On the Capacity of the Half-Duplex Diamond Channel

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Abstract

In this paper, a dual-hop communication system composed of a source $\mathcal{S}$ and a destination $\mathcal{D}$ connected through two non-interfering half-duplex relays, $\mathcal{R}_1$ and $\mathcal{R}_2$, is considered. In the literature of Information Theory, this configuration is known as the *diamond channel*. In this setup, four *transmission modes* are present, namely: 1) $\mathcal{S}$ transmits, and $\mathcal{R}_1$ and $\mathcal{R}_2$ listen (broadcast mode), 2) $\mathcal{S}$ transmits, $\mathcal{R}_1$ listens, and simultaneously, $\mathcal{R}_2$ transmits and $\mathcal{D}$ listens. 3) $\mathcal{S}$ transmits, $\mathcal{R}_2$ listens, and simultaneously, $\mathcal{R}_1$ transmits and $\mathcal{D}$ listens. 4) $\mathcal{R}_1$, $\mathcal{R}_2$ transmit, and $\mathcal{D}$ listens (multiple-access mode). Assuming a constant power constraint for all transmitters, a parameter Δ is defined, which captures some important features of the channel. It is proven that for $\Delta = 0$ the capacity of the channel can be attained by successive relaying, *i.e,* using modes 2 and 3 defined above in a successive manner. This strategy may have an infinite gap from the capacity of the channel when $\Delta \neq 0$. To achieve rates as close as 0.71 bits to the capacity, it is shown that the cases of $\Delta > 0$ and $\Delta < 0$ should be treated differently. Using new upper bounds based on the dual problem of the linear program associated with the cut-set bounds, it is proven that the successive relaying strategy needs to be enhanced by an additional broadcast mode (mode 1), or multiple access mode (mode 4), for the cases of $\Delta < 0$ and $\Delta > 0$, respectively. Furthermore, it is established that under average power constraints the aforementioned strategies achieve rates as close as 3.6 bits to the capacity of the channel.

Index Terms

Capacity, decode-and-forward, diamond channel, dual problem, gap analysis, half-duplex, linear program.

I. INTRODUCTION

A. Motivation

Relay-aided wireless systems, also called multi-hop systems, are implemented to increase the coverage and the throughput of communication systems [1]. These systems are becoming important parts of developing wireless communication standards, such as IEEE 802.16j (also known as WiMAX) [2]. Half-duplex relays, which transmit and receive data in different times and/or frequencies, are proven to be more practical and cost efficient in such standards than full-duplex relays.

From information theoretical point of view, the capacity becomes larger when more relays are added to the system. However, designing optimum strategies, especially in half-duplex systems, is challenging because subtle scheduling, *i.e.*, timing among transmission modes, is required to achieve rates near the capacity of such systems. During the last decade, the main stream of research carried out by several researchers dealt with single relay communication systems (cf. [5] and references therein). A simple model for investigating the potential benefits of a system with multiple relays is a dual-hop configuration with two parallel half-duplex relays (see Fig. 1). This configuration does not cover all two-relay systems because there are no source-destination and relay-relay links. However, it captures the basic difficulty in finding the best strategy in the system. As will be shown in this paper, a single strategy falls short of achieving rates near the capacity of the system for all channel realizations.

B. History

The single relay channel in which the relay facilitates a point-to-point communication was first studied in [3]. Two important coding techniques, *decode-and-forward* and *compress-and-forward*, were proposed in [4]. In the decode-and-forward scheme, the relay decodes the received message. In the compress-and-forward scheme, the relay sends the compressed (quantized) version of the received data to the destination. Following [4], generalizations to multi-relay networks were investigated by several researchers. A comprehensive survey of the progress in this area can be found in [5].

A simple model for understanding some aspects of the multi-relay networks is a network with two parallel relays, as introduced in [6], [7], and Fig. 1. It is assumed that there are no direct links from the source to the destination and also between the relays. This channel is studied in [8]–[16] and [21], and referred to as the *diamond relay channel* in [12].

For full-duplex relays, Schein and Gallager, in [6] and [7], provided upper and lower bounds on the capacity of the diamond channel. In particular, they considered the *amplify-and-forward*, and the decode-and-forward schemes, as well as a hybrid of them based on the time-sharing principle. Kochman, *et al.* proposed a *rematch-and-forward* scheme when different fractions of bandwidth can be allotted to the first and second hops [8]. Rezaei, *et al.* suggested a *combined amplify-and-decode-forward* strategy and proved that their scheme always performs better than the rematch-and-forward scheme [9]. In addition, they showed that the time-sharing between the combined amplify-and-decode-forward and decode-and-forward schemes provides a better achievable rate when compared to the time-sharing between the amplify-and-forward and decode-and-forward, and also between the rematch-and-forward and decode-and-forward, considered in [7], and [8], respectively. Kang and Ulukus employed a combination of the decode-and-forward and compress-and-forward schemes to obtain the capacity of a special class of the diamond channel with a noiseless relay [10]. Ghabeli and Aref in [11] proposed a new achievable rate based on the generalized block Markov encoding [23]. They also showed that their scheme achieves the capacity of a class of deterministic relay networks.

Half-duplex relays are studied in [12]–[18]. Xue and Sandhu in [12] proposed several schemes including the multi-hop with spatial reuse, scale-forward, broadcast-multiaccess with common message, compress-and-forward, and hybrid methods. These authors demonstrated that the *multi-hop with spatial reuse* protocol can achieve the channel capacity if the parallel links have the same capacity. Unlike [6]–[10], [12], [21], which assumed no direct link exists between the relays, [13]–[15] considered such link. More specifically, Chang, *et al.* proposed a combined dirty paper coding and block Markov encoding scheme [14]. Using numerical examples, they showed that the gap between their proposed strategy and the upper bound is relatively small in most cases. Rezaei, *et al.* considered two scheduling algorithms, namely *successive* and *simultaneous* relaying [15]. They derived asymptotic capacity results for the successive relaying and also proposed an achievable rate for the simultaneous relaying using a combination of the amplify-and-forward and decode-and-forward schemes. Other related papers are [17]–[20].

Characterizing the capacity of an information theoretic channel may be difficult. A simpler, yet important approach is to find an achievable scheme that ensures a small gap from the capacity of the channel. Recently, Etkin *et al.* characterized the capacity region of the interference channel to within one bit [26]. Following this new capacity analysis perspective, Avestimehr *et al.* proposed a deterministic model to better analyze the general single-source single-destination and the single-source multi-destination Gaussian networks [16], [21]. Their *quantize-and-map* achievablity scheme is guaranteed to provide a rate that is within a constant number of bits (determined by the graph topology of the network) from the cut-set upper bound.

C. Relation to Previous Works

In this paper, the setup and assumptions used in [12], with no link between the relays, are followed. In [12], the multi-hop with spatial reuse scheme proved to achieve the capacity of the diamond channel if the capacities of the parallel links in Fig. 1 are equal. This is called the *Multi-hopping Decode-and-Forward* (MDF) scheme. In the MDF scheme, relays successively forward their decoded messages to the destination (see Forward Modes I and II in Fig. 2). By introducing a fundamental parameter of the channel Δ (see Fig. 1), we generalize the optimality condition of the MDF scheme. In particular, we show that whenever $\Delta = 0$, the cut-set upper bound can be achieved. We also show that the MDF scheme cannot have a small gap from the cut-set upper bound for all channel realizations because the optimum strategy is highly related to the value of Δ.

In [16], the aim has been to establish the constant gap argument for the general relay networks with a single source and not to obtain a small gap optimized for a specific channel, such as the diamond channel. For the half-
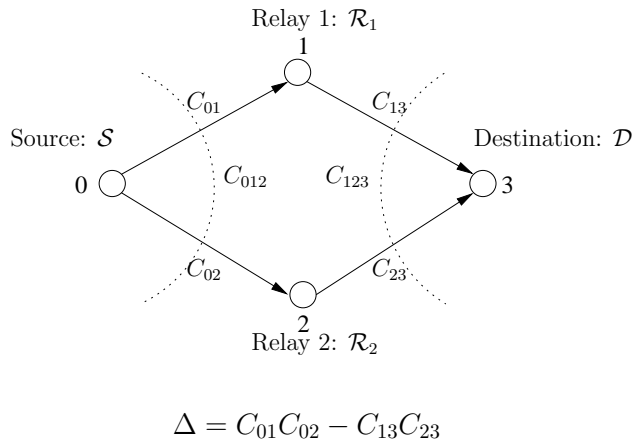
$$\Delta = C_{01}C_{02} - C_{13}C_{23}$$

Fig. 1. The diamond channel with its fundamental parameter Δ.

duplex diamond channel, the expression for the gap derived in [16] results in a 6-bit gap. In this paper, however, we focus on the diamond channel and obtain a smaller gap using our proposed achievablity scheme. In addition, we provide closed-form expressions for the time intervals associated with the transmission modes in the proposed scheduling. Specifically, we show that the expressions are different from those of the cut-set upper bound. This is in contrast to [16], where the constant gap between the cut-set bound and the quantize-and-map scheme was assured for every fixed scheduling, including the optimum scheduling associated with the cut-set upper bound.

In [16], using a different achievablity (a partial decode-and-forward) scheme than the quantize-and-map scheme, Avestimehr *et al.* showed that the capacity of the *full-duplex* diamond channel can be characterized within 1 bit per real dimension, regardless of the values of the channel gains. However, applying this scheme to the *half-duplex* diamond channel does not guarantee a constant gap from the channel capacity. We take one further step by providing an achievable scheme that ensures a small gap from the upper bounds for the half-duplex diamond channel. In particular, we show that the gap is smaller than .71 bits, assuming all transmitters have constant power constraints. We also prove that when transmitters have average power constraints instead, the gap is less than 3.6 bits.

The rest of this paper is organized as follows: Section II introduces the system model, the main ideas and results of this work. Section III presents the MDF scheme, which achieves the channel capacity for $\Delta = 0$. Sections IV and V provide the achievable schemes, upper bounds, and gap analysis for $\Delta < 0$ and $\Delta > 0$ cases, respectively. Section VI concludes the paper. In addition, Appendix A characterizes the Generalized Degrees Of Freedom (GDOF) of the diamond channel to obtain asymptotic capacity of the channel. Finally, Appendix B addresses the diamond channel with average power constraints.

D. Notations

Throughout the paper, $\bar{x} \triangleq 1-x$, and x^* denotes the optimal solution to an optimization problem with an objective function $F(x)$. The transpose of the vector or matrix $\mathbf{A}$ is indicated by $\mathbf{A}^T$. $a \Rightarrow b$ represents the link from node a to node b. Also, $x \leftrightarrow y$ means that the roles of x and y are exchanged in a given function $F(x,y)$. In addition, it is assumed that all logarithms are to base 2. Finally, $\mathcal{C}(P) \triangleq \frac{1}{2}\log(1+P)$.

II. PROBLEM STATEMENT AND MAIN RESULTS

In this work, a dual-hop communication system, depicted in Fig. 1, is considered. The model consists of a source ($\mathcal{S}$), two parallel half-duplex relays ($\mathcal{R}_1$, $\mathcal{R}_2$), and a destination ($\mathcal{D}$), respectively, indexed by 0, 1, 2, and 3 as shown in Fig. 1. No link is assumed between Source and Destination, as well as between the relays. The channel gain between node a and b is assumed to be constant, known to all nodes, and is represented by h_{ab} with magnitude $\sqrt{g_{ab}}$.

Due to the half-duplex constraint, four transmission modes exist in the diamond channel where, in every mode, each relay either transmits data to Destination or receives data from Source (see Fig. 2). In the figure, $X_a^{(i)}$ and
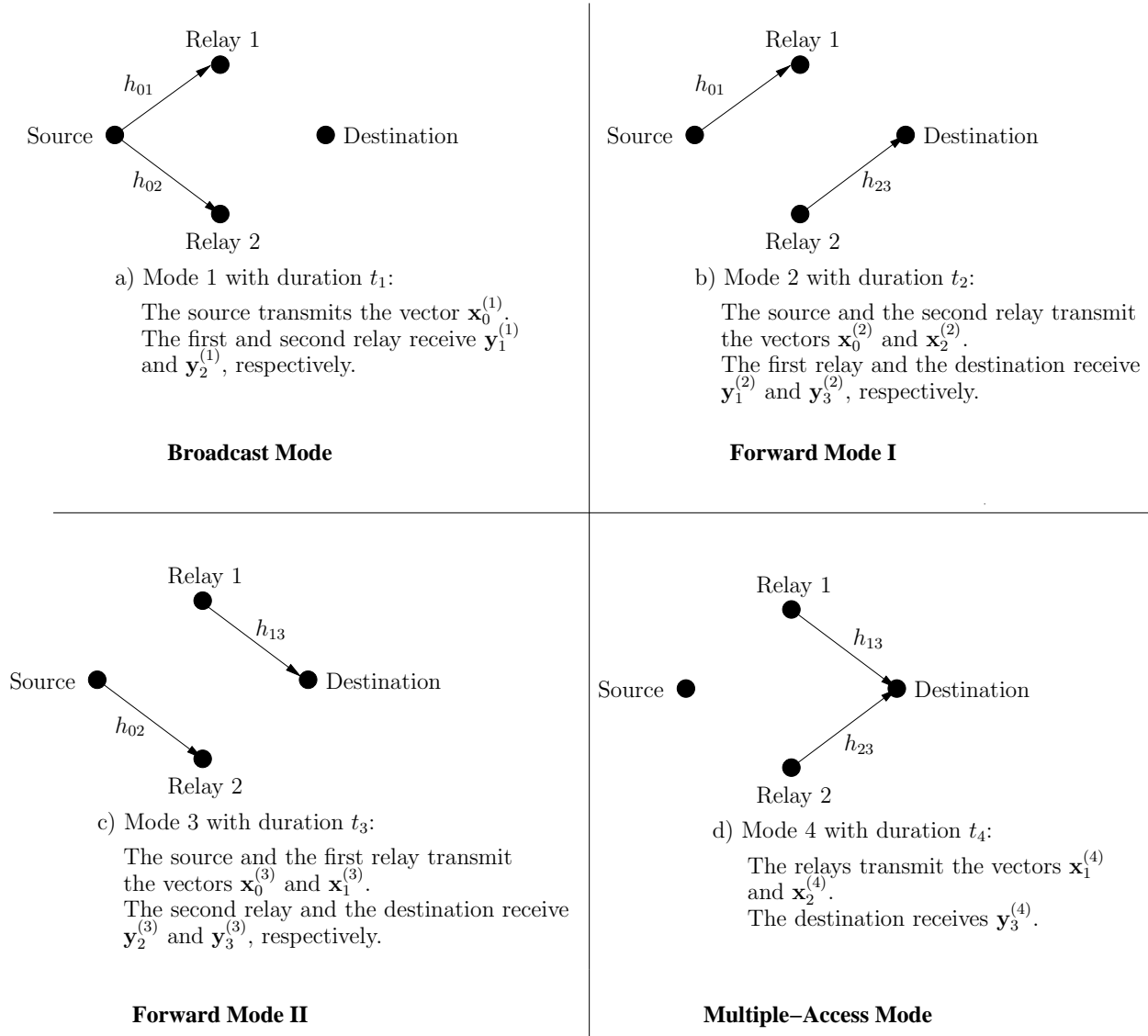
Fig. 2.   Transmission modes for the diamond channel.

