaptive Rectangle Archiving Strategy

In this section, we present an Adaptive Rectangle Archiving (ARA) algorithm that address some of problems with the existing ones. In ARA, we use a self-adaption mechanism to preserve diversity according to the archived information about the objective space. In the "crucial" regions, a solution is allowed to preserved in a narrow (hyper-)rectangle, and then more Pareto solutions are archived. In the unknown, even infinite, regions, some "open" rectangles are used to envelope. This "open" rectangle even may envelop the infinite objective values. Within this "open" rectangles, some Pareto solutions have choices to be archived. The "open" rectangles are specified based on an extended Pareto dominance concept. We define these terminology below, followed by the ARA algorithm.

4.1 Extended Pareto Dominance

Since we need to use an archive of points to approximately dominate the whole objective space, one intuitive solution is allowing some tolerance on dominance. We extend the Pareto dominance to reach this as follows.

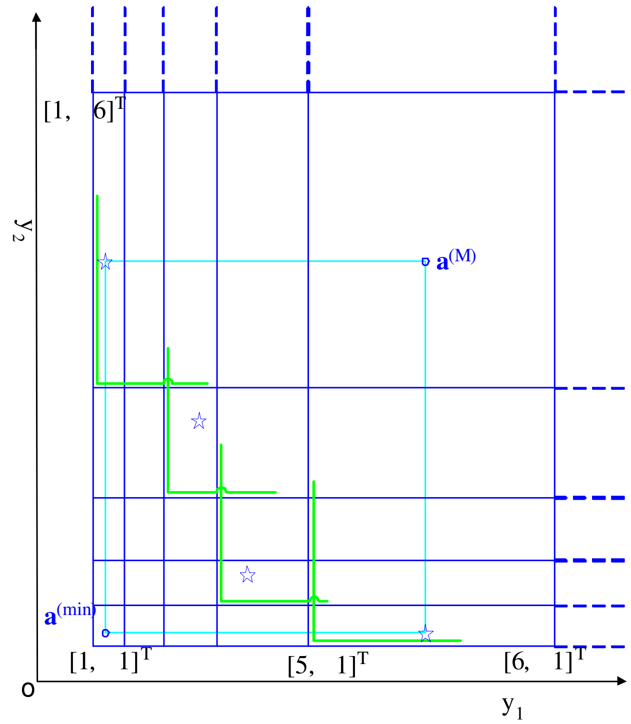

Figure 2: 2-D Adaptive Rectangle Partition. The dashed line segments indicate “open” rectangles. The gray rectangle indicates the crucial region indicated by $\mathbf{a}^{(min)}$ and $\mathbf{a}^{(M)}$. The bold line segments indicate the region E-dominated by a solution indicated by a pentagram.

Definition 1 (E-dominance). *Let $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ be in Y. Then $\mathbf{y}^{(1)}$ is said to* E-dominate *$\mathbf{y}^{(2)}$ for some transferring function,* FUN, *and a constant vector $\mathbf{e}(>0)$, if and only if for all $i \in \{1, \ldots, m\}$*

$$\text{FUN}(y_i^{(1)}) - e_i \leq \text{FUN}(y_i^{(2)}). \tag{6}$$

It is denoted as $\mathbf{y}^{(1)} \preceq_E \mathbf{y}^{(2)}$.

Then transferring function has better be continuous, and monotonously increase. This ensures that E-dominance may be implied by the traditional dominance, i.e., if $\mathbf{y}^{(1)} \prec y^{(2)}$, then $\mathbf{y}^{(1)} \preceq_E y^{(2)}$. Furthermore, it is easy to see that E-dominance relation is transitive.

E-dominance generalizes several existing dominance relations. It becomes t-dominance in [12] as $\text{FUN}(y_i) = \ln(y_i)$ and $e_i = \ln(1+t)$, the additive t-dominance in [12, 13] as $\text{FUN}(y_i) = y_i$ and $e_i = t$, and the Pareto dominance as $\text{FUN}(y_i) = y_i$ and $e_i = 0$.

In order to envelop unknown, possible infinite, objective value, we may employ a nonlinear transferring function, e.g., $\text{FUN}(y_i) = \tan(y_i * scale_i)$. Thus, the infinite point is transferred to $\frac{\pi}{2}$, and may be E-dominated by a limited value, say, $\frac{\arctan(\frac{\pi}{2} - e_i)}{scale_i}$. In the following discussion, we assume E-dominance with such a $\tan$ function. The $scale_i$ will be specified adaptively according to the solutions found so far. The comparison among E-dominance, t-dominance, and Pareto dominance is visualized in Fig. 1. Based on E-dominance relation, we have following definitions.

