ELSEVIER

Wave Motion 36 (2002) 311–333

WAVE MOTION

www.elsevier.com/locate/wavemoti

On the application of the Wiener–Hopf technique to problems in dynamic elasticity☆

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Received 9 January 2002; received in revised form 3 March 2002; accepted 3 March 2002

Abstract

Many problems in linear elastodynamics, or dynamic fracture mechanics, can be reduced to Wiener–Hopf functional equations defined in a strip in a complex transform plane. Apart from a few special cases, the inherent coupling between shear and compressional body motions gives rise to coupled systems of equations, and so the resulting Wiener–Hopf kernels are of matrix form. The key step in the solution of a Wiener–Hopf equation, which is to decompose the kernel into a product of two factors with particular analyticity properties, can be accomplished explicitly for scalar kernels. However, apart from special matrices which yield commutative factorizations, no procedure has yet been devised to factorize exactly general matrix kernels.

This paper shall demonstrate, by way of example, that the Wiener–Hopf approximant matrix (WHAM) procedure for obtaining approximate factors of matrix kernels (recently introduced by the author in [SIAM J. Appl. Math. 57 (2) (1997) 541]) is applicable to the class of matrix kernels found in elasticity, and in particular to problems in QNDE. First, as a motivating example, the kernel arising in the model of diffraction of skew incident elastic waves on a semi-infinite crack in an isotropic elastic space is studied. This was first examined in a seminal work by Achenbach and Gautesen [J. Acoust. Soc. Am. 61 (2) (1977) 413] and here three methods are offered for deriving distinct non-commutative factorizations of the kernel. Second, the WHAM method is employed to factorize the matrix kernel arising in the problem of radiation into an elastic half-space with mixed boundary conditions on its face. Third, brief mention is made of kernel factorization related to the problems of flexural wave diffraction by a crack in a thin (Mindlin) plate, and body wave scattering by an interfacial crack.

PACS: 43.20.Gp; 43.40.Dx; 46.40.Cd; 46.50.+a; 46.70.De; 62.30.+d; 81.70.Cv; 30E10; 47A68; 78A45

Keywords: Elastic waves; Wiener–Hopf technique; Matrix Wiener–Hopf equations; Scattering; Diffraction; Acoustics; Geometrical theory of diffraction; Padé approximants; Non-destructive testing

☆ It is with great honour, and much pleasure, that I present a paper in this special issue of Wave Motion dedicated to Professor Jan D. Achenbach on the occasion of his 65th birthday. His work in theoretical and applied mechanics, with specific focus on the dynamical propagation of disturbances in solids, has been at the very forefront of international research for over 40 years. This output, which continues unabated, remains a lasting source of insight and inspiration to all interested in wave motion in solids.

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PII: S0165-2125(02)00027-6

1. Introduction

This paper will focus on one of the most significant areas that Professor Achenbach has opened up during his long and distinguished career, namely, the diffraction of elastic waves by material defects. Together with students and co-workers he generalized the geometrical theory of diffraction (GTD) from the scalar problems encountered in acoustics or optics to the vectorial models of stress waves found in solids [10,11]. This work directly contributed to the birth of the now enormous field of quantitative non-destructive evaluation (QNDE) of solids using ultrasonic waves. In order to apply the GTD it is necessary to determine the *diffraction coefficients* for the scattered field generated at sharp edges, corners or interfacial boundaries of defects. If the diffraction coefficients are known then the diffracted fields can be readily calculated; the complete scattered field is determined by *adding* the various diffraction contributions from all the edges to the specular components.

In general, the determination of the key diffraction coefficients relies on the solution of canonical *inner* problems based on the *local geometry* and material properties. Thus, for example, take a smooth finite crack with a closed bounding curve, and irradiated by an elastic wave of arbitrary incident direction and wavelength much less than the typical crack lengthscales. Further, the host body could be inhomogeneous also with a typical inhomogeneity *lengthscale* long compared to the incident wavelength. Then the *local* field diffracted by a small segment of the crack's edge is essentially that determined by solution of the problem shown in Fig. 1. That is, scattering of an elastic wave by a *semi-infinite planar crack* in a homogeneous medium. In this *inner* region the crack is of infinite extent along its edge, and the incident field is inclined at an arbitrary angle to this edge so that the model problem is fully three-dimensional in nature. In a series of classic papers, Achenbach and Gautesen [12–14] employed Fourier transform analysis combined with the *Wiener–Hopf technique* to solve the above described inner problem for a variety of different forcings, and thence to employ these to obtain the GTD solution in several cases: including an infinitely long planar crack of finite-width, and a penny shaped crack. Note that the same approach can also be successfully applied to the related area of dynamic fracture—under an applied loading the growth rate of a crack will depend on the *local* behaviour at the crack tip, i.e. only related to the stress-intensity factors (SIFs) of a canonical inner problem (see, e.g., papers [13,21]).

The success of the GTD to QNDE, and the asymptotic scheme employed for dynamic fracture analysis, relies on being able to solve the canonical inner problems. Fortunately, there has evolved, over many years, a wide variety of powerful tools to tackle boundary value problems in canonical domains, and these have been employed, and continue to be employed (e.g. in the context of electromagnetic diffraction by conical bodies, see [15]) with great success in obtaining analytical solutions. Such explicit results often provide deep insight into the physical processes associated with a given model, and are especially convenient for rapid computation. On the other hand, direct numerical schemes for tackling *inner* problems have not proved particularly successful, as they have to accurately cope with singularities at corners and rapid or instantaneous changes in field quantities. Thus, as QNDE is

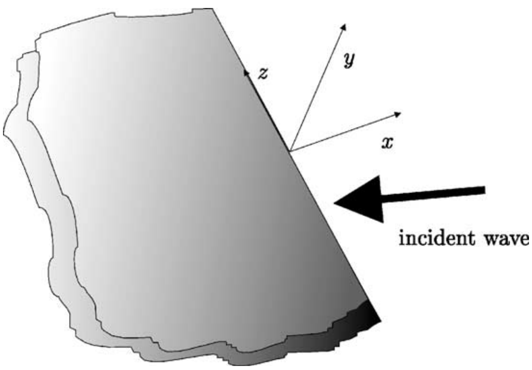

Fig. 1. Semi-infinite crack in an isotropic elastic space.

developed ever further there is a need to extend GTD methods to incorporate increasingly more complex and *realistic* canonical inner problems. For example, the widespread use of composite materials in engineering applications means that the model problem shown in Fig. 1 (discussed in [12]) needs to be extended to include anisotropy [34].

This paper will focus on the Wiener–Hopf technique [33,36] which has proved remarkably successful over some 70 years since its invention, at obtaining analytical solutions for an enormous variety of physical boundary or initial value problems. A bibliography incorporating these application areas may be found in [3]. As noted previously, Achenbach and Gautesen's [12] pioneering work in the area of elastic wave scattering employed the Wiener–Hopf technique. There is, however, a severe limitation to the method. For complicated boundary value problems the solution procedure yields vector rather than scalar Wiener–Hopf equations, for which, as will be shown, there is in general no solution procedure. The presence of both shear and compressional waves in solids in general gives rise to such matrix systems, and this has curtailed the efficacy and hence the success of the Wiener–Hopf technique in the field.

It is the aim of this paper to discuss particular matrix Wiener–Hopf systems that are obtained in elasticity, most of which arise in dynamic situations. There will be an attempt to address the natural canonical problems which are of interest in QNDE and which can be seen as generalizations of Achenbach and Gautesen's [12] original model (Fig. 1). For brevity only the crucial step in the Wiener–Hopf analysis, namely the factorization of the kernel, will be discussed in this paper. In the following section, a short discussion is offered on the kernel product decomposition procedure, and its inherent difficulty for matrix kernels. However, for certain matrix kernels a commutative (or Khrapkov) [28,29] factorization is obtainable, and this is also introduced here. In Section 3, Achenbach and Gautesen's kernel is looked at in some detail, and an explicit factorization determined by several methods. Section 4 is concerned with a particular canonical problem in elastodynamics [2] which offers one of the simplest matrix Wiener–Hopf systems whose kernel does not permit a commutative product factorization. An approximate technique, based on the introduction of Padé approximants, whose accuracy can be increased to very high levels, is employed to effect the factorization. Section 5 makes mention of a couple of kernels from related, and physically important, problems in elasticity, and briefly discusses their factorization. Concluding remarks are offered at the end of Section 5.

2. Scalar and commutative matrix factorization

The Wiener–Hopf technique offers one of the very few approaches to obtaining exact solutions to integral equations. It is appropriate for equations of either first or second kind, defined over a half-line, and which have difference kernels. Physical boundary or initial value problems which have the characteristic feature of data defined on semi-infinite or equivalent planes often reduce to Wiener–Hopf form. The solution method usually proceeds directly from the boundary value problem to an equation in the complex plane defined in an infinite strip. This is accomplished by Fourier transformation (or other appropriate transform) or Green's theorem, and the essential physical details are manifested in the singularity structure of the Fourier transformed difference kernel often just called the Wiener–Hopf kernel, $K(\alpha)$ say [33], where α is the transform parameter. This kernel is usually arranged to be singularity free in a strip of finite-width in the complex α-plane which contains the real line as $|\alpha| \to \infty$. The strip is denoted henceforth as $\mathcal{D}$, but note that it need not necessarily be straight, i.e. it does not have to enclose the whole real line. Solution of the Wiener–Hopf equation is straightforward once $K(\alpha)$ is decomposed into a product of two functions $K^+(\alpha)$ and $K^-(\alpha)$, where $K^+(\alpha)$ is regular and zero-free in the region above and including the strip $\mathcal{D}$, denoted $\mathcal{D}^+$, and $K^-(\alpha)$ is regular and zero-free[1] in the region below and including $\mathcal{D}$, denoted $\mathcal{D}^-$. Further,

[1] This property is necessary as the inverses of the product factors must also be analytic in their respective half-planes in order for a successful rearrangement of a Wiener–Hopf equation [33].

$K^{\pm}(\alpha)$ must have at worst algebraic growth as $|\alpha| \to \infty$ in $\mathcal{D}^{\pm}$, respectively. Cauchy's integral theorem provides a convenient method for obtaining the explicit *sum factorization* of a function, e.g. of the form

$$g(\alpha) = g^{+}(\alpha) + g^{-}(\alpha), \tag{1}$$

where $\pm$ denote the above analyticity properties, and so by exponentiation the following *product factorization* can be deduced (see Theorem C of Noble [33]):

$$K(\alpha) = \exp[g(\alpha)] = K^{+}(\alpha)K^{-}(\alpha), \tag{2}$$

$$K^{\pm}(\alpha) = \exp[g^{\pm}(\alpha)] = \exp\left\{\frac{\pm 1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{\log K(\zeta)}{\zeta-\alpha}\,\mathrm{d}\zeta\right\}, \tag{3}$$

where the integration path lies in $\mathcal{D}$ and α lies above (below) ζ for $K^{+}(\alpha)$ ($K^{-}(\alpha)$). Sometimes a limiting procedure is necessary to ensure convergence of these integrals.