$Y_a^{(i)}$ represent the transmitting and receiving signals at node a corresponding to mode i, respectively. The total transmission time is normalized to one and partitioned into four time intervals (t_1, t_2, t_3, t_4) corresponding to modes $1, 2, 3$, and 4, with the constraint $\sum_{i=1}^{4} t_i = 1$. The discrete-time baseband representation of the received signals at Relay 1, Relay 2, and Destination are respectively given by:

$$Y_1 = h_{01} X_0 + N_1,$$
$$Y_2 = h_{02} X_0 + N_2,$$
$$Y_3 = h_{13} X_1 + h_{23} X_2 + N_3,$$

where N_a is the Gaussian noise at node a with unit variance.

Let us assume Source, Relay 1, and Relay 2 consume, respectively, $P_{\mathcal{S}}^{(i)}$, $P_{\mathcal{R}_1}^{(i)}$, and $P_{\mathcal{R}_2}^{(i)}$ amount of power in

mode i, *i.e.,*

$$\frac{1}{t_i} \sum_{t_i} |X_0|^2 \le P_{\mathcal{S}}^{(i)},$$

$$\frac{1}{t_i} \sum_{t_i} |X_1|^2 \le P_{\mathcal{R}_1}^{(i)},$$

$$\frac{1}{t_i} \sum_{t_i} |X_2|^2 \le P_{\mathcal{R}_2}^{(i)}.$$

The total power constraints for Source, Relay 1, and Relay 2 are $P_{\mathcal{S}}$, $P_{\mathcal{R}_1}$, and $P_{\mathcal{R}_2}$, respectively, and are related to the amount of power spent in each mode as follows:

$$\sum_{i=1}^{4} t_i P_{\mathcal{S}}^{(i)} \le P_{\mathcal{S}},$$

$$\sum_{i=1}^{4} t_i P_{\mathcal{R}_1}^{(i)} \le P_{\mathcal{R}_1},$$

$$\sum_{i=1}^{4} t_i P_{\mathcal{R}_2}^{(i)} \le P_{\mathcal{R}_2}.$$

Due to some practical considerations on the power constraints [12], we mainly consider *constant* power constraints for transmitters, *i.e.,* for $i \in \{1, \cdots, 4\}$,

$$\begin{aligned} P_{\mathcal{S}}^{(i)} &= P_{\mathcal{S}}, \\ P_{\mathcal{R}_1}^{(i)} &= P_{\mathcal{R}_1}, \\ P_{\mathcal{R}_2}^{(i)} &= P_{\mathcal{R}_2}. \end{aligned} \tag{1}$$

Without loss of generality, a unit power constraint is considered for all nodes, *i.e.,* $P_{\mathcal{S}} = P_{\mathcal{R}_1} = P_{\mathcal{R}_2} = 1$. We define the parameters $C_{01}, C_{02}, C_{13}, C_{23}$ as $\mathcal{C}(g_{01}), \mathcal{C}(g_{02}), \mathcal{C}(g_{13}), \mathcal{C}(g_{23})$, respectively. Moreover, C_{012} and C_{123} are defined as:

$$\begin{aligned} C_{012} &\triangleq \mathcal{C}(g_{01} + g_{02}), \\ C_{123} &\triangleq \mathcal{C}\big((\sqrt{g_{13}} + \sqrt{g_{23}})^2\big). \end{aligned} \tag{2}$$

The case in which transmitters have average power constraints instead of constant power constraints is addressed in Appendix B.

In this work, we are interested in finding communication protocols that operate close to the channel capacity. We introduce an important parameter of the channel Δ as:

$$\Delta \triangleq C_{01} C_{02} - C_{13} C_{23}. \tag{3}$$

We categorize all realizations of the diamond channel into three groups based on the sign of Δ (*i.e.,* $\Delta < 0$, $\Delta = 0$, and $\Delta > 0$). As will be shown in the sequel, the sign of Δ plays an important role in designing the optimum scheduling for the channel.

In this setup, the cut-set bounds can be stated in the form of a Linear Program (LP) due to the assumption of constant power constraints for all transmitters. By analyzing the dual program we provide fairly tight upper bounds expressed as single equations corresponding to different channel conditions. Using the dual problem, we prove that when $\Delta = 0$, the MDF scheme achieves the capacity of the diamond channel. Note that $\Delta = 0$ (*i.e.,* $C_{01} C_{02} = C_{13} C_{23}$) includes the previous optimality condition presented in [12] (*i.e.,* $C_{01} = C_{23}$ and $C_{02} = C_{13}$) as a special case. To realize how close the MDF scheme performs to the capacity of the channel when $\Delta \ne 0$, we calculate the gap from the upper bounds. We show that the MDF scheme provides the gap of less than 1.21 bits when applied in the symmetric or some classes of asymmetric diamond channels. More importantly, we explain that the gap can be arbitrarily large for certain ranges of parameters.

By employing new scheduling algorithms we shrink the gap to $.71$ bits for *all* channel conditions. In particular, for $\Delta < 0$ we add Broadcast (BC) Mode (shown in Fig. 2) to the MDF scheme to provide the relays with more

reception time. In this three-mode scheme, referred to as Multi-hopping Decode-and-Forward with Broadcast (*MDF-BC*) scheme, the relays decode what they have received from Source and forward the re-encoded information to Destination in Forward Modes I and II. When $\Delta > 0$, Multiple-Access (MAC) mode (shown in Fig. 2) in which the relays transmit independent information to Destination is added to the MDF scheme. We call this protocol Multi-hopping Decode-and-Forward with Multiple-Access (*MDF-MAC*) scheme.

The mentioned contributions are associated with the case wherein the transmitters are operating under constant power constraints (1). However, for a more general setting in which the transmitters are subject to average power constraints (41), it is shown in Appendix B that the cut-set upper bounds are increased by at most 2.89 bits. Therefore, the proposed achievable schemes guarantee the maximum gap of 3.6 bits from the cut-set upper bounds in the general average power constraint setting.

A. Coding Scheme

The proposed achievable scheme may employ all four transmission modes as follows:

1) **Broadcast Mode**: In t_1 fraction of the transmission time, Source broadcasts independent information to Relays 1 and 2 using the superposition coding technique.

2) **Forward Mode I**: In t_2 fraction of the transmission time, Source transmits new information to Relay 1. At the same time, Relay 2 sends the re-encoded version of part of the data received during Broadcast Mode and/or Forward Mode II of the previous block to Destination.

3) **Forward Mode II**: In t_3 fraction of the transmission time, Source transmits new information to Relay 2. At the same time, Relay 1 sends the re-encoded version of part of what it has received during Broadcast Mode and/or Forward Mode I of the previous block to Destination.

4) **Multiple-Access Mode**: In the remaining t_4 fraction of the transmission time, Relays 1 and 2 simultaneously transmit the residual information (corresponding to the previous block) to Destination where, *joint decoding* is performed to decode the received data.

In Broadcast Mode, superposition coding, which is known to be the optimal transmission scheme for the degraded broadcast channel [28], is used to transmit independent data to the relays. The resulting data-rates u and v, respectively associated with Relay 1 and Relay 2 are:

$$u(\eta) = \begin{cases} \mathcal{C}(\eta g_{01}) & \text{if} \quad g_{02} \le g_{01} \\ C_{01} - \mathcal{C}(\eta g_{01}) & \text{if} \quad g_{01} < g_{02}, \end{cases} \tag{4}$$

$$v(\eta) = \begin{cases} C_{02} - \mathcal{C}(\eta g_{02}) & \text{if} \quad g_{02} \le g_{01} \\ \mathcal{C}(\eta g_{02}) & \text{if} \quad g_{01} < g_{02}. \end{cases} \tag{5}$$

The power allocation parameter η determines the amount of Source power used to transmit information to the relay with better channel quality in Broadcast Mode.

In Multiple-Access Mode, a multiple-access channel exists in which the users (relays) have independent messages for Destination. For this channel, joint decoding is optimum, which provides the following rate region [28]:

$$\begin{aligned} R_1 &\le t_4 C_{13}, \\ R_2 &\le t_4 C_{23}, \\ R_1 + R_2 &\le t_4 C_{\text{MAC}}, \end{aligned} \tag{6}$$

where R_1, R_2 are the rates that Relay 1 and Relay 2 provide to Destination in Multiple-Access Mode, respectively, and C_{MAC} is defined as:

$$C_{\text{MAC}} \triangleq \mathcal{C}(g_{13} + g_{23}). \tag{7}$$

According to the protocol, Relay 1 can receive up to $t_1 u + t_2 C_{01}$ bits per channel use during Broadcast Mode and Forward Mode I. Then the relay has the opportunity to send its received information to Destination in Forward Mode II and Multiple-Access Mode, with the rate $t_3 C_{13} + R_1$. Similarly, Relay 2 can receive and forward messages with the rates $t_1 v + t_3 C_{02}$, and $t_2 C_{23} + R_2$, respectively. Therefore, the maximum achievable rate of the scheme, R, is:

$$R = \max_{\sum_{i=1}^{4} t_i = 1, t_i \ge 0} \left\{ \min\{t_1 u + t_2 C_{01}, t_3 C_{13} + R_1\} + \min\{t_1 v + t_3 C_{02}, t_2 C_{23} + R_2\} \right\}. \tag{8}$$

Sections III-V show that employing Forward Modes I and II for $\Delta = 0$, the first three transmission modes for $\Delta < 0$, and the last three transmission modes for $\Delta > 0$ are sufficient to achieve a small gap from the derived upper bounds.

B. Cut-set Upper Bound and the Dual Program

For general half-duplex networks with K relays, Khojastepour *et al.* proposed a cut-set type upper bound by doing the following steps:

1) Fix the input distribution and scheduling, *i.e.*, $p(X_0, X_1, X_2)$, and t_1, t_2, t_3, t_4 such that $\sum_{i=1}^{4} t_i = 1$.
2) Find the rate $R_{i,j}$ associated with the cut j for each transmission mode i where $i, j \in \{1, \cdots, 2^K\}$.
3) Multiply $R_{i,j}$ by the corresponding time interval t_i.
4) Compute $\sum_{i=1}^{2^K} t_i R_{i,j}$ and minimize it over all cuts.
5) Take the supremum over all input distributions and schedulings.

The preceding procedure can be directly applied to the diamond channel, whose transmission modes are shown in Fig. 2. The best input distribution and scheduling lead to:

$$C_{\text{DC}} \le t_1 I(X_0^{(1)}; Y_1^{(1)}, Y_2^{(1)}) + t_2 I(X_0^{(2)}; Y_1^{(2)}|X_2^{(2)}) + t_3 I(X_0^{(3)}; Y_2^{(3)}|X_1^{(3)}) + t_4.0,$$
$$C_{\text{DC}} \le t_1 I(X_0^{(1)}; Y_1^{(1)}) + t_2 \left(I(X_0^{(2)}; Y_1^{(2)}) + I(X_2^{(2)}; Y_3^{(2)}) \right) + t_3.0 + t_4 I(X_2^{(4)}; Y_3^{(4)}|X_1^{(4)}),$$
$$C_{\text{DC}} \le t_1 I(X_0^{(1)}; Y_2^{(1)}) + t_2.0 + t_3 \left(I(X_0^{(3)}; Y_2^{(3)}) + I(X_1^{(3)}; Y_3^{(3)}) \right) + t_4 I(X_1^{(4)}; Y_3^{(4)}|X_2^{(4)}),$$
$$C_{\text{DC}} \le t_1.0 + t_2 I(X_2^{(2)}; Y_3^{(2)}) + t_3 I(X_1^{(3)}; Y_3^{(3)}) + t_4 I(X_1^{(4)}, X_2^{(4)}; Y_3^{(4)}),$$

where C_{DC} denotes the capacity of the diamond channel. The above bounds do not decrease if each mutual information term is replaced by its maximum value. This substitution simplifies the computation of the upper bound, called R_{up}, by providing the following LP [12]:

$$
\begin{aligned}
\text{maximize} \quad & R_{\text{up}} \\
\text{subject to:} \quad & R_{\text{up}} \le t_1 C_{012} + t_2 C_{01} + t_3 C_{02} + t_4.0 \\
& R_{\text{up}} \le t_1 C_{01} + t_2(C_{01} + C_{23}) + t_3.0 + t_4 C_{23} \\
& R_{\text{up}} \le t_1 C_{02} + t_2.0 + t_3(C_{02} + C_{13}) + t_4 C_{13} \\
& R_{\text{up}} \le t_1.0 + t_2 C_{23} + t_3 C_{13} + t_4 C_{123} \\
& \textstyle\sum_{i=1}^{4} t_i = 1, \ t_i \ge 0.
\end{aligned}
\tag{9}
$$

To obtain appropriate single-equation upper bounds on the capacity, we rely on the fact that every feasible point in the dual program provides an upper bound on the primal. Hence, we develop the desired upper bounds by looking at the dual program. In the sequel, we derive the dual program for the LP (9).