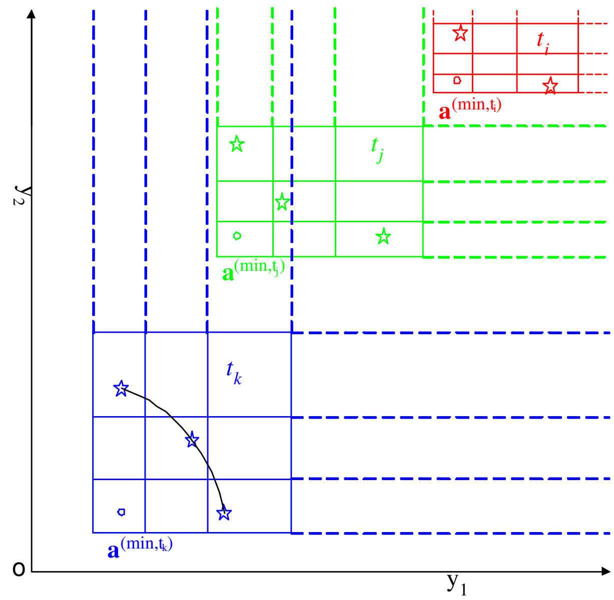

Figure 3: How the adaptive rectangle changes its location and shape in objective space as the vectors in the archive $A^{(t)}$ change through iterations $t_i < t_j < t_k$. The bold line indicates the Pareto front, and pentagrams denotes archived solutions.

Definition 2 (E-approximate Pareto Set). *Let $Y \subseteq \Re^m$ be a set of vectors,* FUN *a monotonically increasing function, and constant vector $e > 0$. Then a set Y_E is called an* E-approximate Pareto set *of Y, if any vector $\mathbf{y} \in Y$ is E-dominated by at least one vector $\mathbf{a} \in Y_E$, i.e.,*

$$\forall \mathbf{y} \in Y : \exists \mathbf{a} \in Y_E \text{ such that } \mathbf{a} \preceq_E \mathbf{y}. \tag{7}$$

The set of all E-approximate Pareto sets of Y is denoted as $EP(Y)$.

Definition 3 (E-Pareto Set). *Let $Y \subseteq \Re^m$ be a set of vectors, and a vector $e > 0$. Then a set $Y_E^* \subseteq Y$ is called a* E-Pareto set *of Y, if*

1. *Y_E^* is an E-approximate Pareto set of Y, i.e., $Y_E^* \in EP(Y)$, and*

2. *Y_E^* only contains Pareto optimal points of Y, i.e., $Y_E^* \subseteq Y^*$*

The set of all E-Pareto sets of Y is denotes as $EP^*(Y)$.

Since finding the Pareto set of an arbitrary set Y is usually not practical because of its size, one needs to be less ambitious in general. Therefore, the E-approximate Pareto set is a practical solution concept as it not only represents all vectors Y but also consists of a smaller number of elements. Of course, a E-Pareto set is more attractive as it consists of Pareto vectors only.

4.2 Archiving Procedure

Our adaptive archiving strategy basically have two concerns. First, we determine the “crucial” solution region adaptively. The second is to find an E-Pareto set based on

Procedure 2. ARA$(\mathbf{y}, A)$ /* *Adaptive Rectangle Archiving* */

1. ***if*** $(\mathbf{y} \prec A^{(min)} \vee \mathbf{a}^{(min)} \nprec \mathbf{y})$ ***then***
2. ***for all*** $i \in \{1, \cdots, m\}$ ***do***
3. ***if*** $(a_i^{(i)} > y_i)$ ***then***
4. $\mathbf{a}^{(i)} := \mathbf{y}$; /* ***Recedes*** */
5. ***else if*** $(\mathbf{y} \prec \mathbf{a}^{(i)})$
6. $\mathbf{a}^{(i)} := \mathbf{y}$; /* ***Dominates*** */
7. ***end if***
8. ***end do***
9. $A^{(ar_2)} := \emptyset$; /* ***Re-forms*** $A^{(ar)}$ */
10. ***for all*** $((\mathbf{a} \in A^{(ar)}) \wedge (A^{(min)} \nprec \mathbf{a}))$ ***do***
11. InsertInRectangles$\big(\mathbf{a}, A^{(ar_2)}, A^{(min)}\big)$;
12. ***end do***
13. $A^{(ar)} := A^{(ar_2)}$;
14. ***else if*** $(A^{(min)} \nprec \mathbf{y})$ /* ***Updates*** $A^{(ar)}$ */
15. InsertInRectangles$\big(\mathbf{y}, A^{(ar)}, A^{(min)}\big)$;
16. ***end if***
17. $A := \{A^{(min)}, A^{(ar)}\}$;