For complicated problems in which $K(\alpha)$ is a scalar function, there are several technical difficulties associated with computing integrals of the above form (3) (see [6]). It is, therefore, often desirable to replace the exact kernel with a simpler one which approximates it accurately in the strip of analyticity, and from which approximate and well behaved product factors, $K^{\pm}(\alpha)$ can be derived. This is an interesting and quite well studied area, see, e.g. [5,17,18,30], and the references mentioned therein, but for brevity it will only be discussed further in Section 4 in the context of matrix factorization.

This paper is concerned with vector functional Wiener–Hopf equations in which the kernel is a square matrix $\mathbf{K}(\alpha)$. This presents two potential obstacles: one has been addressed already by the author and is not considered here [8]; the second is the generalization of the factorization procedure indicated by (2). For any matrix function, $\mathbf{K}(\alpha)$, the logarithm $\mathbf{G}(\alpha) = \log(\mathbf{K}(\alpha))$ can be defined, for example by its power series expansion $\log[\mathbf{I} + (\mathbf{K}(\alpha) - \mathbf{I})] = \mathbf{K}(\alpha) - \mathbf{I} - (1/2)(\mathbf{K}(\alpha) - \mathbf{I})^2 + \cdots$, where $\mathbf{I}$ is the identity matrix, or otherwise. Then the sum split of $\mathbf{G}(\alpha)$ is accomplished by the Cauchy type integrals acting on each element of the matrix. However, the final step to obtain the product factors, i.e.

$$\exp\{\mathbf{G}^{+}(\alpha) + \mathbf{G}^{-}(\alpha)\} = \exp\{\mathbf{G}^{+}(\alpha)\}\exp\{\mathbf{G}^{-}(\alpha)\} \tag{4}$$

is true *if and only if* $\mathbf{G}^{+}(\alpha)$ and $\mathbf{G}^{-}(\alpha)$, and hence the product factors of $\mathbf{K}(\alpha)$, commute:[2]

$$\mathbf{K}^{+}(\alpha)\mathbf{K}^{-}(\alpha) \equiv \exp\{\mathbf{G}^{+}(\alpha)\}\exp\{\mathbf{G}^{-}(\alpha)\} = \exp\{\mathbf{G}^{-}(\alpha)\}\exp\{\mathbf{G}^{+}(\alpha)\} \equiv \mathbf{K}^{-}(\alpha)\mathbf{K}^{+}(\alpha). \tag{5}$$

This difficulty was first discussed by Heins [24] in 1950! For general matrices, $\mathbf{K}(\alpha)$, this commutative property will not hold, and, although no procedure is presently available for tackling such arbitrary matrix Wiener–Hopf problems, Gohberg and Krein [22] have proved the existence of product factors in all cases.

Fortunately, many physically interesting problems naturally give rise to kernels which, although intrinsically matrix in form, have a commutative factorization or can be reworked (by pre- and/or post-multiplication by suitable matrices) into such form. Khrapkov [28,29], in papers concerned with the stresses in elastostatic wedges with notches, was the first author to express the commutative factorization in a form which indicates the sub-algebra associated with this class of kernels, and this is discussed further below. Many other authors have also examined the commutative case, including Daniele [19], Rawlins [35], Hurd [25], and Jones [27]. Meister and Speck [32] introduced an alternative and rather elegant decomposition method to that offered by Khrapkov (which is touched on in the following section), but the precise relationship between the two approaches has yet to be fully established. Citations to other such works are to be found in [3]. As the Khrapkov form will be useful in later sections, it is valuable to briefly discuss here the commutative factorization procedure.

[2] The necessity for commutativity may easily be verified by expanding both sides of (4) using the usual definition of the Taylor series for matrix exponentials.

2.1. Khrapkov factorization

As will now be demonstrated, matrices which permit a commutative factorization can be reduced to scalar factorizations of the above forms (1) and (2). The approach taken here is that employed by Khrapkov [28,29], which is in fact *equivalent* to the exponential form suggested by others, e.g. Heins [24] (see also [33]) and Daniele [19].

For purposes of clarity, the following discussion is restricted to matrix kernels of 2×2 order, but the arguments can, in fact, be generalized to matrices of any size [27]. It is *generally accepted* that any square matrix kernel which permits a commutative factorization can be rearranged (by suitable pre- and post-multiplication by entire matrices) into the following Khrapkov form:

$$\mathbf{K}(\alpha) = a(\alpha)\mathbf{I} + b(\alpha)\mathbf{J}(\alpha), \tag{6}$$

where $\mathbf{I}$ is the identity matrix, $a(\alpha)$, $b(\alpha)$ are arbitrary scalar functions of α analytic in the strip $\mathcal{D}$ and with algebraic behaviour at infinity, and $\mathbf{J}(\alpha)$ is a square matrix with entire elements. Further, $\mathbf{J}(\alpha)$ has the important property that its elements are of algebraic growth at infinity and its square is

$$\mathbf{J}^2(\alpha) = \Delta^2(\alpha)\mathbf{I} \tag{7}$$

in which $\Delta^2(\alpha)$ is a *polynomial* in α. As a consequence of this last property, a commutative product factorization of $\mathbf{K}(\alpha)$ can be posed as

$$\mathbf{K}(\alpha) = \mathbf{Q}^-(\alpha)\mathbf{Q}^+(\alpha), \tag{8}$$

where $\mathbf{Q}^{\pm}(\alpha)$ and their inverses are analytic in the regions $\mathcal{D}^{\pm}$, and these factors take the general form

$$\mathbf{Q}^{\pm}(\alpha) = r_{\pm}(\alpha)\left\{\cosh[\Delta(\alpha)\theta_{\pm}(\alpha)]\mathbf{I} + \frac{1}{\Delta(\alpha)}\sinh[\Delta(\alpha)\theta_{\pm}(\alpha)]\mathbf{J}(\alpha)\right\}. \tag{9}$$

Note that $\Delta(\alpha)$ has branch-points in both half-planes in general, but these are not present in $\mathbf{Q}^{\pm}(\alpha)$ because the latter are in fact functions of the square of $\Delta(\alpha)$. The problem is reduced to solving for the scalar functions $r_{\pm}(\alpha)$, $\theta_{\pm}(\alpha)$ and these can be determined by multiplying $\mathbf{Q}^+(\alpha)$ by $\mathbf{Q}^-(\alpha)$ and equating with (6). It is found that

$$r_+(\alpha)r_-(\alpha)\cosh[\Delta(\alpha)(\theta_+(\alpha)+\theta_-(\alpha))] = a(\alpha), \tag{10}$$

$$\frac{r_+(\alpha)r_-(\alpha)}{\Delta(\alpha)}\sinh[\Delta(\alpha)(\theta_+(\alpha)+\theta_-(\alpha))] = b(\alpha), \tag{11}$$

or rearranging gives

$$[r_+(\alpha)r_-(\alpha)]^2 = a^2(\alpha) - \Delta^2(\alpha)b^2(\alpha), \tag{12}$$

$$\theta_+(\alpha)+\theta_-(\alpha) = \frac{1}{\Delta(\alpha)}\tanh^{-1}\left[\frac{\Delta(\alpha)b(\alpha)}{a(\alpha)}\right]. \tag{13}$$

So, $\theta_{\pm}(\alpha)$ are found from the sum split formula (1)

$$\theta_{\pm}(\alpha) = \frac{\pm 1}{2\pi\mathrm{i}}\int_{\mathcal{C}}\frac{\tanh^{-1}[\Delta(\zeta)f(\zeta)]}{\Delta(\zeta)(\zeta-\alpha)}\,\mathrm{d}\zeta, \quad \pm\alpha \text{ above } \mathcal{C}, \tag{14}$$

and $r_{\pm}(\alpha)$ are given employing the representation (2).

The Khrapkov commutative factorization just described does indeed produce product factors with the requisite analyticity properties. Further, the inverses of $\mathbf{Q}^{\pm}(\alpha)$ are also analytic in $\mathcal{D}^{\pm}$. Unfortunately, as will be shown in later sections, in general a Wiener–Hopf kernel cannot be cast into the form (6). However, it will be shown that an *approximate* factorization can be achieved via a two-stage process: first to cast the kernel into Khrapkov form, in which $\mathbf{J}(\alpha)$ satisfies (7) but is not entire, and then second to remove singularities in respective half-planes arising from an approximation to $\mathbf{J}(\alpha)$.

3. Diffraction by a semi-infinite crack in an elastic solid

Almost certainly the first non-trivial matrix Wiener–Hopf kernel to be studied in the context of elastodynamics was that discussed by Achenbach and Gautesen [12,13]. It is interesting in that, although simple in form, it does not in fact permit a commutative factorization. However, the associated Wiener–Hopf equation can still be solved fairly easily, as will be demonstrated, by deriving product factors $\mathbf{K}^{-}(\alpha)$, $\mathbf{K}^{+}(\alpha)$, analytic in respective overlapping half-planes $\mathcal{D}^{-}$, $\mathcal{D}^{+}$, but whose inverses are *not* analytic in these respective regions. The latter have offending simple poles which are easily *removed* to enable the usual Wiener–Hopf procedure to follow through. This *trick* is not satisfactory in general, because the inverse product factor may contain branch-cut and other complicated singularities in the required domain of analyticity which cannot be easily removed.

Gohberg and Krein's result [22] indicates that a product split definitely exists for which $\mathbf{K}^{\pm}(\alpha)$ and $[\mathbf{K}^{\pm}]^{-1}(\alpha)$ are analytic in $\mathcal{D}^{\pm}$, and so the purpose of this section is to derive their forms here. First, it is instructive to offer Achenbach and Gautesen's *pseudo-factorization*, and this will be followed by the method offered by Meister and Speck [32]. Finally, and not previously presented in the literature, an explicit non-commutative product factorization derived via the Khrapkov form will be discussed in the last subsection.