We start with writing the LP in the standard form as:

$$
\begin{aligned}
\text{maximize} \quad & \mathbf{c}^T \mathbf{x} \\
\text{subject to:} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \\
& \mathbf{x} \ge 0,
\end{aligned}
$$

where the unknown vector $\mathbf{x} = [t_1, t_2, t_3, t_4, R_{\text{up}}]^T$, the vectors of coefficients $\mathbf{b} = \mathbf{c} = [0, 0, 0, 0, 1]^T$, and the matrix of coefficients $\mathbf{A}$ is:

$$
\mathbf{A} =
\begin{pmatrix}
-C_{012} & -C_{01} & -C_{02} & 0 & 1 \\
-C_{01} & -(C_{01} + C_{23}) & 0 & -C_{23} & 1 \\
-C_{02} & 0 & -(C_{02} + C_{13}) & -C_{13} & 1 \\
0 & -C_{23} & -C_{13} & -C_{123} & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}.
$$

Since $\mathbf{A} = \mathbf{A}^T$, it is easy to verify that the primal and dual programs share the same form, *i.e.*,

$$
\begin{aligned}
\text{minimize} \quad & R_{\text{up}} \\
\text{subject to:} \quad & R_{\text{up}} \ge \tau_1 C_{012} + \tau_2 C_{01} + \tau_3 C_{02} + \tau_4.0 \\
& R_{\text{up}} \ge \tau_1 C_{01} + \tau_2(C_{01} + C_{23}) + \tau_3.0 + \tau_4 C_{23} \\
& R_{\text{up}} \ge \tau_1 C_{02} + \tau_2.0 + \tau_3(C_{02} + C_{13}) + \tau_4 C_{13} \\
& R_{\text{up}} \ge \tau_1.0 + \tau_2 C_{23} + \tau_3 C_{13} + \tau_4 C_{123} \\
& \textstyle\sum_{i=1}^{4} \tau_i = 1, \ \tau_i \ge 0.
\end{aligned}
\tag{10}
$$

In the dual program (10), τ_i, for $i \in \{1, \cdots, 4\}$ corresponds to the ith rate constraint in the primal LP (9). Clearly, the LP (9) is *feasible*. Hence, the duality of linear programming ensures that there is no gap between the primal

and the dual solutions [27]. However, the benefit of using the dual problem here is that any feasible choice of the vector $\boldsymbol{\tau}$ provides an upper bound to the rate obtained by solving the original LP. This property is known as the weak duality property of LP [27]. Appropriate vectors (*i.e.*, $\boldsymbol{\tau}$'s) in the dual program (10) are selected to obtain fairly tight upper bounds. In fact, employing such vectors instead of solving the primal LP (9) simplifies the gap analysis. In sections IV and V, these vectors are provided for $\Delta < 0$ and $\Delta > 0$ cases, respectively. In the following sections, we employ the proposed achievable schemes together with the derived upper bounds to characterize the capacity of the diamond channel up to 0.71 bits.

III. MDF Scheme and Achieving the Capacity for $\Delta = 0$

In this section, the MDF scheme is described and then proved to be capacity-achieving when $\Delta = 0$.

A. MDF Scheme

The MDF scheduling algorithm uses two transmission modes: Forward Modes I and II shown in Fig. 2 along with the decode-and-forward strategy and can be described as follows:

1) In λ fraction of the transmission time, Source and Relay 2 transmit to Relay 1 and Destination, respectively.
2) In the remaining $\bar{\lambda}$ fraction of the transmission time, Source and Relay 1 transmit to Relay 2 and Destination, respectively.

The achievable rate of the MDF scheme is the summation of the rates of the first and second parallel paths (branches) from Source to Destination, which can be expressed as [12]:

$$R_{\text{MDF}} = \max_{0 \leq \lambda \leq 1} \left\{ \min\{\lambda C_{01}, \bar{\lambda} C_{13}\} + \min\{\bar{\lambda} C_{02}, \lambda C_{23}\} \right\}.$$

The above LP can be re-written as:

$$
\begin{aligned}
\text{maximize} \quad & R_1 + R_2 \\
\text{subject to:} \quad & R_1 \leq \lambda C_{01} \\
& R_1 \leq \bar{\lambda} C_{13} \\
& R_2 \leq \bar{\lambda} C_{02} \\
& R_2 \leq \lambda C_{23} \\
& 0 \leq \lambda \leq 1,
\end{aligned}
$$

where R_1 and R_2 denote the rate of the upper and the lower branches, respectively. This LP has three unknowns (R_1, R_2, λ) and six inequalities. The solution turns three out of six inequalities to equality. The optimum time interval λ^* can not be equal to 0 or 1, as both solutions give a zero rate. Hence, three out of the first four inequalities should become equality, which leads to the following achievable rates for different channel conditions:

$$R_{\text{MDF}} = \begin{cases} R_{\text{MDF}}^1 = \dfrac{C_{01}(C_{02} + C_{13})}{C_{01} + C_{13}} & \text{if } \Delta \leq 0,\, C_{02} \leq C_{01} \\[2mm] R_{\text{MDF}}^2 = \dfrac{C_{02}(C_{01} + C_{23})}{C_{02} + C_{23}} & \text{if } \Delta \leq 0,\, C_{02} > C_{01} \\[2mm] R_{\text{MDF}}^3 = \dfrac{C_{13}(C_{01} + C_{23})}{C_{01} + C_{13}} & \text{if } \Delta > 0,\, C_{23} \leq C_{13} \\[2mm] R_{\text{MDF}}^4 = \dfrac{C_{23}(C_{02} + C_{13})}{C_{02} + C_{23}} & \text{if } \Delta > 0,\, C_{23} > C_{13}. \end{cases} \tag{11}$$

In particular, the achievable rate for the *symmetric* diamond channel, in which $C_{01} = C_{02}$ and $C_{13} = C_{23}$, is:

$$R_{\text{MDF}}^{\text{sym}} = \min\{C_{01}, C_{13}\}.$$

The optimum time interval λ^* is either equal to λ_1^* or λ_2^* defined below:

$$\lambda^* = \begin{cases} \dfrac{C_{13}}{C_{01} + C_{13}} \triangleq \lambda_1^*, \\ \quad\text{or} \\ \dfrac{C_{02}}{C_{02} + C_{23}} \triangleq \lambda_2^*. \end{cases}$$

Note that if $\lambda^* = \lambda_1^*$, then $\lambda_1^* C_{01} = \bar{\lambda}_1^* C_{13}$. Similarly, $\lambda^* = \lambda_2^*$ leads to $\bar{\lambda}_2^* C_{02} = \lambda_2^* C_{23}$. In other words, λ_i^* for $i \in \{1, 2\}$ makes the maximum amount of data that can be received by Relay i equal to the maximum amount of

data that can be forwarded by Relay i. In this case, branch i (composed of $0 \Rightarrow i \Rightarrow 3$ links) is said to be *fully utilized*.

It is interesting to consider that the case fully utilizing branch 1 or branch 2 leads to the same data-rate. This case occurs when one of the following happens:

$$\begin{cases} \Delta & = 0, \\ C_{01} & = C_{02} & \text{if } \Delta < 0, \\ C_{13} & = C_{23} & \text{if } \Delta > 0. \end{cases} \tag{12}$$

In these situations, one can use either λ_1^* or λ_2^* fraction of the transmission time for Forward Mode I and the remaining fraction for Forward Mode II and achieve the same data-rate. It will be shown later that the MDF scheme achieves the capacity of the diamond channel if $\Delta = 0$ and is at most 1.21 bits less than the capacity for the other two cases. It is remarked that $\Delta = 0$ makes both branches fully utilized and all four rates in Eq. (11) equal.

B. MDF is Optimal for $\Delta = 0$

Here, it is explained that R_{up}^*, found by solving the dual-program (10), is the same as the MDF rate given in Eq. (11) for $\Delta = 0$. It is easy to observe that

$$\boldsymbol{\tau}^* = \left[0, \frac{C_{13}}{C_{01} + C_{13}}, \frac{C_{23}}{C_{02} + C_{23}}, 0 \right] \tag{13}$$

makes all four rate constraints in the dual-program (10) equal to the rate obtained in Eq. (11) and satisfies $\sum_{i=1}^{4} \tau_i = 1$. Therefore, the upper bound provided by vector $\boldsymbol{\tau}$ is indeed the capacity of the channel and equals to:

$$C_{\text{DC}} = \frac{C_{01} C_{13}}{C_{01} + C_{13}} + \frac{C_{02} C_{23}}{C_{02} + C_{23}}. \tag{14}$$

The result is valid for the Gaussian multiple antenna as well as discrete memoryless channels, and therefore $\Delta = 0$ ensures the optimality of the MDF scheme for those channels too.

C. MDF Gap Analysis

To investigate how close the MDF scheme performs to the capacity of the diamond channel when $\Delta \neq 0$, the appropriate upper bounds are required, which will be derived in sections IV and V. Therefore, the detailed gap analysis for the MDF scheme is deferred to Appendix C, where it is shown that although a small gap is achievable for some channel conditions, the gap can be large in general. In the following sections, Broadcast and Multiple-Access Modes are added to the MDF algorithm to achieve 0.71 bits of the capacity for $\Delta > 0$ and $\Delta < 0$ cases, respectively.

IV. MDF-BC SCHEME AND ACHIEVING WITHIN 0.71 BITS OF THE CAPACITY FOR $\Delta < 0$

In the MDF scheme, since both branches cannot be fully utilized when $\Delta < 0$ simultaneously, there exists some unused capacity in the second hop. To efficiently make use of the available resources, Broadcast Mode is added to the MDF scheme. This mode provides the relays with an additional reception time.

A. Achievable Scheme

The modified protocol uses Broadcast Mode together with Forward Modes I and II. Therefore, by setting $t_4 = 0$ in Eq. (8) the maximum achievable rate of the scheme as a function of the power allocation parameter η used in superposition coding is:

$$R_{\text{BC}}(\eta) = \max_{\sum_{i=1}^{3} t_i = 1, t_i \geq 0} \left\{ \min\{t_1 u(\eta) + t_2 C_{01}, t_3 C_{13}\} + \min\{t_1 v(\eta) + t_3 C_{02}, t_2 C_{23}\} \right\}.$$

Recall that u, and v, defined respectively in Eqs. (4) and (5), are the rates associated with Relays 1 and 2 in Broadcast Mode. First, the optimal schedule is obtained, assuming a fixed η, and later an appropriate value for η

will be selected. The achievable rate can be written as the following LP:

$$\text{maximize } R_{\text{BC}}$$

$$\text{subject to: } R_{\text{BC}} \leq t_1(u + v) + t_2 C_{01} + t_3 C_{02} \tag{15}$$

$$R_{\text{BC}} \leq t_1 u + t_2(C_{01} + C_{23}) + t_3.0 \tag{16}$$

$$R_{\text{BC}} \leq t_1 v + t_2.0 + t_3(C_{02} + C_{13}) \tag{17}$$

$$R_{\text{BC}} \leq t_2 C_{23} + t_3 C_{13} \tag{18}$$

$$\sum_{i=1}^{3} t_i = 1 \tag{19}$$

$$t_i \geq 0. \tag{20}$$

For a feasible LP, the solution is at one of the extreme points of the constraint set. One of the extreme points can be obtained by solving a set of linear equations containing Eq. (19) and inequalities (15)-(17) considered as equalities. The solution becomes:

$$t_1 = \frac{-\Delta}{(C_{01} + C_{13})v + (C_{02} + C_{23})u - \Delta},$$

$$t_2 = \frac{C_{13}v + C_{02}u}{(C_{01} + C_{13})v + (C_{02} + C_{23})u - \Delta},$$

$$t_3 = \frac{C_{01}v + C_{23}u}{(C_{01} + C_{13})v + (C_{02} + C_{23})u - \Delta},$$

$$R_{\text{BC}}(\eta) = \frac{C_{13}(C_{01} + C_{23})v(\eta) + C_{23}(C_{02} + C_{13})u(\eta)}{(C_{01} + C_{13})v(\eta) + (C_{02} + C_{23})u(\eta) - \Delta}. \tag{21}$$

It is easy to verify that $\Delta = 0$ makes $t_1 = 0$, and hence leads to the MDF algorithm. Note that in addition to inequalities (15)-(17), the above extreme point also turns inequality (18) into equality. Now, this extreme point is proven to be the solution to the above LP. If one of the elements of vector **t** is increased, at least one of the conditions (15)-(17) provides a smaller rate, compared to the rate obtained by the extreme point. For instance, if t_1 in Eq. (21) is increased, then, because of Eq. (19), at least one of t_2 and t_3 should be decreased, which in turn reduces the rate associated with the inequality (18). This confirms that the extreme point is the optimal solution to the LP with constraints (15)-(20).

