MODULES OF FINITE HOMOLOGICAL DIMENSION WITH RESPECT TO A SEMIDUALIZING MODULE

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Abstract. We prove versions of results of Foxby and Holm about modules of finite (Gorenstein) injective dimension and finite (Gorenstein) projective dimension with respect to a semidualizing module. We also verify two special cases of a question of Takahashi and White.

Introduction

Let R be a commutative noetherian ring. It is well-known that, if R is Gorenstein and local, then every module with finite projective dimension has finite injective dimension. Conversely, Foxby [4, 5] showed that, if R is local and admits a finitely generated module of finite projective dimension and finite injective dimension, then R is Gorenstein. More recently, Holm [12] proved that, if M is an R-module of finite projective dimension and finite *Gorenstein* injective dimension, then M has finite injective dimension, and so the localization $R_\mathfrak{p}$ is Gorenstein for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$. See Section 1 for terminology and notation.

In this paper, we prove analogues of these results for homological dimensions defined in terms of semidualizing R-modules. For instance, the following result is proved in (2.1). Other variants of this result are also given in Section 2. It should be noted that our proof of this result is different from Holm's proof for the special case $C = R$. In particular, this paper also provides a new proof of Holm's result.

Theorem A. *Let C be a semidualizing R-module, and let M be an R-module with $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) < \infty$ and $\operatorname{Gid}_R(M) < \infty$. Then $\operatorname{id}_R(M) = \operatorname{Gid}_R(M) < \infty$ and, for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$, the $R_\mathfrak{p}$-module $C_\mathfrak{p}$ is dualizing.*

Takahashi and White [16] posed another generalization of Vasconcelos' question: When R is a local Cohen-Macaulay ring admitting a dualizing module and C is a semidualizing R-module, if M is an R-module of finite depth such that $\mathcal{P}_C\text{-}\operatorname{pd}_R(M)$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(M)$ are finite, must R be Gorenstein? Our techniques allow us to verify the question in the affirmative in two special cases. The first one is contained in the next result which we prove in (2.11).

Theorem B. *If C is a semidualizing R-module and M is an R-module such that $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) = 0$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(M) < \infty$, then $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$.*

Date: June 4, 2009.
2000 *Mathematics Subject Classification.* 13C05, 13D05, 13H10.
Key words and phrases. dualizing modules, Gorenstein homological dimensions, Gorenstein injective dimension, Gorenstein projective dimension, Gorenstein rings, semidualizing modules,
This research was conducted while S.S.-W. visited the IPM in Tehran during July 2008. The research of S.Y. was supported in part by a grant from the IPM (No. 87130211).

1. Semidualizing Modules and Related Homological Dimensions

Throughout this paper R is a commutative noetherian ring.

Definition 1.1. Let $\mathcal{X}$ be a class of R-modules and M an R-module. An $\mathcal{X}$-*resolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots \xrightarrow{\partial^X_1} X_0 \to 0$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \geqslant 1$. The $\mathcal{X}$*-projective dimension* of M is the quantity

$$\mathcal{X}\text{-}\operatorname{pd}_R(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

The modules of $\mathcal{X}$-projective dimension 0 are the nonzero modules of $\mathcal{X}$.

Dually, an $\mathcal{X}$*-coresolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = \quad 0 \to X_0 \xrightarrow{\partial^X_0} X_{-1} \xrightarrow{\partial^X_{-1}} \cdots \xrightarrow{\partial^X_{1+n}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \leqslant -1$. The $\mathcal{X}$*-injective dimension* of M is the quantity

$$\mathcal{X}\text{-}\operatorname{id}_R(M) = \inf\{\sup\{-n \geqslant 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-coresolution of } M\}.$$

The modules of $\mathcal{X}$-injective dimension 0 are the nonzero modules of $\mathcal{X}$.

When $\mathcal{X}$ is the class of projective R-modules, we write $\operatorname{pd}_R(M)$ for the associated homological dimension and call it the *projective dimension* of M. Similarly, the *flat and injective dimensions* of M are denoted $\operatorname{fd}_R(M)$ and $\operatorname{id}_R(M)$, respectively.

The homological dimensions of interest in this paper are built from semidualizing modules and their associated projective and injective classes, defined next. Semidualizing modules occur in the literature with several different names, e.g., in the work of Foxby [3], Golod [9], Mantese and Reiten [15], Vasconcelos [17] and Wakamatsu [18]. The prototypical semidualizing modules are the dualizing (or canonical) modules of Groethendieck and Hartshorne [10].

