Apollonian Circle Packings: Geometry and Group Theory III. Higher Dimensions

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ABSTRACT

Apollonian circle packings arise by repeatedly filling the interstices between four mutually tangent circles with further tangent circles. Such packings can be specified in terms of the Descartes configurations they contain, where an (ordered) Descartes configuration is an ordered set of four mutually tangent circles having disjoint interiors, on the Riemann sphere. Parts I and II considered groups of transformations preserving Apollonian packings or the Descartes configurations in such packings. The paper considers generalizations of these results to dimensions $n \geq 3$, for Apollonian ensembles of n-dimensional Descartes configurations. These are no longer packings for $n \geq 4$, but there are analogues of most of the properties in parts I and II for such ensembles of n-dimensional Descartes configurations. An Apollonian sphere ensemble is strongly rational if every sphere in it has a rational curvature and a rational center. We show that strongly rational Apollonian sphere ensembles exist in dimension n if and only if $n = 2k^2$ or $n = (2k+1)^2$ for some positive integer k.

Keywords: Circle packings, Apollonian circles, Diophantine equations, Lorentz group, Coxeter group

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1. Introduction

In parts I and II we studied Apollonian packings of circles in two-dimensional Euclidean space in terms of the Descartes configurations they contain. A Descartes configuration is an arrangement of four mutually tangent circles on the Riemann sphere which have disjoint interiors. We identify Apollonian packings $\mathcal{P}$ with the set $\mathbb{D}(\mathcal{P})$ of all ordered Descartes configurations they contain. We studied various groups acting on the ensemble of Descartes configurations, one action coming from the conformal group Möb(2) acting on the Riemann sphere, inducing an action on Descartes configurations, and a linear action on Descartes configurations represented by 4×3 matrices in "curvature-center coordinates", as described in part I. The group associated to the latter is the group $Iso^{\uparrow}(Q_2)$ which is a subgroup of index 2 in the group $Iso(Q_2)$ of real automorphs of the Descartes quadratic form $Q_2 = I_4 - \frac{1}{2}\mathbf{1}_4^T\mathbf{1}_4$. A certain discrete subgroup of the group $Iso^{\uparrow}(Q_2)$, which we called the Apollonian group, leaves Apollonian packings invariant. We also considered a larger group, the super-Apollonian group, that can be used to define super-Apollonian packings. These groups had a representation using 4×4 integer matrices, and we found there were distiguished circle packings in which all curvatures were integral, which we called integral Apollonian circle packings, and other packings where the curvatures were integral and the centers×curvatures were also integral vectors, which we called strongly integral Apollonian packings.

In this paper we generalize these results to higher dimensions. We call any set $\mathcal{D} = (C_1, C_2, ..., C_{n+2})$ of $n+2$ mutually tangent spheres in n-dimensions, having disjoint interiors, an *n-dimensional Descartes configuration.* Given any set of $n+1$ mutually tangent (n-1)-spheres having disjoint interiors, there are exactly two spheres tangent to all of them, cf. Pedoe [25]. Such a set gives rise to two n-dimensional Descartes configurations. There is an inversion operation, given by an n-dimensional Möbius transformation in the n-dimensional conformal group Möb(n), mapping one to the other, cf. Pedoe [25, pp. 630-631]. Starting with an initial Descartes configuration, we can now obtain an ensemble of spheres in n-dimensions by successively adding spheres by such reflection operations. We call the completed set of spheres an *Apollonian sphere ensemble*. It is a sphere packing in dimensions 2 and 3, but for $n \geq 4$ the spheres overlap and it is not a packing, cf. Boyd [3]. However the *Apollonian cluster ensemble* consisting of all n-dimensional Descartes configurations generated by the underlying group of inversions, makes sense in all dimensions. We show that most of the results which hold for 2-dimensional Descartes configurations and Apollonian circle packings viewed as sets of Descartes configurations have n-dimensional analogues.

In §2 we prove characterizations of n-dimensional Descartes configurations which generalize the Descartes circle theorem. We use the n-dimensional version of the curvature-center coordinates introduced in parts I and II. For related results in spherical and hyperbolic space, see [21].

In §3 we show that the group-theoretic constructions of parts I and II have n-dimensional analogues, even though the associated collections of Descartes configurations no longer correspond to packings. We call them *Apollonian sphere ensembles*. We construct the n-dimensional analogues of the Apollonian group, and super-Apollonian group. These groups consist of integer matrices in dimensions 2 and 3 but not for dimensions 4 and higher. However a related group, the dual Apollonian group, is a group of integer matrices in all dimensions. Given a finite set of primes S, an S-integer is any rational number whose denominator is divisible only by powers of primes in S. The entries of the Apollonian group and super-Apollonian group are S-integers where S consists of the prime divisors of $n-1$ if n is even and $\frac{1}{2}(n-1)$ if n is odd.

In §4 we consider integral and rational Apollonian sphere ensembles. In all dimensions $n \geq 2$ there exist Apollonian sphere ensembles in which all spheres have curvatures which are S-integers, where S consists of the prime divisors of $n-1$ if n is even and $\frac{1}{2}(n-1)$ if n is odd. An Apollonian sphere ensemble is *strongly rational* if the curvature of every sphere in the packing is rational, and the center of every sphere is a rational vector. We show that a necessary and sufficient condition for a strongly rational Apollonian sphere ensembles to exist in dimension n is that $n = 2k^2$ or $n = (2k+1)^2$ for some positive integer k. In these dimensions there exist Apollonian sphere ensembles in which all curvatures and curvature-center quantities are S-integers for some fixed S. We do not determine an explicit choice of S, for allowable $n > 2$, however.

In §5 we consider a higher-dimensional analogue of the duality operation introduced in part II. The two-dimensional duality operator studied in part II was a geometric operation which led to a symmetry relating the generators of the super-Apollonian group under the transpose operation. In dimensions 3 and higher the "duality" operation no longer respects packings, and the generators of the associated super-Apollonian group are not preserved by the transpose operation. We show however that in higher dimensions the geometric analogue of the duality operation encodes an "equiangularity" property instead.

In §6 we briefly study the n-dimensional variant of Coxeter's study of loxodromic sequences of tangent spheres, inside an Apollonian sphere ensemble.

In the conclusion §7 we state some open problems.

Notation. In this paper, following earlier notation, the symbol C refers to an n-dimensional sphere ("n-dimensional circle"). The notion of augmented matrix $\tilde{N}_{\mathcal{D}}$ introduced in §2 adds the augmentation in the last column, while that used in [21] adds the augmentation as the second column of the matrix. For a row vector $\mathbf{x}$ in $\mathbb{R}^n$, its squared norm is $|\mathbf{x}|^2 = \mathbf{x}\mathbf{x}^T = \sum_{i=1}^n x_i^2$.

2. Generalized Descartes Theorem

The Descartes circle theorem generalizes to n-dimensional Euclidean space. A 3-dimensional analogue of the Descartes formula was found in 1886 by Lachlan [19, p. 498] and rediscovered in 1936 by Soddy [28]. The result of Soddy was extended to n-dimensions by Gossett [14]. It relates the (oriented) curvatures of $n + 2$ mutually tangent n-spheres, forming an oriented Descartes configuration. Here an *orientation* of a sphere consists of a unit normal direction, pointing inward or outward. The *oriented curvature* of an oriented sphere is $a_i = \frac{1}{r_i}$ if it is inwardly oriented and is $a_i = -\frac{1}{r_i}$ if it is outwardly oriented. We define the *interior* of an oriented sphere to be either its interior or exterior according to the orientation being inward or outward, respectively.

Definition 2.1. An *oriented Descartes configuration* is a set of $n + 2$ oriented spheres in $\mathbb{R}^n$, which are mutually tangent, such that either (i) each pair of oriented interiors are disjoint, or

(ii) each pair of oriented interiors intersect. We call these two cases (i) the positively oriented case and (ii) the negatively oriented case.

Given a positively oriented Descartes configuration, one obtains a negatively oriented Descartes configuration by reversing all orientations, and vice versa.

Theorem 2.1 (Soddy-Gossett Theorem) *Given an oriented Descartes configuration $\mathcal{D}$ in $\mathbb{R}^n$, its oriented curvatures $\{a_i \ : \ 1 \le i \le n+2\}$ satisfy*

$$\sum_{i=1}^{n+2} a_i^2 = \frac{1}{n}(\sum_{i=1}^{n+2} a_i)^2. \tag{2.1}$$

A proof of (2.2) can be found in Pedoe [25]; it also follows from Theorem 3.3 of [21]. This result can be rewritten as

$$\mathbf{a}^T \mathbf{Q}_n \mathbf{a} = 0, \tag{2.2}$$

where $\mathbf{a} := (a_1, \dots, a_{n+2})$ and $\mathbf{Q}_n$ is the symmetric matrix of the Descartes quadratic form, given by

$$\mathbf{Q}_n := I_{n+2} - \frac{1}{n}\mathbf{1}_{n+2}\mathbf{1}_{n+2}^T, \tag{2.3}$$

with $\mathbf{1}_{n+2}$ denoting a column of $n+2$ 1's.

Theorem 2.2 (Converse to Soddy-Gossett Theorem) *(i) Each nonzero real column vector $\mathbf{a} = (a_1, ..., a_{n+2})$ that satisfies the Descartes relation*

$$\sum_{i=1}^{n+2} a_i^2 = \frac{1}{n}(\sum_{i=1}^{n+2} a_i)^2. \tag{2.4}$$

is the set of oriented curvatures of some oriented Descartes configuration $\mathcal{D}$ in $\mathbb{R}^n$.

(ii) Any two oriented Descartes configurations with the same curvature vector are congruent, i.e. there is a Euclidean motion taking one to the other.

We will prove this result at the end of this section, after we have established some more general characterizations of oriented Descartes configurations.

For the next result we let $\mathcal{D}$ denote a general configuration of $n+2$ oriented spheres in $\mathbb{R}^n$, not necessarily an oriented Descartes configuration. If it is an oriented Descartes configuration with $\sum_{i=1}^{n+2} a_i > 0$, then one of the following holds. (i) all of $a_1, a_1, ..., a_{n+2}$ are positive; (ii) $n+1$ are positive and one is negative; (iii) $n+1$ are positive and one is zero; or (iv) n are positive and equal and the other two are zero. These four cases correspond respectively to the following configurations of mutually tangent spheres: (i) $n+1$ spheres, with another in the curvilinear simplex that they enclose; (ii) $n+1$ spheres inscribed inside another larger sphere; (iii) n spheres with a common tangent plane (the $(n+1)$-st "sphere"), with another sphere between them; (iv) n equal spheres with two common parallel tangent planes.

Definition 2.2. (i) Given an oriented sphere C in $\mathbb{R}^n$ with oriented curvature $a = a(C)$, and center $(x_1, x_2, ..., x_n)$ its *curvature-center coordinates* $\mathbf{w}(C)$ are given by the row vector

$$\mathbf{w}(C) := (a, \mathbf{c}) = (a, ax_1, ax_2, ..., ax_n). \tag{2.5}$$

where $\mathbf{c} = (a(C)x_1, ..., a(C)x_n)$.

(ii) We regard a hyperplane as a "degenerate" sphere. Given an oriented hyperplane H with specified unit normal vector $\mathbf{h} := (h_1, h_2, \dots, h_n)$, its *curvature-center coordinates* $\mathbf{w}(H)$ are given by

$$\mathbf{w}(H) := (0, h_1, h_2, \dots, h_n). \tag{2.6}$$

Definition 2.3. Given a configuration $\mathcal{D} = (C_1, C_2, ..., C_{n+2})$ of $n+2$ oriented spheres in $\mathbb{R}^n$ (allowing some to be hyperplanes), define its *curvature-center matrix* $N_{\mathcal{D}}$ to be the $(n+2) \times (n+1)$ matrix whose rows are

$$(N_{\mathcal{D}})_i := \mathbf{w}(C_i) = (a(C_i), a(C_i)x_{i,1}, ..., a(C_i)x_{i,n}). \tag{2.7}$$

It is easy to see that the matrix $N = N_{\mathcal{D}}$ determines the oriented sphere configuration $\mathcal{D}$ uniquely.