In the following, instead of searching for η^*, which maximizes $R_{BC}(\eta)$, an appropriate value for η is found that not only provides a small gap from the upper bounds, but also simplifies the gap analysis of section IV-C. The power allocation parameter η is selected to be either $\eta_1 \triangleq \frac{1}{g_{01}+1}$, or $\eta_2 \triangleq \frac{1}{g_{02}+1}$ for $C_{02} \geq C_{01}$ and $C_{01} \geq C_{02}$ conditions, respectively. As it will be shown in Appendix A, the chosen η produces the same GDOF as the corresponding upper bound, which is a necessary condition in obtaining a small gap. The corresponding u and v for η_1 are:

$$u(\eta_1) = C_{01} - \zeta_1,$$
$$v(\eta_1) = C_{012} - C_{01}, \tag{22}$$

and for η_2 are:

$$u(\eta_2) = C_{012} - C_{02},$$
$$v(\eta_2) = C_{02} - \zeta_2. \tag{23}$$

In the above,

$$\zeta_1 \triangleq \mathcal{C}(\frac{g_{01}}{g_{01}+1}) \leq \frac{1}{2}, \tag{24}$$

$$\zeta_2 \triangleq \mathcal{C}(\frac{g_{02}}{g_{02}+1}) \leq \frac{1}{2}. \tag{25}$$

The selected η divides the source power between u and v (considered as the rates of two virtual users in the broadcast channel consisting of $\mathcal{S} \Rightarrow \mathcal{R}_1$ and $\mathcal{S} \Rightarrow \mathcal{R}_2$ links) in such a way that:

1) the sum data-rate (*i.e.*, $u + v$) in the broadcast channel is close to the sum-capacity of the broadcast channel (*i.e.*, $\max\{C_{01}, C_{02}\}$),

2) the weaker user's rate is close to its capacity. For instance, if $C_{01} \leq C_{02}$, then $u \approx C_{01}$.

Substituting u and v from Eqs. (22) and (23) into (21) leads to the following achievable rates $R^1_{\text{MDF-BC}}$ and $R^2_{\text{MDF-BC}}$ corresponding to η_1 and η_2:

$$R^1_{\text{MDF-BC}} = \frac{C_{13}(C_{01} + C_{23})C_{012} - C_{01}^2 C_{13} + C_{01}C_{02}C_{23} - \zeta_1 C_{23}(C_{02} + C_{13})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23}) - \zeta_1(C_{02} + C_{23})},$$

$$R^2_{\text{MDF-BC}} = \frac{C_{23}(C_{02} + C_{13})C_{012} - C_{02}^2 C_{23} + C_{01}C_{02}C_{13} - \zeta_2 C_{13}(C_{01} + C_{23})}{(C_{02} + C_{23})(C_{012} - C_{02} + C_{13}) - \zeta_2(C_{01} + C_{13})}. \tag{26}$$

B. Upper Bound

Following the discussion in section II-B, we select one of the extreme points of the constraint set (10) to obtain a fairly tight upper bound. Below, some insights on how to find an appropriate extreme point are given.

First, Forward Modes I and II play an important role in data transfer from Source to Destination. These two modes let both Source and Destination be simultaneously active, which is important for efficient communication. This implies that *generally* t_2^* and t_3^* are not zero in the original LP (9). In addition, $\Delta < 0$ roughly means that the second hop is better than the first hop. In this case, Broadcast Mode helps the relays to collect more data which will be sent to Destination using Forward Modes I and II later. Therefore, Multiple-Access Mode is less important when $\Delta < 0$ and consequently t_4 can be set to zero. Using the complementary slackness theorem of linear programming (cf. [27]), having non-zero t_1, t_2, and t_3 in the original LP translates into having the first three inequalities in the dual program satisfied with equality. Now looking at the dual problem (10) with the same structure as the original LP, in order to achieve a smaller objective function, we set τ_2 or τ_3 to zero. This is in contrast to the claim for having both of t_2 and t_3 non-zero in the original LP with the maximization objective. Therefore, the vector $\boldsymbol{\tau}$ with the following properties is selected:

1) Either τ_2 or τ_3 is zero.

2) The first three inequalities are satisfied with equality.

To have a valid $\boldsymbol{\tau}$, we need to make sure that all the elements of vector $\boldsymbol{\tau}$ are non-negative and that $\boldsymbol{\tau}$ satisfies the last condition.

As mentioned earlier, either τ_2 or τ_3 can be set to zero in the dual program (10). For instance, setting $\tau_2 = 0$ in the dual-program gives the following LP:

$$\begin{aligned}
\text{minimize} \quad & \widetilde{R} \\
\text{subject to:} \quad & \widetilde{R} \geq \tau_1 C_{012} + \tau_3 C_{02} + \tau_4.0 \\
& \widetilde{R} \geq \tau_1 C_{01} + \tau_3.0 + \tau_4 C_{23} \\
& \widetilde{R} \geq \tau_1 C_{02} + \tau_3(C_{02} + C_{13}) + \tau_4 C_{13} \\
& \widetilde{R} \geq \tau_1.0 + \tau_3 C_{13} + \tau_4 C_{123} \\
& \sum_{i=1, i \neq 2}^{4} \tau_i = 1, \ \tau_i \geq 0.
\end{aligned} \tag{27}$$

Setting the first three inequalities to equalities gives:

$$\tau_1^* = \frac{C_{13}}{C_{012} - C_{02} + C_{13}},$$

$$\tau_3^* = \frac{C_{23}(C_{012} - C_{02}) - C_{13}(C_{012} - C_{01})}{(C_{02} + C_{23})(C_{012} - C_{02} + C_{13})},$$

$$\tau_4^* = \frac{C_{13}(C_{012} - C_{01}) + C_{02}(C_{012} - C_{02})}{(C_{02} + C_{23})(C_{012} - C_{02} + C_{13})},$$

$$\widetilde{R}^* = \frac{(C_{02} + C_{13})C_{23}}{C_{02} + C_{23}} + \frac{C_{13}(C_{01}C_{02} - C_{13}C_{23})}{(C_{02} + C_{23})(C_{012} - C_{02} + C_{13})}. \tag{28}$$

For obtaining a valid result, the following conditions have to be ensured:

1) $\tau_3^* \geq 0$.

   Since $C_{02} \leq C_{012}$, the denominator of τ_3^* is non-negative, therefore, the non-negativity of the nominator has

to be guaranteed. This imposes the constraint $\Gamma \geq 0$ on the values of channel parameters, where Γ is defined as:

$$\Gamma \triangleq C_{23}[C_{012} - C_{02}] - C_{13}[C_{012} - C_{01}]. \tag{29}$$

2) $\widetilde{R}^* \geq \tau_3^* C_{13} + \tau_4^* C_{123}$.

To satisfy the following condition:

$$\widetilde{R}^* = \tau_1^* C_{02} + \tau_3^*(C_{02} + C_{13}) + \tau_4^* C_{13} \geq \tau_3^* C_{13} + \tau_4^* C_{123},$$

it is sufficient to show:

$$(\tau_1^* + \tau_3^*)C_{02} \geq \tau_4^*(C_{123} - C_{13}),$$

which can be equivalently represented as:

$$\frac{C_{02}}{C_{123} - C_{13} + C_{02}} \geq \tau_4^*.$$

The following lemma proves the preceding inequality.

Lemma 1 $\tau_4^* \leq \frac{C_{02}}{C_{123} - C_{13} + C_{02}}$ *for* $C_{123} \leq C_{13} + C_{23}$.

Proof: See Appendix D-A . ∎

Lemma 1 requires $C_{123} \leq C_{13} + C_{23}$, which is not true for $g_{13}g_{23} \leq 4$. To be able to use Lemma 1 for the case of $g_{13}g_{23} \leq 4$, we replace either C_{13} by $\hat{C}_{13} \triangleq C_{13} + \delta$ or C_{23} by $\hat{C}_{23} = C_{23} + \delta$ with δ defined as:

$$\delta \triangleq \max\{C_{123} - (C_{13} + C_{23}), 0\}. \tag{30}$$

This change provides the desired inequality (*i.e.,* $C_{123} \leq \hat{C}_{13} + C_{23}$ or $C_{123} \leq C_{13} + \hat{C}_{23}$) at the expense of increasing the upper bound. However, we will show in Lemma 2 that this increase is always less than δ. We will prove that δ itself is bounded in Lemma 3.

Continuing the derivation of the upper bound from the LP (27), if $C_{123} \geq C_{13} + C_{23}$, then C_{23} is replaced by $\hat{C}_{23}$. In this case, the dual program (27) remains unchanged except for C_{23}. Hence, the set of solutions (28) can be used by replacing C_{23} with $\hat{C}_{23}$ and thus the upper bound becomes:

$$\hat{\widetilde{R}}^* = \frac{(C_{23} + \delta)(C_{02} + C_{13})}{C_{02} + C_{23} + \delta} + \frac{C_{13}\big(C_{01}C_{02} - C_{13}(C_{23} + \delta)\big)}{(C_{02} + C_{23} + \delta)(C_{012} - C_{02} + C_{13})}. \tag{31}$$

Note that the inequality $\hat{\tau}_3^* \geq 0$ holds because $\hat{\Gamma} \geq 0$ simply follows from $\Gamma \geq 0$[1]. According to Lemma 1, since $C_{123} = C_{13} + \hat{C}_{23}$, the condition $\hat{\widetilde{R}}^* \geq \hat{\tau}_3^* C_{13} + \hat{\tau}_4^* C_{123}$ is satisfied. Lemma 2 shows that the enlarged upper bound $\hat{\widetilde{R}}^*$ (Eq. (31)) is at most δ bits greater than the upper bound of (28).

Lemma 2 *If* $C_{123} \geq C_{13} + C_{23}$*, then* $\hat{\widetilde{R}}^* - \widetilde{R}^* \leq \delta$.

Proof: See Appendix D-B. ∎

Therefore, the proposed upper bound for $\Delta \leq 0$ and $\Gamma > 0$ is:

$$R_{\text{up}}^2 = \frac{C_{23}(C_{02} + C_{13})}{C_{02} + C_{23}} + \frac{C_{13}\Delta}{(C_{012} - C_{02} + C_{13})(C_{02} + C_{23})} + \delta.$$

---

[1]The superscript ˆ is used to indicate parameters associated with $\hat{C}_{23}$. For instance, $\hat{\Gamma}$ has the same formula as Γ in Eq. (29), with C_{23} replaced by $\hat{C}_{23}$.

Similarly, when $\Delta \leq 0$ and $\Gamma \leq 0$, τ_3 is set to zero and again the first three inequalities are assumed to be satisfied with equality in the dual-program (10). Following the same procedure, the subsequent results are achieved:

$$\tau_1^* = \frac{C_{23}}{C_{012} - C_{01} + C_{23}},$$

$$\tau_2^* = \frac{C_{13}(C_{012} - C_{01}) - C_{23}(C_{012} - C_{02})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})},$$

$$\tau_4^* = \frac{C_{23}(C_{012} - C_{02}) + C_{01}(C_{012} - C_{01})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})},$$

$$\widetilde{R}^* = \frac{(C_{01} + C_{23})C_{13}}{C_{01} + C_{13}} + \frac{C_{23}(C_{01}C_{02} - C_{13}C_{23})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})},$$

$$R_{\text{up}}^1 = \frac{C_{13}(C_{01} + C_{23})}{C_{01} + C_{13}} + \frac{C_{23}\Delta}{(C_{012} - C_{01} + C_{23})(C_{01} + C_{13})} + \delta. \tag{32}$$

In this case, when $C_{123} \geq C_{13} + C_{23}$, $\hat{C}_{13}$ is replaced by $C_{13} + \delta$, it is easy to see that the preceding results can be obtained by exchanging the roles of $C_{01} \leftrightarrow C_{02}$, $C_{13} \leftrightarrow C_{23}$, and $\tau_2 \leftrightarrow \tau_3$ in the results derived for the case of $\Delta \leq 0$ and $\Gamma > 0$.

In order to be able to achieve a small gap from the upper bounds, δ should be bounded. Lemma 3 proves that δ is smaller than 0.21 bits.

Lemma 3 $\delta \leq \frac{1}{2}\log(\frac{4}{3})$.

    Proof: See Appendix D-C. ∎

C. Gap Analysis

The MDF-BC scheme is proposed for the following regions:

1) $\Delta < 0$, $\Gamma \leq 0$, $C_{02} \geq C_{01}$, and $C_{01} \geq 1$
2) $\Delta < 0$, $\Gamma \geq 0$, $C_{01} \geq C_{02}$, and $C_{02} \geq 1$

For $\Delta < 0$, Appendix C shows that the MDF scheme provides a small gap from the upper bounds for the remaining regions. Here, the first case is considered. The gap $\kappa_{\text{MDF-BC}}^1$ between the achievable rate $R_{\text{MDF-BC}}^1$ and the upper bound R_{up}^1 is:

$$\kappa_{\text{MDF-BC}}^1 = \frac{-\zeta_1\left((C_{012} - C_{01} + C_{23})(C_{13} - C_{23}) + C_{23}(C_{02} + C_{23})\right)\Delta}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})\left((C_{01} + C_{13})(C_{012} - C_{01} + C_{23}) - \zeta_1(C_{02} + C_{23})\right)} + \delta.$$

In the following lemma, the gap $\kappa_{\text{MDF-BC}}^1$ is proved to be smaller than $\frac{1}{2} + \delta$ bits.

Lemma 4 $\kappa_{MDF\text{-}BC}^1 \leq \frac{1}{2} + \delta$.

    Proof: See Appendix D-D. ∎

By exchanging the roles of $g_{01} \leftrightarrow g_{02}$ and $g_{13} \leftrightarrow g_{23}$, the gap for the second case can be easily derived and shown to be less than $\frac{1}{2} + \delta$ bits.

V. MDF-MAC Scheme and Achieving within 0.71 Bits of the Capacity for $\Delta > 0$

Similar to section IV, a third mode is added to the MDF scheme when $\Delta > 0$ to effectively utilize the unused capacity of the first hop.