Definition 1.2. A finitely generated R-module C is *semidualzing* if the natural homothety morphism $R \to \operatorname{Hom}_R(C,C)$ is an isomorphism and $\operatorname{Ext}^{\geqslant 1}_R(C,C) = 0$. An R-module D is *dualizing* if it is semidualizing and has finite injective dimension.

Let C be a semidualizing R-module. We set

$\mathcal{P}_C(R) =$ the subcategory of modules $P \otimes_R C$ where P is R-projective

$\mathcal{I}_C(R) =$ the subcategory of modules $\operatorname{Hom}_R(C,I)$ where I is R-injective.

Modules in $\mathcal{P}_C(R)$ are called C*-projective*, and those in $\mathcal{I}_C(R)$ are called C*-injective*.

Fact 1.3. Let C be a semidualizing R-module. It is straightforward to show that, if $P \in \mathcal{P}_C(R)$ and $I \in \mathcal{I}_C(R)$, then $P_\mathfrak{p} \in \mathcal{P}_{C_\mathfrak{p}}(R_\mathfrak{p})$ and $I_\mathfrak{p} \in \mathcal{I}_{C_\mathfrak{p}}(R_\mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$. It follows that we have $\mathcal{P}_{C_\mathfrak{p}}\text{-}\operatorname{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(M)$ and $\mathcal{I}_{C_\mathfrak{p}}\text{-}\operatorname{id}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \mathcal{I}_C\text{-}\operatorname{id}_R(M)$ for each R-module M.

The next classes are central to our proofs and were introduced by Foxby [6].

Definition 1.4. Let C be a semidualizing R-module. The *Auslander class* of C is the class $\mathcal{A}_C(R)$ of R-modules M such that

(1) $\operatorname{Tor}^R_{\geqslant 1}(C,M) = 0 = \operatorname{Ext}^{\geqslant 1}_R(C, C \otimes_R M)$, and
(2) the natural map $M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* of C is the class $\mathcal{B}_C(R)$ of R-modules M such that

(1) $\mathrm{Ext}_R^{\geqslant 1}(C, M) = 0 = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, M))$, and
(2) the natural evaluation map $C \otimes_R \mathrm{Hom}_R(C, M) \to M$ is an isomorphism.

Fact 1.5. Let C be a semidualizing R-module. The categories $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ are closed under extensions, kernels of epimorphisms and cokernels of monomorphism; see [14, Cor. 6.3]. The category $\mathcal{A}_C(R)$ contains all modules of finite flat dimension and those of finite $\mathcal{I}_C$-injective dimension, and the category $\mathcal{B}_C(R)$ contains all modules of finite injective dimension and those of finite $\mathcal{P}_C$-projective dimension by [14, Cors. 6.1 and 6.2].

The next definitions are due to Holm and Jørgensen [13] in this generality.

Definition 1.6. Let C be a semidualizing R-module. A *complete $\mathcal{I}_C\mathcal{I}$-resolution* is a complex Y of R-modules satisfying the following:

(1) Y is exact and $\mathrm{Hom}_R(I, Y)$ is exact for each $I \in \mathcal{I}_C(R)$, and
(2) $Y_i \in \mathcal{I}_C(R)$ for all $i \geqslant 0$ and $Y_i \in \mathcal{I}(R)$ for all $i < 0$.

An R-module H is *G_C-injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-resolution Y such that $H \cong \mathrm{Im}(\partial_1^Y)$, in which case Y is a *complete $\mathcal{I}_C\mathcal{I}$-resolution of H*. We set

$$\mathcal{GI}_C(R) = \text{the class of } \mathrm{G}_C\text{-injective } R\text{-modules.}$$

In the special case $C = R$, we set $\mathcal{GI}(R) = \mathcal{GI}_R(R)$ and $\mathrm{Gid}_R(M) = \mathcal{GI}_R\text{-}\,\mathrm{id}_R(M)$, and we write "complete injective resolution" instead of "complete $\mathcal{I}_R\mathcal{I}$-resolution".

A *complete $\mathcal{PP}_C$-resolution* is a complex X of R-modules satisfying the following.

(1) X is exact and $\mathrm{Hom}_R(X, P)$ is exact for each $P \in \mathcal{P}_C(R)$, and
(2) $X_i \in \mathcal{P}(R)$ for all $i \geqslant 0$ and $X_i \in \mathcal{P}_C(R)$ for all $i < 0$.