Fig. 1 illustrates the model examined by Achenbach and Gautesen [12]. A longitudinal plane wave, of radian frequency ω, is incident upon a semi-infinite crack (occupying the plane $y = 0$, $-\infty < z < \infty$, $x < 0$ as shown) in an isotropic homogeneous elastic solid. For steady-state motions, the material is defined in terms of the density, ρ, and wavenumbers, k_{L} and k_{T}, of longitudinal and transverse body waves, respectively. The ratio of these wavenumbers is defined as

$$k = \frac{k_{\mathrm{T}}}{k_{\mathrm{L}}}, \tag{15}$$

which is always greater than unity. The forcing wave is skew incident onto the crack edge, and so its wavenumber component in the z-direction, defined as

$$k_{\mathrm{L}}\eta, \tag{16}$$

say, is non-zero. Note that $0 < \eta \leq 1$. If a Fourier transform in the x-direction is taken, with transform variable α scaled on k_{L}, then the boundary value problem can be reduced to two Wiener–Hopf equations, one scalar equation for motions symmetric about the plane $y = 0$ and the other a coupled pair of equations relating antisymmetric strains and displacements. The kernel of the latter Wiener–Hopf system, of interest in this article, can be written as the 2×2 matrix

$$\mathbf{K}(\alpha) = \frac{1}{\eta^2+\alpha^2}\begin{pmatrix} \eta^2\delta(\sqrt{\eta^2+\alpha^2}) + \alpha^2 d(\sqrt{\eta^2+\alpha^2}) & \alpha\eta(\delta(\sqrt{\eta^2+\alpha^2}) - d(\sqrt{\eta^2+\alpha^2})) \\ \alpha\eta(\delta(\sqrt{\eta^2+\alpha^2}) - d(\sqrt{\eta^2+\alpha^2})) & \alpha^2\delta(\sqrt{\eta^2+\alpha^2}) + \eta^2 d(\sqrt{\eta^2+\alpha^2}) \end{pmatrix}, \tag{17}$$

where

$$d(\zeta) = \frac{-R(\zeta)}{k^2\delta(\zeta)}, \tag{18}$$

in which $\zeta = \sqrt{\eta^2+\alpha^2}$,

$$R(\zeta) = (2\zeta^2 - k^2)^2 - 4\zeta^2\gamma(\zeta)\delta(\zeta) \tag{19}$$

is the Rayleigh function and

$$\gamma(\zeta) = (\zeta^2-1)^{1/2}, \qquad \delta(\zeta) = (\zeta^2-k^2)^{1/2}. \tag{20}$$

The latter functions have branch-points, respectively, at $\zeta = \pm 1$, $\zeta = \pm k$, and the associated cuts are taken to $\pm i\infty$. Thus, the strip of analyticity for the kernel (17) is defined to enclose the real line in the α-plane, indented to pass *below* the cuts running from $+\sqrt{1-\eta^2}$, $+\sqrt{k^2-\eta^2}$, and above the cuts emanating from $-\sqrt{1-\eta^2}$, $-\sqrt{k^2-\eta^2}$. The functions $\gamma(\zeta)$, $\delta(\zeta)$ are made single-valued by choosing the Riemann surface such that

$$\gamma(0) = -\mathrm{i}, \qquad \delta(0) = -\mathrm{i}k. \tag{21}$$

With this choice the Rayleigh function (19) has just two zeros in the ζ-plane, located on the real line at

$$\zeta = \pm k_0, \tag{22}$$

where it can be shown that

$$k_0 > k > 1, \tag{23}$$

and so $R(\sqrt{\eta^2+\alpha^2})$ has zeros at

$$\alpha = \alpha_0 = \pm\sqrt{k_0^2-\eta^2}, \tag{24}$$

which are also real.

The kernel (17) is identical to that studied by Achenbach and Gautesen [12] apart from a scalar constant. However, the notation is slightly different to that employed there in order to be consistent with later sections of this article. Three approaches are now presented to factorize this kernel, the last being useful to following sections which introduce the Wiener–Hopf approximant matrix (WHAM) method.

3.1. Achenbach and Gautesen's factorization

The product factorization of the matrix kernel (17) offered by Achenbach and Gautesen [12] is

$$\mathbf{K}(\alpha) = \hat{\mathbf{K}}^-(\alpha)\hat{\mathbf{K}}^+(\alpha), \tag{25}$$

where

$$\hat{\mathbf{K}}^+(\alpha) = \frac{1}{\eta-\mathrm{i}\alpha}\mathbf{D}^+(\alpha)\mathbf{C}(\alpha), \qquad \hat{\mathbf{K}}^-(\alpha) = \frac{1}{\eta+\mathrm{i}\alpha}\mathbf{C}(\alpha)\mathbf{D}^-(\alpha), \tag{26}$$

$$\mathbf{C}(\alpha) = \begin{pmatrix} \alpha & -\eta \\ -\eta & -\alpha \end{pmatrix}, \qquad \mathbf{D}^\pm(\alpha) = \begin{pmatrix} d^\pm & 0 \\ 0 & \delta^\pm \end{pmatrix}, \tag{27}$$

in which $d^\pm$ and $\delta^\pm$ are the scalar product factors of the scalar functions $d(\sqrt{\eta^2+\alpha^2})$ and $\delta(\sqrt{\eta^2+\alpha^2})$, respectively. Clearly, $\hat{\mathbf{K}}^-(\alpha)$ is analytic in the lower half-plane with an explicit simple pole in $\mathcal{D}^+$ at $\alpha = \mathrm{i}\eta$, but its inverse is

$$[\hat{\mathbf{K}}^-]^{-1}(\alpha) = \frac{1}{\eta-\mathrm{i}\alpha}\begin{pmatrix} \dfrac{1}{d^-} & 0 \\ 0 & \dfrac{1}{\delta^-} \end{pmatrix}\mathbf{C}(\alpha), \tag{28}$$

which is not analytic in $\mathcal{D}^-$ because of the pole at $\alpha = -\mathrm{i}\eta$. Similarly, $\hat{\mathbf{K}}^+(\alpha)$ is analytic in $\mathcal{D}^+$ but $[\hat{\mathbf{K}}^+(\alpha)]^{-1}$ has a simple pole at $\alpha = +\mathrm{i}\eta$. In contrast, the following two procedures will yield *proper* product factorizations of $\mathbf{K}(\alpha)$.

3.2. Meister and Speck's factorization

It is interesting to offer a factorization proposed by Meister and Speck [32] because it relies on writing $\mathbf{K}(\alpha)$ as a sum of the two singular matrices

$$\mathbf{A}(\alpha) = \frac{1}{\eta^2+\alpha^2}\begin{pmatrix} \alpha^2 & -\alpha\eta \\ -\alpha\eta & \eta^2 \end{pmatrix}, \qquad \mathbf{B}(\alpha) = \frac{1}{\eta^2+\alpha^2}\begin{pmatrix} \eta^2 & \alpha\eta \\ \alpha\eta & \alpha^2 \end{pmatrix}. \tag{29}$$

The latter have the useful properties

$$\mathbf{AB} = \mathbf{BA} = \mathbf{0}, \tag{30}$$

and

$$\mathbf{A}^2 = \mathbf{A}, \qquad \mathbf{B}^2 = \mathbf{B}. \tag{31}$$

Hence, by inspection

$$\mathbf{K}(\alpha) = d(\sqrt{\eta^2+\alpha^2})\mathbf{A}(\alpha) + \delta(\sqrt{\eta^2+\alpha^2})\mathbf{B}(\alpha), \tag{32}$$

which may be expressed as

$$\mathbf{K}(\alpha) = (d^-\mathbf{A}(\alpha) + q_2\delta^-\mathbf{B}(\alpha))\left(\frac{1}{q_1}\mathbf{A}(\alpha) + \frac{1}{q_2}\mathbf{B}(\alpha)\right)(d^+q_1\mathbf{A}(\alpha) + \delta^+\mathbf{B}(\alpha)) \tag{33}$$

for arbitrary constants q_1, q_2, and $d^{\pm}$, $\delta^{\pm}$ are given as before. Note that the left most matrix in (33) is analytic in $\mathcal{D}^-$ if q_2 is chosen to *remove* the pole at $\alpha = -\mathrm{i}\eta$ occurring in both **A** and **B**. It is easy to show that the required value is

$$q_2 = \left.\frac{d^-}{\delta^-}\right|_{\alpha\to-\mathrm{i}\eta}, \tag{34}$$

and similarly to keep the right most matrix pole free in the upper half-plane then

$$q_1 = \left.\frac{\delta^+}{d^+}\right|_{\alpha\to+\mathrm{i}\eta}. \tag{35}$$

Now, $d(\sqrt{\eta^2+\alpha^2})$ and $\delta(\sqrt{\eta^2+\alpha^2})$ are even functions of α and so

$$q_1 = \frac{1}{q_2} = \lambda, \tag{36}$$

say. It now remains, to effect a successful factorization of $\mathbf{K}(\alpha)$, to factorize the inner matrix. This is achieved by Meister and Speck [32] more-or-less by inspection, which is possible in this case because there are just two simple poles at $\alpha = \pm\mathrm{i}\eta$. Here, for brevity, just the result is stated

$$\left(\frac{1}{\lambda}\mathbf{A}(\alpha) + \lambda\mathbf{B}(\alpha)\right) = \frac{1}{\lambda(1+\lambda^2)}\begin{pmatrix} \dfrac{\alpha-\mathrm{i}\lambda^2\eta}{\alpha-\mathrm{i}\eta} & \mathrm{i} \\ \mathrm{i}\dfrac{\lambda^2\alpha-\mathrm{i}\eta}{\alpha-\mathrm{i}\eta} & 1 \end{pmatrix}\begin{pmatrix} \dfrac{\alpha+\mathrm{i}\lambda^2\eta}{\alpha+\mathrm{i}\eta} & -\mathrm{i}\dfrac{\lambda^2\alpha+\mathrm{i}\eta}{\alpha+\mathrm{i}\eta} \\ -\mathrm{i}\lambda^2 & \lambda^2 \end{pmatrix}, \tag{37}$$

where the left (right) hand matrix together with its inverse is analytic in $\mathcal{D}^-$ ($\mathcal{D}^+$). Multiplying the relevant terms in (33) and (37) yields Meister and Speck's factorization

$$\mathbf{K}(\alpha) = \mathbf{K}^-(\alpha)\mathbf{K}^+(\alpha), \tag{38}$$

$$\mathbf{K}^-(\alpha) = \frac{1}{\sqrt{\lambda(1+\lambda^2)}}\begin{pmatrix} \dfrac{\alpha d^- - \mathrm{i}\lambda\eta\delta^-}{\alpha-\mathrm{i}\eta} & \dfrac{\mathrm{i}}{\lambda}\dfrac{\alpha\lambda d^- + \mathrm{i}\eta\delta^-}{\alpha+\mathrm{i}\eta} \\ \mathrm{i}\dfrac{\alpha\lambda\delta^- - \mathrm{i}\eta d^-}{\alpha-\mathrm{i}\eta} & \dfrac{1}{\lambda}\dfrac{\alpha\delta^- + \mathrm{i}\lambda\eta d^-}{\alpha+\mathrm{i}\eta} \end{pmatrix}, \tag{39}$$

$$\mathbf{K}^+(\alpha) = \frac{1}{\sqrt{\lambda(1+\lambda^2)}}\begin{pmatrix} \dfrac{\alpha d^+ + \mathrm{i}\lambda\eta\delta^+}{\alpha+\mathrm{i}\eta} & -\mathrm{i}\dfrac{\alpha\lambda\delta^+ + \mathrm{i}\eta d^+}{\alpha+\mathrm{i}\eta} \\ -\mathrm{i}\lambda\dfrac{\alpha\lambda d^+ - \mathrm{i}\eta\delta^+}{\alpha-\mathrm{i}\eta} & \lambda\dfrac{\alpha\delta^+ - \mathrm{i}\lambda\eta d^+}{\alpha-\mathrm{i}\eta} \end{pmatrix}. \tag{40}$$

3.3. Modified Khrapkov factorization

The first step in the factorization procedure is to rearrange the kernel (17) into *Khrapkov-like* form. This may be achieved by choosing

$$a(\alpha) = \tfrac{1}{2}(d + \delta), \qquad b(\alpha) = \tfrac{1}{2}(d - \delta) \tag{41}$$

in (6), and

$$\mathbf{J}(\alpha) = \frac{1}{\alpha^2 + \eta^2} \begin{pmatrix} \alpha^2 - \eta^2 & -2\alpha\eta \\ -2\alpha\eta & -(\alpha^2 - \eta^2) \end{pmatrix} \tag{42}$$

so that

$$\mathbf{J}^2(\alpha) = \mathbf{I}, \tag{43}$$

i.e. $\Delta(\alpha) \equiv 1$ in the split functions $\mathbf{Q}^{\pm}(\alpha)$ (9). The other scalar quantities can be shown to be

$$r^{\pm}(\alpha) = \sqrt{d^{\pm}\delta^{\pm}}, \qquad \theta_{\pm}(\alpha) = \frac{1}{2}\log\left(\frac{d^{\pm}}{\delta^{\pm}}\right). \tag{44}$$