A. Achievable Scheme

Here, Multiple-Access Mode is added to the MDF scheme with independent messages sent from the relays to Destination. This mode provides the relays with an increased transmission time. The modified protocol uses three transmission modes, *i.e.*, Multiple-Access Mode and Forward Modes I and II. Therefore, by setting $t_1 = 0$ in Eq. (8) the maximum achievable rate of the scheme, R_{MAC} is:

$$R_{\text{MAC}} = \max_{\sum_{i=2}^{4} t_i = 1, t_i \geq 0} \left\{ \min\{t_2 C_{01}, t_3 C_{13} + R_1\} + \min\{t_3 C_{02}, t_2 C_{23} + R_2\} \right\}, \tag{33}$$

where R_1 and R_2 are the rates that Relays 1 and 2 provide to Destination in Multiple-Access Mode, respectively. These rates satisfy the multiple-access constraints in (6). Lemma 5 presents achievable rates, which will be shown to be smaller than the capacity, by at most .71 bits, in section V-C.

Lemma 5 *The achievable rates for $\Delta > 0$ together with their corresponding scheduling are as follows:*

$$R_{MDF\text{-}MAC}^1 = \frac{C_{01}(C_{02} + C_{13})}{C_{01} + C_{13}} - \frac{C_{02}\Delta}{(C_{01} + C_{13})(C_{MAC} - C_{13} + C_{02})} \quad \text{for} \quad \Delta > 0, \ \Gamma' \leq 0,$$

$$R_{MDF\text{-}MAC}^2 = \frac{C_{02}(C_{01} + C_{23})}{C_{02} + C_{23}} - \frac{C_{01}\Delta}{(C_{02} + C_{23})(C_{MAC} - C_{23} + C_{01})} \quad \text{for} \quad \Delta > 0, \ \Gamma' > 0, \tag{34}$$

$\Gamma' \leq 0$		$\Gamma' > 0$	
t_2	$= \frac{C_{13}}{C_{01} + C_{13}},$	t_2	$= \frac{C_{02}(C_{MAC} - C_{23}) + C_{13}C_{23}}{(C_{02} + C_{23})(C_{MAC} - C_{23} + C_{01})},$
t_3	$= \frac{C_{01}(C_{MAC} - C_{13}) + C_{13}C_{23}}{(C_{01} + C_{13})(C_{MAC} - C_{13} + C_{02})},$	t_3	$= \frac{C_{23}}{C_{02} + C_{23}},$
t_4	$= \frac{\Delta}{(C_{01} + C_{13})(C_{MAC} - C_{13} + C_{02})},$	t_4	$= \frac{\Delta}{(C_{02} + C_{23})(C_{MAC} - C_{23} + C_{01})},$

where

$$\Gamma' \triangleq C_{02}[C_{123} - C_{23}] - C_{01}[C_{123} - C_{13}]. \tag{35}$$

Proof: See Appendix D-E. ■

It is noted that if $\Delta = 0$, t_4 becomes zero and the scheme is converted to the MDF scheme.

B. Upper Bound

Following the same procedure as section IV-B, the upper bound for the case of $\Delta \geq 0$, $\Gamma' \geq 0$ is attained from (28) by exchanging the roles of $C_{01} \leftrightarrow C_{13}$, $C_{02} \leftrightarrow C_{23}$, $\tau_2 \leftrightarrow \tau_3$, and $\tau_1 \leftrightarrow \tau_4$. Similarly, when $\Delta \geq 0$ and $\Gamma' \leq 0$, swapping the positions of $C_{01} \leftrightarrow C_{23}$, $C_{02} \leftrightarrow C_{13}$, and $\tau_1 \leftrightarrow \tau_4$ in (28) provides the upper bound. Therefore:

$$R_{\text{up}}^3 = \frac{C_{01}(C_{02} + C_{13})}{C_{01} + C_{13}} + \frac{-C_{02}\Delta}{(C_{123} - C_{13} + C_{02})(C_{01} + C_{13})} + \delta \text{ for } \Gamma' \leq 0,$$

$$R_{\text{up}}^4 = \frac{C_{02}(C_{01} + C_{23})}{C_{02} + C_{23}} + \frac{-C_{01}\Delta}{(C_{123} - C_{23} + C_{01})(C_{02} + C_{23})} + \delta \text{ for } \Gamma' > 0. \tag{36}$$

C. Gap Analysis

By comparing the achievable rates (34) and the upper bounds (36), the gaps κ_{MAC}^1 and κ_{MAC}^2 are respectively calculated for $\Gamma' \leq 0$ and $\Gamma' > 0$ cases as:

$$\kappa_{\text{MAC}}^1 \triangleq R_{\text{up}}^3 - R_{MDF\text{-}MAC}^1 = \frac{C_{02}(C_{123} - C_{\text{MAC}})\Delta}{(C_{01} + C_{13})(C_{\text{MAC}} - C_{13} + C_{02})(C_{123} - C_{13} + C_{02})} + \delta,$$

$$\kappa_{\text{MAC}}^2 \triangleq R_{\text{up}}^4 - R_{MDF\text{-}MAC}^2 = \frac{C_{01}(C_{123} - C_{\text{MAC}})\Delta}{(C_{02} + C_{23})(C_{\text{MAC}} - C_{23} + C_{01})(C_{123} - C_{23} + C_{01})} + \delta.$$

To show that the above gaps are small, Lemma 6 is employed.

Lemma 6 $C_{123} - C_{MAC} \leq \frac{1}{2}$.

Proof: See Appendix D-F. ∎

Considering Lemma 6, it is straightforward to show that the gap is at most $\frac{1}{2}+\delta$ bits. Therefore, adding Multiple-Access Mode, with independent messages sent from the relays to Destination, to the MDF scheme ensures the gap of less than .71 bits from the upper bounds for $\Delta > 0$.

VI. CONCLUSION

In this work, we considered a dual-hop network with two parallel relays in which each transmitting node has a constant power constraint. We categorized the network into three classes based on the fundamental parameter of the network Δ, defined in this paper. We derived explicit upper bounds for the different classes using the cut-set bound. Based on the upper bounds, we proved that the MDF scheme, which employs two transmission modes (Forward Modes I and II), achieves the capacity of the channel when $\Delta=0$. Furthermore, we analyzed the gap between the achievable rate of the MDF scheme and the upper bounds, showing that the gap can be large in some ranges of parameters when $\Delta\neq0$. To guarantee the gap of at most 0.71 bits from the bounds, we added an extra broadcast or multiple-access mode to the baseline MDF scheme for the cases of $\Delta<0$ and $\Delta>0$, respectively. In addition, we provided the asymptotic capacity analysis in the high SNR regime. Finally, we argued that when the transmitting nodes operate under average power constraints, the gap between the achievable scheme and the cut-set upper bound is at most 3.6 bits.

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APPENDIX A
GENERALIZED DEGREES OF FREEDOM CHARACTERIZATION

It is interesting to consider the asymptotic capacity of the diamond channel in the high SNR regime. A useful parameter in studying this capacity is the GDOF (cf. [16], [26]) defined as:

$$\text{GDOF}(\boldsymbol{\alpha}) \triangleq \lim_{P\to\infty} \frac{R}{\log P},$$

where R is the data-rate, P is a channel parameter (can be considered as SNR), and $\boldsymbol{\alpha} = \{\alpha_{01}, \alpha_{02}, \alpha_{13}, \alpha_{23}\}$ with

$$\alpha_{ij} \triangleq \lim_{P\to\infty} \frac{\log(g_{ij})}{\log P} \quad \text{for } i \in \{0,1,2\}, \text{and } j \in \{1,2,3\}.$$

The vector $\boldsymbol{\alpha}$ shows how channel gains scale with P. Based on the above definition, the following approximations are valid:

$$C_{ij}= \frac{1}{2}\log(1+g_{ij}) \approx \frac{1}{2}\alpha_{ij}\log P,$$

$$C_{012}= \frac{1}{2}\log(1+g_{01}+g_{02}) \approx \frac{1}{2}\max\{\alpha_{01},\alpha_{02}\}\log P,$$

$$C_{123}= \frac{1}{2}\log\left(1+(\sqrt{g_{13}}+\sqrt{g_{23}})^2\right) \approx \frac{1}{2}\max\{\alpha_{13},\alpha_{23}\}\log P,$$

$$C_{\text{MAC}}= \frac{1}{2}\log(1+g_{13}+g_{23}) \approx \frac{1}{2}\max\{\alpha_{13},\alpha_{23}\}\log P,$$

$$\Gamma\approx \left\{\alpha_{23}(\max\{\alpha_{01},\alpha_{02}\}-\alpha_{02})-\alpha_{13}(\max\{\alpha_{01},\alpha_{02}\}-\alpha_{01})\right\}(\log P)^2 + \sigma\log(P),$$

$$\Gamma'\approx \left\{\alpha_{02}(\max\{\alpha_{13},\alpha_{23}\}-\alpha_{23})-\alpha_{01}(\max\{\alpha_{13},\alpha_{23}\}-\alpha_{13})\right\}(\log P)^2 + \sigma'\log(P),$$

where σ and σ' are positive constants. In the following analysis, it is assumed that $(\log P)^2$ terms are dominant, *i.e.*, the coefficients of $(\log P)^2$ for Γ and Γ' are not zero. If this assumption is not valid, MDF scheme achieves the optimum GDOF of the channel. According to the above approximations, it is easy to infer:

$$\begin{cases} \Gamma \leq 0, & \text{if } \alpha_{01} \leq \alpha_{02}; \\ \Gamma > 0, & \text{if } \alpha_{01} > \alpha_{02}; \\ \Gamma' \leq 0, & \text{if } \alpha_{13} \leq \alpha_{23}; \\ \Gamma' > 0, & \text{if } \alpha_{13} > \alpha_{23}. \end{cases}$$

Therefore, the GDOF associated with the upper bounds is:

$$\begin{aligned}
\text{GDOF}_{\text{up}}^1 &= \frac{\alpha_{13}(\alpha_{01} + \alpha_{23})}{\alpha_{01} + \alpha_{13}} + \frac{\alpha_{23}(\alpha_{01}\alpha_{02} - \alpha_{13}\alpha_{23})}{(\alpha_{01} + \alpha_{13})(\alpha_{02} - \alpha_{01} + \alpha_{23})}, \\
\text{GDOF}_{\text{up}}^2 &= \frac{\alpha_{23}(\alpha_{02} + \alpha_{13})}{\alpha_{02} + \alpha_{23}} + \frac{\alpha_{13}(\alpha_{01}\alpha_{02} - \alpha_{13}\alpha_{23})}{(\alpha_{02} + \alpha_{23})(\alpha_{01} - \alpha_{02} + \alpha_{13})}, \\
\text{GDOF}_{\text{up}}^3 &= \frac{\alpha_{01}(\alpha_{02} + \alpha_{13})}{\alpha_{01} + \alpha_{13}} + \frac{-\alpha_{02}(\alpha_{01}\alpha_{02} - \alpha_{13}\alpha_{23})}{(\alpha_{01} + \alpha_{13})(\alpha_{23} - \alpha_{13} + \alpha_{02})}, \\
\text{GDOF}_{\text{up}}^4 &= \frac{\alpha_{02}(\alpha_{01} + \alpha_{23})}{\alpha_{02} + \alpha_{23}} + \frac{-\alpha_{01}(\alpha_{01}\alpha_{02} - \alpha_{13}\alpha_{23})}{(\alpha_{02} + \alpha_{23})(\alpha_{13} - \alpha_{23} + \alpha_{01})}.
\end{aligned} \tag{37}$$

The GDOF for different achievablity schemes is as follows:
MDF:

$$\begin{aligned}
\text{GDOF}_{\text{MDF}}^1 &= \frac{\alpha_{01}(\alpha_{02} + \alpha_{13})}{\alpha_{01} + \alpha_{13}}, \\
\text{GDOF}_{\text{MDF}}^2 &= \frac{\alpha_{02}(\alpha_{01} + \alpha_{23})}{\alpha_{02} + \alpha_{23}}, \\
\text{GDOF}_{\text{MDF}}^3 &= \frac{\alpha_{13}(\alpha_{01} + \alpha_{23})}{\alpha_{01} + \alpha_{13}}, \\
\text{GDOF}_{\text{MDF}}^4 &= \frac{\alpha_{23}(\alpha_{02} + \alpha_{13})}{\alpha_{02} + \alpha_{23}}.
\end{aligned} \tag{38}$$

MDF-BC:

$$\begin{aligned}
\text{GDOF}_{\text{MDF-BC}}^1 &= \frac{\alpha_{02}\alpha_{13}(\alpha_{01} + \alpha_{23}) - \alpha_{01}^2\alpha_{13} + \alpha_{01}\alpha_{02}\alpha_{23}}{(\alpha_{01} + \alpha_{13})(\alpha_{02} - \alpha_{01} + \alpha_{23})}, \\
\text{GDOF}_{\text{MDF-BC}}^2 &= \frac{\alpha_{01}\alpha_{23}(\alpha_{02} + \alpha_{13}) - \alpha_{02}^2\alpha_{23} + \alpha_{01}\alpha_{02}\alpha_{13}}{(\alpha_{02} + \alpha_{23})(\alpha_{01} - \alpha_{02} + \alpha_{13})}.
\end{aligned} \tag{39}$$

MDF-MAC:

$$\begin{aligned}
\text{GDOF}_{\text{MDF-MAC}}^1 &= \frac{\alpha_{01}(\alpha_{02} + \alpha_{13})}{\alpha_{01} + \alpha_{13}} - \frac{\alpha_{02}(\alpha_{01}\alpha_{02} - \alpha_{13}\alpha_{23})}{(\alpha_{01} + \alpha_{13})(\alpha_{23} - \alpha_{13} + \alpha_{02})}, \\
\text{GDOF}_{\text{MDF-MAC}}^2 &= \frac{\alpha_{02}(\alpha_{01} + \alpha_{23})}{\alpha_{02} + \alpha_{23}} - \frac{\alpha_{01}(\alpha_{01}\alpha_{02} - \alpha_{13}\alpha_{23})}{(\alpha_{02} + \alpha_{23})(\alpha_{13} - \alpha_{23} + \alpha_{01})}.
\end{aligned} \tag{40}$$

By comparing the upper bounds on the GDOF and the achievable GDOFs, it is easy to see that MDF-BC and MDF-MAC achieve the optimum GDOF of the channel, while the MDF cannot achieve it for all channel parameters.