Procedure 3. InsertInRectangles$(\mathbf{y}, A^{(ar)}, A^{(min)})$

1. $D := \{\mathbf{a} \in A^{(ar)} | \text{Rect}(\mathbf{a}, A^{(min)}) \prec \text{Rect}(\mathbf{a}, A^{(min)})\}$;
2. ***if*** $D \neq \emptyset$ ***then***
3. $A^{(ar)} := A^{(ar)} \bigcup \mathbf{y} \backslash D$; /* ***Inter_Rect_Dom*** */
4. ***else if*** $\exists \mathbf{a} \in A^{(ar)} : (\text{Rect}(\mathbf{a}, A^{(min)}) = \text{Rect}(\mathbf{y}, A^{(min)})) \wedge (\mathbf{y} \prec \mathbf{a})$ ***then***
5. $A^{(ar)} := A^{(ar)} \bigcup \{\mathbf{y}\} \backslash \{\mathbf{a}\}$ /* ***Intra_Rect_Dom*** */
6. ***else if*** $\forall \mathbf{a} \in A^{(ar)} : \text{Rect}(\mathbf{a}, A^{(min)}) \sim \text{Rect}(\mathbf{y}, A^{(min)})$
7. $A^{(ar)} := A^{(ar)} \bigcup \{\mathbf{y}\}$ /* ***Occupies*** *a rectangle* */
8. ***else***
9. $A^{(ar)} := A^{(ar)}$; /* ***Steady_state*** */
10. ***end if***

the E-dominance. For the sake of description, we partition the archive in ARA into two parts : $A = \{A^{(min)}, A^{(ar)}\}$ $(A^{(t)} = A^{(min,t)}, A^{(ar,t)})$. The purpose of $A^{(ar)}$ is to maintain an E-Pareto set according to the solution space information collected in $A^{(min)}$. $A^{(min)}$ is an array, $A^{(min)} = [\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \cdots, \mathbf{a}^{(m)}]$. Each member is initialized to be infinite, and has the minimal objective value found so far, i.e., $\mathbf{a}_i^{(i)} = \min_{\mathbf{a} \in A}\{a_i\}$. Furthermore, we introduce two vectors associated with $A^{(min)}$ to describe the "crucial" region: $\mathbf{a}^{(min)}$ with $a_i^{(min)} = \min_{\mathbf{a} \in A}\{a_i\}$ and $\mathbf{a}^{(M)}$ with $\mathbf{a}_i^{(M)} = \max_{\mathbf{a} \in A^{(min)}}\{a_i\}$. The region whose member dominates $\mathbf{a}^{(M)}$ but is dominated by $\mathbf{a}^{(min)}$ contains most Pareto optimal solutions generated so far, and it is *crucial* for archiving. For example, all solutions dominated by $\mathbf{a}^{(M)}$ are not Pareto optimal. Especially, all Pareto optimal solutions are located in the crucial region in 2-D case. The gray rectangle in Fig. 2 indicates the crucial region, and envelops all four Pareto solutions, indicated by pentagram.

The pseudo code of ARA is given in Procedure 2. At east step, the algorithm first check whether the crucial region should be updated. If the new objective value is less than the archived one, **Recedes** will accept the new vector; If the new vector dominates a vector in $A^{(min)}$, **Dominates** will replace the old vector with the new one. If the crucial region is updated, the solutions in $A^{(ar)}$ have to be archived again (**Re-forms** $A^{(ar)}$). Thus, the less minimal objective value is certainly archived, and the solutions in $A^{(ar)}$ are chosen based on the current $A^{(min)}$.

There is little choice that, when the condition in Line 1 of Procedure 2 holds, neither **Recedes** nor **Dominates** is executed. In fact, the unique exception is that $(\mathbf{y} = \mathbf{a}^{(min)}) \wedge (\mathbf{a}^{(i)} = \mathbf{y} \forall i)$. However, the exception is rare, since multiobjective problems normally have more than one solution.