The commutative product split (8) would complete the factorization were if not for the fact that $\mathbf{J}(\alpha)$ is not entire as required, but has poles at $\alpha = \pm \mathrm{i}\eta$. Thus, it is necessary to remove the pole in $\mathbf{Q}^{\pm}(\alpha)$ in the upper (lower) half-plane, and this can be achieved in the following straightforward manner. A meromorphic matrix function is introduced into the factorization (8)

$$\mathbf{K}(\alpha) = \mathbf{Q}^{-}(\alpha)\mathbf{M}(\alpha)\mathbf{M}^{-1}(\alpha)\mathbf{Q}^{+}(\alpha), \tag{45}$$

so that

$$\mathbf{K}^{-}(\alpha) = \mathbf{Q}^{-}(\alpha)\mathbf{M}(\alpha), \qquad \mathbf{K}^{+}(\alpha) = \mathbf{M}^{-1}(\alpha)\mathbf{Q}^{+}(\alpha), \tag{46}$$

are analytic in $\mathcal{D}^{-}$, $\mathcal{D}^{+}$, respectively. Note that the introduction of $\mathbf{M}(\alpha)$ in this particular fashion leads to a *non-commutative factorization* as clearly

$$\mathbf{K}^{+}(\alpha)\mathbf{K}^{-}(\alpha) \neq \mathbf{K}(\alpha). \tag{47}$$

The form of $\mathbf{M}(\alpha)$ is chosen to have the same singularities as $\mathbf{J}(\alpha)$, to have the same reflection properties as $\mathbf{J}(\alpha)$, i.e. diagonal elements even in α and off-diagonal elements odd in α, and to tend to the identity matrix as $|\alpha| \to \infty$. Thus, an acceptable form of $\mathbf{M}(\alpha)$ is assumed to be

$$\mathbf{M}(\alpha) = \begin{pmatrix} 1 + \dfrac{a}{\alpha^2 + \eta^2} & \dfrac{\alpha b}{\alpha^2 + \eta^2} \\ \dfrac{\alpha c}{\alpha^2 + \eta^2} & 1 + \dfrac{d}{\alpha^2 + \eta^2} \end{pmatrix}, \tag{48}$$

where a, b, c, and d have to be evaluated. Pre-multiplying this matrix by $\mathbf{Q}^{-}(\alpha)$ and expanding and collecting terms with double poles at $\alpha = \pm \mathrm{i}\eta$ gives

$$\frac{\sinh\theta^{-}(\alpha)}{(\alpha^2 + \eta^2)^2} \begin{pmatrix} (\alpha^2 - \eta^2)a - 2\alpha^2\eta c & \alpha(\alpha^2 - \eta^2)b - 2\alpha\eta d \\ -\alpha(\alpha^2 - \eta^2)c - 2\alpha\eta a & -(\alpha^2 - \eta^2)d - 2\alpha^2\eta b \end{pmatrix}. \tag{49}$$

It is clear that this expression has simple poles rather than double poles if and only if

$$a = \eta c, \qquad d = -\eta b, \tag{50}$$

in which case $\mathbf{Q}^-(\alpha)\mathbf{M}(\alpha)$ can be shown to have the singular terms

$$\frac{\sinh\theta^-}{\alpha^2+\eta^2}\begin{pmatrix} \eta c\coth\theta^- + (\alpha^2-\eta^2) - \eta c & \alpha b\coth\theta^- - (2\eta - b)\alpha \\ \alpha c\coth\theta^- - (2\eta + c)\alpha & -\eta b\coth\theta^- - (\alpha^2-\eta^2) - \eta b \end{pmatrix}. \tag{51}$$

The pole at $\alpha = -\mathrm{i}\eta$ (in $\mathcal{D}^-$) can be removed from the left most column by setting

$$c = \frac{2\eta}{\coth\theta^+(\mathrm{i}\eta) - 1}, \tag{52}$$

where the relation $\theta^-(-\mathrm{i}\eta) = \theta^+(+\mathrm{i}\eta)$ has been utilized, and similarly the right-hand column gives

$$b = \frac{2\eta}{\coth\theta^+(\mathrm{i}\eta) + 1}. \tag{53}$$

Thus, $\mathbf{M}(\alpha)$ has been explicitly determined, and the specified values (50), (52) and (53) yield $\mathbf{K}^-(\alpha)$ analytic in $\mathcal{D}^-$ as required.

It now remains to deduce two points: first that $[\mathbf{K}^-(\alpha)]^{-1}$ is also analytic in $\mathcal{D}^-$, and second that this choice of $\mathbf{M}(\alpha)$ also renders $\mathbf{K}^+(\alpha)$ (46) and its inverse regular in $\mathcal{D}^+$. As regards the former, as $\mathbf{K}^-(\alpha)$ is analytic in $\mathcal{D}^-$ then $[\mathbf{K}^-(\alpha)]^{-1}$ will be too unless $|\mathbf{K}^-(\alpha)|$ introduces spurious singularities into $\mathcal{D}^-$. Clearly, $\mathbf{Q}^-(\alpha)$ does not do this as $|\mathbf{Q}^-(\alpha)| = (r^-(\alpha))^2$, and

$$|\mathbf{M}(\alpha)| = \left(1 + \frac{\eta c}{\alpha^2+\eta^2}\right)\left(1 - \frac{\eta b}{\alpha^2+\eta^2}\right) - \frac{\eta^2 bc}{(\alpha^2+\eta^2)^2} \equiv 1, \tag{54}$$

in view of (50), (52) and (53). Thus, $|\mathbf{K}^-(\alpha)| = |\mathbf{Q}^-(\alpha)||\mathbf{M}(\alpha)|$ is regular in $\mathcal{D}^-$ as required. Note that the particular form of $\mathbf{M}(\alpha)$ (48) is *chosen* to enable the determinant to take the constant value unity. As regards $\mathbf{K}^+(\alpha)$, it can be shown to be regular in $\mathcal{D}^+$ by multiplying out $[\mathbf{M}(\alpha)]^{-1}\mathbf{Q}^+(\alpha)$ and checking the coefficients of the simple and double poles. However, it is easier to note from the symmetry in $\mathbf{K}(\alpha)$, $\mathbf{Q}^+(\alpha)$ and $\mathbf{M}(\alpha)$, that $\mathbf{K}^+(\alpha)$ can be expressed as

$$\mathbf{K}^+(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}[\mathbf{K}^-(-\alpha)]^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{55}$$

from which it follows immediately that $\mathbf{K}^+(\alpha)$ and $[\mathbf{K}^+(\alpha)]^{-1}$ are both analytic in $\mathcal{D}^+$.

In summary, three different factorization procedures have been presented for the matrix kernel $\mathbf{K}(\alpha)$ given in (17). The first does not yield inverse product factors with the correct analyticity properties, and so cannot be seen as a successful split in the sense defined by Gohberg and Krein [22]. Note that all three product factorizations are *non-commutative*, and are surprisingly different in their forms. Only the last approach, which introduces a meromorphic matrix $\mathbf{M}(\alpha)$ offers a constructive algorithmic approach to the factorization procedure, and, as shall now be shown, is generalizable to more complex kernels.

4. The elastic dock problem

The aim of this section is to demonstrate the WHAM procedure recently proposed by the author [3] for obtaining explicit *approximate* product factors for a general class of non-commutative matrix kernels $\mathbf{K}(\alpha)$. It must be emphasized that the only other constructive approach offered in the literature for general matrix factorization is that presented by Wickham [37] and Lewis et al. [31]. However, their approach, although having several aspects in common with the present method, employs integral equation methods which do not yield explicit kernel factors. It is also much more difficult to apply in practice. Usually matrices yield non-commutative factorizations because they contain multiple branch-cut functions, infinite sequences of poles or zeros, or a combination of both of these singularity structures. In certain special cases the factorization procedure can be completed by subtraction of a finite or infinite family of poles from both sides of the Wiener–Hopf equation (see the previous section for an example of the former, and for the latter, see works by Idemen [26], Abrahams [1], Abrahams and Wickham [9] and references quoted therein). The method to be expounded here is based on the replacement of certain scalar components of the matrix function by *Padé approximants*. This allows the level of approximation to be increased to very high accuracies whilst at the same time offering a surprisingly simple factorization form. Note that the actual boundary value problem that the modified kernel satisfies is immaterial (although this topic will be addressed in a forthcoming article by the author); all that is required is that the exact and modified kernels take values which are close to each other in the strip of analyticity $\mathcal{D}$ and that the approximate kernel be factorizable by some means.

The clearest method for illustrating the factorization procedure is to examine a specific example, in this case the diffraction of monochromatic elastic waves by a discontinuity in boundary conditions on an elastic half-space. The forcing shall be taken to induce only displacements in the (x, y)-plane (plane strain), where the body occupies the region $y > 0$, $-\infty < z < \infty$, and the boundary is taken as traction-free on $y = 0$, $x > 0$ but fixed (displacements zero) on $y = 0$, $x < 0$. This is the elastic analogue of the classical semi-infinite dock problem in linear water wave theory; here the Rayleigh surface waves on the traction-free part of the surface are reflected and scattered into body waves by the rigid portion of the boundary. Following [2], Fourier transforms can be employed to reduce the boundary value problem to a matrix Wiener–Hopf equation in which the kernel has the form

$$\mathbf{K}(\alpha) = \frac{1}{R(\alpha)}\begin{pmatrix} -\mathrm{i}\alpha[2\alpha^2 - k^2 - 2\gamma(\alpha)\delta(\alpha)] & -k^2\delta(\alpha) \\ k^2\gamma(\alpha) & -\mathrm{i}\alpha[2\alpha^2 - k^2 - 2\gamma(\alpha)\delta(\alpha)] \end{pmatrix}. \tag{56}$$

Here, as before, α is the transform parameter, and k, $R(\alpha)$, $\gamma(\alpha)$ and $\delta(\alpha)$ are given in (15), (19) and (20).

In elastodynamic problems, the presence in the α-plane of two branch-cut functions $\gamma(\alpha)$, $\delta(\alpha)$, leads in general to matrix Wiener–Hopf equations with non-commutative kernels. The boundary value problem discussed herein is merely the simplest non-trivial example of this class, and other more interesting physical cases will be mentioned in the following section. The approximate approach to be employed here is applicable to all such problems, and was first propounded for a model concerned with diffraction at an interface between two (scalar) acoustic materials [3]. The procedure followed is along the lines indicated at the end of the previous section, but with one extra step, namely the *approximation* of the kernel. Firstly, the kernel (56) is rearranged into a form which contains a scalar function with finite branch-cuts. This function is then replaced by its Padé approximant, and an upper bound on the error in taking the approximate kernel can be estimated. A partial factorization is achieved using the Khrapkov (commutative) formulation, but the approximate factors achieved in this way contain poles in their indicated half-planes of analyticity. Finally, by insertion of suitable meromorphic matrices, the offending poles are removed. This yields an explicit, but approximate non-commutative matrix decomposition.