APPENDIX B
DIAMOND CHANNEL WITH AVERAGE POWER CONSTRAINTS

In this appendix, it is shown that if the transmitting nodes are subject to average power constraints, each of the cut-set bounds in Eq. (9) is increased at most by $\frac{2}{\ln 2}$ bits. This analysis confirms that the achievable schemes proposed in this paper with constant power constraints are still valid. In other words, they provide a gap of at most $.71 + \frac{2}{\ln 2} \leq 3.6$ bits from the cut-set bounds.

Let $P_{\mathcal{S}}^{(i)*}$, $P_{\mathcal{R}_1}^{(i)*}$, and $P_{\mathcal{R}_2}^{(i)*}$, for $i \in \{1, \cdots, 4\}$ be the optimum power allocated to Source, Relay 1, and Relay 2 in transmission mode i with the corresponding time interval t_i^* leading to the cut-set bound R_0. The following

constraints are in effect[2]:

$$\sum_{i=1}^{4} t_i^* P_{\mathcal{S}}^{(i)*} \le P_{\mathcal{S}},$$

$$\sum_{i=1}^{4} t_i^* P_{\mathcal{R}_1}^{(i)*} \le P_{\mathcal{R}_1}, \tag{41}$$

$$\sum_{i=1}^{4} t_i^* P_{\mathcal{R}_2}^{(i)*} \le P_{\mathcal{R}_2}.$$

Therefore, the cut-set upper bound R_0 satisfies the following constraints:

$$
\begin{aligned}
R_0 &\le t_1^* \mathcal{C}\big((g_{01} + g_{02}) P_{\mathcal{S}}^{(1)*}\big) + t_2^* \mathcal{C}(g_{01} P_{\mathcal{S}}^{(2)*}) + t_3^* \mathcal{C}(g_{02} P_{\mathcal{S}}^{(3)*}), \\
R_0 &\le t_1^* \mathcal{C}(g_{01} P_{\mathcal{S}}^{(1)*}) + t_2^* \Big(\mathcal{C}(g_{01} P_{\mathcal{S}}^{(2)*}) + \mathcal{C}(g_{23} P_{\mathcal{R}_2}^{(2)*})\Big) + t_4^* \mathcal{C}(g_{23} P_{\mathcal{R}_2}^{(4)*}), \\
R_0 &\le t_1^* \mathcal{C}(g_{02} P_{\mathcal{S}}^{(1)*}) + t_3^* \Big(\mathcal{C}(g_{02} P_{\mathcal{S}}^{(3)*}) + \mathcal{C}(g_{13} P_{\mathcal{R}_1}^{(3)*})\Big) + t_4^* \mathcal{C}(g_{13} P_{\mathcal{R}_1}^{(4)*}), \\
R_0 &\le t_2^* \mathcal{C}(g_{23} P_{\mathcal{R}_2}^{(2)*}) + t_3^* \mathcal{C}(g_{13} P_{\mathcal{R}_1}^{(3)*}) + t_4^* \mathcal{C}\Big(\big(\sqrt{g_{13} P_{\mathcal{R}_1}^{(4)*}} + \sqrt{g_{23} P_{\mathcal{R}_2}^{(4)*}}\big)^2\Big).
\end{aligned}
\tag{42}
$$

Suppose that the vector $\mathbf{t}'$ is the solution to the LP (9) leading to the rate R_1. If the vector $\mathbf{t}^*$ is used instead of $\mathbf{t}'$ in the LP (9), the resulting rate that satisfies the conditions of the LP, called R_2, becomes smaller than R_1. It is clear that the increase in the cut-set bound due to the *average* instead of the *constant* power constraints (compare Eq. (1) to Eq. (41)), *i.e.*, $R_0 - R_1$ is smaller than $R_0 - R_2$. Here, it is proved that $R_0 - R_2 \le \frac{2}{\ln 2}$.

Consider each component term in the form of $t_i^* \mathcal{C}(.)$ present in the inequality set (42). For instance, consider $R_{c,0} \triangleq t_1^* \mathcal{C}(g_{02} P_{\mathcal{S}}^{(1)*})$. The corresponding term in constructing R_2 is $R_{c,2} \triangleq t_1^* \mathcal{C}(g_{02} P_{\mathcal{S}})$. Because of the power constraints (41), $R_{c,0} \le t_1^* \mathcal{C}(g_{02} \frac{P_{\mathcal{S}}}{t_1^*})$. Therefore, it is easy to show:

$$
\begin{aligned}
R_{c,0} - R_{c,2} &\le t_1^* \mathcal{C}\big(\frac{g_{02} P_{\mathcal{S}}(1 - t_1^*)}{(1 + g_{02} P_{\mathcal{S}}) t_1^*}\big) \\
&\overset{(a)}{\le} \frac{g_{02} P_{\mathcal{S}}(1 - t_1^*)}{2(1 + g_{02} P_{\mathcal{S}}) \ln 2} \\
&\le \frac{1}{2 \ln 2},
\end{aligned}
$$

where (a) is due to the fact that $\mathcal{C}(x) \le \frac{x}{2 \ln 2}$ for any $x \ge 0$. Similar analysis applies to each component term. It is observed that the first and fourth cut-set bounds in inequality set (42) have three component terms and the second and third cut-set bounds have four component terms. Therefore, $R_0 - R_2 \le \frac{2}{\ln 2}$.

APPENDIX C
MDF GAP ANALYSIS

We investigate how close the MDF scheme performs to the upper bounds when $\Delta \neq 0$. First, the gap between the MDF scheme and the upper bound is calculated for regions specified in Table I. Then, two special cases are considered.

General Case. We calculate the difference, named κ, between the upper bounds and the rate offered by the MDF scheme from Eq. (11) for the cases shown in Table I (see Appendix E):

---

[2]For the purpose of clarity, here the average powers are not set to unity.

$$\kappa_1 = \frac{-(C_{012} - C_{01})\Delta}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})} + \delta,$$

$$\kappa_2 = \frac{-(C_{012} - C_{02})\Delta}{(C_{02} + C_{23})(C_{012} - C_{02} + C_{13})} + \delta,$$

$$\kappa_3 = \frac{(C_{123} - C_{13})\Delta}{(C_{01} + C_{13})(C_{123} - C_{13} + C_{02})} + \delta,$$

$$\kappa_4 = \frac{(C_{123} - C_{23})\Delta}{(C_{02} + C_{23})(C_{123} - C_{23} + C_{01})} + \delta,$$

$$\kappa_5 = \frac{-\Delta}{C_{01} + C_{13}} \left(\frac{C_{01} + C_{23}}{C_{02} + C_{23}} - \frac{C_{23}}{C_{012} - C_{01} + C_{23}} \right) + \delta,$$

$$\kappa_6 = \frac{-\Delta}{C_{02} + C_{23}} \left(\frac{C_{02} + C_{13}}{C_{01} + C_{13}} - \frac{C_{13}}{C_{012} - C_{02} + C_{13}} \right) + \delta,$$

$$\kappa_7 = \frac{\Delta}{C_{01} + C_{13}} \left(\frac{C_{02} + C_{13}}{C_{02} + C_{23}} - \frac{C_{02}}{C_{123} - C_{13} + C_{02}} \right),$$

$$\kappa_8 = \frac{\Delta}{C_{02} + C_{23}} \left(\frac{C_{01} + C_{23}}{C_{01} + C_{13}} - \frac{C_{01}}{C_{123} - C_{23} + C_{01}} \right).$$

Note that for the regions associated with κ_7 and κ_8 specified in Table I, $C_{123} \leq C_{13} + C_{23}$ and hence, $\delta = 0$.

To prove that κ_i for $i \in \{1, \cdots, 4\}$ are small, the following lemma is needed:

Lemma 7

$$C_{012} - \max\{C_{01}, C_{02}\} \leq \frac{1}{2},$$

$$C_{123} - \max\{C_{13}, C_{23}\} \leq 1.$$

Proof: See Appendix D-G. ■

For instance, following $\kappa_1 \leq \frac{1}{2} + \delta$ is proved:

$$\kappa_1 = \frac{(C_{13}C_{23} - C_{01}C_{02})(C_{012} - C_{01})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})} + \delta$$

$$\overset{(a)}{\leq} \frac{C_{13}C_{23}(C_{012} - C_{01})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})} + \delta$$

$$\overset{(b)}{\leq} \frac{1}{2} \frac{C_{13}C_{23}}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})} + \delta$$

$$= \frac{1}{2} \frac{C_{13}}{C_{01} + C_{13}} \times \frac{C_{23}}{C_{012} - C_{01} + C_{23}} + \delta$$

$$\leq \frac{1}{2} + \delta,$$

where (a) comes from the fact that $\Delta > 0$ for this case. According to the corresponding region shown in Table I, $C_{02} \leq C_{01}$ and therefore (b) is true based on Lemma 7.

Lemmas 8 and 9 prove that $\kappa_5 \leq \frac{1}{2} + \delta$ and $\kappa_7 \leq 1$, respectively. The proof techniques can be easily adopted to correspondingly show that $\kappa_6 \leq \frac{1}{2} + \delta$, and $\kappa_8 \leq 1$.

Lemma 8 $\kappa_5 \leq \frac{1}{2} + \delta$.

Proof: See Appendix D-H. ■

Lemma 9 $\kappa_7 \leq 1$.

Proof: See Appendix D-I. ■

Two special cases are also considered:

Symmetric Case. When $C_{01} = C_{02}$ and $C_{13} = C_{23}$, $\Gamma = \Gamma' = 0$ and it can be seen from Table I that the MDF scheme offers a data-rate that is, at most, $1 + \delta$ bits less than the corresponding upper bound.

Partially Symmetric Case. When either $C_{01} = C_{02}$ with $\Delta < 0$, or $C_{13} = C_{23}$ with $\Delta > 0$ occurs, it was seen in section III-A that fully utilizing branch 1 or branch 2 gives the same achievable rate. Table I shows that in such cases, the gap is less than $1 + \delta$ bits.

Discussion. Multiplexing Gain (MG) of a scheme is defined in [24], [25] as:

$$\text{MG} \triangleq \lim_{\text{SNR} \to \infty} \frac{R}{0.5 \log(\text{SNR})},$$

where R is the achievable rate of the scheme. Using Eq. (11), it can be shown that the MDF scheme achieves the multiplexing gain of 1. Avestimehr, *et.al* proposed a broadcast mutiple-access scheme for the full-duplex diamond channel and proved that the scheme is within one bit from the cut-set bound [21]. In the half-duplex case, the multiplexing gain of 1 is lost if this approach is followed, leading to an infinite gap between the achievable rate and the upper bound.

It is easy to show that, for the remaining cases (shown in Table I), the gap can be large. For instance, suppose $C_{02} = x$, $C_{13} = C_{23} = \alpha x$ and $C_{01} = \beta x$, with $\alpha > \beta > 1$. In this case $\Delta < 0$, and $\Gamma > 0$ and therefore, the gap κ is:

$$\kappa = \frac{-\Delta}{C_{02} + C_{23}} \left(\frac{C_{02} + C_{13}}{C_{01} + C_{13}} - \frac{C_{13}}{C_{012} - C_{02} + C_{13}} \right) + \delta$$

$$= \frac{-\Delta}{C_{02} + C_{23}} \left(\frac{C_{02}(C_{012} - C_{02}) + C_{13}(C_{012} - C_{01})}{(C_{01} + C_{13})(C_{012} - C_{02} + C_{13})} \right) + \delta$$

$$\overset{(a)}{\geq} \frac{-\Delta}{C_{02} + C_{23}} \left(\frac{C_{02}(C_{012} - C_{02})}{(C_{01} + C_{13})(C_{012} - C_{02} + C_{13})} \right) + \delta$$

$$\overset{(b)}{\geq} \frac{-\Delta}{C_{02} + C_{23}} \left(\frac{C_{02}(C_{01} - C_{02})}{(C_{01} + C_{13})(C_{012} - C_{02} + C_{13})} \right) + \delta$$

$$\overset{(c)}{\geq} \frac{-\Delta}{C_{02} + C_{23}} \left(\frac{C_{02}(C_{01} - C_{02})}{(C_{01} + C_{13})^2} \right) + \delta$$

$$\overset{(d)}{=} \frac{(\alpha^2 - \beta)(\beta - 1)}{(\alpha + \beta)^2(\alpha + 1)} x + \delta,$$

where in (a) the nominator is decreased by $C_{13}(C_{012} - C_{01})$. To obtain (b), C_{012} in the nominator is replaced by the smaller quantity C_{01}. For (c), C_{012} is substituted by the larger term $C_{01} + C_{02}$ in the denominator. In (d), the assumed values of the capacities in terms of x are substituted. It is clear that the gap increases as x becomes large. GDOF analysis of Appendix A also confirms that the MDF scheme can have a large gap from the upper bound.

APPENDIX D
PROOFS

In this appendix, the proofs of the lemmas used in this paper are provided.