If the new vector $\mathbf{y}$ neither has less objective values nor dominates any vector in $A^{(min)}$, it is input into the procedure InsertInRectangle, as described in Procedure 3. The procedure mainly chooses representative Pareto optimal solutions according to the crucial region specified by $A^{(min)}$. It has a two level concept. On the coarse level, the search space is discretized by a division into (hyper-)rectangles (see Function 4), where each vector uniquely belongs to one rectangle. Using the proposed E-dominance relation on these rectangles, the algorithm always maintains a set of nondominated rectangles (**Inter_Rect_Dom** and **Occupies**), thus guaranteeing the E-approximation property. On the fine level, at most one element is kept in each rectangle. Within a rectangle, each representative vector can only be replaced by a dominating one (**Intra_Rect_Dom**) (similar to t-Pareto set algorithm in [12]), thus guaranteeing convergence.

Now let us see how the function Rect, given in Function 4, partition the crucial region finely while place the unknown regions into "open" rectangles. Since it is difficult to automatically detect the maximal objective values in the Pareto front [10], we simply view it as infinite. As shown in Lines 2∼4 in Rect, $a_i^{(min)}$ and $+\infty$ are mapped into 1 and $\left\lfloor \frac{\frac{\pi}{2}}{e_i} + 1.5 \right\rfloor$. The scale calculated in Line 2 reflects the distance between $a_i^{(M)}$ and $a_i^{(min)}$. The farther away $a_i^{(M)}$ is

Function 4. RECT($\mathbf{y}$, $A^{(min)}$)

1. ***for all*** $i \in \{1, \cdots, m\}$ ***do***
2. $scale_i = \frac{\arctan\left(\frac{\pi}{2} - e_i\right)}{a_i^{(M)} - a_i^{(min)}}$;
3. $\alpha_i := \tan\left(\left(y_i - a_i^{(min)}\right) * scale_i\right)$;
4. $r_i := \left\lfloor \frac{\alpha_i}{e_i} + 1.5 \right\rfloor$;
5. ***end do***
6. ***output****: return* $\mathbf{r} = [r_1, \cdots, r_m]^T$.

from $a_i^{(min)}$, the larger the scale value is. Furthermore, this scale, together with the value 1.5 in Line 4, enable $a_i^{(M)}$ to be mapped into $\left(\left\lfloor \frac{\frac{\pi}{2}}{e_i} + 1.5 \right\rfloor - 1\right)$, which is next to that corresponding to $+\infty$. Therefore, if $e_i < \frac{\pi}{4}$, $a_i^{(min)}$, $a_i^{(M)}$, and $+\infty$ are located in different rectangles. Furthermore, there are $\left(\left\lfloor \frac{\frac{\pi}{2}}{e_i} + 1.5 \right\rfloor - 2\right)$ rectangles between $a_i^{(min)}$ and $a_i^{(M)}$. Less e_i is, more finely the crucial region is partitioned. An example with $e_i = \frac{\pi}{10}$ is illustrated in Fig. 2. The unknown region is enveloped by some "open" rectangles, as indicated by dashed line segments. Clearly, the crucial region is finely divided, and more Pareto optimal solutions will be archived in $A^{(ar)}$.

5 Convergence analysis

We now give some theorems to support that our archiving strategy converges to the Pareto set while preserving diversity of solution vectors at the same time. We first give theoretical analysis on each iteration of Procedures 2 and 3.

We first show that the lower boundaries of archive $A^{(t)}$, i.e., $\mathbf{a}^{(min,t)}$, can retain the minimal objective values generated so far.

Theorem 1. *Let* $Y^{(\tau)} = \bigcup_{t=1}^{\tau} \{\mathbf{y}^{(t)}\}$ *be the set of objective vectors created in* EVOLUTION. *Then the archive* $A^{(\tau)}$ *contains the minimal objective values of* $Y^{(\tau)}$. *That is,* $a_i^{(min,\tau)} = \min_{t=1}^{\tau}\{y_i^{(t)}\}$.

Proof. We need to prove two cases: *Case 1*. the minimal objective values generated-so-far will enter the archive; *Case 2*. the objective vectors with the minimal objective values will not lose.

