

# Navigation on Self-Organized Networks

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**Abstract**—On a locally finite point set, a navigation defines a path through the point set from a point to another. The set of paths leading to a given point defines a tree, the navigation tree. In this article, we analyze the properties of the navigation tree when the point set is a Poisson point process on  $\mathbb{R}^d$ . We examine the distribution of stable functionals, the local weak convergence of the navigation tree, the asymptotic average of a functional along a path, the shape of the navigation tree and its topological ends. We illustrate our work in the small world graphs, and new results are established. This work is motivated by applications in computational geometry and in self-organizing networks.

## I. INTRODUCTION

### A. Navigation: definition and perspective

The development of large scale self-organized networks has created a variety of new problems. The mathematical analysis may have two major contributions in this field: the performance analysis of current structures and the design of new networks more adapted to large scale. In this work, we examine decentralized navigation algorithms on random graphs.

Before reviewing the existing literature, we define what will be called in this paper a navigation. Let  $N$  be a locally finite point set and  $O$  a point in  $\mathbb{R}^d$  taken as the origin sometimes denoted by 0. For  $x, y \in \mathbb{R}^d$ ,  $|x|$  will denote the Euclidean norm and  $\langle x, y \rangle$  the usual scalar product.  $B(X, r)$  is the open ball of radius  $r$  and center  $x$ , and  $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$  is the  $d$ -dimensional hyper-sphere.

*Definition 1:* Assume that  $O \in N$ , a navigation (with root  $O$ ) is a mapping  $\mathcal{A}$  from  $N$  to  $N$  such that for all  $X$  in  $N$  there exists a finite  $k$  satisfying  $\mathcal{A}^k(X) = O$ . A navigation on a graph  $\mathcal{G} = (N, E)$  is a navigation such that  $(X, \mathcal{A}(X)) \in E$ .

With a navigation with root  $O$ , we can define a navigation with root  $Y$  by  $\mathcal{A}_Y(X; N) = Y + \mathcal{A}(X - Y; S_{-Y} \circ N)$  where  $S_x$  is the translation by  $x$ : if  $B \subset \mathbb{R}^d$ ,  $S_x B = \{y : y - x \in B\}$ .

In this work we will analyze the *decentralized navigation* algorithms. For a navigation defined on a graph  $\mathcal{G}$ , a decentralized navigation is such that  $\mathcal{A}(X)$  depends only on  $X$  and the set of vertices adjacent to  $X$  in  $\mathcal{G}$ . A navigation is always decentralized on the complete graph, so the meaning of this definition is unclear and it is not intrinsic to  $\mathcal{A}$ , we will give later a better definition.

Navigation algorithms have emerged recently in papers in four different classes of problems (at least). A first class of problem which has recently drawn much attention is the small world phenomenon. As it is pointed by Kleinberg [15], the small world phenomenon relies on the existence of shortcuts in a decentralized navigation on a small world graph.

Extension and refinements of his results have been carried out by Franceschetti and Meester [10], Ganesh et al. [12], [8].

A second field of application is computational geometry. Kranakis, Singh and Urrutia [17] have introduced the compass routing (some numerous variants exist). The Ph.D. Thesis of Morin [20] gives a review of this class of problems. Computer scientists do not analyze the probabilistic properties of navigation algorithms, they rather examine if a given algorithm is a proper navigation, that is if it converges in a finite number of hops to its root.

The ideas of computational geometry may benefit the design of real world networks. A first field of application is sensor and ad-hoc networks, see for example the survey papers of Akyildiz et al. [1] and Ko and Vaidya [16]. A second application is self-organized overlay and peer-to-peer networks. Each node in the network receives a virtual coordinate in some naming space, and the messages are routed along a geometric navigation algorithm, see Plaxton, Rajaraman and Richa [25], Liebeherr, Nahas and Si [19] or Kermarrec, Massoulié and Ganesh [11].