An R-module M is *G_C-projective* if there exists a complete $\mathcal{PP}_C$-resolution X such that $M \cong \mathrm{Im}(\partial_1^X)$, in which case X is a *complete $\mathcal{PP}_C$-resolution of M*. We set

$$\mathcal{GP}_C(R) = \text{the class of } \mathrm{G}_C\text{-projective } R\text{-modules.}$$

In the case $C = R$, we set $\mathcal{GP}(R) = \mathcal{GP}_R(R)$ and $\mathrm{Gpd}_R(M) = \mathcal{GP}_R\text{-}\,\mathrm{pd}_R(M)$.

The next two lemmas are proved as in [2, (2.17),(2.18)] using tools from [19].

Lemma 1.7. *Let M be an R-module with $\mathcal{GP}_C\text{-}\,\mathrm{pd}_R(M) < \infty$. There is an exact sequence of R-modules*

$$0 \to M \to P \to M' \to 0$$

such that $M' \in \mathcal{GP}_C(R)$ and $\mathcal{P}_C\text{-}\,\mathrm{pd}_R(P) = \mathcal{GP}_C\text{-}\,\mathrm{pd}_R(M)$. □

Lemma 1.8. *Let M be an R-module with $\mathcal{GI}_C\text{-}\,\mathrm{id}_R(M) < \infty$. There is an exact sequence of R-modules*

$$0 \to M' \to E \to M \to 0$$

such that $\mathcal{I}_C\text{-}\,\mathrm{id}_R(E) = \mathcal{GI}_C\text{-}\,\mathrm{id}_R(M)$ and $M' \in \mathcal{GI}_C(R)$. □

Definition 1.9. Assume that R is local with residue field k. The *depth* of a (not necessarily finitely generated) R-module M is

$$\mathrm{depth}_R(M) = \inf\{n \geqslant 0 \mid \mathrm{Ext}^n_R(k, M) \neq 0\}.$$

2. Main Results

2.1. *Proof of Theorem A.* Set $d = \mathcal{P}_C\text{-}\operatorname{pd}_R(M)$. As $\operatorname{Gid}_R(M)$ is finite, Lemma 1.8 yields an exact sequence of R-modules

$$(*) \qquad 0 \to M' \to E \to M \to 0$$

such that $\operatorname{id}_R(E) < \infty$ and M' is G-injective. The finiteness of $\mathcal{P}_C\text{-}\operatorname{pd}_R(M)$ and $\operatorname{id}_R(E)$ implies that $M, E \in \mathcal{B}_C(R)$, and so $M' \in \mathcal{B}_C(R)$; see Fact 1.5.

We claim that $\operatorname{Ext}_R^{\geqslant 1}(M, M') = 0$. To see this, let Y be a complete injective resolution of M' and set $M^{(i)} = \operatorname{Im}(\partial_Y^i)$ for each $i \in \mathbb{Z}$. Since $M', Y_i \in \mathcal{B}_C(R)$ for each $i \in \mathbb{Z}$, we have $M^{(i)} \in \mathcal{B}_C(R)$ for each i, and so $\operatorname{Ext}_R^{\geqslant 1}(C, M^{(i)}) = 0$. Hence

$$\operatorname{Ext}_R^{\geqslant 1}(P \otimes_R C, M^{(i)}) \cong \operatorname{Hom}_R(P, \operatorname{Ext}_R^{\geqslant 1}(C, M^{(i)})) = 0$$

for each projective R-module P and each i. Using a bounded $\mathcal{P}_C$-resolution of M, a dimension-shifting argument shows that $\operatorname{Ext}_R^{\geqslant d+1}(M, M^{(i)}) = 0$ for each i. Another dimension-shifting argument using the complete injective resolution of M' yields the following

$$\operatorname{Ext}_R^{\geqslant 1}(M, M') \cong \operatorname{Ext}_R^{\geqslant 1}(M, M^{(1)}) \cong \operatorname{Ext}_R^{\geqslant d+1}(M, M^{(d)}) = 0$$

as claimed.

The previous paragraph shows that the sequence $(*)$ splits. Hence, we have

$$\sup\{\operatorname{id}_R(M), \operatorname{id}_R(M')\} = \operatorname{id}_R(E) < \infty$$

and so $\operatorname{id}_R(M) < \infty$. The equality $\operatorname{id}_R(M) = \operatorname{Gid}_R(M)$ now follows from the result dual to [11, (2.27)].