Theorem 2.3 (n-Dimensional Euclidean Descartes Theorem) *An $(n+2) \times (n+1)$ real curvature-center matrix N has $N = N_{\mathcal{D}}$ for some oriented Descartes configuration $\mathcal{D}$ if and only if*

$$N^T \mathbf{Q}_n N = \begin{bmatrix} 0 & 0 \\ 0 & 2I_n \end{bmatrix} = diag(0,\ 2,\ 2, ...,\ 2). \tag{2.8}$$

Futhermore this Descartes configuration is positively oriented if and only if

$$\sum_{i=1}^{n+2} N_{i,1} > 0. \tag{2.9}$$

Remark. The Soddy-Gossett theorem (2.1)appears as the $(1,1)$-entry of the matrix equation (2.8). Note that in (2.9) the value $N_{i,1} = a_i$ is the oriented curvature of the i-th sphere in the oriented Descartes configuration.

Proof. Let $\mathcal{D}$ be an oriented Descartes configuration, and we must prove (2.8). We first treat the case where no curvature vanishes, i.e. the Descartes configuration contains no hyperplanes. Later we obtain the remaining cases by a limiting process. We use matrix notation. Recall that $J = \mathbf{1}\mathbf{1}^T$, where $\mathbf{1} = (1,1,...,1)^T$ is an $(n+2) \times 1$ column vector. Let $X = [x_{i,j}]$ be the $(n+2) \times n$ matrix of sphere centers, and set $R = diag(r_1, r_2, \cdots, r_{n+2})$, where the r_i are the oriented radii of the spheres. We are assuming that all radii r_i are finite, so R is invertible. Note that one radius is assigned a negative sign if the sphere corresponding to it encloses the other spheres. Then $A := R^{-1}$ is the diagonal matrix of curvatures. and we have $N_{\mathcal{D}} = [R^{-1}\mathbf{1}, R^{-1}X] = [A\mathbf{1}, AX]$.

Without loss of generality we may rescale all coordinates by a positive constant factor λ, sending x_j to λx_j and r_j to λr_j. This rescales the first column of $N_{\mathcal{D}}$, leaving the other columns unchanged, and the relation (2.8) is preserved because the first column is an isotropic vector with respect to the indefinite bilinear form given by Q_n, i.e. it has inner product zero with all vectors. We choose the rescaling to make

$$\mathbf{1}^T A \mathbf{1} = \sum_{i=1}^{n+2} \frac{1}{r_i} = n. \tag{2.10}$$

The Soddy-Gossett relation, which is the $(1,1)$-entry of (2.8), then implies that

$$\mathbf{1}^T A^T A \mathbf{1} = \sum_{i=1}^{n+2} \frac{1}{r_i^2} = \frac{1}{n} (\sum_{i=1}^{n+2} \frac{1}{r_i})^2 = n, \tag{2.11}$$

see Theorem 3.3 of [21].

We next note that (2.8) is preserved under a translation of all sphere centers, because this subtracts a multiple of the first column of $N_{\mathcal{D}}$ from each other column, and this leaves

$N_{\mathcal{D}}^T Q_n N_{\mathcal{D}}$ unchanged, again because the first column is an isotropic vector. Without loss of generality we may now translate all the spheres to make

$$\mathbf{a}^T X = \mathbf{1}^T A X = \mathbf{0}^T, \tag{2.12}$$

and since the curvatures don't change, (2.10) and (2.11) still hold. It therefore suffices to prove the theorem in this special case.

Assuming that (2.10)- (2.12) all hold, we have

$$\begin{aligned} N_{\mathcal{D}}^T Q_n N_{\mathcal{D}} &= [A\mathbf{1}, AX]^T \left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T \right) [A\mathbf{1}, AX] \\ &= \begin{bmatrix} \mathbf{1}^T A^T A \mathbf{1} - \frac{1}{n}(\mathbf{1}^T A \mathbf{1})^2 & \mathbf{1}^T A^T A X - \frac{1}{n}(\mathbf{1}^T A^T \mathbf{1})(\mathbf{1}^T A X) \\ X^T A^T A \mathbf{1} - \frac{1}{n}(X^T A \mathbf{1})(\mathbf{1}^T A X) & X^T A^T A X - \frac{1}{n}(X^T A \mathbf{1})(\mathbf{1}^T A X) \end{bmatrix} \\ &= \begin{bmatrix} n - \frac{1}{n}(n^2) & \mathbf{1}^T A^T A X \\ X^T A^T A \mathbf{1} & X^T A^T A X \end{bmatrix}. \end{aligned} \tag{2.13}$$

The upper left corner of this block-partitioned matrix is zero, so to prove that it equals $diag(0, 2, 2, \dots, 2)$ it remains to prove that

$$\mathbf{1}^T A^T A X = \mathbf{a}^T A X = \mathbf{0}^T, \tag{2.14}$$

and

$$X^T A^2 X = 2 I_n. \tag{2.15}$$

The condition that two spheres with radii r_i and r_j touch is that

$$|\mathbf{x}_i - \mathbf{x}_j|^2 = (r_i + r_j)^2. \tag{2.16}$$

If we set

$$D = \text{diag}(X X^T) = \text{diag}(|\mathbf{x}_1|^2, ..., |\mathbf{x}_{n+2}|^2). \tag{2.17}$$

then the condition that all the spheres mutually touch is the matrix equality

$$D\mathbf{1}\mathbf{1}^T - 2XX^T + \mathbf{1}\mathbf{1}^T D = R^2 \mathbf{1}\mathbf{1}^T + 2R\mathbf{1}\mathbf{1}^T R + \mathbf{1}\mathbf{1}^T R^2 - 4R^2, \tag{2.18}$$

in which the (i, j)-th entry is (2.16). Multiplying by A :=R^{-1} on the left and on the right, we may rewrite this as

$$AD\mathbf{1}\mathbf{a}^T - 2AXX^TA + \mathbf{a}\mathbf{1}^TDA = A^{-1}\mathbf{1}\mathbf{a}^T + 2\mathbf{1}\mathbf{1}^T + \mathbf{a}\mathbf{1}^TA^{-1} - 4I. \tag{2.19}$$

Note that from (2.10),

$$\mathbf{f} := \frac{1}{\sqrt{2}}(\mathbf{1} - \mathbf{a}); \quad \mathbf{g} := \frac{1}{\sqrt{n}}\ \mathbf{a} \tag{2.20}$$

are orthogonal unit vectors. Now define

$$\alpha := \mathbf{1}^TR\mathbf{1} = \mathbf{1}^TA^{-1}\mathbf{1}. \tag{2.21}$$

Pre-multiplying (2.19) by $\mathbf{1}^T$ and post-multiplying by $\mathbf{1}$, and using (2.10) and (2.12), we find that

$$\mathbf{a}^TD\mathbf{1} = \alpha + \ n + \ 2. \tag{2.22}$$

Similarly, pre-multiplying (2.19) by $\mathbf{1}^T$ and post-multiplying by $\mathbf{a}$, we obtain

$$\mathbf{a}^TD\mathbf{1} + \mathbf{1}^TDA\mathbf{a} = \alpha + \ 3n + \ 2. \tag{2.23}$$

and pre-multiplying (2.19) by $\mathbf{a}^T$ and post-multiplying by $\mathbf{a}$, we obtain

$$\mathbf{a}^TAD\mathbf{1} - \frac{1}{n}\mathbf{a}^TAXX^TA\mathbf{a} = 2n. \tag{2.24}$$

From (2.22) and (2.23), we obtain

$$\mathbf{1}^TDA\mathbf{a} = 2n, \tag{2.25}$$

so from (2.24) we get $\mathbf{a}^TAXX^TA\mathbf{a} = 0$, i.e.

$$\mathbf{a}^TAX = \mathbf{0}^T. \tag{2.26}$$

Now post-multiplying (2.19) by $\mathbf{1}$, we find

$$A^{-1}\mathbf{1} - AD\mathbf{1} = \frac{n+2}{n}\mathbf{a} - 2\ \mathbf{1}, \tag{2.27}$$

whence from (2.19), we have

$$AXX^TA = 2I_{n+2} - (\mathbf{1} - \mathbf{a})(\mathbf{1}^T - \mathbf{a}^T) - \frac{2}{n}\mathbf{a}\mathbf{a}^T = 2(I_{n+2} - \mathbf{f}\mathbf{f}^T - \mathbf{g}\mathbf{g}^T). \tag{2.28}$$

Thus $\frac{1}{\sqrt{2}}AX$ is a $(n+2) \times n$ section of an orthogonal matrix, so

$$X^T A^2 X = 2I_n, \tag{2.29}$$

which completes the proof that an n-dimensional Descartes configuration satisfies (2.8), when all curvatures are nonzero.

It remains to consider the limiting cases of configurations in which one sphere has curvature zero, say $a = 0$, i.e. it is a hyperplane H, with equation $\mathbf{x}^T\mathbf{h} = p$, where $\mathbf{h}$ is the oriented unit normal vector, pointing to the correct half-space. Choosing $\mathbf{x} \in H$, one can obtain H as a limit of a sequence of spheres with radius r centered at $\mathbf{x} + r\mathbf{h}$ as $r \to \infty$. The curvature×center of these spheres converges to $(0, h_1, \dots h_n)$, independent of the choice of $\mathbf{x}$. Thus all is well provided we define $\mathbf{a}^T A := \mathbf{h}^T$. If two curvatures $a = b = 0$, then the remaining sphere centers must lie at the vertices of a regular simplex (with side $\frac{2}{c}$) and (2.8) is trivial in this case.

If a Descartes configuration is inwardly oriented, then either all curvatures are nonnegative, or exactly one is negative, corresponding to one sphere enclosing the others. Certainly if $\mathcal{D}$ has all (oriented) curvatures are positive or zero, then (2.9) holds. If one is negative, i.e. its sphere encloses the others, then the sphere having negative oriented curvature has the smallest curvature in absolute value, so condition (2.9) is satisfied. An outwardly oriented configuration reverses all signs of an inward one, so (2.9) does not hold.

That the conditions (2.8) and (2.9) always yield an n-dimensional Descartes configuration follows by reversing the above argument. First assume the first column of $N_{\mathcal{D}}$ has no zero entries. Given (2.8) holds, the curvatures satisfy the Soddy-Gossett relation (2.1), and by rescaling and translating as necessary we may assume (2.10) and (2.12) both hold. Here the rescaling is by a positive λ since (2.9) holds. Then $\mathbf{f}$ and $\mathbf{g}$ are orthogonal unit vectors. We now need to prove (2.19). From (2.8) we have (2.26), so that

$$\mathbf{f}^T AX = \mathbf{g}^T AX = \mathbf{0}. \tag{2.30}$$

From (2.8) again we have (2.29), so that the $(n+2) \times (n+2)$ matrix

$$[\mathbf{f}, \quad \mathbf{g}, \quad \frac{1}{\sqrt{2}}AX\ ] \tag{2.31}$$

is orthogonal, hence (2.28) holds. The diagonal of the matrix AXX^TA is

$$ADA = 2I_{n+2} - 2 \cdot \frac{1}{2}\,(I_{n+2} - A)^2 - 2 \cdot \frac{1}{n}A^2 \tag{2.32}$$

and it follows that

$$AD\mathbf{1} = A^{-1}\mathbf{1} + 2\ \mathbf{1} - \frac{n+2}{n}\ \mathbf{a}, \tag{2.33}$$

as required. This proves (2.19) in the case that no curvature vanishes.

In the remaining case where an element in the first column of $N_{\mathcal{D}}$ vanishes, any solution M satisfying (2.8) and (2.9) arises as a limit of such $N_{\mathcal{D}}$ in which all elements of the first column are nonzero. The limit of the corresponding Descartes configuration exists and gives the Descartes configuration corresponding to $N_{\mathcal{D}}$. ∎

Theorem 2.3 has a further generalization, which extends the $(n+2) \times (n+1)$ matrix $N_{\mathcal{D}}$ to an $(n+2) \times (n+2)$ augmented matrix $\tilde{N}_{\mathcal{D}}$ obtained by adding an additional column. The augmented matrix $\tilde{N}_{\mathcal{D}}$ involves information concerning two (oriented) Descartes configurations, the original one and one obtained from it by inversion in the unit sphere, as we now explain. This construction, which appears unmotivated here, was originally discovered in generalizing the Descartes theorem to spherical and hyperbolic geometry, as described in Lagarias, Mallows and Wilks [21, Section 4].