Lastly, in the probabilistic literature a few authors have examined decentralized navigation algorithms (under other names). Baccelli, Tchoumatchenko and Zuyev [5] have analyzed a navigation on the Delaunay graph. Others examples include the Poisson Forest of Ferrari, Landim and Thorisson [9] and the Directed Spanning Forest introduced by Gangopadhyay Roy and Sarkar [13] (see also Penrose and Wade [21] and Baccelli and Bordenave [3]).

The aim of the present paper is to find a unified approach to these problems. The mathematical material used in this work is a natural extension of the ideas developed in [4]. Proofs and details may be found in [7], [6].

We give three canonical examples of navigation. Among those three, only the last will draw our attention. These examples are nevertheless useful to understand the context better.

A natural navigation is the *shortest path* on a connected graph  $\mathcal{G} = (N, E)$ . Let  $g$  be a functional on  $E$  ( $g$  is a cost function) and let  $\Pi(X, Y)$  denote the set of paths in  $\mathcal{G}$  from  $X$  to  $Y$ , i.e. the finite sequences of vertices in  $N$   $(X_0, \dots, X_k)$  such that  $X_0 = X$ ,  $(X_i, X_{i+1}) \in E$   $0 \leq i \leq k-1$  and  $X_k = Y$ . Provided that it is uniquely defined the shortest path is

$$\pi^*(X, Y) = \arg \inf_{(X_0, \dots, X_k) \in \Pi(X, Y)} \sum_{i=0}^{k-1} g(X_i, X_{i+1}).$$

(a tie-breaking rule may always be enforced to guarantee unic-

ity). If  $\pi^*(X, Y) = (X_0^*, \dots, X_k^*)$ , the shortest path navigation is:  $\mathcal{A}_Y^*(X) = X_1^*$ . For  $g = 1$ , the shortest path is a path which minimizes the number of hops from  $X$  to  $Y$  in  $\mathcal{G}$ : its length corresponds to the graph distance between  $X$  and  $Y$ . Shortest paths on specific graphs have drawn much attention. In particular Vahidi-Asl and Wierman ([26],[27]) have studied the shortest path on the Delaunay graph of a Poisson point process for  $g = 1$ , see also Pimentel [24]. On the complete graph of a Poisson point process and  $g(X, Y) = |X - Y|^\beta, \beta > 2$  an in-depth analysis was performed by Howard and Newman in [14]. The shortest path is the continuum analog of the celebrated *first passage percolation* on the regular  $\mathbb{Z}^d$ -lattice. The shortest path navigation has poor decentralization properties, nevertheless it gives the best achievable performance of a decentralized navigation.

A *random walk* on  $\mathcal{G}$  is a decentralized navigation provided that the random walk is recurrent: the length of the path is the hitting time of  $Y$  starting from  $X$ . In the recurrent case, this hitting time is almost surely finite for all pairs  $(X, Y)$ . However, on an infinite graph, even in the recurrent case, one might expect that the walk is null recurrent: the expectation of the length is infinite. Therefore random walks will not provide interesting navigation algorithms. More efficient decentralized navigation algorithms exist.

An important decentralized navigation is the *maximal progress navigation*. If  $\mathcal{A}$  is a navigation, the progress is defined as:

$$P(X) = |X| - |\mathcal{A}(X)|.$$

In words, the progress is the effective distance that is performed to the root  $O$ . An appealing class of decentralized navigation is the subclass of navigation such that the progress is non-negative for all  $X$ . On a graph  $\mathcal{G} = (N, E)$ , the maximal progress navigation is the navigation which maximizes the progress:  $\mathcal{A}(X) = Y$  if  $(X, Y) \in E$  and  $|Y|$  is minimal. Note that the maximal progress navigation will not be a proper navigation on all graphs, some additional properties on the graph (or on the point set) have to be added. Some breaking ties rules should also be defined to guarantee the uniqueness of this navigation.