Now, let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$. It follows that $\mathcal{P}_{C_\mathfrak{p}}\text{-}\operatorname{pd}_{R_\mathfrak{p}}(M_\mathfrak{p})$ and $\operatorname{id}_{R_\mathfrak{p}}(M_\mathfrak{p})$ are finite. The finiteness of $\mathcal{P}_{C_\mathfrak{p}}\text{-}\operatorname{pd}_{R_\mathfrak{p}}(M_\mathfrak{p})$ implies $M_\mathfrak{p} \in \mathcal{B}_{C_\mathfrak{p}}(R_\mathfrak{p})$ and thus $\operatorname{Ext}_{R_\mathfrak{p}}^{\geqslant 1}(C_\mathfrak{p}, M_\mathfrak{p}) = 0$. It also implies

$$\operatorname{fd}_{R_\mathfrak{p}}(\operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, M_\mathfrak{p})) \leqslant \operatorname{pd}_{R_\mathfrak{p}}(\operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, M_\mathfrak{p})) = \mathcal{P}_{C_\mathfrak{p}}\text{-}\operatorname{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$$

by [16, (2.11)]. Hence, it follows from [1, (8.2)] that $C_\mathfrak{p}$ is dualizing for $R_\mathfrak{p}$. □

Corollary 2.2. *Assume that R is local, and let C be a semidualizing R-module. The following conditions are equivalent:*

(i) *C is a dualizing R-module;*
(ii) *there exists a finitely generated R-module $M \neq 0$ such that $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) < \infty$ and $\operatorname{id}_R(M) < \infty$;*
(iii) *there exists an R-module $M \neq 0$ of finite depth such that $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) < \infty$ and $\operatorname{Gid}_R(M) < \infty$.*

Proof. The implication (ii) $\Longrightarrow$ (iii) is straightforward, and (iii) $\Longrightarrow$ (i) follows from Theorem A. For (i) $\Longrightarrow$ (ii), note that $\mathcal{P}_C\text{-}\operatorname{pd}_R(C) < \infty$ and $\operatorname{id}_R(C) < \infty$ since C is dualizing for R. □

The following versions of Theorem A and Corollary 2.2 are proved similarly, using Lemmas 1.7 and 1.8.

Theorem 2.3. *Let C a semidualizing R-module, and let M be an R-module with $\operatorname{pd}_R(M) < \infty$ and $\mathcal{GI}_C\text{-}\operatorname{id}_R(M) < \infty$. Then $\mathcal{I}_C\text{-}\operatorname{id}_R(M) = \mathcal{GI}_C\text{-}\operatorname{id}_R(M) < \infty$. Furthermore, for each $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$, the localization $C_\mathfrak{p}$ is a dualizing $R_\mathfrak{p}$-module.* □

Corollary 2.4. *Assume that R is local, and let C be a semidualizing R-module. The following conditions are equivalent:*

(i) *C is a dualizing R-module;*
(ii) *there exists a finitely generated R-module $M \neq 0$ such that $\mathrm{pd}_R(M) < \infty$ and $\mathcal{I}_C\text{-}\mathrm{id}_R(M) < \infty$;*
(iii) *there exists an R-module $M \neq 0$ of finite depth such that $\mathrm{pd}_R(M) < \infty$ and $\mathcal{GI}_C\text{-}\mathrm{id}_R(M) < \infty$.* □

Theorem 2.5. *Let C a semidualizing R-module, and let M be an R-module with $\mathcal{I}_C\text{-}\mathrm{id}_R(M) < \infty$ and $\mathrm{Gpd}_R(M) < \infty$. Then $\mathrm{pd}_R(M) = \mathrm{Gpd}_R(M) < \infty$. Furthermore, for each $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$, the localization $C_\mathfrak{p}$ is a dualizing $R_\mathfrak{p}$-module.* □

Corollary 2.6. *Assume that R is local, and let C be a semidualizing R-module. The following conditions are equivalent:*

(i) *C is a dualizing R-module;*
(ii) *there exists a finitely generated R-module $M \neq 0$ such that $\mathcal{I}_C\text{-}\mathrm{id}_R(M) < \infty$ and $\mathrm{pd}_R(M) < \infty$;*
(iii) *there exists an R-module $M \neq 0$ of finite depth such that $\mathcal{I}_C\text{-}\mathrm{id}_R(M) < \infty$ and $\mathrm{Gpd}_R(M) < \infty$.* □

Remark 2.7. As is noted in [12], when R has finite Krull dimension, we can change $\mathrm{Gpd}_R(M)$ and $\mathrm{pd}_R(M)$ to $\mathrm{Gfd}_R(M)$ and $\mathrm{fd}_R(M)$, respectively, in the previous two results. Similarly, in the next two results, if $\dim(R) < \infty$, then $\mathcal{GP}_C\text{-}\mathrm{pd}_R(M)$ and $\mathcal{P}_C\text{-}\mathrm{pd}_R(M)$ can be changed to $\mathcal{GF}_C\text{-}\mathrm{pd}_R(M)$ and $\mathcal{F}_C\text{-}\mathrm{pd}_R(M)$.