In n-dimensional Euclidean space, the operation of *inversion in the unit sphere* replaces the point $\mathbf{x}$ by $\mathbf{x}/|\mathbf{x}|^2$, where $|\mathbf{x}|^2 = \sum_{j=1}^{n} x_j^2$. Consider a general sphere C with center $\mathbf{x}$ and oriented radius r. Then inversion in the unit sphere takes C to the sphere $\bar{C}$ with center $\bar{\mathbf{x}} = \mathbf{x}/(|\mathbf{x}|^2 - r^2)$ and oriented radius $\bar{r} = r/(|\mathbf{x}|^2 - r^2)$. Note that if $|\mathbf{x}|^2 > r^2$, $\bar{C}$ has the same orientation as C. The inversion may take some spheres to hyperplanes, and vice-versa, as well as sending some hyperplanes to hyperplanes. In all cases,

$$\frac{\mathbf{x}}{r} = \frac{\bar{\mathbf{x}}}{\bar{r}}.$$

Definition 2.4. (i) Given an oriented sphere C with oriented curvature $a = a(C)$ and center $(x_1, ...x_n)$, with inverse oriented sphere $\bar{C}$ having oriented curvature $\bar{a} = a(\bar{C})$. Then its *augmented curvature-center coordinates* are

$$\tilde{\mathbf{w}}(C) := (a(C), \mathbf{c}(C), a(\bar{C})) = (a(C), a(C)x_1, ..., a(C)x_n, \bar{a}). \tag{2.34}$$

(ii) Given an oriented hyperplane H, with inverse $\bar{H}$ in the unit sphere, its *augmented curvature-center coordinates* $\tilde{\mathbf{w}}(H)$ are given by

$$\tilde{\mathbf{w}}(H) := (a(H), \mathbf{c}(H), a(\bar{H}) = (0, h_1, ..., h_n, a(\bar{H})). \tag{2.35}$$

Definition 2.5. Given a configuration $\mathcal{D} = (C_1, C_2, ..., C_{n+2})$ of $(n+2)$ oriented spheres in $\mathbb{R}^n$, in which some spheres may be hyperplanes, its *augmented curvature-center matrix* is the $(n+2) \times (n+2)$ matrix $\tilde{N}_{\mathcal{D}}$ with rows

$$(\tilde{N}_{\mathcal{D}})_i := (\ (N_{\mathcal{D}})_i, \bar{a}_i), \tag{2.36}$$

in which $\bar{a}_i$ is the signed curvature of the inverse sphere $\bar{C}_i$ to C_i.

Given an oriented sphere C, we have

$$\bar{a} = \frac{|\mathbf{x}|^2}{r} - r, \quad \mathbf{c}(\bar{C}) = \mathbf{c}(C) = \frac{\mathbf{x}}{r}. \tag{2.37}$$

Notice that the relation $\tilde{\mathbf{w}}(\bar{C}) = (\bar{a}, \mathbf{c}(C), a(C))$ enables us to extend the definition of $\mathbf{w}(C)$ to degenerate spheres with infinite radius; simply find $\mathbf{w}(\bar{C})$ and interchange the first and last coordinates. If H is a hyperplane containing the origin, then $H = \bar{H}$, $a = \bar{a} = 0$ and $\mathbf{c}$ is a unit vector orthogonal to H.

Theorem 2.4 (Augmented Euclidean Descartes Theorem) *An $(n+2) \times (n+2)$ real matrix $\tilde{N}$ is the augmented curvature$\times$center matrix $\tilde{N}_{\mathcal{D}}$ of some oriented Descartes configuration $\mathcal{D}$ in $\mathbb{R}^n$ if and only if*

$$\tilde{N}^T \mathbf{Q}_n \tilde{N} = \begin{bmatrix} 0 & 0 & -4 \\ 0 & 2I_n & 0 \\ -4 & 0 & 0 \end{bmatrix}. \tag{2.38}$$

The augmented Euclidean Descartes Theorem implies one direction of the n-dimensional Euclidean Descartes Theorem, namely that all oriented Descartes configurations satisfy (2.8). The converse direction, that all sphere configurations satisfying (2.8) are oriented Descartes configurations, requires an additional argument, given in the proof of Theorem 2.3.

We proceed to prove the augmented Euclidean Descartes theorem, via a preliminary lemma. Given a real number λ, define the matrix

$$\mathbf{K}_n(\lambda) := \begin{bmatrix} 0 & 0 & -\lambda \\ 0 & 2I_n & 0 \\ -\lambda & 0 & 0 \end{bmatrix}, \tag{2.39}$$

Note that $\mathbf{K}_n(4)$ appears in the theorem above, and a calculation reveals that

$$\mathbf{K}_n(\lambda)^{-1} = \frac{1}{4}\mathbf{K}_n(\frac{4}{\lambda}). \tag{2.40}$$

Lemma 2.5. *(i) For any $(n+2)$-vector $\tilde{\mathbf{w}}$, there is a sphere (or hyperplane) C in $\mathbb{R}^n$ with $\tilde{\mathbf{w}}(C) = \tilde{\mathbf{w}}$ if and only if*

$$\tilde{\mathbf{w}}\mathbf{K}_n(1)\tilde{\mathbf{w}}^T = 2.$$

(ii) The oriented spheres C and C' are externally tangent if and only if

$$\tilde{\mathbf{w}}(C)\mathbf{K}_n(1)\tilde{\mathbf{w}}(C')^T = -2.$$

Proof. . (i) This restates the relation $b\bar{b} = (|\mathbf{x}|^2 - r^2)/r^2 = |\mathbf{c}|^2 - 1$.

(ii) This is an immediate consequence of $|\mathbf{x} - \mathbf{x}'|^2 = (r + r')^2$. ∎

Proof of the Augmented Euclidean Descartes Theorem. Suppose $\tilde{N} = \tilde{N}_{\mathcal{D}}$ for some configuration of $(n+2)$ oriented spheres. From Lemma 2.5(ii), if the spheres all touch externally, we have

$$\tilde{N}\mathbf{K}_n(1)\tilde{N}^T = 4\mathbf{I}_{n+2} - 2\mathbf{1}_{n+2}\mathbf{1}_{n+2}^T = 4(\mathbf{Q}_n)^{-1} \tag{2.41}$$

Next, recall the the matrix identity that if A, B are symmetric non-singular $n \times n$ matrices satisfying $WAW^T = B$, then [2]

$$W^T B^{-1} W = A^{-1}. \tag{2.42}$$

Apply this identity with $A = \mathbf{K}_n(1)$ and $W = \tilde{N}$ noting that

$$4\mathbf{K}_n(1)^{-1} = \mathbf{K}_n(4), \tag{2.43}$$

to obtain (2.38).

For the converse direction, suppose $\tilde{N}$ satisfies (2.38). This matrix equation implies (2.41), using the identity (2.42). The diagonal terms in (2.41) imply, using Lemma 2.5(i), that $\tilde{N} = \tilde{N}_{\mathcal{D}}$ for some configuration of oriented spheres $\mathcal{D}$. Then Lemma 2.5(ii), implies that the spheres touch externally pairwise. ∎

Proof of Converse to Soddy-Gossett Theorem. (i) We start from a standardized Descartes configuration, which is the one with two parallel hyperplanes at distance 2 from each other, and n-unit spheres in between them, whose centers form a regular $(n-1)$-simplex lying

[2] Clearly W must be nonsingular. Invert both sides, and multiply on the left by W^T and on the right by W.

in the hyperplane parallel to the two hyperplanes in the configuration and halfway between them. Let its augmented matrix be W_0, and oriented curvature vector be $\mathbf{a}_0$. The automorphism group $Aut(\mathbf{Q}_n)$ is the set of all $(n+2)\times(n+2)$ matrices $\mathbf{U}$ such that $\mathbf{U}^T\mathbf{Q}_n\mathbf{U} = \mathbf{Q}_n$. The matrix $\mathbf{U}W_0$ clearly satisfies (2.38) since W_0 does, so by the augmented Euclidean Descartes Theorem, the matrix $\mathbf{U}W_0$ is itself the augmented matrix of some oriented Descartes configuration. In particular its first column $\mathbf{U}\mathbf{a}_0$ are the oriented curvatures of some oriented Descartes configuration.

Now since $\mathbf{Q}_n$ has signature $(1, n+1)$, there exists a real matrix $\mathbf{V}$ such that

$$\mathbf{V}^T\mathbf{Q}_n\mathbf{V} = \mathbf{Q}_{\mathcal{L}} := diag(-1, 1, 1, ..., 1).$$

This quadratic form, the $(n+2)$-dimensional Lorentzian form, has automorphism group $O(1, n+1)$, and it is known that $O(1, n+1)$ acts transitively on the nonzero elements of the *null cone* (or *light cone*)

$$\mathbf{b}^T\mathbf{Q}_{\mathcal{L}}\mathbf{b} = 0$$

of the Lorentzian form. Pulling back to $\mathbf{Q}_n$, we find that

$$Aut(\mathbf{Q}_n) = \mathbf{V}O(1, n+1)\mathbf{V}^{-1}$$

and that the action of $Aut(\mathbf{Q}_n)$ acts transitively on the nonzero solutions to $\mathbf{a}^T\mathbf{Q}_n\mathbf{a} = 0$. That is, for any non-zero vector $\mathbf{a}$ satisfying $\mathbf{a}^T\mathbf{Q}_n\mathbf{a} = 0$, there exists a matrix $\mathbf{U} \in Aut(\mathbf{Q}_n)$ such that $\mathbf{a} = \mathbf{U}\mathbf{a}_0$. Then by the argument at the beginning of the proof, $\mathbf{U}W_0$ is the augmented matrix of an oriented Descartes configuration whose vector of oriented curvatures is $\mathbf{a}$.

(ii) We may assume that the two Descartes configurations with the same curvature vector $\mathbf{a}$ are positively oriented. This is because the orientation of a Descartes configuration is determined by the signs of the a_i's, and any negatively oriented Descartes configuration can be obtained from a positively oriented one by reversing all orientations. By Definition 2.1, one of the following holds for $\mathbf{a} = (a_1, a_2, \ldots, a_{n+2})$. (a) all of $a_1, a_2, \ldots, a_{n+2}$ are positive; (b) $n+1$ are positive and one is negative; (c) $n+1$ are positive and one is zero; or (d) n are positive and equal and the other two are zero. For case (d), the theorem holds trivially. For cases (b) and (c), suppose $a_1 \leq 0$. Let C' be the sphere that is tangent to $C_2, \ldots, C_{n+2}$ but not equal to

C_1. By Soddy-Gossett Theorem, the curvature a' of C' equals $2(a_2 + \cdots + a_{n+2})/(n-1) - a_1$ which is positive and finite. Thus C' is positive oriented with finite radius. Since the position of C_1 is uniquely determined by that of $C_2, \ldots, C_{n+2}, C'$, cases (b) and (c) are reduced to (a).

Now we treat the case (a) in which all spheres have finite radius and positive oriented curvatures. We use the fact that an n-simplex is completely determined, up to congruence, by the lengths of its $\frac{n(n-1)}{2}$ edges (between labelled vertices). Any set of $n+1$ externally touching spheres is rigid, because the set of sphere centers forms an n-simplex in which the distance along the edge from center of S_i to center of S_j is $r_i + r_j$. Given two oriented Descartes configurations that have the same oriented curvature vector, the simplices determined by the first $n+1$ sphere centers are congruent, hence there is a Euclidean motion taking the first $n+1$ spheres of one to the first $n+1$ spheres of the other. Euclidean motions preserve tangencies, and the remaining sphere of the initial configuration must therefore be mapped to a sphere tangent to the other's first $n+1$ spheres. There are only two choices for the image sphere, and the Euclidean motion can map to the wrong image only if the second configuration has two tangent spheres of equal size. But if this happens, there is also a reflection of the second configuration taking this image configuration into the other. This finishes the proof. ∎

3. n-Dimensional Apollonian Ensembles and Group Actions

The n-dimensional (generalized) Möbius group $\text{Möb}(n)$ is the set of conformal isomorphisms of the n-sphere $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ to itself, allowing orientation-reversing maps of the n-sphere. This notion of orientation is unrelated to the notion of an oriented Descartes configuration in §2; in fact the action of the n-dimensional Möbius group preserves orientation of Descartes configurations. In this section all Descartes configurations will be given the positive orientation [3] unless otherwise noted. Each group element $\mathfrak{g}$ acts as a permutation of individual $(n-1)$-spheres, and also induces a permutation action on the set $\mathbb{D}_n$ of all n-dimensional positively oriented Descartes configurations.

The following result is a straightforward generalization of Theorem 2.9 of part I. Given an oriented Descartes configuration $\mathcal{D}$, let $N_{\mathcal{D}}$ denote the $(n+2) \times (n+1)$ matrix assigned to it

[3] This conforms with the notation in parts I and II, which treated positively oriented Descartes configurations only.

in Theorem 2.3, and let $\tilde{N}_{\mathcal{D}}$ denote the $(n+2)\times(n+2)$ augmented matrix associated to $\mathcal{D}$ in Theorem 2.4.