### B. Directed Navigation

A navigation links a point  $X$  to another  $Y$ . When  $X$  is far from  $Y$ , the progress made is roughly equal to  $\langle X - \mathcal{A}_Y(X), X - Y \rangle / |X - Y|$  that is the progress made along an axis with direction  $Y - X$ . Hence in most circumstances, we expect that a navigation behaves asymptotically as a *directed navigation*. Let  $e_1 \in S^{d-1}$ , a directed navigation with direction  $e_1$  is a mapping  $\mathcal{A}_{e_1}$  from  $N$  to  $N$  such that for all  $X$  in  $N$ ,  $\lim_k \langle \mathcal{A}_{e_1}^k(X), e_1 \rangle = +\infty$ . On a graph  $\mathcal{G} = (N, E)$ , a directed navigation is such that for all  $X \in N$ ,  $(X, \mathcal{A}_{e_1}(X)) \in E$ .

As pointed above, directed navigation will appear as natural limiting objects for navigation. We will actually see later what type of convergence has to be considered.

The *directed progress* is defined as:

$$P_{e_1}(X) = \langle \mathcal{A}_{e_1}(X), e_1 \rangle - \langle X, e_1 \rangle.$$

A few examples of decentralized directed navigation may be found in the literature: directed path on the Delaunay tessellation [5], the Poisson forest [9], the directed spanning forest [13], [21].

On a graph, we also define the *maximal directed progress navigation* as the navigation which maximizes the directed progress. The maximal directed progress navigation is the limit mapping of the maximal progress navigation.

### C. Navigation Tree and Navigation Graph

Assume that  $O \in N$ , a navigation  $\mathcal{A}$  to the origin  $O$  defines a graph: the *navigation tree* which will be denoted by  $\mathcal{T}_0 = (N, E_0)$ . It is defined by

$$(X, Y) \in E_0 \text{ if } \mathcal{A}(X) = Y \text{ or } \mathcal{A}(Y) = X.$$

It is easily checked that  $\mathcal{T}_0$  is actually a tree: if there were a loop it would be contradictory with the assumption that  $\mathcal{A}(X)^k = O$  for  $k$  large enough.  $\mathcal{T}_0$  is the union of all the paths from  $X \in N$  to  $O$ . Note that  $\mathcal{T}_0$  is a *spanning tree* of  $N$ .

For a directed navigation, we define similarly the *directed navigation forest*,  $\mathcal{T}_{e_1} = (N, E_{e_1})$  by

$$(X, Y) \in E_{e_1} \text{ if } \mathcal{A}_{e_1}(X) = Y \text{ or } \mathcal{A}_{e_1}(Y) = X.$$

We check similarly that  $\mathcal{T}_{e_1}$  is a forest. We will prove that  $\mathcal{T}_{e_1}$  is the natural limit of  $\mathcal{T}_0$  for the local weak convergence of Aldous and Steele [2].

Extending the navigation tree to the origin to any point of  $N$ , we can also define the *navigation graph*  $\cup_{Y \in N} \mathcal{T}_Y$  and the directed navigation graph  $\cup_{e_1 \in S^{d-1}} \mathcal{T}_{e_1}$ . These two graphs record the set of possible navigation from one point to another (or in a direction).

We can now state an intrinsic definition for a decentralized navigation:

*Definition 2:* A navigation  $\mathcal{A}$  (to the root  $O$ ) is *decentralized* if  $\mathcal{A}(X)$  depends only of  $X$ ,  $O$  and the edges adjacent to  $X$  in  $\mathcal{T}_0$ .

Note that the set of edges adjacent to  $X$  is  $\mathcal{A}(X) \cup \mathcal{A}^{-1}(X)$ . Hence, with this definition, a navigation with positive progress is always decentralized. A shortest path navigation is not a decentralized algorithm, whereas a maximal progress navigation is decentralized. This definition will not be used in the sequel, another concept related but not equivalent will be introduced when the point set is an instance of a Poisson point process.