Case 1. To prove this point, we only to prove $a_i^{(min,t)} = y_i^{(t)}$ when a less objective value is generated for some $i \in \{1, \ldots, m\}$ and $t < \tau$, i.e, when $y_i^{(t)} < a_i^{(min,t-1)}$. We have $\mathbf{y}^t \prec \mathbf{a}^{(min,t-1)}$ or $\mathbf{y}^t \sim \mathbf{a}^{(min,t-1)}$. Since $a_i^{(min,t-1)} = a_i^{(i,t-1)}$), we have either $\mathbf{y}^t \prec \mathbf{a}^{(i,t-1)}$ or $\mathbf{y}^t \sim \mathbf{a}^{(i,t-1)}$. For the former, $\mathbf{y}^t \prec A^{(min,t-1)}$ and the rule, **Dominates**, will execute. For the latter, if $\mathbf{y}^{(t)} \prec \mathbf{a}^{(min,t-1)}$, then $\mathbf{y}^{(t)} \prec \mathbf{a}^{(min,t-1)} \prec \mathbf{a}^{(i,t-1)}$. It contradicts $\mathbf{y}^t \sim \mathbf{a}^{(i,t-1)}$. So, $\mathbf{y}^t \sim \mathbf{a}^{(min,t-1)}$, and **Recedes** executes. For both, $\mathbf{a}^{(i,t)} = \mathbf{y}^{(t)}$. Thus, $a_i^{(min,t)} = y_i^{(t)}$.

Case 2. We have to prove $\mathbf{a}_i^{(min,t)} = \mathbf{a}_i^{(min,t-1)}$ if $\mathbf{a}_i^{(min,t-1)} < y_i^{(t)}$. Since $\mathbf{a}_i^{(min,t-1)} < y_i^{(t)}$, we know $\mathbf{a}_i^{(i,t-1)} < y_i^{(t)}$ and $\mathbf{y}^{(t)} \not\prec \mathbf{a}^{(i,t)}$. So, both **Dominates** and **Recedes** will not execute, and $\mathbf{a}^{(i,t)} = \mathbf{a}^{(i,t-1)}$. □

As described in Procedures 2 and 3, one solution dominated by archived solutions is impossible to enter the archive. Furthermore, as required in Lines 10 and 14, if a solution dominates $A^{(min)}$, it cannot enter $A^{(ar)}$. In addition, since solutions in $A^{(min)}$ must have one minimal objective value generated so far, the solution in $A^{(ar)}$ also cannot dominate $A^{(min)}$. Thus, we have the following nondominated relations among between the solutions in the archive.

Lemma 1. *Members in* $A^{(t)}$ *are either nondominated or equal to one another, i.e.,* $\forall \mathbf{a}^0, \mathbf{a}^1 \in A^{(t)}$, $(\mathbf{a}^0 \sim \mathbf{a}^1) \vee (\mathbf{a}^0 = \mathbf{a}^1)$.

Similar with Theorem 1, $A^{(ar,t)}$ also collects the Pareto solutions iteratively, as stated in the following theorem.

Theorem 2. *The archive* $A^{(ar,\tau)} (\neq \emptyset)$ *is an E-Pareto set of* $Y^{(\tau_0,\tau)} = \bigcup_{t=\tau_0}^{\tau} \{\mathbf{y}^{(t)}\} \bigcup A^{(ar,\tau_0)}$ *if* $A^{(min,\tau_0)} = A^{(min,\tau)}$.

Due to the space limitation, we only sketch the proof. When $A^{(min,t)}$ does not change, the generated solutions are input to INSERTINRECTANGLE. Any solution $\mathbf{y}$ must be E-dominated by $A^{(ar,t)}$, or enters the archive. Once archived, it will not be deleted only if it is replaced by a new one that E-dominated it. The solutions E-dominated by $\mathbf{y}$ are transitively E-dominated by the new one. So, $A^{(ar,t)}$ still E-dominates these solutions. Similarly, if an archived solution $\mathbf{a}$ is not Pareto optimal, it always be replaced by a Pareto optimal one by execute **Inter_Rect_Dom** or **Intra_Rect_Dom**. So, $A^{(ar,t)}$ must be an E-Pareto set of the solutions input into INSERTINRECTANGLE.

Proof. First, we consider an extreme situation: $\mathbf{a}^{(1,t)} = \ldots = \mathbf{a}^{(m,t)} = \mathbf{a}^{(min,t)}$. At this time, $A^{(ar,t)}$ can only contain $\mathbf{y} = \mathbf{a}^{(min,t)}$. Otherwise, if $\mathbf{y}(\in A^{(ar,t)}) \neq \mathbf{a}^{(min,t)}$, then $(\mathbf{y} \sim \mathbf{a}^{(min,t)})$. The condition in Line 1 satisfies, **Updates** $A^{(ar)}$ will not execute, and $\mathbf{y}$ cannot enter $A^{(ar,t)}$, which contradicts $\mathbf{y} \in A^{(ar,t)}$. According to Theorem 1, $\mathbf{a}^{(min,t)}$ dominates all solutions generated-so-far. So, $A^{(ar,t)}$ dominates, then E-dominates, the solutions generated-so-far. The theorem is correct.