Theorem 3.1. *There is a representation $\rho_n : M\ddot{o}b(n) \to GL(n+2,\mathbb{R})$ with each $\rho_n(\mathfrak{g}) = G_{\mathfrak{g}}$ having determinant ± 1, such that for all n-dimensional oriented Descartes configurations $\mathcal{D}$,*

$$\tilde{N}_{\mathfrak{g}(\mathcal{D})} = \tilde{N}_{\mathcal{D}} G_{\mathfrak{g}}^T. \tag{3.1}$$

Proof. It suffices to verify Theorem 3.1 on a set of generators of Möb(n). A general element of Möb(n) is either a similarity of $\mathbb{R}^n$ or the product of an inversion and an isometry of $\mathbb{R}^n$, cf. Wilker [31, §5, Corollary 1]. That is, Möb(n) is generated by the following operators: dilatations $\mathfrak{d}_k(v) = kv$ where $k \in \mathbb{R}$, $k \neq 0$, translations $\mathfrak{t}_{v_0}(v) = v + v_0$, where $v_0 \in \mathbb{R}^n$ is a row vector, rotations $\mathfrak{r}_O(v) = Ov$, where O is an orthogonal matrix of size n, and the inversion in the unit circle $\mathfrak{j}_C(v) = \frac{v}{|v|^2}$.

Direct computation shows that formula (3.1) holds for $\mathfrak{d}_k$, $\mathfrak{t}_{v_0}$, $\mathfrak{r}_O$, and $\mathfrak{j}_C$, where the right multiplication are given by the matrices

$$G_{\mathfrak{d}_k}^T := \begin{bmatrix} \frac{1}{k} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & k \end{bmatrix},$$

$$G_{\mathfrak{t}_{v_0}}^T := \begin{bmatrix} 1 & v_0 & |v_0|^2 \\ 0 & I_n & 2v_0^T \\ 0 & 0 & 1 \end{bmatrix},$$

$$G_{\mathfrak{r}_O}^T := \begin{bmatrix} 1 & 0 & 0 \\ 0 & O & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $G_{\mathfrak{j}_C}^T := P_{1,n+2}$, the permutation matrix which permutes the first and $(n+2)$th entries. ■

Given an n-dimensional Descartes configuration $\mathcal{D} = \{C_1, C_2, \ldots, C_{n+2}\}$ one can generate new Descartes configurations using *reflection operators* $\mathfrak{s}_i = \mathfrak{s}_i[\mathcal{D}] \in$ Möb(n) for $1 \leq i \leq n+2$, in which $\mathfrak{s}_i$ is the unique Möbius transformation that maps the sphere C_i to the other sphere C_i' which is tangent to all the remaining C_j, while leaving the other C_j invariant, cf. Pedoe [25, p. 630] and Wilker [31, Theorem 3]. The Möbius transformation $\mathfrak{s}_i[\mathcal{D}] := \mathfrak{j}_{C_i^\perp}$ is inversion with respect to the unique $(n-1)$-sphere $C_i^\perp$ which passes through the $\frac{n(n+1)}{2}$ points of tangency of the other $n+1$ circles $\{C_j : j \neq i\}$. The existence of $C_i^\perp$ is given by the following well-known result.

Proposition 3.2. *Given $n+1$ mutually tangent $(n-1)$-spheres $\{C_i : 1 \le j \le n+1\}$ in $\mathbb{R}^n$ having disjoint interiors, there exists a unique $(n-1)$-sphere $C^\perp$ passing through the $\frac{n(n-1)}{2}$ tangency points of these spheres. At each such tangency point the normal to the sphere $C^\perp$ is perpendicular to the normals of the two spheres C_i and C_j tangent there.*

Proof. The assumption of disjoint interiors (we allow interior to be defined as "exterior" for one sphere) is equivalent to all $\frac{n(n-1)}{2}$ tangency points of the spheres being distinct. For dimension $n = 2$ there is a unique circle through any three distinct points. For $n \ge 3$ the conditions are over-determined, since $n+1$ distinct points already determine a unique $(n-1)$-sphere, and the main issue is existence.

Both assertions of the theorem are invariant under Möbius transformations (which preserve angles), and there exists a Möbius transformation taking a set of $n+1$ mutually tangent $(n-1)$-spheres in $\mathbb{R}^n$ having disjoint interiors to any other such set, cf. Wilker [31, Theorem 3]. Thus it suffices to prove the result for a single such configuration, and we consider the configuration of $n+1$ mutually touching spheres of equal radius whose centers are at the vertices of a regular n-simplex, and tangency points of the spheres are the midpoints of its edges. The first assertion of the theorem holds in this case because there is an $(n-1)$- sphere whose center is at the center of gravity of this simplex, which passes through the midpoints of every edge of the simplex. Indeed the isometries preserving an n-simplex fix the center of gravity and act transitively on the edges. Note that for $n = 2$ the simplex is an equlateral triangle and $C^\perp$ is the inscribed circle; however for $n \ge 3$ the sphere $C^\perp$ is neither inscribed nor circumscribed about this simplex.

For the second assertion of the proposition, in this configuration the sphere $C^\perp$ has each edge of the n-simplex lying in a tangent plane to the sphere; so the normal to $C^\perp$ at the midpoint of an edge is perpendicular to that edge. Two spheres C_i and C_j intersect at the midpoint of an edge, and the normal to their tangent planes points along this edge; thus this normal is perpendicular to the normal to $C^\perp$ there. ∎

The second assertion in Proposition 3.2 explains why the sphere $C^\perp$ is termed " orthogonal." In §5 we give formulas for the curvature and center of $C^\perp$.

Definition 3.1. (i) The the *configuration group* $\mathcal{G}^0_{\mathcal{D}}$ of the Descartes configuration $\mathcal{D}$ is the group generated by the operators $\mathfrak{s}_i$ associated to $\mathcal{D}$, i.e.

$$\mathcal{G}^0_{\mathcal{D}} := \langle \mathfrak{s}_1[\mathcal{D}], \ldots, \mathfrak{s}_{n+2}[\mathcal{D}] \rangle \subseteq \text{Möb}(n) \ . \tag{3.2}$$

(ii) The *ordered configuration group* $\mathcal{G}_{\mathcal{D}}$ of $\mathcal{D}$ is the group obtained from $\mathcal{G}^0_{\mathcal{D}}$ by adjoining as generators the set of $(n+2)!$ different Möbius transformations that permute the spheres in $\mathcal{D}$.

The configuration group $\mathcal{G}^0_{\mathcal{D}}$ satisfies the relations

$$\mathfrak{s}_i[\mathcal{D}]^2 = I, \quad \text{for} \quad 1 \le i \le n+2 \ , \tag{3.3}$$

but may satisfy additional relations for certain $n \ge 3$. For $n = 3$ it satisfies the extra relations

$$(\mathfrak{s}_i \mathfrak{s}_j)^3 = 1, \qquad \text{when} \qquad i \ne j. \tag{3.4}$$

Definition 3.2. Given an n-dimensional Descartes configuration $\mathcal{D} = \{C_1, C_2, \ldots C_{n+2}\}$, the *$n$-dimensional Apollonian sphere ensemble* $\mathcal{P}_{\mathcal{D}}$ is the set of $(n-1)-$spheres,

$$\mathcal{P}_{\mathcal{D}} := \{\mathfrak{s}(C_i) : \mathfrak{s} \in \mathcal{G}^0_{\mathcal{D}} \ \text{ and } C_i \in \mathcal{D}, 1 \le i \le n+2\} \ . \tag{3.5}$$

The *n-dimensional Apollonian cluster ensemble* $\mathbb{D}(\mathcal{P}_{\mathcal{D}})$ associated to $\mathcal{D}$ is the set of Descartes configurations

$$\mathbb{D}(\mathcal{P}_{\mathcal{D}}) = \{\mathfrak{s}(\mathcal{D}) : \mathfrak{s} \in \mathcal{G}_{\mathcal{D}}\}. \tag{3.6}$$

Boyd [3, Theorem 5] observed that the Apollonian sphere ensemble has spheres with disjoint interiors, thus giving an Apollonian sphere packing, if and only if the dimension is 2 or 3. The spheres overlap in higher dimensions, which motivates calling it an "ensemble", rather than a packing. In certain dimensions $n \ge 3$ Boyd [5] constructs configurations of $n+2$ spheres, not all touching, in which an associated group of inversions generates a packing of disjoint spheres. He finds examples up to dimension 9. Later Maxwell [24] classified the possible reflection groups involved.

On the level of Descartes configurations the Apollonian cluster ensembles $\mathbb{D}(\mathcal{P}_{\mathcal{D}})$ above make sense [4] in all dimensions. Furthermore we can recursively calculate the spheres appearing

[4] The set $\mathbb{D}(\mathcal{P}_{\mathcal{D}})$ is contained in the set $\mathbb{D}'(\mathcal{P}_{\mathcal{D}})$ of all Descartes configurations that consist of $n+2$ spheres from the set $\mathcal{P}_{\mathcal{D}}$, but it remains to be decided whether it is the entire set of such configurations.

in such an ensemble using the tree structure enumerating the elemants of $\mathcal{G}_{\mathcal{D}}$. That is, given a Descartes configuration $\mathcal{D}$ we can go to a neighboring configuration $\mathcal{D}'$ in the ensemble by deleting one sphere and adding a new sphere. The coordinates of the new sphere are easily calculated by linear operations, using the following result, derived from the n-dimensional Euclidean Descartes theorem 2.3.

Theorem 3.3. *(i) Given a configuration of $n+1$ tangent spheres $(C_1, C_2, ..., C_{n+1})$ in $\mathbb{R}^n$, with all tangents distinct, there are exactly two spheres, call them C_{n+2} and $C_{n+2}^{'}$, tangent to each of the $n+1$ spheres.*

(ii) Let $\mathcal{D}$ and $\mathcal{D}^{'}$ denote the positively oriented Descartes configurations associated to $(C_1, C_2, ..., C_{n+1})$ with C_{n+2} and $C_{n+2}^{'}$ added, respectively. Then the curvatures a, a' and centers $\mathbf{x}$ and $\mathbf{x}^{'}$ of C_{n+2} and $C_{n+2}^{'}$ are related by

$$a + a^{'} = \frac{2}{n-1}(a_1 + ... + a_{n+1}) \tag{3.7}$$

and

$$a\mathbf{x} + a'\mathbf{x}^{'} = \frac{2}{n-1}(a_1\mathbf{x}_1 + ... + a_{n+1}\mathbf{x}_{n+1}). \tag{3.8}$$

Proof. (i) This result is established in Pedoe [25], who gives references to earlier work, and who observes that in dimensions $n \geq 3$ it is a result in real algebraic geometry, rather than complex algebraic geometry. (In dimension $n \geq 3$ there may be more than two complex circles tangent to such a configuration.

(ii). This follows from the n-dimensional Euclidean Descartes theorem 2.3. If N and N' are the matrices corresponding to $\mathcal{D}$ and $\mathcal{D}'$, then Theorem 2.3 gives

$$N^T \mathbf{Q}_n N = (N')^T \mathbf{Q}_n N' = diag(0, 2I_n),$$

and their first $n+1$ rows agree.