### D. Poisson Point Process and Poisson Weighted Infinite Tree

We will pay attention to  $\mathcal{A}^k(X)$  on a locally finite point set containing  $X$  and  $0$ , and respectively for a directed navigation, to  $\mathcal{A}_{e_1}^k(0)$  where  $e_1 \in S^{d-1}$  and  $0$  is a point of the point set. In our analysis, we will prove convergence results for two types of probabilistic models.

The first model is the usual *Poisson point process (PPP)*,  $N$ , of intensity one on  $\mathbb{R}^d$ . We will denote:  $N^0 = N + \delta_0$

and  $N^{0,X} = N + \delta_X + \delta_0$ . From Slyvniak Theorem,  $N^0$  (resp.  $N^{0,X}$ ) is a PPP on its Palm version at 0 (resp.  $(0, X)$ ). Intuitively,  $N^0$  (resp.  $N^{0,X}$ ) can be understood as a PPP conditioned on having an atom at 0 (resp. atoms at 0 and  $X$ ). It is not a restriction to assume that the intensity of the PPP is one, with a proper rescaling, our results extend to any positive intensity. Indeed, if  $N = \sum_{n \in \mathbb{N}} \delta_{T_n}$  is a realization a PPP of intensity one, then  $N_\lambda = \sum_{n \in \mathbb{N}} \delta_{\lambda^{-1/d} T_n}$  is a PPP of intensity  $\lambda > 0$ .

The second model is the *Poisson Weighted Infinite Tree*. Following the brilliant approach of Meester and Franceschetti in [10], we will try to understand the intrinsic behavior of a navigation through a virtual model which is the simplest possible probabilistic model. To this end we build a Poisson weighted infinite tree (PWIT) which is a slight variation of Aldous' PWIT [2]. We fix a root  $X \in \mathbb{R}^d$  and define the PWIT  $T^{0,X}$  as follows. The points of  $N^{0,X} \setminus \{X\}$  are the vertices of first generation in  $T^X$ .  $T^{0,X}$  is defined iteratively at each generation: at each vertex  $Y$  the subtree rooted at  $Y$  consisting of all descendants of  $Y$  is a PWIT  $T^{0,Y}$  and the Poisson point processes are drawn independently of the others. Note that there is a vertex located at 0 at each positive generation. Thus each generation has a different copy of the origin in order to guarantee that  $T^{0,X}$  is indeed a tree.

In the examples outlined in the next Section, it is important to note that the distribution of  $(X, \mathcal{A}(X))$  is the same in the PWIT  $T^{0,X}$  and in the PPP  $N^{0,X}$ . However the joint distribution of  $(\mathcal{A}^k(X))_{k \in \mathbb{N}}$  is not the same in the PWIT and the PPP. It is much simpler on the PWIT.

For a directed navigation  $\mathcal{A}_{e_1}$ , let  $X_k = \mathcal{A}_{e_1}^k(X)$  and  $\mathcal{F}_k = \sigma\{X_0, \dots, X_k\}$ . A key feature of the PWIT is the relation

$$\mathbb{P}(X_{k+1} - X_k \in \cdot | \mathcal{F}_k) = \mathbb{P}(\mathcal{A}_{e_1}(O) \in \cdot). \quad (1)$$

This last property is the (*spatial*) *memoryless* property of the directed navigation on the PWIT. Similarly, for a navigation  $\mathcal{A}$  and  $X_k = \mathcal{A}^k(X)$ , we have:

$$\mathbb{P}(X_{k+1} \in \cdot | \mathcal{F}_k) = \mathbb{P}(X_{k+1} \in \cdot | X_k), \quad (2)$$

the sequence  $(X_k)_{k \in \mathbb{N}}$  is a Markov chain with 0 as absorbing state. With an abuse of terminology we will call also this property the *memoryless* property of a navigation on a PWIT. More generally for a navigation on a PPP, we introduce the two following definitions which are the core of this work.