4.1. *Approximate kernel*

Following the approach of Section 3.3, the kernel for the elastic dock problem (56) can be made to look to be of Khrapkov form if it is rewritten as

$$\mathbf{K}(\alpha) = a(\alpha)\mathbf{I} + b(\alpha)\mathbf{J}(\alpha), \tag{57}$$

where

$$a(\alpha) = \frac{-\mathrm{i}\alpha}{R(\alpha)}[2\alpha^2 - k^2 - 2\gamma(\alpha)\delta(\alpha)], \qquad b(\alpha) = \frac{k^2 f(\alpha)\delta(\alpha)}{R(\alpha)}, \tag{58}$$

$$\mathbf{J}(\alpha) = \begin{pmatrix} 0 & -\dfrac{1}{f(\alpha)} \\ f(\alpha) & 0 \end{pmatrix}, \tag{59}$$

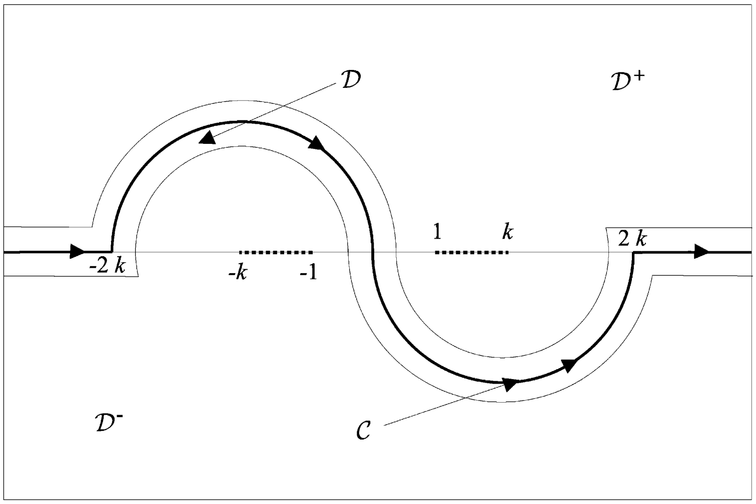

Fig. 2. The complex α-plane illustrating the branch-points of $f(\alpha)$ (60) at ± 1, $\pm k$ and the position of the cuts. Also shown are the upper and lower regions $\mathcal{D}^{\pm}$ with a strip of common analyticity $\mathcal{D}$ inside which lies the inverse contour $\mathcal{C}$ of the integrals in (82) and (85). The strip $\mathcal{D}$ is suitably chosen so that $\mathcal{C}$ is bounded away from the branch-points.

and

$$f(\alpha)=\sqrt{\frac{\gamma(\alpha)}{\delta(\alpha)}}=\left(\frac{\alpha^2-1}{\alpha^2-k^2}\right)^{1/4}. \tag{60}$$

As required

$$\mathbf{J}^2(\alpha)=-\mathbf{I}, \tag{61}$$

but $\mathbf{J}(\alpha)$ is not itself entire as it contains $f(\alpha)$, which has two finite branch-cuts at $\alpha\in[1,k]$, $\alpha\in[-1,-k]$ joining the branch-points at $1, k$ and $-1, -k$, respectively (see Fig. 2). Also, $f(\alpha)\to 1$ as $|\alpha|\to\infty$ in any direction. This function is replaced by its $[N/N]$ Padé approximant, denoted $f_N(\alpha)$, where

$$f(\alpha)\approx f_N(\alpha)=\frac{P_N(\alpha)}{Q_N(\alpha)}, \tag{62}$$

and $P_N(\alpha)$, $Q_N(\alpha)$ are the polynomials

$$P_N(\alpha)=a_0+a_1\alpha^2+a_2\alpha^4+\cdots+a_N\alpha^{2N}, \tag{63}$$

$$Q_N(\alpha)=1+b_1\alpha^2+b_2\alpha^4+\cdots+b_N\alpha^{2N}, \tag{64}$$

in which N is any positive integer. The notation employed here is slightly different from that given in [16] but is consistent with [3], in which details regarding the properties of Padé approximants are given. The essential points to mention here are that the coefficients of $P_N(\alpha)$, $Q_N(\alpha)$, namely a_n, b_n are determined *uniquely* by equating the Taylor series expansion of $f(\alpha)$, (60), with that of $f_N(\alpha)$, (62), up to order α^{4N} at any point(s) of regularity. Here, in fact, two-point Padé approximants are employed (discussed in some detail in [5]), where the points at the origin and infinity are chosen for convenience. This ensures that $f_N(\infty)=1$, which means that the approximate factorization is *exact* at infinity. Further, the $2N$ zeros and $2N$ poles of $f_N(\alpha)$ are simple and all lie on the real line segments $\alpha\in[1,k]$, $\alpha\in[-1,-k]$. The N positive real zeros of $P_N(\alpha)$, $Q_N(\alpha)$ are denoted as p_n, and q_n, respectively, are ordered as

$$1<p_1<p_2<p_3<\cdots<p_N<k,\qquad 1<q_1<q_2<q_3<\cdots<q_N<k, \tag{65}$$

and the negative zeros are positioned at $-p_n$, and $-q_n$, respectively. Thus, the Padé approximant $f_N(\alpha)$ replaces the finite branch-cuts of $f(\alpha)$ with a distribution of zeros and poles along the same line segments, and, as will be shown, approximates the function very closely away from these singularities.

The crucial point to note here is that the approximate kernel

$$\mathbf{K}_N(\alpha) = a(\alpha)\mathbf{I} + b(\alpha)\mathbf{J}_N(\alpha), \tag{66}$$

where the scalar $f(\alpha)$ has been replaced by its Padé approximant $f_N(\alpha)$ in

$$\mathbf{J}_N(\alpha) = \begin{pmatrix} 0 & -\dfrac{1}{f_N(\alpha)} \\ f_N(\alpha) & 0 \end{pmatrix}, \tag{67}$$

and $a(\alpha)$, $b(\alpha)$ are given in (58), will be shown to be *exactly factorizable* (cf. the kernel (17)). However, first an argument must be given as to why these factors do indeed approximate the exact factors of $\mathbf{K}(\alpha)$. If $\mathbf{K}^+(\alpha)$ is the product factor of $\mathbf{K}(\alpha)$, analytic, of algebraic growth and with a regular inverse in $\mathcal{D}^+$, then because of the properties of functions of a complex variable, its maximum modulus on $\mathcal{D}^+$ must occur on its boundary, namely $\partial\mathcal{D}^+$ (which includes the point at infinity). The approximation to $\mathbf{K}^+(\alpha)$, $\mathbf{K}_N^+(\alpha)$, which has the same analyticity property and is exact at infinity, also has its maximum modulus on $\partial\mathcal{D}^+$. Therefore, because the difference function $\mathbf{K}^+(\alpha) - \mathbf{K}_N^+(\alpha)$ is also analytic in $\mathcal{D}^+$ so it attains its maximum modulus on $\partial\mathcal{D}^+ \equiv \partial\mathcal{D}$, which is at a *finite point* in the strip $\mathcal{D}$. By the same arguments $\mathbf{K}^-(\alpha) - \mathbf{K}_N^-(\alpha)$ is analytic in $\mathcal{D}^-$ and attains its maximum modulus at a finite point on its bounding strip in $\mathcal{D}$, which for the purposes of this argument, is also taken as $\partial\mathcal{D}$.

Now, suppose the approximate kernel has the known factors

$$\mathbf{K}_N(\alpha) = \mathbf{K}_N^+(\alpha)\mathbf{K}_N^-(\alpha), \tag{68}$$

and by definition $\mathbf{K}(\alpha)$ and $\mathbf{K}_N(\alpha)$ tend to the same value as $|\alpha| \to \infty$ in $\mathcal{D}$. Further, the particular choice of strip $\mathcal{D}$ (see Fig. 2), which is bounded by unity away from the branch-cuts of $f(\alpha)$, implies that the absolute error, $|1 - [\mathbf{K}(\alpha)]^{-1}/\mathbf{K}_N(\alpha)|$ is bounded by a small constant, ε say for $\alpha \in \partial\mathcal{D}$. Thus,

$$\mathbf{K}(\alpha) - \mathbf{K}_N(\alpha) = \varepsilon\mathbf{K}(\alpha)\mathbf{G}(\alpha), \quad 0 < \varepsilon \ll 1, \tag{69}$$

$$\max|\mathbf{G}(\alpha)| = 1, \quad \alpha \in \partial\mathcal{D}. \tag{70}$$

It is now possible to show, from the above considerations, that

$$\mathbf{K}_N^\pm(\alpha) = \mathbf{K}^\pm(\alpha)(1 + \mathrm{o}(1)), \quad \alpha \in \partial\mathcal{D}, \tag{71}$$

but it is expected, in fact, that the error in the approximation is actually $\mathrm{O}(\varepsilon)$ in the strip $\mathcal{D}$ (which can be proved for certain classes of scalar kernels [5]). Furthermore, the error in approximating $\mathbf{K}^\pm(\alpha)$ by $\mathbf{K}_N^\pm(\alpha)$ actually *decreases* as α moves into the interior regions of analyticity $\mathcal{D}^\pm$ away from the point of maximum error on $\mathcal{D}$. Hence it can be expected, and indeed this is borne out by numerical experiment [2], that for a given Padé number the actual error in the factorization at an interior point of interest is in fact a good deal smaller than that suggested by the upper bound ϵ.

To determine the size of maximum error, ϵ, for a given Padé number it is simplest to define a (percentage) error function as

$$e_N(\alpha) = 100 \times \left|1 - \frac{f_N(\alpha)}{f(\alpha)}\right|, \quad \alpha \in \mathcal{D}. \tag{72}$$

As $f(\alpha)$ is an even function, it is sufficient to confine attention to values of $e_N(\alpha)$ on the right half of a typical contour $\mathcal{C}$ lying in $\mathcal{D}$ (Fig. 2), henceforth denoted as $\mathcal{C}_1$. A convenient parametric representation of the semi-infinite