Suppose, more generally, that $\mathbf{y}, \mathbf{z}, \mathbf{g}_1, \ldots, \mathbf{g}_{n+1}$ are row vectors in $\mathbb{R}^{n+1}$, and define the $(n+2) \times (n+1)$ matrices Y and Z by

$$Y^T := [\mathbf{g}_1^T, \ldots, \mathbf{g}_{n+1}^T, \mathbf{y}^T],$$
$$Z^T := [\mathbf{g}_1^T, \ldots, \mathbf{g}_{n+1}^T, \mathbf{z}^T].$$

Then we claim that

$$Y^T \mathbf{Q}_n Y = Z^T \mathbf{Q}_n Z \tag{3.9}$$

if and only if either $\mathbf{y} = \mathbf{z}$ or

$$\mathbf{y} + \mathbf{z} = \frac{2}{n-1}(\mathbf{g}_1 + \cdots + \mathbf{g}_{n+1}). \tag{3.10}$$

To see this, let $\mathbf{f}$ be an arbitrary $n+1$ row vector. Then $\mathbf{f}Y^T\mathbf{Q}_nY\mathbf{f}^T = \mathbf{f}Z^T\mathbf{Q}_nZ\mathbf{f}^T = c$ where c is a constant, so that

$$\begin{aligned} n((\mathbf{f}^T\mathbf{y})^2 + \sum_{i=1}^{n+1}(\mathbf{f}^T\mathbf{g}_i)^2) &- (\mathbf{f}^T\mathbf{y} + \sum_{i=1}^{n+1}\mathbf{f}^T\mathbf{g}_i)^2 = c, \\ n((\mathbf{f}^T\mathbf{z})^2 + \sum_{i=1}^{n+1}(\mathbf{f}^T\mathbf{g}_i)^2) &- (\mathbf{f}^T\mathbf{z} + \sum_{i=1}^{n+1}\mathbf{f}^T\mathbf{g}_i)^2 = c. \end{aligned}$$

That is, both $\mathbf{f}^T\mathbf{y}$ and $\mathbf{f}^T\mathbf{z}$ are solutions of the equation

$$n(x^2 + \sum_{i=1}^{n+1}(\mathbf{f}^T\mathbf{g}_i)^2) - (x + \sum_{i=1}^{n+1}\mathbf{f}^T\mathbf{g}_i)^2 = c.$$

It follows that either $\mathbf{f}^T\mathbf{y} = \mathbf{f}^T\mathbf{z}$ or

$$\mathbf{f}^T(\mathbf{y} + \mathbf{z}) = \frac{2}{n-1}\mathbf{f}^T(\mathbf{g}_1 + \cdots + \mathbf{g}_{n+1}).$$

for all $\mathbf{f} \in \mathbb{R}^{n+1}$. For $n \geq 2$ this is possible if and only if either $\mathbf{y} = \mathbf{z}$ or the equation (3.10) holds. ■

This theorem shows that starting with the curvatures and centers of a set of $n+2$ mutually tangent spheres, we can step along a sequence of spheres, each tangent to some set of $n+1$ of the preceding spheres, simply by updating the a's and $a\mathbf{x}$'s using this linear recurrence. The special role of dimensions $n = 2$ and $n = 3$ is apparant here, in terms of integrality properties of this recurrence. In two or three dimensions, if we start with integer values of $\mathbf{a}$ and AX then all succeeding values will be integers. This fails to hold in higher dimensions, because then $\frac{2}{n-1}$ is not an integer.

Most results in parts I and II for Apollonian packings have n-dimensional analogues for Apollonian sphere ensembles. By definition the elements of $\mathcal{G}^0_{\mathcal{D}}$ leave the Apollonian sphere ensemble $\mathcal{P}_{\mathcal{D}}$ invariant. If a Descartes configuration $\mathcal{D}' \in \mathcal{P}_{\mathcal{D}}$ then $\mathcal{P}_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}'}$. The automorphism group $Aut(\mathcal{P}) \subseteq \text{Möb}(n)$ acts sharply transitively on the set $\mathbb{D}(\mathcal{P})$ of ordered Descartes configurations in the Apollonian cluster ensemble $\mathbb{D}(\mathcal{P})$ in $\mathbb{R}^n$. Under Möbius transformations all Apollonian sphere ensembles (resp. cluster ensembles) in $\mathbb{R}^n$ are the same: there exists $\mathfrak{g} \in \text{Möb}(n)$ with $\mathfrak{g}(\mathcal{P}) = \mathcal{P}'$, and $\mathfrak{g}(\mathbb{D}(\mathcal{P})) = \mathbb{D}(\mathcal{P}')$. This follows from Wilker [31, Theorem 3,p. 394].

One can also define Möbius transformations that move between Apollonian ensembles, which have natural geometric meanings. The *inversion operators* $\mathfrak{s}_i^{\perp} := \mathfrak{j}_{C_i}$ are the inversions with respect to the circles C_i, for $1 \le i \le n+2$. The *dual operator* does not generalize to $n \ge 3$ as a Möbius transformation, however.

We next consider the n-dimensional analogue of the Apollonian group. Recall that the *Descartes quadratic form* Q_n is

$$Q_n(\mathbf{x}) := \mathbf{x}^T \mathbf{Q}_n \mathbf{x} = \mathbf{x}^T (I_{n+2} - \frac{1}{n} \mathbf{1}_{n+2} \mathbf{1}_{n+2}^T) \mathbf{x} \tag{3.11}$$

where $\mathbf{1}_{n+2}^T = (1, 1, \ldots, 1)$. This is a rational quadratic form with $\det(Q_n) = -\frac{2}{n}$ (see Lemma 4.3 below) and it has signature $(1, n+1)$. Let $Iso^{\uparrow}(Q_n)$ denote the group

$$Iso^{\uparrow}(Q_n) := \{ M \in GL(n+2, \mathbb{R}) : M^T \mathbf{Q}_n M = \mathbf{Q}_n, \text{ and } \mathbf{1}_{n+2}^T M \mathbf{1}_{n+2} > 0 \} . \tag{3.12}$$

Theorem 2.3 shows that "curvature-center coordinates" describe an (oriented) Descartes configuration $\mathcal{D}$ in $\hat{\mathbb{R}}^n$ by an $(n+2) \times (n+1)$ matrix $N_{\mathcal{D}}$. Theorem 2.6 of part I generalizes as follows.

Theorem 3.4. *The group $Iso^{\uparrow}(Q_n)$ is sharply transitive on the set $\mathbb{D}_n$ of all ordered (positively oriented) n-dimensional Descartes configurations $\mathcal{D}$.*

This result is analogous to that of Wilker [31, Theorem 3, p.394], and we omit a proof.

The action of $Iso^{\uparrow}(Q_n)$ can be extended to the augmented matrices $\tilde{N}_{\mathcal{D}}$ by left linear multiplications. Similar to the 2-dimensional case, we have

Theorem 3.5. *The actions of $Iso^{\uparrow}(Q_n)$ and Möb(n) on the set $\{\tilde{N}_{\mathcal{D}} : \mathcal{D} \in \mathbb{D}_n\}$ commute with each other.*

The proof is similar to that of Theorem 2.8 of part I, and follows from Theorem 3.1.

Definition 3.3. The (unordered) *n-dimensional Apollonian group* $\mathcal{A}_n^0$ is the group of $(n+2)\times(n+2)$ matrices generated by

$$\mathcal{A}_n^0 = \langle S_1, S_2, \ldots, S_{n+2} \rangle, \tag{3.13}$$

in which

$$S_1 = \left[\begin{array}{c|cccc} -1 & \frac{2}{n-1} & \frac{2}{n-1} & \cdots & \frac{2}{n-1} \\ \hline 0 & & & & \\ 0 & & & & \\ \vdots & & I_{n+1} & & \\ 0 & & & & \end{array}\right] \tag{3.14}$$

and $S_i = P_{(1i)} S_1 P_{(1i)}$, where $P_{(1i)}$ is the permutation matrix for $(1i)$.

The action of S_1 on a Descartes configuration $\mathcal{D} = \{C_1, C_2, \ldots, C_{n+2}\}$ is to send it to the unique Descartes configuration $\mathcal{D}' = \{C_1', C_2, \ldots, C_{n+2}\}$ having $C_1' \neq C_1$, i.e. $S_1 N_{\mathcal{D}} = N_{\mathcal{D}'}$. It is easy to check that

$$S_i^T Q_n S_i = Q_n \ , \tag{3.15}$$

and $\mathbf{1}^T S_i \mathbf{1} > 0$, so that

$$\mathcal{A}_n^0 \subseteq Iso^{\uparrow}(Q_n) \ .$$

The group $\mathcal{A}_n^0$ preserves all Apollonian cluster ensembles in the sense that if $M \in \mathcal{A}_n^0$ and $\mathcal{D}$ is a Descartes configuration, then

$$M(\mathbb{D}(\mathcal{P}_{\mathcal{D}})) = \mathbb{D}(\mathcal{P}_{\mathcal{D}}) \ . \tag{3.16}$$

As in the two-dimensional case, one can find "integral" n-dimensional Apollonian ensembles all of whose curvature×center coordinates lie in $\mathbb{Z}[\frac{2}{n-2}]$. It remains to study number-theoretic properties of such packings, generalizing §5 and results in [18].

The n-dimensional Apollonian group $\mathcal{A}_n^0$ satisfies different relations than the two-dimensional case. For $n = 3$ its generators satisfy relations associated to the "Hexlet" noted by Soddy [29], [30]:

$$(S_i S_j)^3 = I \quad \text{for } i \neq j. \tag{3.17}$$

The results of Maxwell [24, Table I, p. 91] imply that for $n = 3$ the Apollonian group is a Coxeter group with the defining relations above. As far as we know it is an open problem to determine a complete set of relations for the generators of $\mathcal{A}_n^0$ for $n \geq 4$.

Definition 3.4. The (unordered) *inverse-Apollonian group* $(\mathcal{A}_n^0)^{\perp}$ on $\mathbb{R}^n$ is the group of $(n+2) \times (n+2)$ matrices

$$(\mathcal{A}_n^0)^{\perp} = \langle S_1^{\perp}, S_2^{\perp}, \dots S_{n+2}^{\perp} \rangle \tag{3.18}$$

in which

$$S_1^{\perp} = \left[\begin{array}{c|cccc} -1 & 0 & 0 & \dots & 0 \\ \hline 2 & & & & \\ 2 & & & & \\ \vdots & & I_{n+1} & & \\ 2 & & & & \end{array} \right] \tag{3.19}$$

and $S_i^{\perp} = P_{(1i)}^T S_1^{\perp} P_{(1i)}$, where $P_{(1i)}$ is the permutation matrix for $(1i)$.

The action of $S_1^{\perp}$ on the Descartes configuration $\mathcal{D} = \{C_1, C_2, \dots, C_{n+2}\}$ is to send it to $\mathcal{D}'' = \{C_1, C_2'', \dots, C_n''\}$ where C_j'' denotes the inversion of C_j in the circle C_1, i.e.

$$S_1^{\perp} N_{\mathcal{D}} = N_{\mathcal{D}''} \ .$$

One can directly check that

$$(S_i^{\perp})^T Q_n S_i^{\perp} = Q_n \tag{3.20}$$

and $\mathbf{1}^T S_i \mathbf{1} > 0$, so that $(\mathcal{A}_n^0)^{\perp} \subseteq Iso^{\uparrow}(Q_n)$.

In dimension $n = 2$ one has the special relation

$$S_i^{\perp} = S_i^T \qquad \text{for} \qquad 1 \leq i \leq 4,$$

so that $(\mathcal{A}_2^0)^{\perp} = (\mathcal{A}_2^0)^T$. This symmetry, given by the transpose, no longer holds for $n \geq 3$. We also note that $(\mathcal{A}_n^0)^{\perp}$ is a group of integer matrices in all dimensions, while $\mathcal{A}_n^0$ is a group of integer matrices only for $n \leq 3$.

Definition 3.5. (i) The *(unordered) n-dimensional super-Apollonian group* $\tilde{\mathcal{A}}_n^0$ is the group generated by $\mathcal{A}^0$ and $(\mathcal{A}_n^0)^\perp$, with generators

$$\tilde{\mathcal{A}}_n^0 := \langle S_1, S_2, \dots, S_{n+2}, S_1^\perp, S_2^\perp, \dots, S_{n+2}^\perp \rangle \ . \qquad (3.21)$$

(ii) The *(ordered) n-dimensional super-Apollonian group* $\tilde{\mathcal{A}}_n$ is obtained by adjoining to $\tilde{\mathcal{A}}_n^0$ the permutation matrices $\{P_\sigma : \ \sigma \in Sym(n+2)\}$.

The super-Apollonian group $\tilde{\mathcal{A}}_n^0$ is contained in the group of automorphs $Aut(Q_n, \mathbb{Z}[\frac{2}{n-2}])$ of Q_n with coefficients in the ring $\mathbb{Z}[\frac{2}{n-2}]$ by (3.15) and (3.20). It is natural ask whether it is a finite index subgroup of $Aut(Q_n, \mathbb{Z}[\frac{2}{n-2}])$ and, if so, to determine its index. It is also an open problem to find a complete set of relations among the generators of $\tilde{\mathcal{A}}_n^0$, for $n \geq 3$.

Finally, one may consider *n-dimensional super-packings*, which we define to be the set of n-dimensional Descartes configurations in the orbit of a single Descartes configuration under the action of the super-Apollonian group. The n-dimensional super-Apollonian group $\tilde{\mathcal{A}}_n$ lies in $Mat_{(n+2)\times(n+2)}(\mathbb{Z}[\frac{2}{n-2}])$. Considering curvatures alone, we can start with a Descartes configuration having curvatures $(0, 0, 1, 1, \dots, 1)$ and construct a super-packing from it under the action of $\tilde{\mathcal{A}}_n^0$. We conjecture that this super-packing contains Descartes $(n+2)$-tuples similar to all integral Descartes $(n+2)$-tuples.