Except the extreme case above, we have, if neither (**Dominates** nor **Recedes**) execute, **Update** $A^{(ar)}$ must execute. Since $A^{(min,\tau_0)} = A^{(min,\tau)}$, there is no change in $A^{(min,t)}$ for $t = \{\tau_0, \ldots, \tau\}$. That is, neither **Dominates** nor **Recedes** execute. So, for all $\mathbf{y}^{(t)}(t = \{\tau_0, \ldots, \tau\})$, Procedure 3 must be called.

Note that each update on $A^{(min,t)}$ (**Dominates** or **Recedes**), $A^{(ar,t)}$ will be updated using Procedure 3 with the new $A^{(min,t)}$ (**Re-forms** $A^{(ar)}$). So we can only consider Procedure 3 with $A^{(ar,\tau_0)} = \emptyset$.

If the conclusion is not correct, i.e., $A^{(ar,\tau)} \in EP^*(Y^{(\tau_0,\tau)})$ is not true, for some t. According to Definition 3, this occurs only if some $\mathbf{y}^{(\tau)}(\tau < t)$ is not E-dominated by any member of $Y^{(\tau_0,\tau)}$ and not in $A^{(ar,\tau)}$ (*Case 1*), or in $A^{(ar,t)}$ but not in the Pareto set of $Y^{(\tau_0,\tau)}$ (*Case 2*).

Case 1. For $\mathbf{y}^{(\tau)}$ not being in $A^{(ar,\tau)}$, it can either have been rejected at $t = \tau$ or accepted at $t = \tau$ and removed later on. However, $\mathbf{y}^{(\tau)}$ will only be rejected if there is another $\mathbf{y}^{(0)} \in A^{(ar,t)}$ with $\text{RECT}(\mathbf{y}^{(0)}, A^{(min,\tau)}) \prec \text{RECT}(\mathbf{y}^{(\tau)}, A^{(min,\tau)})$ or the same rectangle value and that is not dominated by $\mathbf{y}^{(\tau)}$ (**Steady_state**). Since both relations are transitive, and since they both imply E-dominance, $\mathbf{y}^{(0)}$, E-dominates $\mathbf{y}^{(\tau)}$, and can only be replaced by accepting elements that also E-dominate $\mathbf{y}^{(\tau)}$. There will always be an element in $A^{(ar,t)}$ that E-dominates $\mathbf{y}^{(\tau)}$, which contradicts the assumption. On the other hand, removal only takes place when some new $\mathbf{y}$ enters $A^{(ar,t)}$, which dominates $\mathbf{y}^{(\tau)}$ (**Intra_Rect_Dom**) or whose RECT value dominates that of $\mathbf{y}^{(\tau)}$ (**Inter_Rect_Dom**). Since both relations are transitive, and since they both imply E-dominance, there will always be an element in $A^{(ar,t)}$ that E-dominates $\mathbf{y}^{(\tau)}$, which contradicts the assumption.

Case 2. Since $\mathbf{y}^{(\tau)}$ is not in the Pareto set of $Y^{(\tau_0,\tau)}$, there exists $\mathbf{y}^{(\tau_1)}(\tau_0 \leq \tau_1 \neq \tau \leq \tau)$ in the Pareto set of $Y^{(\tau_0,\tau)}$ with $\mathbf{y}^{(\tau_1)} \prec \mathbf{y}^{(\tau)}$. This implies $\text{RECT}(\mathbf{y}^{(\tau_1)}, A^{(min,\tau)}) \prec \text{RECT}(\mathbf{y}^{(\tau)}, A^{(min,\tau)})$ or $\text{RECT}(\mathbf{y}^{(\tau_1)}, A^{(min,\tau)}) = \text{RECT}(\mathbf{y}^{(\tau)}, A^{(min,\tau)})$. Hence, if $\tau_1 < \tau$, $\mathbf{y}^{(\tau)}$ would not have been accepted (**Steady_state**). If $\tau_1 > \tau$, $\mathbf{y}^{(\tau)}$ would have been removed from $A^{(ar,\tau_1)}$ (**Inter_Rect_Dom** or **Intra_Rect_Dom**). Thus, $\mathbf{y}^{(\tau)}$ is not in $A^{(ar,\tau)}$, which contradicts the assumption. This completes the proof. □

Theorems 1 and 2 states that, in ARA, the archive retains the lowest objective values and the E-Pareto solutions of the objective vectors generated so far. The archive retains some best-so-far solutions, and this point allows the MOEA with the proposed archiving technique stops anytime. Using these properties for iterations, we give the convergence results based on an assumption of the EVOLUTION, that the archiving algorithm may reach the crucial region of the whole objective space, and then approximately dominate it.