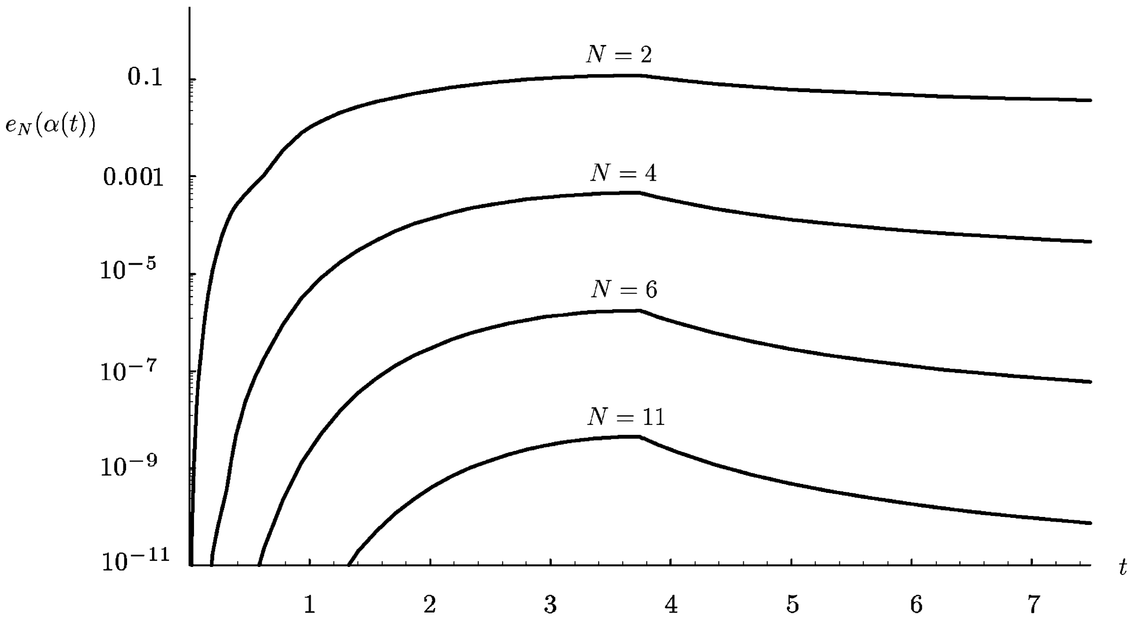

Fig. 3. The error $e_N(\alpha(t))$, (72), plotted on a log-scale versus the parameter t, (73), for $N = 2, 4, 6, 11$ and $\nu = 0.3$.

contour $\mathcal{C}_1$ is

$$\alpha = \begin{cases} k\left(1+\exp\left\{\dfrac{\mathrm{i}\pi(t-2k)}{2k}\right\}\right), & 0 < t < 2k, \\ t, & t > 2k, \end{cases} \tag{73}$$

and in Fig. 3, $e_N(\alpha(t))$ is illustrated for a range of approximant numbers and typical Poisson's ratio $\nu = 0.3$ ($k = 1.8703$). The ordinate is given on a logarithmic scale because of the dramatic increase in accuracy with increasing N. From this graph remarkable accuracy can be seen for modest Padé number; for example one can be confident that a factorization is accurate to better than 5×10^{-4}% error with a Padé approximation $N = 4$, or 2×10^{-6}% for $N = 6$, for this value of Poisson's ratio. Thus, it can be concluded that a satisfactory approximate non-commutative factorization of $\mathbf{K}(\alpha)$ (56) can be obtained to any specified accuracy as long as the factors $\mathbf{K}_N^{\pm}(\alpha)$ of $\mathbf{K}_N(\alpha)$ (66) can be determined explicitly.

4.2. Khrapkov partial decomposition

In this section, a commutative factorization of $\mathbf{K}_N(\alpha)$ is obtained which, whilst removing most singularities from the respected regions of imposed analyticity, will contain poles at the zeros of $P_N(\alpha)$, $Q_N(\alpha)$ (63) and (64). These will then be removed in the final step of the factorization procedure. For mathematical convenience, it is more useful to factorize $\mathbf{L}_N(\alpha)$, where

$$\mathbf{L}_N(\alpha) = b(\alpha)\mathbf{I} - a(\alpha)\mathbf{J}_N(\alpha), \tag{74}$$

and so

$$\mathbf{K}_N(\alpha) = \mathbf{L}_N(\alpha)\mathbf{J}_N(\alpha) \tag{75}$$

in view of (61). Thus, the Khrapkov factors of $\mathbf{L}_N(\alpha)$ take the form (see (9))

$$\mathbf{L}_N^{\pm}(\alpha) = r_{\pm}(\alpha)[\cos\theta_{\pm}(\alpha)\mathbf{I} + \sin\theta_{\pm}(\alpha)\mathbf{J}_N(\alpha)], \tag{76}$$

in which $\Delta(\alpha)$ has taken the value i, so that (13) and (14) become, after simplification,

$$[r_+(\alpha)r_-(\alpha)]^2 = \frac{\gamma(\alpha)\delta(\alpha) - \alpha^2}{R(\alpha)}, \tag{77}$$

$$\theta_+(\alpha) + \theta_-(\alpha) = \tan^{-1}\left(\frac{\mathrm{i}\alpha[2\alpha^2 - k^2 - 2\gamma(\alpha)\delta(\alpha)]}{k^2 f(\alpha)\delta(\alpha)}\right) \tag{78}$$

for $\alpha \in \mathcal{D}$. Thus, the scalar sum and product factorizations of the functions on the right-hand sides of (77) and (78), respectively, are required. To obtain explicit integral expressions for $\theta_\pm(\alpha)$ it is convenient to factorize the *even* function

$$c(\alpha) = \frac{1}{\alpha}\tan^{-1} g(\alpha), \tag{79}$$

where

$$g(\alpha) = \frac{\mathrm{i}\alpha[2\alpha^2 - k^2 - 2\gamma(\alpha)\delta(\alpha)]}{k^2 f(\alpha)\delta(\alpha)}, \tag{80}$$

and

$$c(\alpha) \sim \frac{\mathrm{i}}{|\alpha|}\tanh^{-1}\left(\frac{1}{k^2}\right), \quad \alpha \to \pm\infty, \quad \alpha \in \mathcal{D}. \tag{81}$$

Therefore, Cauchy's integral theorem can be employed (see Theorem B, p. 13 of Noble [33]) to obtain after simplification

$$\theta_\pm(\alpha) = \frac{\pm\alpha^2}{\pi \mathrm{i}}\int_{\mathcal{C}_1} \frac{\tan^{-1} g(\zeta)}{\zeta(\zeta^2 - \alpha^2)}\,\mathrm{d}\zeta, \tag{82}$$

where α lies above (below) the contour ζ for $\theta_+(\alpha)$ ($\theta_-(\alpha)$) and $\mathcal{C}_1$ is that half of the contour $\mathcal{C}$ (Fig. 2) which goes from 0 to $+\infty$. Note that

$$\theta_-(\alpha) = -\theta_+(-\alpha), \tag{83}$$

and it can be shown that

$$\theta_+(\alpha) \sim \frac{1}{\pi}\tanh^{-1}\left(\frac{1}{k^2}\right)\log(2\alpha) + B + \mathrm{o}(1) \tag{84}$$

as $|\alpha| \to \infty$, $\alpha \in \mathcal{D}^+$, where B is the constant

$$B = \frac{\mathrm{i}}{\pi}\int_{\mathcal{C}_1}\left(\frac{\tan^{-1} g(\zeta)}{\zeta} - \frac{\mathrm{i}\tanh^{-1}(1/k^2)}{\gamma(\zeta)}\right)\mathrm{d}\zeta. \tag{85}$$

In view of the fact that the coefficient of the logarithm function in (84) is real, it is easily proved that $\cos\theta_+(\alpha)$ and $\sin\theta_+(\alpha)$ are bounded as $|\alpha| \to \infty$ in $\mathcal{D}^+$.

The product factorization of (77) can be achieved in a manner similar to that outlined in Section 2.5.2 of Freund [21]. Omitting all details it is found that

$$r_-(\alpha) = r_+(-\alpha), \tag{86}$$

$$r_+(\alpha) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}(\alpha + k_0)^{1/2}}\left(\frac{k^2+1}{k^2-1}\right)^{1/4}\mathrm{e}^{\ell_+(\alpha)}, \quad \alpha \in \mathcal{D}^+, \tag{87}$$

where

$$\ell_+(\alpha) = -\frac{1}{2\pi}\int_1^k \tan^{-1}\left(\frac{\sqrt{\zeta^2-1}\sqrt{k^2-\zeta^2}}{\zeta^2}\right)\frac{d\zeta}{\zeta+\alpha} + \frac{1}{2\pi}\int_1^k \tan^{-1}\left(\frac{4\zeta^2\sqrt{\zeta^2-1}\sqrt{k^2-\zeta^2}}{(2\zeta^2-k^2)^2}\right)\frac{d\zeta}{\zeta+\alpha}, \tag{88}$$

and k_0 is the Rayleigh wavenumber given by the positive real zero of $R(\alpha)$ (19). By inspection, $\ell_+(\alpha) = O(\alpha^{-1})$ as $|\alpha| \to \infty$ in $\mathcal{D}^+$, and so

$$r_+(\alpha) \sim \frac{e^{-i\pi/4}}{\sqrt{2}}\left(\frac{k^2+1}{k^2-1}\right)^{1/4}\alpha^{-1/2}, \quad |\alpha| \to \infty, \ \alpha \in \mathcal{D}^+. \tag{89}$$

This completes the commutative partial decomposition of $\mathbf{K}_N(\alpha)$.

4.3. Non-commutative factorization

Suppose an *upper* × *lower* factorization of (56) is required rather than a *lower* × *upper* split derived in the last section for kernel (17). Then the matrix factors $\mathbf{L}_N^+(\alpha)$ and $\mathbf{L}_N^-(\alpha)\mathbf{J}_N(\alpha)$ of $\mathbf{K}_N(\alpha)$ (75) are analytic in $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively, except for simple poles at the zeros of $P_N(\alpha)$, $Q_N(\alpha)$ occurring in $f_N(\alpha)$. These must be removed, in an identical fashion to that discussed in Section 3.3, in order to complete the factorization procedure. Thus, as before, a regularizing matrix, $\mathbf{M}(\alpha)$, is introduced such that (cf. Eq. (46))

$$\mathbf{K}_N^+(\alpha) = \mathbf{L}_N^+(\alpha)\mathbf{M}(\alpha), \tag{90}$$

$$\mathbf{K}_N^-(\alpha) = \mathbf{M}^{-1}(\alpha)\mathbf{L}_N^-(\alpha)\mathbf{J}_N(\alpha), \tag{91}$$

so that

$$\mathbf{K}_N(\alpha) = \mathbf{K}_N^+(\alpha)\mathbf{K}_N^-(\alpha) = \mathbf{L}_N^+(\alpha)\mathbf{L}_N^-(\alpha)\mathbf{J}_N(\alpha). \tag{92}$$

As before, the introduction of $\mathbf{M}(\alpha)$ leads to a non-commutative factorization as

$$\mathbf{K}_N^-(\alpha)\mathbf{K}_N^+(\alpha) \neq \mathbf{K}_N(\alpha). \tag{93}$$

The regularizing matrix $\mathbf{M}(\alpha)$ must suppress *all* the simple poles occurring in the half-planes of intended analyticity, and it therefore must consist of meromorphic elements which are generalizations of that used previously (48). The procedure described in Appendix A of Abrahams [3] can be employed here to *construct* the following ansatz for $\mathbf{M}(\alpha)$:

$$\mathbf{M}(\alpha) = \begin{pmatrix} 1 + \sum_{n=1}^{N}\frac{A_n}{\alpha-p_n} + \sum_{n=1}^{N}\frac{B_n}{\alpha+p_n} & \sum_{n=1}^{N}\frac{E_n}{\alpha-p_n} + \sum_{n=1}^{N}\frac{F_n}{\alpha+p_n} \\ -\sum_{n=1}^{N}\frac{C_n}{\alpha-q_n} - \sum_{n=1}^{N}\frac{D_n}{\alpha+q_n} & 1 - \sum_{n=1}^{N}\frac{G_n}{\alpha-q_n} - \sum_{n=1}^{N}\frac{H_n}{\alpha+q_n} \end{pmatrix} \tag{94}$$

which, in fact, is more general than that required for the kernel in [3]. It should be stated that, as Wiener–Hopf factorization is unique only up to arbitrary constant matrix factors, different forms for $\mathbf{M}(\alpha)$ could have been equally well employed.