A. Proof of Lemma 1

We start with the fact that $C_{01} + C_{02} \geq C_{012}$. Rearranging the terms, and multiplying both sides of the inequality by C_{13} give:

$$C_{13}C_{02} \geq C_{13}(C_{012} - C_{01}).$$

By adding $C_{02}(C_{012} - C_{02})$ to both sides and then dividing both sides by $C_{012} - C_{02} + C_{13}$, we obtain:

$$C_{02} \geq \frac{C_{13}(C_{012} - C_{01}) + C_{02}(C_{012} - C_{02})}{C_{012} - C_{02} + C_{13}}.$$

Assuming $C_{123} \leq C_{13} + C_{23}$, we divide the Right Hand Side (RHS) by $C_{02} + C_{23}$ and the Left Hand Side (LHS) by the smaller quantity $C_{123} - C_{13} + C_{02}$ to achieve:

$$\frac{C_{02}}{C_{123} - C_{13} + C_{02}} \geq \frac{C_{13}(C_{012} - C_{01}) + C_{02}(C_{012} - C_{02})}{(C_{012} - C_{02} + C_{13})(C_{02} + C_{23})} = \tau_4^*.$$

This completes the proof.

B. Proof of Lemma 2

$$\widehat{\widetilde{R}}^{*} - \widetilde{R}^{*} = \frac{\delta C_{02}\Big((C_{02}+C_{13})(C_{012}-C_{02}+C_{13})-C_{13}(C_{01}+C_{13})\Big)}{(C_{02}+C_{23})(C_{02}+C_{23}+\delta)(C_{012}-C_{02}+C_{13})}$$

$$\overset{(a)}{\leq} \frac{\delta C_{02}^2}{(C_{02}+C_{23})^2}$$

$$\leq \delta,$$

where in (a), the nominator is increased by replacing $C_{012}-C_{02}$ with C_{01}, using the fact that $C_{012}-C_{02} \leq C_{01}$ (see Eq. (2)). In addition, the denominator is decreased by removing δ.

C. Proof of Lemma 3

$$\delta = C_{123} - (C_{13}+C_{23})$$

$$= \frac{1}{2}\log\left(\frac{1+g_{13}+g_{23}+2\sqrt{g_{13}g_{23}}}{1+g_{13}+g_{23}+g_{13}g_{23}}\right)$$

$$\overset{(a)}{\leq} \frac{1}{2}\log\left(1+\frac{2\sqrt{g_{13}g_{23}}-g_{13}g_{23}}{1+2\sqrt{g_{13}g_{23}}+g_{13}g_{23}}\right)$$

$$\overset{(b)}{\leq} \frac{1}{2}\log(\frac{4}{3}),$$

where in (a) the denominator is decreased by replacing $g_{13}+g_{23}$ with the smaller term $2\sqrt{g_{13}g_{23}}$. Defining $x \triangleq \sqrt{g_{13}g_{23}}$, it is easy to show that the maximum of $\log(1+\frac{2x-x^2}{1+2x+x^2})$, for $0 \leq x \leq 2$, is $x^* = \frac{1}{2}$, i.e., $g_{13}^* g_{23}^* = \frac{1}{4}$, which proves (b).

D. Proof of Lemma 4

It is known that $C_{01}, C_{02} \leq C_{012}$, which proves $0 \leq C_{23}(C_{012}-C_{02})$ and $0 \leq C_{01}(C_{012}-C_{01})$. Since both terms are positive, the sum of them is also positive, i.e., $0 \leq C_{23}(C_{012}-C_{02})+C_{01}(C_{012}-C_{01})$. By adding and subtracting $(C_{012}-C_{01}+C_{23})C_{13}+C_{01}C_{13}$, the inequality can be rearranged to:

$$0 \leq (C_{012}-C_{01}+C_{23})(C_{01}+C_{13})+(C_{012}-C_{01})(C_{23}-C_{13})-C_{23}(C_{02}+C_{13}).$$

As mentioned earlier, Broadcast Mode is used for $\Delta \leq 0$, i.e., $C_{01}C_{02} \leq C_{13}C_{23}$. Therefore, both sides are multiplied by the positive term $-\Delta$ to acquire:

$$0 \leq (C_{13}C_{23}-C_{01}C_{02})\big((C_{012}-C_{01}+C_{23})(C_{01}+C_{13})+(C_{012}-C_{01})(C_{23}-C_{13})-C_{23}(C_{02}+C_{13})\big).$$

Now, the positive term $(C_{012}-C_{01}+C_{23})(C_{012}-C_{01})(C_{01}+C_{13})^2$ can be added to the RHS of the inequality to achieve:

$$0 \leq (C_{13}C_{23}-C_{01}C_{02})\big((C_{012}-C_{01}+C_{23})(C_{01}+C_{13})+(C_{012}-C_{01})(C_{23}-C_{13})-C_{23}(C_{02}+C_{13})\big)$$

$$+(C_{012}-C_{01}+C_{23})(C_{012}-C_{01})(C_{01}+C_{13})^2.$$

The above inequality can be equivalently stated as:

$$(C_{13}C_{23}-C_{01}C_{02})\Big((C_{012}-C_{01}+C_{23})(C_{13}-C_{23})+C_{23}(C_{02}+C_{23})\Big)+$$

$$C_{01}(C_{02}+C_{23})(C_{01}+C_{13})(C_{012}-C_{01}+C_{23}) \leq (C_{012}-C_{01}+C_{23})^2(C_{01}+C_{13})^2.$$

Since $1 \leq C_{01}$, the LHS becomes smaller if $C_{01}(C_{02}+C_{23})$ is replaced by $(C_{02}+C_{23})$, leading to:

$$(C_{13}C_{23}-C_{01}C_{02})\Big((C_{012}-C_{01}+C_{23})(C_{13}-C_{23})+C_{23}(C_{02}+C_{23})\Big)+$$

$$(C_{02}+C_{23})(C_{01}+C_{13})(C_{012}-C_{01}+C_{23}) \leq (C_{012}-C_{01}+C_{23})^2(C_{01}+C_{13})^2.$$

Now as $\zeta_1 \leq \frac{1}{2}$ (see Eq. (25)), the following inequality is also true:

$$\zeta_1 \Bigg\{ 2 \times (C_{13}C_{23} - C_{01}C_{02})\Big((C_{012} - C_{01} + C_{23})(C_{13} - C_{23}) + C_{23}(C_{02} + C_{23})\Big) +$$

$$(C_{02} + C_{23})(C_{01} + C_{13})(C_{012} - C_{01} + C_{23}) \Bigg\} \leq (C_{012} - C_{01} + C_{23})^2(C_{01} + C_{13})^2.$$

By rearranging the preceding inequality

$$\frac{\zeta_1(C_{13}C_{23} - C_{01}C_{02})\Big((C_{012} - C_{01} + C_{23})(C_{13} - C_{23}) + C_{23}(C_{02} + C_{23})\Big)}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})\Big((C_{01} + C_{13})(C_{012} - C_{01} + C_{23}) - \zeta_1(C_{02} + C_{23})\Big)} \leq \frac{1}{2},$$

which completes the proof.

E. Proof of Lemma 5

The optimization (33) is an LP and together with the multiple-access constraints (6) can be written as follows:

$$\text{maximize } R_{\text{MAC}}$$
$$\text{subject to: } R_{\text{MAC}} \leq t_2 C_{01} + t_3 C_{02}$$
$$R_{\text{MAC}} - R_1 \leq t_3(C_{02} + C_{13})$$
$$R_{\text{MAC}} - R_2 \leq t_2(C_{01} + C_{23})$$
$$R_{\text{MAC}} - (R_1 + R_2) \leq t_2 C_{23} + t_3 C_{13}$$
$$R_1 \leq t_4 C_{13}$$
$$R_2 \leq t_4 C_{23}$$
$$R_1 + R_2 \leq t_4 C_{\text{MAC}}$$
$$\sum_{i=2}^{4} t_i = 1, \ t_i \geq 0.$$

Using Fourier-Motzkin elimination [27], the LP can be equivalently stated as:

$$\text{maximize } R_{\text{MAC}}$$
$$\text{subject to: } R_{\text{MAC}} \leq t_2 C_{01} + t_3 C_{02} \tag{43}$$
$$R_{\text{MAC}} \leq t_3(C_{02} + C_{13}) + t_4 C_{13} \tag{44}$$
$$R_{\text{MAC}} \leq t_2(C_{01} + C_{23}) + t_4 C_{23} \tag{45}$$
$$R_{\text{MAC}} \leq t_2 C_{23} + t_3 C_{13} + t_4 C_{\text{MAC}} \tag{46}$$
$$R_{\text{MAC}} \leq t_2 C_{23} + t_3 C_{13} + t_4(C_{13} + C_{23}) \tag{47}$$
$$2R_{\text{MAC}} \leq t_2(C_{01} + C_{23}) + t_3(C_{02} + C_{13}) + t_4 C_{\text{MAC}} \tag{48}$$
$$2R_{\text{MAC}} \leq t_2 C_{23} + t_3(C_{02} + 2C_{13}) + t_4(C_{13} + C_{\text{MAC}}) \tag{49}$$
$$\sum_{i=2}^{4} t_i = 1, \ t_i \geq 0. \tag{50}$$

Now, it is shown that inequalities (47)-(49) are redundant. First, since $C_{\text{MAC}} \leq (C_{13} + C_{23})$, the RHS of inequality (47) is greater than the RHS of inequality (46). Therefore, inequality (47) is redundant. Second, inequalities (48) and (49) are simply obtained by adding inequalities (43, 46) and (44, 46), respectively. Therefore, the following

LP gives the maximum achievable rate of this scheme:

$$\text{maximize } R_{\text{MAC}}$$

$$\text{subject to: } R_{\text{MAC}} \le t_2 C_{01} + t_3 C_{02} \tag{51}$$

$$R_{\text{MAC}} \le t_3(C_{02} + C_{13}) + t_4 C_{13} \tag{52}$$

$$R_{\text{MAC}} \le t_2(C_{01} + C_{23}) + t_4 C_{23} \tag{53}$$

$$R_{\text{MAC}} \le t_2 C_{23} + t_3 C_{13} + t_4 C_{\text{MAC}} \tag{54}$$

$$\sum_{i=2}^{4} t_i = 1, \ t_i \ge 0. \tag{55}$$

Instead of solving the above LP, a feasible solution that satisfies all the constraints is found. This solution is not necessarily optimum, however it provides us with an achievable rate. For $\Gamma' \le 0$ inequalities (51), (52), and (54) are set to equalities, leading to:

$$t_2 = \frac{C_{13}}{C_{01} + C_{13}},$$

$$t_3 = \frac{C_{01}(C_{\text{MAC}} - C_{13}) + C_{13}C_{23}}{(C_{01} + C_{13})(C_{\text{MAC}} - C_{13} + C_{02})},$$

$$t_4 = \frac{\Delta}{(C_{01} + C_{13})(C_{\text{MAC}} - C_{13} + C_{02})},$$

$$R_{\text{MDF-MAC}}^1 = \frac{C_{01}(C_{02} + C_{13})}{C_{01} + C_{13}} - \frac{C_{02}\Delta}{(C_{01} + C_{13})(C_{\text{MAC}} - C_{13} + C_{02})}. \tag{56}$$

To ensure that the above results are valid, the inequality (53) has to be satisfied. Considering inequalities (51) and (53), it is sufficient to show that $t_3 C_{02} \le \bar{t}_3 C_{23}$. Using the values obtained in Eq. (56), this is equivalent to prove:

$$C_{02}\big(C_{01}(C_{\text{MAC}} - C_{13}) + C_{13}C_{23}\big) \le C_{23}\big(\Delta + C_{13}(C_{\text{MAC}} - C_{13} + C_{02})\big).$$

By re-ordering the terms and using the definition of Δ, the above inequality can be alternatively written as:

$$C_{\text{MAC}}\Delta \le (C_{13} + C_{23})\Delta,$$

which is true since $\Delta > 0$, and $C_{\text{MAC}} = \mathcal{C}(g_{13} + g_{23})$.