Theorem 2.8. *Let C be a semidualizing R-module, and let M be an R-module with $\mathrm{id}_R(M) < \infty$ and $\mathcal{GP}_C\text{-}\mathrm{pd}_R(M) < \infty$. Then $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) = \mathcal{GP}_C\text{-}\mathrm{pd}_R(M) < \infty$. Furthermore, for each $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$, the localization $C_\mathfrak{p}$ is a dualizing $R_\mathfrak{p}$-module.* □

Corollary 2.9. *Assume that R is local, and let C be a semidualizing R-module. The following conditions are equivalent:*

(i) *C is a dualizing R-module;*
(ii) *there exists a finitely generated R-module $M \neq 0$ such that $\mathrm{id}_R(M) < \infty$ and $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) < \infty$;*
(iii) *there exists an R-module $M \neq 0$ of finite depth such that $\mathrm{id}_R(M) < \infty$ and $\mathcal{GP}_C\text{-}\mathrm{pd}_R(M) < \infty$.* □

Remark 2.10. Holm proves his results in a more general setting than ours, namely, over associative rings. While the Gorenstein projective dimension and Gorenstein injective dimension have been well-studied in this setting, the same cannot be said for G_C-projective dimension and G_C-injective dimension. Some of the foundation has been laid by Holm and White [14]. To prove our results in this setting, though, would require a development of these ideas that is outside the scope of this paper.

2.11. *Proof of Theorem B.* Let $\mathfrak{p} \in \mathrm{Supp}_R(M)$, and replace R with $R_\mathfrak{p}$ to assume that R is local. In particular, every projective R-module is free, and so $M \cong C \oplus M'$

for some $M' \in \mathcal{P}_C(R)$. In the next sequence, the final equality is from [16, (2.11.b)]

$$\begin{aligned}\sup\{\mathrm{id}_R(C \otimes_R C), \mathrm{id}_R(C \otimes_R M')\} &= \mathrm{id}_R((C \otimes_R C) \oplus (C \otimes_R M')) \\ &= \mathrm{id}_R(C \otimes_R (C \oplus M')) \\ &= \mathrm{id}_R(C \otimes_R M) \\ &= \mathcal{I}_C\text{-}\mathrm{id}_R(M)\end{aligned}$$

and so $\mathcal{I}_C\text{-}\mathrm{id}_R(C) \leqslant \mathcal{I}_C\text{-}\mathrm{id}_R(M) < \infty$. It follows that $C \in \mathcal{A}_C(R)$. By definition, this includes the condition $\mathrm{Tor}^R_{\geqslant 1}(C, C) = 0$, and so [8, (3.8)] implies that $C \otimes_R C$ is a semidualizing R-module. Similarly, from [7, (3.2)] we conclude that $C \cong R$. It follows that $\infty > \mathcal{I}_C\text{-}\mathrm{id}_R(C) = \mathrm{id}_R(R)$ and so R is Gorenstein as desired. □

The final result of this paper contains another partial answer to the question of Takahashi and White. We include the proof because it is different from the proof of Theorem B.

Theorem 2.12. *If C is a semidualizing R-module and M is an R-module such that $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) < \infty$ and $\mathcal{I}_C\text{-}\mathrm{id}_R(M) = 0$, then $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$.*

Proof. As $M \in \mathcal{I}_C(R)$, we have $M \cong \mathrm{Hom}_R(C, E)$ for some injective R-module E.