To evaluate the as yet unknown coefficients A_n through H_n, the right-hand sides of (90) and (91) are expanded in order to suppress the spurious poles. For example, the (1, 1) element of $\mathbf{K}_N^+(\alpha)$ is given, from (76) and (94), by

$$r_+(\alpha)\cos\theta_+(\alpha)\left\{1+\sum_{n=1}^{N}\frac{A_n}{\alpha-p_n} + \sum_{n=1}^{N}\frac{B_n}{\alpha+p_n} + \frac{1}{f_N(\alpha)}\tan\theta_+(\alpha)\left(\sum_{n=1}^{N}\frac{C_n}{\alpha-q_n}+\sum_{n=1}^{N}\frac{D_n}{\alpha+q_n}\right)\right\}, \tag{95}$$

which has simple poles in $\mathcal{D}^+$ at $\alpha = p_n$, $n = 1, \dots, N$. It does not have poles at $\alpha = q_n$ because $1/f_N(\alpha)$ has zeros at these points. Writing $f_N(\alpha)$ and its inverse as Mittag–Leffler expansions

$$f_N(\alpha) = 1 + \sum_{n=1}^{N} \frac{\alpha_n}{\alpha^2 - q_n^2}, \qquad \frac{1}{f_N(\alpha)} = 1 + \sum_{n=1}^{N} \frac{\beta_n}{\alpha^2 - p_n^2}, \tag{96}$$

where the partial sum coefficients are

$$\alpha_n = \frac{2q_n P_N(q_n)}{Q_N'(q_n)}, \qquad \beta_n = \frac{2p_n Q_N(p_n)}{P_N'(p_n)} \tag{97}$$

with $'$ denoting differentiation with respect to α, then (95) is pole free in $\mathcal{D}^+$ if and only if

$$A_m = -\frac{\beta_m}{2p_m} \tan\theta_+(p_m) \sum_{m=1}^{N} \left(\frac{C_n}{p_m - q_n} + \frac{D_n}{p_m + q_n} \right), \quad m = 1, \dots, N. \tag{98}$$

Similarly the (1, 1) element of $[\mathbf{K}_N^-(\alpha)]^{-1}$ is

$$-\frac{1}{r_+(-\alpha)} \sin\theta_+(-\alpha) \left\{ 1 + \sum_{n=1}^{N} \frac{A_n}{\alpha - p_n} + \sum_{n=1}^{N} \frac{B_n}{\alpha + p_n} + \frac{1}{f_N(\alpha)} \cot\theta_+(-\alpha) \left(\sum_{n=1}^{N} \frac{C_n}{\alpha - q_n} + \sum_{n=1}^{N} \frac{D_n}{\alpha + q_n} \right) \right\}, \tag{99}$$

which is free of poles in $\mathcal{D}^-$ if and only if

$$B_m = +\frac{\beta_m}{2p_m} \cot\theta_+(p_m) \sum_{m=1}^{N} \left(\frac{C_n}{p_m + q_n} + \frac{D_n}{p_m - q_n} \right), \quad m = 1, \dots, N. \tag{100}$$

Note that use has been made of the symmetry properties (83) and (86) in the above equations. Repeating this procedure for the (2, 1) elements of $\mathbf{K}_N^+(\alpha)$, $[\mathbf{K}_N^-(\alpha)]^{-1}$ gives two more equations relating A_m through D_m. Concatenating all four equations into matrix form allows them to be written as the following pair of equations:

$$\mathbf{YA} = \mathbf{XC}, \qquad \mathbf{ZC} = \mathbf{1} + \mathbf{X}^{\mathrm{T}}\mathbf{A}, \tag{101}$$

where

$$\mathbf{A} = (A_1, A_2, \dots, A_N, B_1, \dots, B_N)^{\mathrm{T}}, \qquad \mathbf{C} = (C_1, \dots, C_N, D_1, \dots, D_N)^{\mathrm{T}}, \tag{102}$$

$\mathbf{Y}$ is a diagonal matrix with elements

$$\frac{2p_1}{\beta_1} \cot\theta_+(p_1), \dots, \frac{2p_N}{\beta_N} \cot\theta_+(p_N), \qquad \frac{2p_1}{\beta_1} \tan\theta_+(p_1), \dots, \frac{2p_N}{\beta_N} \tan\theta_+(p_N), \tag{103}$$

$\mathbf{Z}$ is a diagonal matrix with elements

$$\frac{2q_1}{\alpha_1} \cot\theta_+(q_1), \dots, \frac{2q_N}{\alpha_N} \cot\theta_+(q_N), \qquad \frac{2q_1}{\alpha_1} \tan\theta_+(q_1), \dots, \frac{2q_N}{\alpha_N} \tan\theta_+(q_N), \tag{104}$$

$$\mathbf{X}=\begin{pmatrix} \dfrac{-1}{p_1-q_1} & \dfrac{-1}{p_1-q_2} & \cdots & \dfrac{-1}{p_1-q_n} & \dfrac{-1}{p_1+q_1} & \dfrac{-1}{p_1+q_2} & \cdots & \dfrac{-1}{p_1+q_N} \\ \dfrac{-1}{p_2-q_1} & \dfrac{-1}{p_2-q_2} & \cdots & \dfrac{-1}{p_2-q_N} & \dfrac{-1}{p_2+q_1} & \dfrac{-1}{p_2+q_2} & \cdots & \dfrac{-1}{p_2+q_N} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ \dfrac{-1}{p_N-q_1} & & \cdots & \dfrac{-1}{p_N-q_N} & \dfrac{-1}{p_N+q_1} & & \cdots & \dfrac{-1}{p_N+q_N} \\ \dfrac{1}{p_1+q_1} & \dfrac{1}{p_1+q_2} & \cdots & \dfrac{1}{p_1+q_N} & \dfrac{1}{p_1-q_1} & \dfrac{1}{p_1-q_2} & \cdots & \dfrac{1}{p_1-q_N} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ \dfrac{1}{p_N+q_1} & & \cdots & \dfrac{1}{p_N+q_N} & \dfrac{1}{p_N-q_1} & & \cdots & \dfrac{1}{p_N-q_N} \end{pmatrix}, \tag{105}$$

and

$$\mathbf{1}=(1,1,1,\ldots,1)^{\mathrm{T}}, \tag{106}$$

which is a constant column vector of length $2N$. It is a simple matter to solve the coupled system

$$\mathbf{C}=(\mathbf{Z}-\mathbf{X}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{X})^{-1}\mathbf{1}, \qquad \mathbf{A}=\mathbf{Y}^{-1}\mathbf{X}\mathbf{C}. \tag{107}$$

Eliminating the poles from the $(2,1)$, and $(2,2)$ elements of the matrix factors leads to the complementary system of equations for the remaining coefficients

$$\mathbf{E}=(\mathbf{Y}-\mathbf{X}\mathbf{Z}^{-1}\mathbf{X}^{\mathrm{T}})^{-1}\mathbf{1}, \qquad \mathbf{G}=\mathbf{Z}^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{E}, \tag{108}$$

where

$$\mathbf{E}=(E_1,\ldots,E_N,F_1,\ldots,F_N)^{\mathrm{T}}, \qquad \mathbf{G}=(G_1,\ldots,G_N,H_1,\ldots,H_N)^{\mathrm{T}}. \tag{109}$$

This completes the evaluation of $\mathbf{M}(\alpha)$. From (107) and (108) it is straightforward to show, as before, that

$$\det(\mathbf{M}(\alpha))=1, \tag{110}$$

and so $\mathbf{M}^{-1}(\alpha)$ does not introduce spurious singularities. The product factors $\mathbf{K}_N^{\pm}(\alpha)$ have now been determined explicitly in terms of the commutative partial factors $\mathbf{L}_N^{\pm}(\alpha)$ and the regularizing matrix $\mathbf{M}(\alpha)$ or its inverse. It is easy to check that $\mathbf{K}_N^{\pm}(\alpha)$ have all the required properties; they are analytic in $\mathcal{D}^{\pm}$, have well-defined inverses in $\mathcal{D}^{\pm}$ and finally have algebraic growth at infinity in $\mathcal{D}^{\pm}$. In particular, results (76), (84), (89) and (94) reveal that

$$\mathbf{K}_N^{\pm}(\alpha)\sim \mathrm{O}(\alpha^{-1/2}), \quad |\alpha|\to\infty, \quad \alpha\in\mathcal{D}^{\pm}. \tag{111}$$

As already mentioned, the approximate factors tend to the exact result as $N\to\infty$, i.e.

$$\mathbf{K}_N^{\pm}(\alpha)\to\mathbf{K}^{\pm}(\alpha) \tag{112}$$

as long as α lies in the respective half-plane of analyticity. If $\mathbf{K}^{+}(\alpha)$ is required in $\mathcal{D}^{-}$, say, then inaccurate results may be obtained, even for large N, unless it is approximated by

$$\mathbf{K}^{+}(\alpha)\approx\mathbf{K}(\alpha)\mathbf{K}_N^{-}(\alpha), \quad \alpha\in\mathcal{D}^{-}, \tag{113}$$

where $\mathbf{K}(\alpha)$ is the *exact* kernel (56).

5. Further model problems and conclusions

The procedure outlined in the previous section can be utilized for a large number of canonical problems in dynamic elasticity, whether in fracture dynamics or in QNDE. The Wiener–Hopf kernels for two such examples are briefly discussed in this section.