4. Integral and Rational Apollonian Sphere Ensembles

We now consider integrality and rationality properties for Apollonian sphere ensembles. The Apollonian group has an integral structure in dimensions 2 and 3, and retains an S-integral structure in all dimensions. Here S is a given finite set of primes and a rational number is *S-integral* if its denominator is divisible only by powers of primes in S.

Definition 4.1. An Apollonian sphere ensemble is *S-integral* if the curvature of every sphere in the ensemble is S-integral.

The recurrence relation between curvatures of two adjacent Descartes configurations, given in Theorem 3.3 as

$$a_1 + a_1' = \frac{2}{n-1}(a_2 + ... + a_{n+2}).$$

shows that S-integrality is preserved under this operation, for any S containing all primes dividing the denominator of $\frac{2}{n-1}$. More generally all entries of all matrices in the Apollonian group are S-integral, where S consists of the primes dividing the denominator of $\frac{2}{n-1}$. The same property persists for the super-Apollonian group in n-dimensions, since its extra generators are all integral matrices.

Theorem 4.1. *In each dimension $n \geq 2$ there exists an S-integral Apollonian sphere ensembles with S being the set of primes dividing $n-1$ if n is even and dividing $\frac{n-1}{2}$ if n is odd.*

Proof. It suffices to show that the Descartes equation

$$\mathbf{a}^T \mathbf{Q}_n \mathbf{a} = 0 \tag{4.22}$$

has a non-zero S-integral solution $\mathbf{a}$ for each $n \geq 2$. There is such a configuration $\mathcal{D}$ which is not only S-integral, but integral, with curvatures $(0, 0, 1, 1, ..., 1)$. It consists of two parallel hyperplanes separated by distance 2 together with n unit spheres whose centers comprise the vertices an $(n-1)$-dimensional simplex in a hyperplane parallel to the two hyperplanes in the configuration, and lying midway between them.

The other Descartes configurations in the Apollonian sphere ensemble and super-Apollonian sphere ensemble generated by this configuration are S-integral, where S is the set of primes dividing the denominator of $\frac{2}{n-1}$, since they have associated matrices $GN_{\mathcal{D}}$ for some G in the Apollonian group. ■

The Apollonian group and super-Apollonian group act on the set of S-integral Apollonian packings. For dimension $n = 2$ various number-theoretic questions related to the integers appearing in such packings were studied in [18]; in dimensions $n \geq 3$ the corresponding problems all remain open.

Next we consider S-integrality involving the sphere centers as well.

Definition 4.2. (i) An oriented Descartes configuration is *strongly S-integral* if if its associated matrix $N_{\mathcal{D}}$ has all entries S-integers.

(ii) An oriented Descartes configuration $\mathcal{D}$ is *super-strongly S-integral* if its associated augmented matrix $\tilde{N}_{\mathcal{D}}$ is S-integral.

We extend these definitions to Apollonian packings.

Definition 4.3. (i) An Apollonian sphere ensemble is *strongly S-integral* if every Descartes configuration $\mathcal{D}$ in the packing has S-integral matrix $N_{\mathcal{D}}$.

(ii) An Apollonian sphere ensemble is *super-strongly S-integral* if every Descartes configuration $\mathcal{D}$ in the packing has S-integral augmented matrix $\tilde{N}_{\mathcal{D}}$.

If a single Descartes configuration is strongly (resp. super-strongly) S-integral, then the Apollonian packing it generates is strongly (resp. super-strongly) S'-integral, where S' consists of S together with all primes dividing the denominator of $\frac{2}{n-1}$. For this reason it suffices to consider S-integrality for individual Descartes configurations.

For dimension $n = 2$, in part II we showed that strongly S-integral Descartes configurations existed, with $S = 1$, and that strongly integral Apollonian packings also existed. We also completely classified them, in the sense that we showed [17, Theorem 3.5] that under the action of the super-Apollonian group, the set of all strongly integral Descartes configurations formed exactly eight orbits ([17, Theorem 3.5]) .

In dimension $n = 2$, every strongly integral Descartes configuration is actually super-strongly integral! To show this it suffices to consider one Descartes configuration in each of the eight orbits above and verify that it has an integral augmented matrix $\tilde{N}_{\mathcal{D}}$, because the super-strong integrality property is preserved under the action of the super-Apollonian group ($n = 2$). For example, the strongly integral Descartes matrix

$$N_{\mathcal{D}} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix},$$

extends to the augmented Descartes matrix

$$\tilde{N}_{\mathcal{D}} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 2 & 1 \end{bmatrix}.$$

Similar integrality formulae hold for the other seven cases.

The existence of stongly S-integral Descartes configurations for some S is the same as the existence of Descartes configurations $\mathcal{D}$ having a rational augmented matrix $\tilde{N}_{\mathcal{D}}$.

Definition 4.4. A Descartes configuration $\mathcal{D}$ is *rational* if and only if its non-augmented matrix $N_{\mathcal{D}}$ is a rational matrix.

In this definition we could, alternatively, require that the augmented matrix $\tilde{N}_{\mathcal{D}}$ be rational, because the matrix $N_{\mathcal{D}}$ is rational if and only if the augmented matrix $\tilde{N}_{\mathcal{D}}$ is rational. Indeed, the last column of $\tilde{N}_{\mathcal{D}}$ is calculated from the entries of $N_{\mathcal{D}}$ using inversion in the unit circle, and this map sends the set of spheres with rational centers and rational curvatures into itself.

According to the augmented Euclidean Descartes theorem 2.4, rational Descartes configurations occur exactly in those dimensions n in which there exists an invertible rational matrix $\tilde{N}$ such that

$$\tilde{N}^T \mathbf{Q}_n \tilde{N} = \tilde{\mathbf{Q}}_n := \begin{bmatrix} 0 & 0 & -4 \\ 0 & 2I_n & 0 \\ -4 & 0 & 0 \end{bmatrix}, \tag{4.23}$$

that is, the quadratic form $\mathbf{Q}_n$ is rationally equivalent to the form $\tilde{\mathbf{Q}}_n$. We use this fact to show that in most higher dimensions rational Descartes configurations do not exist.

Theorem 4.2. *A necessary condition for a rational Descartes configuration to exist in dimension n is that $n = 2k^2$ or $(2k-1)^2$ for some positive integer k.*

To establish this result, we use the following lemma.

Lemma 4.3. *Given a Descartes configuration $\mathcal{D}$ in $\mathbb{R}^n$ its associated augmented matrix $\tilde{N}_{\mathcal{D}}$ has determinant satisfying*

$$\det(\tilde{N}_{\mathcal{D}})^2 = n2^{n+3}. \tag{4.24}$$

Proof. This follows from taking determinants in (2.38), since the right side has determinant -2^{n+4} while the left side has determinant $\det(\tilde{N}_{\mathcal{D}})^2 \det(\mathbf{Q}_n)$ and

$$\det(\mathbf{Q}_n) = -\frac{2}{n}. \tag{4.25}$$

To verify this last statement, we apply the following row operations to the matrix $\mathbf{Q}_n$. Add rows 2 through $n+2$ to the first row, to get a new first row that has all entries $-\frac{2}{n}$. Then add this row multiplied by $-\frac{1}{2}$ to each of the other rows. Aside from the first row, the first column is zero, and the lower right $(n+1) \times (n+1)$ matrix is the identity. But this matrix obviously has determinant $-\frac{2}{n}$. ∎

Proof of Theorem 4.2. A necessary condition for the existence of a Descartes configuration $\mathcal{D}$ whose augmented matrix $\tilde{N}_{\mathcal{D}}$ has rational entries is that $\det(\tilde{N}_{\mathcal{D}})$ be rational. This requires that $n2^{n+3}$ be the square of a rational number. By Lemma 4.3, this holds for even n if and only if n is twice a square, and for odd n if and only if n is an (odd) square. ■

We now prove the converse to Theorem 4.2.

Theorem 4.4. *In each dimension $n \geq 2$ which has n of the form $n = 2k^2$ or $(2k-1)^2$ for some positive integer k, there exists a rational Descartes configuration.*

This theorem is proved using the well-developed theory of equivalence of rational quadratic forms, cf. Cassels [6] or Conway [8]. We write $\mathbf{Q} \simeq_{\mathbb{Q}} \mathbf{Q}'$ to mean that the (rational) quadratic form $\mathbf{Q}$ is rationally equivalent to $\mathbf{Q}'$. To apply the decision procedure, we first diagonalize $\mathbf{Q}_n$ over the rationals, which we do for all $n \geq 2$.

Lemma 4.5. *For each $n \geq 2$, the Descartes quadratic form $\mathbf{Q}_n = I_{n+2} - \frac{1}{n}\mathbf{1}_{n+2}\mathbf{1}_{n+2}^T$ has*

$$\mathbf{Q}_n \ \simeq_{\mathbb{Q}} \ diag(\frac{n-1}{n}, \frac{n-2}{n-1}, \cdots, \frac{2}{3}, 2, 2, 2, -2). \tag{4.26}$$

Proof. We diagonalize the quadratic form as in Conway [8, pp. 92–94]. Set

$$M^{(n+2)} := \mathbf{Q}_n = (x_0 + y_0)I_{n+2} - y_0\mathbf{1}_{n+2}\mathbf{1}_{n+2}^T\,,$$

where $x_0 = \frac{n-1}{n}$, $y_0 = \frac{1}{n}$. At the j-th stage of reduction we will have

$$\mathbf{Q}_n \simeq_{\mathbb{Q}} \mathrm{diag}(d_1, d_2, \ldots, d_j, M^{(n+2-j)})\,,$$

where

$$M^{(n+2-j)} = (x_j + y_j)I_{n+2-j} - y_j\mathbf{1}_{n+2-j}\mathbf{1}_{n+2-j}^T \tag{4.27}$$

for certain x_j, y_j. The reduction step is

$$(W^{(j)})^T M^{(n+2-j)} W^{(j)} = \mathrm{diag}(d_{j+1}, M^{(n+1-j)})\,. \tag{4.28}$$

To specify $W^{(j)}$ we first let $W_m(\alpha)$ be the $m \times m$ real matrix

$$W_m(\alpha) = \begin{bmatrix} 1 & \alpha \cdots \alpha \\ & \\ \mathbf{0} & I_{m-1} \end{bmatrix}.$$

and we set

$$W^{(j)} := W_{m+2-j}\begin{pmatrix} y_j \\ x_j \end{pmatrix}. \tag{4.29}$$

Substituting this in (4.28), its left side yields a matrix with the form of the right side with

$$d_{j+1} = x_j\,,$$

and with x_{j+1}, y_{j+1} given by the recursion

$$y_{j+1} = y_j + \frac{y_j^2}{x_j}\,, \tag{4.30}$$

$$x_{j+1} + y_{j+1} = x_j - \frac{y_j^2}{x_j}\,. \tag{4.31}$$

Solving this recursion, by induction on j, one obtains

$$\begin{aligned} x_j &= \frac{n-j-1}{n-j}\,, & 0 \le j \le n-2, \\ y_j &= \frac{1}{n-j}\,, & 0 \le j \le n-2\,. \end{aligned}$$

This yields the diagonal elements

$$d_j = \frac{n-j-1}{n-j}, \quad 1 \le j \le n-3\,, \tag{4.32}$$

with

$$\mathbf{Q}_n \simeq_{\mathbb{Q}} \operatorname{diag}(\frac{n-1}{n}, \ldots, \frac{2}{3}, d_2, M^{(4)})\,.$$

We find $d_2 = x_3 = \frac{2}{3}$ and

$$M^{(4)} = (x_{n-2} + y_{n-2})I_4 - y_{n-2}\mathbf{1}_4\mathbf{1}_4^T = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = \mathbf{Q}_2\,.$$

For the final step in the reduction we use

$$W^T\,(\mathbf{Q}_2)\,W = \operatorname{diag}(2, 2, 2, -2) \tag{4.33}$$

with

$$W = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

This completes the reduction. ∎

Proof of Theorem 4.4. The theorem is equivalent to proving that if $n = 2k^2$ and $n = (2k-1)^2$ then

$$\mathbf{Q}_n \simeq_{\mathbb{Q}} \tilde{\mathbf{Q}}_n := \begin{bmatrix} 0 & 0 & -4 \\ 0 & 2I_n & 0 \\ -4 & 0 & 0 \end{bmatrix}.$$

We begin by noting the rational equivalence

$$\tilde{\mathbf{Q}}_n \simeq_{\mathbb{Q}} \mathrm{diag}(2, 2, \cdots, 2, -2) = \mathrm{diag}(2I_{n+1}, -2) \tag{4.34}$$

via the matrix

$$W_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2I_n & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Thus the theorem is equivalent to showing that $\mathbf{Q}_n$ is rationally equivalent to $\mathrm{diag}(2, 2, 2, ..., -2)$. Lemma 4.5 gives

$$\begin{aligned} \mathbf{Q}_n &\simeq_{\mathbb{Q}} \left(\frac{n-1}{n}, \frac{n-2}{n-1}, \dots, \frac{3}{2}, 2, 2, 2, -2\right), \\ &\simeq_{\mathbb{Q}} (n(n-1), (n-1)(n-2), \dots, 3 \cdot 2, 2, 2, 2, -2), \end{aligned} \tag{4.35}$$

using at the last step a conjugacy by $W = \mathrm{diag}(n, n-1, \dots, 2, 1, 1, 1, 1)$.