Theorem 3. *If the* EVOLUTION *procedure gives every possible solution in the search space with a positive minimum probability, then*

1. *the lower boundaries of archive $A^{(t)}$, $\mathbf{a}^{(min,t)}$, converge to the global minimal objective values,*

2. *$A^{(min,t)}$ converges to a Pareto optimal set*

$$\{\mathbf{a} \in Y | a_i = \min_{\mathbf{y^0} \in Y} \{y_i^0\} \wedge (\mathbf{a} \prec \mathbf{y} \wedge y_i = a_i) \text{ for an } i\} \quad (8)$$

with probability 1 as $t \to \infty$.

Proof. 1. Since the EVOLUTION can generate every possible solution with a positive minimum probability, according to the Borel-Cantelli Lemma (see e.g., [5, p. 201]), it is guaranteed that arbitrary solution will be generated infinitely often and that the waiting time for the first occurrence as well as for the second, and so forth will be finite with probability 1. Thus, there exists $t_i < +\infty$ such that $y_i^{(t_i)} = \min_{\mathbf{y} \in Y} \{y_i\}$. According to Lemma 1, $a_i^{(min,t)} = \min_{\mathbf{y} \in Y} \{y_i\}$ for all $t > t_i$. Therefore, when $t > \tau_{c_1} \triangleq \max_{i=1,\ldots,m} \{t_i\}$, each element of $\mathbf{a}^{(min,t)}$ reaches the minimal objective value and will not change.

2. If $A^{(min,t)}(t > \tau_{c_1})$ is not Pareto optimal in Y, there must exist $\mathbf{y}^*(\in Y^*)$ such that $\mathbf{y}^* \prec \mathbf{a}^{(i,t)} \wedge y_i^* = \min_{\mathbf{y^0} \in Y} \{y_i^0\}$ for some i. There exists $t_{i_2}(> t)$ such that $\mathbf{y}^{(t_{i_2})} = \mathbf{y}^*$. $\mathbf{y}^{(t_{i_2})} \prec \mathbf{a}^{(i,t_{i_2}-1)}$, then $\mathbf{y}^{(t_{i_2})} \prec A^{(t_{i_2}-1)}$. Thus, **Dominates** executes, and $\mathbf{a}^{(i,t_{i_2}-1)}$ is replaced by $\mathbf{y}^{(t_{i_2})}$. Once $t > \tau_{c_2} \triangleq \max_{i=\{1,\ldots,m\}} \{t_{i_2}\}$, $A^{(min,t)}$ reaches a set as given in Eq.(8).

Once $A^{(min,t)}$, as in Eq.(8), is Pareto optimal in Y and each member at least has a minimal objective value, there is not a vector $\mathbf{y}$ that either dominates $A^{(min,t)}$ or $y_i < a_i^{(i,t)}$. The condition in Line 1 of Procedure 2 is not satisfied. Neither **Dominates** nor **Recedes** will execute. Therefore, $A^{(min,t)}$ will not change. This completes the proof. □

The assumption about EVOLUTION is quite common in theoretical analysis of evolutionary computation [10, 14]. It is true whenever, for example, a mutation is applied to every bit in a binary string with some small probability, the standard method of generating a new point in a random mutation hillclimber [10]. Based on this weak assumption, we give another convergence property of our archiving strategy below.