5.1. Scattering by a crack in a moderately thin plate

First, consideration is given to scattering of flexural waves by a semi-infinite crack in a *moderately* thin elastic plate. Such a problem is important for the inspection of thin component structures, such as airplane or other vehicle fuselages, rotor blades in jet engines, etc., to determine defects or inclusions. Achenbach and Gautesen's [12] model may be considered as the limiting case when the specimen's thickness tends to infinity; here the thickness is moderately small compared to characteristic wavelengths of monochromatic elastic waves (of radian frequency ω, say) and so a plate theory that in addition to flexural motion includes the effects of transverse shear and rotary inertia (usually referred to as Mindlin theory [23]) is employed. Omitting all details for brevity [7], motions antisymmetric about the line of the crack satisfy a Wiener–Hopf equation whose kernel is

$$\mathbf{K}(\alpha) = \frac{1}{\gamma_3(\alpha)(k_1^2 - k_2^2)} \begin{pmatrix} K_{11}(\alpha) & K_{12}(\alpha) \\ -K_{12}(\alpha) & K_{22}(\alpha) \end{pmatrix}, \tag{114}$$

where

$$K_{11}(\alpha) = k_3^{-4}\{-(2\alpha^2 - k_3^2)^2(k_1^2 - k_2^2) + 2(1-\nu)k_3^2\alpha^2\gamma_3(\alpha)(\gamma_1(\alpha) - \gamma_2(\alpha)) \\ -4\alpha^2\gamma_3(\alpha)(k_2^2\gamma_1(\alpha) - k_1^2\gamma_2(\alpha))\}, \tag{115}$$

$$K_{22}(\alpha) = \alpha^2(k_1^2 - k_2^2) - \frac{k_1^2k_2^2k^2}{k_\mathrm{f}^4}\gamma_3(\alpha)(\gamma_1(\alpha) - \gamma_2(\alpha)) + \frac{k^2}{k_\mathrm{f}^4}\gamma_3(\alpha)(k_2^2\gamma_1(\alpha) - k_1^2\gamma_2(\alpha)), \tag{116}$$

$$K_{12}(\alpha) = \alpha k_3^{-2}\{(2\alpha^2 - k_3^2)(k_1^2 - k_2^2) + 2\gamma_3(\alpha)(k_2^2\gamma_1(\alpha) - k_1^2\gamma_2(\alpha))\}. \tag{117}$$

Here, as before, α is the Fourier transform variable related to the direction parallel to the crack, non-dimensionalized with respect to the wavenumber of longitudinal waves, and the three (non-dimensionalized) bulk wavenumbers k_1, k_2 and k_3 depend upon ω but are independent of α. They are given in terms of longitudinal and transverse wavenumbers, k_L and k_T, and the wavenumber for non-dimensionalized flexural (bending) waves k_f, via

$$k_1^2 + k_2^2 = k^2 + 1, \qquad k_1^2k_2^2 = k^2 - k_\mathrm{f}^4, \qquad k_3^2 = \kappa^2k_1^2k_2^2, \tag{118}$$

where, as before, $k = k_\mathrm{T}/k_\mathrm{L}$. The constant ν is Poisson's ratio and κ, introduced in Mindlin plate theory to better approximate the shear forces, may be chosen according to different criteria, but normally $\kappa^2 < 1$ [23]. Finally, the kernel elements contain the branch-cut functions

$$\gamma_j = (\alpha^2 - k_j^2)^{1/2}, \quad \mathrm{Re}\,\gamma_j \geq 0, \quad j = 1, 2, 3, \tag{119}$$

which fully specifies the matrix kernel (114).

Algebraic manipulation allows the kernel to be rearranged into Khrapkov form

$$\mathbf{K}(\alpha) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{Q}(\alpha)\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{120}$$

in which $\mathbf{Q}(\alpha)$ is

$$\mathbf{Q}(\alpha) = \delta(\alpha)\mathbf{I} + \sqrt{\beta(\alpha)\beta(-\alpha)}\mathbf{J}(\alpha) \tag{121}$$

with $\mathbf{I}$ the identity,

$$\beta(\alpha) = \frac{1}{2\gamma_3(\alpha)(k_1^2 - k_2^2)}(K_{11}(\alpha) - K_{22}(\alpha) + 2K_{12}(\alpha)), \tag{122}$$

$$\delta(\alpha) = \frac{1}{2\gamma_3(\alpha)(k_1^2 - k_2^2)}(K_{11}(\alpha) + K_{22}(\alpha)). \tag{123}$$

Also,

$$\mathbf{J}(\alpha) = \begin{pmatrix} 0 & f(\alpha) \\ \dfrac{1}{f(\alpha)} & 0 \end{pmatrix}, \qquad \mathbf{J}^2(\alpha) = \mathbf{I}, \tag{124}$$

and the scalar function to be approximated is

$$f(\alpha) = \sqrt{\frac{\beta(\alpha)}{\beta(-\alpha)}}. \tag{125}$$

Solution by the WHAM can now be carried out in the manner discussed previously; i.e., $f(\alpha)$ is chosen to tend to unity at infinity, and so is replaced by its $[N/N]$ Padé approximant $f_N(\alpha)$ in $\mathbf{Q}(\alpha)$. However, here the form of $f(\alpha)$ is a little different, exhibiting the reflectional property

$$f(\alpha) = \frac{1}{f(-\alpha)}, \tag{126}$$

and so its approximant takes a similar form. Suppose now that $f_N(\alpha)$ has P poles in the upper half-plane at $\alpha = p_n$, $n = 1, 2, \ldots, P$ ($p_n \notin \mathcal{D}^-$) and Q poles in the region below the strip at $\alpha = -q_n$, $n = 1, 2, \ldots, Q$. Thus, there is, in total $P + Q = N$ simple poles in the complex plane, and due to the symmetry (126) there are N simple zeros of $f_N(\alpha)$ at

$$\alpha = -p_n, \quad n = 1, 2, \ldots, P, \qquad \alpha = q_n, \quad n = 1, 2, \ldots, Q \tag{127}$$

in the lower and upper regions, respectively. Thus, $f_N(\alpha)$ and its inverse may be expressed as Mittag–Leffler expansions

$$f_N(\alpha) = 1 + \sum_{n=1}^{P} \frac{\alpha_n}{p_n - \alpha} + \sum_{n=1}^{Q} \frac{\beta_n}{q_n + \alpha}, \qquad P + Q = N, \tag{128}$$

$$\frac{1}{f_N(\alpha)} = 1 + \sum_{n=1}^{P} \frac{\alpha_n}{p_n + \alpha} + \sum_{n=1}^{Q} \frac{\beta_n}{q_n - \alpha}, \tag{129}$$

where both tend to unity at infinity by virtue of $f_N(\alpha)$ being a two-point Padé approximant of $f(\alpha)$. The coefficients α_n, and β_n are easily determined and by inspection of the way $f_N(\alpha)$ appears in $\mathbf{Q}_N(\alpha)$ (the approximation of $\mathbf{Q}(\alpha)$), the ansatz for $\mathbf{M}(\alpha)$ is now posed

$$\mathbf{M}(\alpha) = \begin{pmatrix} 1 + \displaystyle\sum_{n=1}^{P} \frac{A_n}{p_n - \alpha} + \sum_{n=1}^{Q} \frac{B_n}{q_n + \alpha} & -\displaystyle\sum_{n=1}^{P} \frac{C_n}{p_n - \alpha} - \sum_{n=1}^{Q} \frac{D_n}{q_n + \alpha} \\ \displaystyle\sum_{n=1}^{P} \frac{C_n}{p_n + \alpha} + \sum_{n=1}^{Q} \frac{D_n}{q_n - \alpha} & 1 + \displaystyle\sum_{n=1}^{P} \frac{A_n}{p_n + \alpha} + \sum_{n=1}^{Q} \frac{B_n}{q_n - \alpha} \end{pmatrix}, \tag{130}$$

where A_n, B_n, C_n, and D_n are constants that must be determined as previously. With the information provided above an approximate explicit factorization can be accomplished without too much difficulty [7].

5.2. Scattering by an interfacial crack

The second example illustrating applications of the WHAM is scattering by a crack in an interface between two *dissimilar* elastic materials. The model is identical to that discussed by Achenbach and Gautesen [12] except that the region above the crack ($y > 0$) has material properties denoted by a suffix 2, whereas below the crack ($y < 0$) the material is denoted by the suffix 1. Then, the boundary value problem can be shown to reduce to a matrix Wiener–Hopf equation with kernel

$$\mathbf{K}(\alpha) = \begin{pmatrix} 0 & -\mathrm{i}\alpha & \mathrm{i}\eta \\ 0 & \mathrm{i}\eta & \mathrm{i}\alpha \\ 1 & 0 & 0 \end{pmatrix} \mathbf{Q}^{-1}(\alpha) \begin{pmatrix} -\mathrm{i}\alpha & 0 & \mathrm{i}\eta \\ \mathrm{i}\eta & 0 & \mathrm{i}\alpha \\ 0 & 1 & 0 \end{pmatrix}^{-1}, \tag{131}$$

in which

$$\mathbf{Q}(\alpha) = \begin{pmatrix} \mathbf{N}^{(2)} - \mathbf{N}^{(1)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & g(\alpha) \end{pmatrix}, \tag{132}$$

where $g(\alpha)$ is the scalar function

$$g(\alpha) = -\frac{1}{\mu_1\delta_1} - \frac{1}{\mu_2\delta_2}, \tag{133}$$

and $\mathbf{N}^{(j)}(\alpha)$ is the 2×2 matrix

$$\mathbf{N}^{(j)}(\alpha) = \frac{1}{\mu_j R_j} \begin{pmatrix} \alpha^2 + \eta^2 + \delta_j^2 - 2\gamma_j\delta_j & (-1)^j k_j^2 \delta_j \\ (-1)^j k_j^2 \gamma_j & (\alpha^2 + \eta^2)(\alpha^2 + \eta^2 + \delta_j^2 - 2\gamma_j\delta_j) \end{pmatrix}. \tag{134}$$

The notation is the same as that employed in Section 3, namely δ_j, γ_j and R_j are given in (19) and (20) with parameter $k \to k_j$ and argument $\sqrt{\alpha^2 + \eta^2}$. Also, μ_j is the shear modulus in medium j.

It can be shown that this 3×3 kernel (131) can be rearranged into Khrapkov form with a scalar function which is approximated in the usual way. For brevity this is not discussed further, but instead it is clear that the kernel in (131) must reduce to that in (17) when medium 2 and medium 1 are identical. When $k_1 = k_2 = k$, by inspection $\mathbf{Q}(\alpha)$ reduces to

$$\mathbf{Q}(\alpha) = \frac{2k^2}{\mu R} \begin{pmatrix} 0 & \delta & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & d \end{pmatrix}, \tag{135}$$

and so multiplying out the matrices in (131) gives

$$\mathbf{K}(\alpha) = \frac{-\mu}{2(\alpha^2 + \eta^2)} \begin{pmatrix} \eta^2\delta + \alpha^2 d & \alpha\eta(\delta - d) & 0 \\ \alpha\eta(\delta - d) & \alpha^2\delta + \eta^2 d & 0 \\ 0 & 0 & \dfrac{\delta d}{\gamma} \end{pmatrix}. \tag{136}$$

Clearly, this 3×3 kernel has reduced to a 2×2 matrix (top left) which is exactly that given in (17) except for a constant, plus a scalar function. The former is the kernel of the Wiener–Hopf system for antisymmetric motions in y, the direction perpendicular to the crack, and the latter that for symmetric motions.

5.3. Concluding remarks

This paper has offered a brief summary of the role of the Wiener–Hopf technique in elastodynamics, relevant to QNDE and fracture dynamics, and in particular has demonstrated a practical and constructive method for the approximate factorization of matrix kernels arising from such problems. A fuller description of the method which employs Padé approximants, and further applications, can be found in other papers by the author [2–5]. As mentioned in Section 2, the replacement of scalar kernels by Padé generated rational functions can be seen as a direct generalization of Koiter's approach [30]. However, the WHAM method has several important advantages over Koiter's, and other, approximate and exact factorization techniques (e.g. contrast the approach of the author [4] with that offered by Daniele [20]). First the approximation can be extended to any order, hence improving accuracy almost indefinitely. Second, the automatic generation of the polynomial coefficients from the Taylor series expansion seems to present an optimal choice in regard to accuracy; to illustrate this, Fig. 3 revealed an error of less than 10^{-8}% for the modest Padé number $N = 11$! Third, the Padé coefficients are generated by inversion of a linear algebraic system, which means that the procedure is quick and accurate using packages such as Matlab or Mathematica. In [5] several physically important scalar kernels are discussed, and the advantage, in terms of ease of use and increase in computational speed, of the approximate kernels is demonstrated. Lastly, and most importantly, the WHAM method is the only approximate approach which is directly applicable to matrix as well as scalar systems.

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