The Hasse-Minkowski theorem says that two rational quadratic forms of the same dimension are equivalent if and only they have the same signature, the ratio of their determinants is a nonzero square, and they are p-adically equivalent for all primes p, cf. Conway [8, p. 96ff]. Lemma 4.5 shows that the signatures of $\mathbf{Q}_n$ and $\mathrm{diag}(2, 2, 2, ..., 2, -2)$ agree, and the hypothesis $n = 2k^2$ or $n = (2k-1)^2$ is exactly the condition that the ratio of their determinants is a square of a rational, and it remains to check the p-adic invariants.

The p-adic invariants $\sigma_p(\mathbf{Q})$ are defined $(\bmod\ 8)$, and for a diagonal form $\mathbf{Q} = \mathrm{diag}(d_1, d_2, \dots, d_n)$, one has

$$\sigma_p(\mathbf{Q}) \equiv \sum_{j=1}^{n} \sigma_p(d_j) \pmod 8. \tag{4.36}$$

We recall formulas for $\sigma_p(d)$ when $d \in \mathbb{Z}$, cf. Conway [8, pp. 94–96]. Writer $d = bp^l$ with $(b, p) = 1$. For $p \geq 3$, and an even power $l = 2j$,

$$\sigma_p(d) \equiv p^{2j} \equiv 1 \pmod 8, \tag{4.37}$$

while for an odd power $l = 2j + 1$,

$$\sigma_p(d) \equiv \begin{cases} p \pmod 8 & \text{if } \left(\frac{b}{p}\right) = 1\,, \\ p + 4 \pmod 8 & \text{if } \left(\frac{b}{p}\right) = -1\,. \end{cases} \tag{4.38}$$

If $p = 2$ then for an even power $l = 2j$,

$$\sigma_2(d) \equiv b \pmod 8\,, \tag{4.39}$$

while for an odd power$l = 2j + 1$,

$$\sigma_2(d) \equiv \begin{cases} b & \text{if } b \equiv \pm 1 \pmod 8\,, \\ b + 4 & \text{if } b \equiv \pm 3 \pmod 8\,. \end{cases} \tag{4.40}$$

Now (4.35) gives

$$\sigma_p(\mathbf{Q}_n) \equiv \sum_{j=0}^{n-3} \sigma_p((n-j)(n-j-1)) + 3\sigma_p(2) + \sigma_p(-2) \pmod 8$$

while (4.34) gives

$$\sigma_p(\tilde{\mathbf{Q}}_n) \equiv \sum_{j=0}^{n-3} \sigma_p(2) + 3\sigma_p(2) + \sigma_p(-2) \pmod 8\,.$$

To show equality of these, it suffices to show that for all p,

$$\sum_{j=0}^{n-3} \sigma_p(2) \equiv \sum_{j=0}^{n-3} \sigma_p((n-j)(n-j-1)) \pmod 8 \tag{4.41}$$

holds whenever $n = 2k^2$ or $n = (2k-1)^2$.

Consider first the case that $p \geq 3$ is odd. Then each $\sigma_p(2) = 1$, so

$$\sum_{j=0}^{n-3} \sigma_p(2) \equiv n - 2 \pmod 8\,. \tag{4.42}$$

Now if $p \nmid (n-j)(n-j-1)$ then $\sigma_p((n-j)(n-j-1)) = 1$. The terms divisible by p occur in blocks of two consecutive terms, and we claim that if p divides j then

$$\sigma_p((j+1)j) + \sigma_p(j(j-1)) \equiv 2 \pmod 8. \tag{4.43}$$

Suppose $j = bp^l$, with where $(b, p) = 1$ and $l \geq 1$. If l is even, both terms on the left side of (4.43) are 1 (mod 8) by (4.37), while if K is odd, then if $p \equiv 1 \pmod 4$, then $\left(\frac{-1}{p}\right) = 1$, so the two terms both have values p (resp. $p+4$) according as $\left(\frac{b}{p}\right) = 1$ (resp. -1), and their sum is $2p \equiv 2 \pmod{8}$. If $p \equiv 3 \pmod{4}$, then $\left(\frac{-1}{p}\right) = -1$, so exactly one of $\left(\frac{\pm b}{p}\right)$ takes the value -1, and the two terms add up to $2p + 4 \equiv 2 \pmod{8}$. Thus (4.43) follows. Thus adding up the right side of (4.41) and grouping terms divisible by p in consecutive pairs gives

$$\sum_{j=0}^{n-3} \sigma_p((n-j)(n-j-1)) \equiv \sum_{j=0}^{n-3} 1 \equiv n-2 \ (\mathrm{mod} 8)\,. \qquad (4.44)$$

There remains an exceptional case where $p|n$, in which case $n(n-1)$ is divisible by p and is an un-paired term. Since $n = 2k^2$ or $(2k-1)^2$, thus $p^l \| n$ with l even, hence $\sigma_p(n(n-1)) = 1$ in this case, and (4.44) holds. This establishes (4.41) for $p \geq 3$.

Now consider the case $p = 2$. Certainly $\sigma_2(2) = 1$ so (4.42) holds. We claim that

$$\sigma_2((2j+1)2j) + \sigma_2(2j(2j-1)) \equiv 0 \pmod{8}\,. \qquad (4.45)$$

Write $2j = 2^l b$ with b odd, and by checking all possible cases using (4.39) and (4.40), one verifies (4.45). Suppose $n = 2k^2$. Then in the right side of (4.41) all terms pair except the first and last, and (4.45) yields

$$\begin{aligned} \sum_{j=0}^{n-3} \sigma_2((n-j(n-j-1)) &\equiv \sigma_2(n(n-1)) + \sigma_2(3 \cdot 2) \\ &= \begin{cases} -1 + -1 & \text{if} \quad k \equiv 0 \pmod{2}\,, \\ \ \ 1 + -1 & \text{if} \quad k \equiv 1 \pmod{2} \end{cases} \\ &= n-2 \pmod{8}\,, \end{aligned}$$

so (4.41) holds. If $n = (2k-1)^2 \equiv 1 \pmod{8}$ then all term pair except the last term, and (4.45) yields

$$\sum_{j=0}^{n-3} \sigma_2((n-j)(n-j-1)) = \sigma_2(3 \cdot 2) \equiv -1 \pmod{8}\,,$$

so (4.41) holds in this case. ■

In the dimensions covered by Theorem 4.4, rational Descartes configurations exist. Therefore there exist finite sets S for which S-integral configurations exist. As long as such an S is enlarged to include all prime divisors of $n-1$, all configurations in the Apollonian cluster ensemble generated by such a configuration will also be S-integral. One can then raise the question of classifying such ensembles; this appears difficult.

Theorem 4.4 establishes the existence of rational Descartes configurations in the given dimensions, but does not give a bound for the denominators of the rationals appearing in these configurations, i.e. an explicit value for S. As far as we know, it could be that in dimensions $n = 2k^2$ and $(2k+1)^2$ there exist super-strongly integral Descartes configurations, i.e. one could take $S = 1$. (Note that, if such configurations exist for $n > 2$, the Apollonian packing containing them would not inherit the super-strong integrality property.) We leave this as an open problem; results on it should be attainable using the theory of integral quadratic forms.

5. Duality Operator

In two dimensions we studied in part II a duality operation $\mathfrak{d}$ based on orthogonal spheres. This operator had an analogue operator D which was contained in the normalizer of the super-Apollonian group.

The duality operation based on orthogonal spheres generalizes to higher dimensions as follows. Given $n+1$ mutually tangent spheres in n dimensions, there is a unique sphere through their points of tangency, and this sphere is orthogonal to each of the given $n+1$ spheres, see Proposition 3.2. Thus, given a Descartes configuration of $n+2$ spheres C_i, we get a system of $n+2$ "orthogonal" spheres

$$\mathcal{D}^{\perp} := \{C_1^{\perp}, \ldots , C_{n+2}^{\perp}\},$$

where $C_i^{\perp}$ is associated to the $n+1$ spheres obtained by deleting C_i. When $n = 2$ the new spheres are mutually tangent and give a new Descartes configuration; this gives the "duality" operation D studied in parts I and II. For $n \geq 3$, however, the spheres are not mutually tangent. In fact for all n their curvatures satisfy a relation similar in form to the original (two-dimensional) Descartes relation, namely

$$\sum_{i=1}^{n+2} q_i^2 = \frac{1}{2}(\sum_{i=1}^{n+2} q_i)^2, \tag{5.1}$$

and not the Soddy-Gossett relation (2.1). (We omit a proof of this formula.) In particular, for $n \geq 3$ given a Descartes configuration $\mathcal{D}$, the set $\mathcal{D}^{\perp} := \{C_1^{\perp}, \ldots , C_{n+2}^{\perp}\}$ of orthogonal spheres is *not* a Descartes configuration, and the duality operation is *not* in $Iso^{\uparrow}(Q_n)$.

The question arises, are these $n+2$ "orthogonal" spheres in any special relation to one another? We answer this in terms of an inversive invariant of two arbitrary (not necessarily tangent) oriented spheres.

Definition 5.1. (i) The *separation* between two oriented spheres C_1 and C_2 with finite radii r_1 and r_2, and with centers distance d apart, as

$$\Delta(C_1, C_2) := \frac{d^2 - r_1^2 - r_2^2}{2r_1 r_2}. \tag{5.2}$$

provided both spheres are inwardly oriented or outwardly oriented, and is otherwise the negative of the right side of this formula.

(ii) The *separation* of an oriented sphere C_1 of finite radius r_1 and an oriented hyperplane C_2 is

$$\Delta(C_1, C_2) := \frac{d}{r_1}. \tag{5.3}$$

where d is the (signed) distance from the center $\mathbf{a}_1$ of C_1 to C_2, measured so that $d \geq 0$ if $\mathbf{a}_1$ is not in the interior of C_2 and C_1 is inwardly oriented, or if $\mathbf{a}_1$ is in the interior of C_2 and C_1 is outwardly oriented, and $d < 0$ otherwise.

(iii) The *separation* between two oriented hyperplanes C_1 and C_2 is

$$\Delta(C_1, C_2) := -\cos\theta. \tag{5.4}$$

where θ is the dihedral angle between the designated normals at a point of intersection.

The separation of two spheres is an inversive invariant ; that is,

$$\Delta(\mathfrak{g}(C_1), \mathfrak{g}(C_2)) = \Delta(C_1, C_2), \tag{5.5}$$

holds for any Möbius transformation $\mathfrak{g}$. This concept appears in Boyd [3], who introduced the term separation for it, but the concept [5] was used earlier by Mauldon [23] in 1962, who used the term *inclination* to mean the negative of $\Delta(C_1, C_2)$, and showed it was an inversive invariant.

[5] The idea of considering such an inversive invariant traces back to work of Clifford [7] in 1868 and of Darboux [13] in 1872. However, neither Clifford's nor Darboux' definition was precisely $\Delta(C_1, C_2)$. Clifford defines the *power of two spheres* to be the square distance of their centers less the sum of the squares of their radii, i.e., $d^2 - r_1^2 - r_2^2$, and Darboux also uses the same quantity, [13, p.350].