We first show that the assumption $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ implies that $\mathfrak{p} \in \mathrm{Ass}_R(M)$. The fact that C is finitely generated and E is injective yields the next isomorphisms

$$\begin{aligned}\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) &\cong \mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, \mathrm{Hom}_R(C, E)_\mathfrak{p}) \\ &\cong \mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, \mathrm{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, E_\mathfrak{p})) \\ &\cong \mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{Tor}^{R_\mathfrak{p}}_i(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, C_\mathfrak{p}), E_\mathfrak{p}).\end{aligned}$$

Each module $\mathrm{Tor}^{R_\mathfrak{p}}_i(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, C_\mathfrak{p})$ is a finite-dimensional vector space over $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Furthermore, we have $\mathrm{Tor}^{R_\mathfrak{p}}_0(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, C_\mathfrak{p}) \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_{R_\mathfrak{p}} C_\mathfrak{p} \neq 0$ since $C_\mathfrak{p}$ is nonzero and finitely generated over $R_\mathfrak{p}$. Since $\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) \neq 0$ for some i, by hypothesis, it therefore follows that $\mathrm{Ext}^0_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) \neq 0$, and so $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$, that is, $\mathfrak{p} \in \mathrm{Ass}_R(M)$ as claimed.

Write $E \cong \oplus_\mathfrak{q} E_R(R/\mathfrak{q})^{(\mu_\mathfrak{q})}$, where the direct sum is taken over all $\mathfrak{q} \in \mathrm{Spec}(R)$. It follows that there are equalities

$$\mathrm{Ass}_R(M) = \mathrm{Supp}_R(C) \cap \mathrm{Ass}_R(E) = \mathrm{Spec}(R) \cap \mathrm{Ass}_R(E) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mu_\mathfrak{q} \neq 0\}.$$

Since $\mathfrak{p} \in \mathrm{Ass}_R(M)$, this implies $E \cong E_R(R/\mathfrak{p}) \oplus E'$ for some injective R-module E'. It follows that $M \cong \mathrm{Hom}_R(C, E_R(R/\mathfrak{p})) \oplus \mathrm{Hom}_R(C, E')$. As in the proof of Theorem B, we see that $\mathcal{P}_C\text{-}\mathrm{pd}_R(\mathrm{Hom}_R(C, E_R(R/\mathfrak{p}))) < \infty$. If follows that we may replace R with $R_\mathfrak{p}$ and M with $\mathrm{Hom}_R(C, E_R(R/\mathfrak{p}))_\mathfrak{p}$ to assume that R is local with maximal ideal $\mathfrak{m}$ and $M \cong \mathrm{Hom}_R(C, E)$ where $E = E_R(R/\mathfrak{m})$.

Let $\widehat{R}$ denote the completion of R. It is straightforward to show that the condition $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) < \infty$ implies $\mathcal{P}_{\widehat{C}}\text{-}\mathrm{pd}_{\widehat{R}}(M \otimes_R \widehat{R}) < \infty$. Also, we have isomorphisms

$$M \otimes_R \widehat{R} \cong \mathrm{Hom}_R(C, E_R(R/\mathfrak{m})) \otimes_R \widehat{R} \cong \mathrm{Hom}_{\widehat{R}}(\widehat{C}, E_{\widehat{R}}(\widehat{R}/\mathfrak{m}\widehat{R}))$$

and so $M \otimes_R \widehat{R} \in \mathcal{I}_{\widehat{C}}(\widehat{R})$. It follows that we may replace R with $\widehat{R}$ and M with $M \otimes_R \widehat{R}$ to assume that R is complete.

To complete the proof, we show that $\mathcal{I}_C\text{-}\operatorname{id}_R(C) < \infty$; the desired conclusion then follows from Theorem B since $\mathcal{P}_C\text{-}\operatorname{pd}_R(C) = 0$. The module M admits a bounded $\mathcal{P}_C$-resolution

$$0 \to C \otimes_R P_n \to \cdots \to C \otimes_R P_0 \to 0.$$

Augmenting this resolution and applying the functor $\operatorname{Hom}_R(-, E)$ yields an exact sequence

$$0 \to \underbrace{\operatorname{Hom}_R(M, E)}_{\cong C} \to \operatorname{Hom}_R(C \otimes_R P_0, E) \to \cdots \to \operatorname{Hom}_R(C \otimes_R P_n, E) \to 0.$$

Since each P_i is projective, each module $\operatorname{Hom}_R(P_i, E)$ is injective, and so

$$\operatorname{Hom}_R(C \otimes_R P_i, E) \cong \operatorname{Hom}_R(C, \operatorname{Hom}_R(P_i, E)) \in \mathcal{I}_C(R).$$

It follows that the displayed exact sequence is an augmented $\mathcal{I}_C$-coresolution of C, and so $\mathcal{I}_C\text{-}\operatorname{id}_R(C) < \infty$, as desired. □

Acknowledgments

S.S.-W. would like to thank the IPM for its generosity and hospitality during his visit when these results were proved.

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