For $\Gamma' > 0$, inequalities (51), (53), and (54) are set to equality. In this case, the time intervals and the achievable rate become:

$$t_2 = \frac{C_{02}(C_{\text{MAC}} - C_{23}) + C_{13}C_{23}}{(C_{02} + C_{23})(C_{\text{MAC}} - C_{23} + C_{01})},$$

$$t_3 = \frac{C_{23}}{C_{02} + C_{23}},$$

$$t_4 = \frac{\Delta}{(C_{02} + C_{23})(C_{\text{MAC}} - C_{23} + C_{01})},$$

$$R_{\text{MDF-MAC}}^2 = \frac{C_{02}(C_{01} + C_{23})}{C_{02} + C_{23}} - \frac{C_{01}\Delta}{(C_{02} + C_{23})(C_{\text{MAC}} - C_{23} + C_{01})}. \tag{57}$$

F. Proof of Lemma 6

$$
\begin{aligned}
C_{123} - C_{\text{MAC}} &= \frac{1}{2}\log\left(\frac{1 + (\sqrt{g_{13}} + \sqrt{g_{23}})^2}{1 + g_{13} + g_{23}}\right)\\
&= \frac{1}{2}\log\left(1 + \frac{2\sqrt{g_{13}\,g_{23}}}{1 + g_{13} + g_{23}}\right)\\
&\leq \frac{1}{2}\log\left(1 + \frac{g_{13} + g_{23}}{1 + g_{13} + g_{23}}\right)\\
&\leq \frac{1}{2}.
\end{aligned}
$$

G. Proof of Lemma 7

$$
\begin{aligned}
C_{012} - \max\{C_{01}, C_{02}\} &= \frac{1}{2}\log\left(\frac{1 + g_{01} + g_{02}}{1 + \max\{g_{01}, g_{02}\}}\right)\\
&= \frac{1}{2}\log\left(1 + \frac{\min\{g_{01}, g_{02}\}}{1 + \max\{g_{01}, g_{02}\}}\right)\\
&\leq \frac{1}{2}\log\left(1 + \frac{\max\{g_{01}, g_{02}\}}{1 + \max\{g_{01}, g_{02}\}}\right)\\
&\leq \frac{1}{2},
\end{aligned}
$$

$$
\begin{aligned}
C_{123} - \max\{C_{13}, C_{23}\} &= \frac{1}{2}\log\left(\frac{1 + (\sqrt{g_{13}} + \sqrt{g_{23}})^2}{1 + \max\{g_{13}, g_{23}\}}\right)\\
&= \frac{1}{2}\log\left(1 + \frac{\min\{g_{13}, g_{23}\} + 2\sqrt{g_{13}g_{23}}}{1 + \max\{g_{13}, g_{23}\}}\right)\\
&\leq \frac{1}{2}\log\left(1 + \frac{3\sqrt{g_{13}g_{23}}}{1 + \max\{g_{13}, g_{23}\}}\right)\\
&\leq \frac{1}{2}\log\left(1 + \frac{3\sqrt{g_{13}g_{23}}}{1 + \sqrt{g_{13}g_{23}}}\right)\\
&\leq 1.
\end{aligned}
$$

H. Proof of Lemma 8

In this region, $C_{01} \leq 1$ and $C_{01} \leq C_{02}$, therefore, $0 \leq C_{13}C_{23}(C_{02} - C_{01})(1 - C_{01})$. It is easy to verify that the following inequality is valid:

$$2C_{13}C_{23}\big(C_{01}(C_{02} - C_{01}) + 0.5(C_{01} + C_{23})\big) \leq (C_{01} + C_{13})(C_{02} + C_{23})\big(C_{23} + .5 + C_{01}(C_{02} - C_{01})\big). \quad (58)$$

Replacing $C_{13}C_{23}$ by the smaller quantity $(C_{13}C_{23} - C_{01}C_{02})$ in the LHS of the above inequality results in:

$$2(C_{13}C_{23} - C_{01}C_{02})\big(C_{01}(C_{02} - C_{01}) + 0.5(C_{01} + C_{23})\big) \leq (C_{01} + C_{13})(C_{02} + C_{23})\big(C_{23} + .5 + C_{01}(C_{02} - C_{01})\big). \quad (59)$$

Since $C_{01} \leq 1$ in the RHS, $C_{01}(C_{02} - C_{01})$ can be substituted by the larger term $(C_{02} - C_{01})$. Hence, the following inequality is true:

$$-2\Delta\big(C_{01}(C_{02} - C_{01}) + 0.5(C_{01} + C_{23})\big) \leq (C_{01} + C_{13})(C_{02} + C_{23})\big(C_{23} + .5 + (C_{02} - C_{01})\big). \quad (60)$$

Rearranging the terms leads to:

$$\frac{-\Delta}{C_{01} + C_{13}}\left(\frac{C_{01} + C_{23}}{C_{02} + C_{23}} - \frac{C_{23}}{C_{02} + 0.5 - C_{01} + C_{23}}\right) \leq \frac{1}{2}. \quad (61)$$

The gap can be further increased by replacing $C_{02} + 0.5$ with the smaller term C_{012} according to Lemma 7. Therefore:

$$\frac{-\Delta}{C_{01} + C_{13}}\Big(\frac{C_{01} + C_{23}}{C_{02} + C_{23}} - \frac{C_{23}}{C_{012} - C_{01} + C_{23}}\Big) \leq \frac{1}{2}, \tag{62}$$

which completes the proof.

I. Proof of Lemma 9

$$
\begin{aligned}
\kappa_7 &= \frac{\Delta}{C_{01} + C_{13}}\Big(\frac{C_{02} + C_{13}}{C_{02} + C_{23}} - \frac{C_{02}}{C_{123} - C_{13} + C_{02}}\Big) \\
&\overset{(a)}{\leq} \frac{\Delta}{C_{01} + C_{13}} \times \frac{C_{13}}{C_{02} + C_{23}} + \delta \\
&\overset{(b)}{\leq} \frac{\Delta}{(C_{01} + C_{13})(C_{02} + C_{23})} + \delta \\
&\overset{(c)}{\leq} \frac{C_{01}}{C_{01} + C_{13}} \times \frac{C_{02}}{C_{02} + C_{23}} + \delta \\
&\leq 1 + \delta.
\end{aligned}
$$

As $C_{123} \leq C_{13} + C_{23}$ in this region, $C_{123} - C_{13}$ is replaced by the larger quantity C_{23} to obtain (a). (b) is valid since $C_{13} \leq 1$ for this scenario. In (c), Δ is substituted by the larger term $C_{01}C_{02}$.

APPENDIX E

GAP ANALYSIS SUMMARY

The results related to gap analysis are compactly shown in Table I. For each region specified by some conditions on the link capacities, the corresponding symbols for the upper bound, the achievable rate, and the gap, (*i.e.*, the difference between the upper bound and the achievable rate) are shown [3]. In addition, an upper bound on the value of the gap is given. For instance, for the region specified by $\Delta \leq 0, \Gamma \leq 0$, and $C_{02} \leq C_{01}$ conditions, the upper bound, the achievable rate, and the gap are respectively represented by R_{up}^1, R_{MDF}^1, and κ_1. Using the achievable scheme that leads to R_{MDF}^1, the gap from the upper bound R_{up}^1 is less than $\frac{1}{2} + \delta$. Our results, summarized in Table I, indicate that sending *independent* information during each mode together with the decode-and-forward scheme are sufficient to operate close to the capacity of the channel.

REFERENCES

[1] R. Pabst, B. H. Walke, D. C. Schultz, P. Herhold, H. Yanikomeroglu, S. Mukherjeee, H. Viswanathan, M. Lott, W. Zirwas, M. Dohler, D. D. Falconer , H. Aghvami, and G. P. Fettweis, "Relay-based deployment concepts for wireless and mobile broadband radio", *IEEE Commun. Mag.*, vol. 42, no. 9, pp. 80–89, Sep. 2004.
[2] IEEE 802.16's Relay Task Group, available at: http://wirelessman.org/relay.
[3] E. C. van-der Meulen, "Three-terminal communication channels", *Adv. Appl. Prob.*, vol. 3, pp. 120-154, 1971.
[4] T. M. Cover and A. El Gamal, "Capacity theorems for the relay channel", *IEEE Trans. Inf. Theory*, vol. 25, no. 5, pp. 572–584, Sep. 1979.
[5] G. Kramer, M. Gastpar, P. Gupta, "Cooperative strategies and capacity theorems for relay networks", *IEEE Trans. Inf. Theory*, vol. 51, no. 9, pp. 3037–3063, Sep. 2005.
[6] B. Schein and R. Gallager, "The Gaussian parallel relay network", in *Proc. IEEE Int. Symp. Inf. Theory*, 2000.
[7] B. Schein, "Distributed coordination in network information theory", *Ph.D. dissertation, MIT, Cambridge, MA*, 2001.
[8] Y. Kochman, A. Khina, U. Erez, and R. Zamir, "Rematch and forward for parallel relay networks", in *Proc. IEEE Int. Symp. Inf. Theory*, 2008.
[9] S. S. Changiz Rezaei, S. Oveis Gharan, and A. K. Khandani, "A new achievable rate for the Gaussian parallel relay channel", available at: http://arxiv.org/abs/0902.1734.
[10] W. Kang and S. Ulukus, "Capacity of a class of diamond channels", *Submitted to IEEE Trans. Inf. Theory*, Jul. 2008, available at: http://terpconnect.umd.edu/~wkang/TIT08_sub.pdf.
[11] L. Ghabeli and M. R. Aref, "A new achievable rate for relay networks based on parallel relaying", in *Proc. IEEE Int. Symp. Inf. Theory*, 2008.
[12] F. Xue and S. Sandhu, "Cooperation in a half-duplex Gaussian diamond relay channel", *IEEE Trans. Inf. Theory*, vol. 53, no. 10, pp. 3806–3814, Oct. 2007.

[3] The characterizing equation for each symbol used in the table is given in the body of the paper.

TABLE I
SUMMARY OF THE RESULTS: GAP ANALYSIS FOR DIFFERENT REGIONS

Region				Achievable Rate	Gap	Upper Bound on the Gap	Upper Bound on the Capacity
$\Delta \leq 0$	$\Gamma \leq 0$	$C_{02} \leq C_{01}$		R^1_{MDF}	κ_1	$\frac{1}{2} + \delta$	R^1_{up}
		$C_{02} \geq C_{01}$	$C_{01} \leq 1$	R^2_{MDF}	κ_5		
			$C_{01} \geq 1$	$R^1_{\text{MDF-BC}}$	$\kappa^1_{\text{MDF-BC}}$		
$\Delta \leq 0$	$\Gamma > 0$	$C_{01} \leq C_{02}$		R^2_{MDF}	κ_2	$\frac{1}{2} + \delta$	R^2_{up}
		$C_{01} \geq C_{02}$	$C_{02} \leq 1$	R^1_{MDF}	κ_6		
			$C_{02} \geq 1$	$R^2_{\text{MDF-BC}}$	$\kappa^2_{\text{MDF-BC}}$		
$\Delta > 0$	$\Gamma' \leq 0$	$C_{23} \leq C_{13}$		R^3_{MDF}	κ_3	$1 + \delta$	R^3_{up}
				$R^1_{\text{MDF-MAC}}$	$\kappa^1_{\text{MDF-MAC}}$	$\frac{1}{2} + \delta$	
		$C_{23} \geq C_{13}$	$C_{13} \leq 1, C_{123} \leq C_{13} + C_{23}$	R^4_{MDF}	κ_7	1	
			$C_{13} \leq 1, C_{123} \geq C_{13} + C_{23}$	$R^1_{\text{MDF-MAC}}$	$\kappa^1_{\text{MDF-MAC}}$	$\frac{1}{2}$	
			$C_{13} \geq 1$			$\frac{1}{2} + \delta$	
$\Delta > 0$	$\Gamma' > 0$	$C_{13} \leq C_{23}$		R^4_{MDF}	κ_4	$1 + \delta$	R^4_{up}
				$R^2_{\text{MDF-MAC}}$	$\kappa^2_{\text{MDF-MAC}}$	$\frac{1}{2} + \delta$	
		$C_{13} \geq C_{23}$	$C_{23} \leq 1, C_{123} \leq C_{13} + C_{23}$	R^3_{MDF}	κ_8	1	
			$C_{23} \leq 1, C_{123} \geq C_{13} + C_{23}$	$R^2_{\text{MDF-MAC}}$	$\kappa^2_{\text{MDF-MAC}}$	$\frac{1}{2}$	
			$C_{23} \geq 1$			$\frac{1}{2} + \delta$	

[13] B. Rankov and A. Wittneben, "Spectral efficient signaling for half-duplex relay channels", *IEEE J. Sel. Areas Commun.*, vol. 25, no. 2, pp. 379–389, Feb. 2007.

[14] W. Chang, S. Chung, and Y. H. Lee, "Capacity bounds for alternating two-path relay channels", in *Proc. 45th Allerton Conference on Communication, Control, and Computing*, 2007.

[15] S. S. Changiz Rezaei, S. Oveis Gharan, and A. K. Khandani, "Cooperative strategies for half-duplex parallel relay channel: simultaneous relaying versus successive relaying", Tech. Rep. UW-ECE 2008-02, University of Waterloo, 2008.

[16] A. S. Avestimehr, "Wireless network information flow: a deterministic approach", *Ph.D. dissertation, University of California at Berkeley, Berkeley, CA*, 2008, available at: http://www.eecs.berkeley.edu/Pubs/TechRpts/2008/EECS-2008-128.html.

[17] S. Oveis Gharan, A. Bayesteh, and A. K. Khandani, "On diversity-multiplexing tradeoff in multiple-relay networks", available at: http://arxiv.org/abs/0803.3117.

[18] K. Sreeram, S. Birenjith, P. V. Kumar, "DMT of multi-hop cooperative networks - part II: half-duplex networks with full-duplex performance", available at: http://arxiv.org/abs/0808.0235.

[19] M. Gastpar and M. Vetterli, "On asymptotic capacity of gaussian relay networks," in *Proc. IEEE Int. Symp. Inf. Theory*, 2002.

[20] Y. Fan, C. Wang, J. S. Thompson, and H. V. Poor, "Recovering multiplexing loss through successive relaying using simple repetition coding", *IEEE Trans. Wireless Commun.*, vol. 6, no. 12, pp. 4484–4493, Dec. 2007.

[21] A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse, "A deterministic approach to wireless relay networks", in *Proc. 45th Allerton Conference on Communication, Control, and Computing*, 2007.

[22] M. A. Khojastepour, A. Sabharwal, and B. Aazhang, "Bounds on achievable rates for general multi-terminal networks with practical constraints", in *Inf. Process. Sens. Netw.: Second Int. Work.*, Palo Alto, CA, Apr. 2003.

[23] A. El Gamal and M. Aref, "The capacity of the semi-deterministic relay channel", *IEEE Trans. Inf. Theory*, vol. 28, no. 3, pp. 536, May. 1982.

[24] K. Sreeram, S. Birenjith, P. V. Kumar, "DMT of multi-hop cooperative networks - part I: basic results", available at: http://arxiv.org/abs/0808.0234.

[25] A. Host-Madsen and A. Nostratinia, "The multiplexing gain of wireless networks," in *Proc. IEEE Int. Symp. Inf. Theory*, 2005.

[26] R. Etkin, D. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit", *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.

[27] V. Chvátal, *Linear programming*. W.H. Freeman, 1983.

[28] T. Cover and J. Thomas, *Elements of information theory*. NY, John Wiley, 1991.