Theorem 4. *If the* EVOLUTION *procedure gives every possible solution in the search space with a positive minimum probability, the archive sequence $\{A^{(ar,t)}\}$ converges to a well-distributed E-Pareto set of the search space with bounded size with probability one as $t \longrightarrow +\infty$, i.e.,*

- *$A^{(ar,t)} \in EP^*(Y)$;*
- *$2 \leq |A^{(ar,t)}| \leq \dfrac{\prod_{i=1}^{m} \left\lfloor \frac{\frac{\pi}{2}}{e_i} + 1.5 \right\rfloor}{\max_{i=\{1,\ldots,m\}} \left\lfloor \frac{\frac{\pi}{2}}{e_i} + 1.5 \right\rfloor}$ for any given $\mathbf{e}$ with $0 < e_i < \frac{\pi}{4}$.*

Proof. According to Theorem 3, $A^{(min,t)}$ converges to a Pareto optimal set when $t > \tau_{c_2}$. According to Theorem 4, $A^{(ar,\tau)}$ is an E-Pareto set of $\bigcup_{t=\tau_{c_2}}^{\tau} \{\mathbf{y}^{(t)}\} \bigcup A^{(ar,\tau_{c_2})}$. EVOLUTION generates any solution infinitely often and that the waiting time for the first occurrence as well as for the second, and so forth will be finite with probability 1, so, for each solution $\mathbf{y} \in Y$, there exists $t_{\mathbf{y}}(\tau_{c_2} < t < +\infty)$ such that $\mathbf{y}^{(\mathbf{t_y})} = \mathbf{y}$. Then $A^{(ar,t_{\mathbf{y}}+1)}$ must E-dominates $\mathbf{y}$. Since Y is finite, $\tau_{c_3} \triangleq \max_{\mathbf{y} \in Y} \{t_{\mathbf{y}}\} < +\infty$. Thus, $A^{(ar,t)}$ is an E-Pareto set of Y as $t > \tau_{c_3}$.

We consider $t > \tau_{c_2}$ (Theorem 3). The rectangles envelop the archived vectors in $A^{(min,t)}$ must envelop a solution in $A^{(ar,t)}$ as t increases (Otherwise, the solution in $A^{(min,t)}$ will occupy the rectangle.). For each objective i, the coordinates of these rectangles must have two different values: 1 and $\left\lfloor \frac{\frac{\pi}{2}}{e_i} + 1.5 \right\rfloor - 1$, because they corresponds $a_i^{(min,t)}$ and $a_i^{(M,t)}$. So, $|A^{(ar,t)}| \leq 2$ as $t \longrightarrow \infty$.

As we can observed in RECT (Function 4), the i^{th} objective value is divided into $\left\lfloor \frac{\frac{\pi}{2}}{e_i} + 1.5 \right\rfloor$. The objective space is divided into $\prod_{i=1}^{m}\left\lfloor \frac{\frac{\pi}{2}}{e_i} + 1.5 \right\rfloor$ hyper-rectangles. From each hyper-rectangle, at most one solution can be in $A^{(ar,t)}$ at the same time. Now consider the equivalence classes of hyper-rectangles where, without loss of generality, in each class the hyper-rectangles have the same coordinates in all but one dimension. There are at most $\max_{i=\{1,\ldots,m\}}\left\lfloor \frac{\frac{\pi}{2}}{e_i} + 1.5 \right\rfloor$ different hyper-rectangles in each class constituting a chain of dominating boxes. Hence, only one solution from each of these classes can be a member of $A^{(ar,t)}$ at the same time. This completes the proof. □

This theorem states that the size of E-Pareto set is bounded, given the constant vector $\mathbf{e}$. In addition, there are at least two vectors in the archive. This point is different from t-Pareto set, which may retain only one solution [12].

6 Conclusion

In this paper, we have introduced the E-(approximate) Pareto set as a novel solution concept for evolutionary multiobjective optimization. It is

- theoretically attractive as it helps to construct algorithms with the desired convergence and distribution properties, and it generalizes the Pareto dominance concept in the MOEAs literature,
- It practically important as it works with a solution set with bounded size and with little prior knowledge about the target multiobjective problem.
- We have constructed the adaptive archiving strategy that can be used in any evolutionary algorithms and allow for the desired convergence properties, while at the same time, guaranteeing an optimal distribution of solutions (Theorems 1 and 2.
- Our archiving strategy, with appropriate assumption on the solution generation procedure, can retain the minimal objective value and a well distributed approximate of the whole Pareto front with probability 1 (Theorem 3 and 4).

When the distribution knowledge of the multiobjective values is available, the archiving strategy in the work [12] is a good choice. Otherwise, the user can set the vector $\mathbf{e}$ and the adaptive archiving strategy may provide a representative, well-distribution Pareto optimal set. In our future work, we will give some guidance on how to set the parameter. We may apply other transferring function, instead of $\tan$, in order to treat different solution regions more fairly. Our theoretical analysis is based on the assumption of finite search space, however, the E-Pareto set concept is applicable to more complicated situations. We also leave these for future work.

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