The separation $\Delta(C_1, C_2)$ of two spheres can be expressed in terms of their augmented curvature-center coordinates as

$$\begin{aligned}\Delta(C_1, C_2) &= \frac{1}{2}\tilde{\mathbf{w}}(C_1)^T \mathbf{K}_n(1)\tilde{\mathbf{w}}(C_2) \\ &= -\frac{1}{2}(\bar{a}(C_1)a(C_2) + a(C_1)\bar{a}(C_2)) + a(C_1)a(C_2)\sum_{j=1}^{n} x_j(C_1)x_j(C_2), \end{aligned} \quad (5.6)$$

where $\mathbf{K}_n(1)$ is given in (2.39). This formula can be proved by a simple algebraic calculation, cf. [20]. Using it, one can check that for two tangent spheres C_1 and C_2, $\Delta(C_1, C_2) = 1$, if (1) C_1 and C_2 are externally tangent, and both are inwardly oriented or outwardly oriented, or (2) C_1 and C_2 are internally tangent and one is inwardly oriented, the other is outwardly oriented. In all other cases two tangent spheres have $\Delta(C_1, C_2) = -1$, and orthogonal spheres are those with $\Delta(C_1, C_2) = 0$.

From Proposition 3.2 one obtains

$$\Delta(C^{\perp}, C_j) = 0 \qquad \text{for} \qquad 1 \le j \le n+1, \quad (5.7)$$

using (5.4), and these relations determine $C^{\perp}$ up to orientation. It can also be shown that if a set of tangent spheres $\{C_1, ..., C_{n+1}\}$ have oriented curvatures $\mathbf{a}_{n+1} = (a_1, ..., a_{n+1})$, and centers $\mathbf{x}_j$, then for either orientation the orthogonal sphere $C^{\perp}$ has oriented curvature q satisfying

$$q^2 = \frac{1}{2}\left(\frac{1}{n-1}(\sum_{j=1}^{n+1} a_j)^2 - \sum_{j=1}^{n+1} a_j^2\right), \quad (5.8)$$

and (oriented) center $\mathbf{x}$ satisfying

$$q\mathbf{x} = -\mathbf{a}_{n+1}(\frac{1}{2}\mathbf{Q}_{n-1})\mathbf{C}, \quad (5.9)$$

in which $\mathbf{C}$ is an $(n+1) \times n$ matrix whose j-th row is $a_j\mathbf{x}_j$, and $\mathbf{Q}_{n-1}$ is the Descartes form.

An oriented Descartes configuration in $\mathbb{R}^n$ is characterized in terms of separation as a set of $n+2$ oriented spheres each pair of which has $\Delta(C_i, C_j) = 1$, when $i \ne j$. Thus such a configuration has the following property.

Definition 5.2. A collection of oriented spheres is *equiseparated* if all values $\Delta(C_j, C_k)$ with $j \ne k$ are equal.

The equiseparation property can also be viewed as an *equiangularity* property, because for two oriented circles that intersect or touch one has

$$\Delta(C_1, C_2) = -\cos\theta, \tag{5.10}$$

where θ is the angle between oriented normals at a point of intersection of the two circles. We now show the duality operation preserves equiseparability in all dimensions; a further generalization appears in [20].

Theorem 5.1 (Equiseparation Theorem) *Given an oriented Descartes configuration $\mathcal{D} = (C_1, C_2, ..., C_{n+2})$ in $\mathbb{R}^n$, if the dual spheres are properly oriented then the (oriented) dual configuration $(C_1^\perp, C_2^\perp, ..., C_{n+2}^\perp)$ is equiseparated, with*

$$\Delta(C_j^\perp, C_k^\perp) = \frac{1}{n-1} \qquad \text{if} \qquad j \neq k. \tag{5.11}$$

Proof. In this result, the orientation assigned to the dual spheres in the theorem depends on all $n+2$ spheres in the Descartes configuration, and the orientation of $C_j^\perp$ cannot be consistently assigned from the $n+1$ oriented spheres $\{C_i : i \neq j\}$ alone. If all $n+2$ spheres C_j are inwardly oriented, then $n+1$ of the spheres $C_j^\perp$ will be inwardly oriented and one outwardly oriented, the last being the one of largest radius. If all but one of the $n+2$ spheres are inwardly oriented, and one outwardly oriented, then all $n+2$ spheres $C_j^\perp$ will be inwardly oriented.

Since the result is invariant under inversion, it suffices to prove it for a single Descartes configuration. We consider the special oriented Descartes configuration where the curvatures are (0,0,1,1, ... , 1). Here we have two parallel planes, which we take as $x_1 = \pm 1$, and n unit spheres, all with centers on the plane $x_1 = 0$. Their centers form a regular simplex in this plane. We may take one of these centers at $(0, \xi, 0, 0, \dots)$ where $\xi^2 = 2(n-1)/n$. Consider the "orthogonal" spheres that pass through the point $T = (1, \xi, 0, 0, \dots, 0)$. There are n such, and all but one of them is a plane containing T, $(-1, \xi, 0, 0 \dots, 0)$, and the centers of all but one of the original unit spheres. Since these centers are the vertices of a regular simplex, these $n-1$ "orthogonal" planes are equiangular satisfying (5.10), where θ is the angle between the normals of two facets of a regular n-simplex. It follows that these orthogonal planes satisfy (5.11). The final "orthogonal" sphere through T is orthogonal to the plane $x_1 = 1$ and all the n original unit spheres. Its center is thus $(1, 0, 0, \dots)$ and its radius is ξ. Hence it is also equiangular with the $n-1$ "orthogonal" planes, with $\cos\theta = -\frac{1}{n-1}$. (These angles are all eaqual to the one formed by connecting the vertices of a regular simplex to its center, i.e. the angle in a triangle of sides ξ, ξ and 2.) Finally, the last two "orthogonal" spheres meet at the same angle in the plane $x_1 = 0$. ■

6. Loxodromic Sequences of Tangent Spheres

As our final topic we turn to a concept studied by Coxeter [11], concerning loxodromic sequences of tangent spheres. Coxeter defines a *loxodromic sequence* of spheres as being a sequence where each successive set of $n+2$ spheres are mutually tangent. Thus if the sequence of curvatures is

$$\ldots\ a_{-2},\ a_{-1},\ a_0,\ a_1,\ a_2,\ \ldots$$

then each successive set of $n+2$ spheres satisfies the Soddy relation $\mathbf{a}^T Q_n \mathbf{a} = 0$, so that by Theorem 2.3 they also satisfy a linear recurrence

$$a_i + a_{i+n+2} = \frac{2}{n-1}(a_{i+1} + \ldots + a_{i+n+1}). \tag{6.1}$$

For $n = 3$, Coxeter proves in [12] that the sequence

$$\ldots \Delta_{0,-2}\ ,\quad \Delta_{0,-1}\ ,\quad \Delta_{0,0}\ ,\quad \Delta_{0,1}\ ,\quad \Delta_{0,2}\ ,\ \ldots$$

where $\Delta_{ij} = \Delta(C_i, C_j)$ is the separation between circles C_i and C_j, also satisfies the linear recurrence (6.1). This is slightly unexpected, since while Δ is dimensionless, it involves the square of the distance between the centers, while the other quantities that obey the recurrence are the curvatures a_i and $a_i\mathbf{x}_i$, which is the product of curvatures and centers. We prove a slightly more general result.

Theorem 6.1 (Separation Formula) *Given $n+1$ mutually tangent spheres $C_1, \ldots C_{n+1}$ with disjoint interiors, let C_0 and C_{n+2} be the two spheres that are tangent to each of these. Let C' be an arbitrary sphere, and let Δ_i be the separation between C_i and C'. Then*

$$\Delta_0 + \Delta_{n+2} = \frac{2}{n-1}(\Delta_1 + \ldots + \Delta_{n+1}). \tag{6.2}$$

Proof. Without loss of generality, we may assume that all the curvatures qare non-zero, because the separation is invariant under inversions. Furthermore, we may assume that the center of C_0 is the origin, and that its curvature a_0 is not zero. Let the curvatures of $C_1, \ldots, C_{n+2}, C'$ be $a_1, \ldots, a_{n+2}, a'$, and let their centers be $\mathbf{x}_1, \ldots, \mathbf{x}_{n+2}, \mathbf{x}'$. Then

$$\begin{aligned} \Delta_i &= \frac{1}{2}\left(a_i a' |\mathbf{x}_i - \mathbf{x}'|^2 - \frac{a_i}{a'} - \frac{a'}{a_i}\right) \\ &= \frac{1}{2}\left(a'\left(a_i|\mathbf{x}_i|^2 - \frac{1}{a_i}\right) + a_i\left(a'|\mathbf{x}'|^2 - \frac{1}{a'}\right) - 2a'a_i\mathbf{x}_i^T\mathbf{x}'\right). \end{aligned}$$

We know that each of the sequences $(a_0, \ldots, a_{n+2})$, $(a_0\mathbf{x}_0, \ldots, a_{n+2}\mathbf{x}_{n+2})$ satisfy the linear recurrence, so it is sufficient to prove that the quantities

$$t_i = a_i|\mathbf{x}_i|^2 - \frac{1}{a_i}$$

do also. Notice that the dependence on a' and $\mathbf{x}'$ has been eliminated. Thus $t_0 = -1/a_0$, and for $i > 0$,

$$\begin{aligned} t_i &= a_i\left(\frac{1}{a_0} + \frac{1}{a_i}\right)^2 - \frac{1}{a_i} \\ &= \frac{a_i + 2a_0}{a_0^2}. \end{aligned}$$

Therefore the vector $\mathbf{t}^T = (t_0, t_1, \ldots, t_{n+1})$ is given by

$$\mathbf{t} = \frac{1}{a_0^2}(\mathbf{a} + 2a_0(\mathbf{1}_{n+2} - 2\mathbf{e}_0)),$$

where $\mathbf{a}^T = (a_0, \ldots, a_{n+1})$, ${\mathbf{e}_0}^T = (1, 0, \ldots, 0)$. Simple algebra now verifies that $\mathbf{t}^T Q_n \mathbf{t} = 0$. Hence the t's also satisfy the linear recurrence (6.1). ■

As Coxeter (1997,[12]) points out, if we are in two dimensions and $S' = S_0$, then $(\Delta_0, \Delta_1, \Delta_2, \Delta_3) = (-1, 1, 1, 1)$ and the sequence extends uniquely to

$$-1, 1, 1, 1, 7, 17, 49, 145, 415, 1201, 3473, 10033, 28999, 83809, 242209, 700001, 2023039, \ldots$$

(sequence A045821 in Sloane [27]). In three dimensions the corresponding (unique) sequence is

$$-1, 1, 1, 1, 1, 5, 7, 13, 25, 49, 89, 169, 319, 601, 1129, 2129, 4009, \ldots$$

(this is sequence A027674 [27]).

7. Conclusion

This series of papers studied various group theoretic problems raised by geometrically defined groups associated to Apollonian packings. It obtained fairly complete answers when the dimension $n = 2$, but left many open problems, particularly in dimensions $n \geq 3$.

In the case of the Apollonian group and super-Apollonian groups in n-dimensions there remain a number of open questions. One is the problem of determining their exact normalizers. Another is that of establishing the index of the super-Apollonian group in the automorphism group $Aut(Q_n,\ \mathbb{Z}[\frac{2}{n-1}]\ )$. It is also an open problem to obtain finite presentations for these groups, for $n \geq 3$. We noted that for $n \geq 4$ the Apollonian group no longer produced a sphere-packing. Is this related to the non-integral nature of the matrices in the Apollonian group, for $n \geq 4$? Can one define a discontinuous action of this group on a real space $\times$ p-adic spaces corresponding to primes dividing the denominator of $\frac{2}{n-1}$? There also remain open questions connected with the fact that these groups are integral over the ring $\mathbb{Z}[\frac{2}{n-1}]$. Various number-theoretic questions in this direction are raised in the concluding section of the companion paper [18]. Finally, in §4 we showed that S-integral Descartes configurations exist in dimensions of the form $n = 2k^2$ or $(2k-1)^2$, for some finite set S of primes, which depends on the dimension, but we did not determine an explicit set S that can be used in such dimensions. It is an open problem to find a minimal set S. In particular do there exist Descartes configurations which are super-strongly integral ($S = \{1\}$) in all such dimensions?

This paper treated Apollonian packings in Euclidean space. Such packings can also be constructed in spherical n-space (positive curvature), and in hyperbolic n-space (negative curvature). In spherical and hyperbolic space the notion of center and radius of a sphere change, but there exist suitable analogues of (augmented) curvature-center coordinates for Descartes configurations, see [21]. Various questions raised in this paper may have interesting analogues in these geometries.

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