*Definition 3:* - A navigation  $\mathcal{A}$  is a *memoryless navigation* if Equation (2) holds (and respectively for a directed navigation with Equation (1)).

- A navigation is *regenerative* if there exists a stopping-time (on a enlarged probability space)  $\theta > 0$  such that  $\mathcal{A}^\theta$  is a memoryless navigation and the distribution of  $\theta(X)$  is independent of  $X$  for  $|X_{\theta_n}| \geq x_0$  (and respectively for a directed navigation).

The stopping time  $\theta$  will be called a *regenerative time*. If there exists a regenerative time, there exists an increasing sequence  $(\theta_n), n \in \mathbb{N}$ , which we will call a *regenerative sequence* such that  $\theta_0 = 0$ , the distribution of  $(\theta_{n+1} - \theta_n)_{n \in \mathbb{N}}$

is iid and for  $|X_{\theta_n}| \geq x_0$

$$\mathbb{P}(X_{\theta_{n+1}} \in \cdot | \mathcal{F}_{\theta_n}) = \mathbb{P}(X_{\theta_{n+1}} \in \cdot | X_{\theta_n}).$$

Respectively for a directed navigation, we will have  $\theta_0 = 0$ , the distribution of  $(\theta_{n+1} - \theta_n)_n$  is iid and

$$\mathbb{P}(X_{\theta_{n+1}} - X_{\theta_n} \in \cdot | \mathcal{F}_{\theta_n}) = \mathbb{P}(X_{\theta_1} - X_0 \in \cdot).$$

A memoryless navigation will be much simpler to analyze. We will prove under some assumptions that a navigation on a PPP will contain a regenerative sequence, that is an embedded memoryless navigation. This idea is the cornerstone of this work.

All the examples of navigation algorithms we have in mind satisfy the following property:

$$\mathcal{A}(X) \text{ is } \mathcal{F}_{B(0, |X|)}^N \text{-measurable,}$$

where for a Borel set  $B$ ,  $\mathcal{F}_B^N$  is the smallest  $\sigma$ -algebra such that the point set  $N \cap B$  is measurable. A sufficient condition for this type of navigation to be memoryless is that for all  $t \in \mathbb{N}$  and all Borel sets  $A$ :

$$\text{If } A \subset B(0, |X_k|) \text{ then } \mathbb{P}(N(A) = t | \mathcal{F}_k) = \mathbb{P}(N(A) = t), \quad (3)$$

in other words,  $N \cap B(0, |X_k|)$  is a PPP of intensity 1.

## II. EXAMPLES

### A. Small world graphs

The *small world graph* is a graph  $\mathcal{G} = (N^0, E)$  such that vertices  $X \in N^0$  and  $Y \in N^0$  are connected with probability  $f(|X - Y|)$  independently of the other, and  $f$  is a non-increasing function with value in  $[0, 1]$ . We assume, as  $t$  tends to infinity, that:

$$f(t) \sim ct^{-\beta},$$

with  $c > 0$  and  $\beta > 0$ . More formally, we add marks to  $N$  to obtain a marked point process:  $\overline{N} = \sum_n \delta_{X_n, V_n}$ , where  $V_n = (V_{nm})_{m \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$  is independent of the collection  $N$ ,  $(V_{nm})_{m < n}$  is an iid sequence of uniform random variables on  $[0, 1]$ ,  $V_{nn} = 1$  and  $V_{nm} = V_{mn}$ . For  $X, Y$  in  $N$ , we will write  $V(X, Y)$  for  $V_{nm}$  where  $n$  and  $m$  are the index of  $X$  and  $Y$ . The small world graph is defined by:

$$(X, Y) \in E \text{ if } V(X, Y) \leq f(|X - Y|).$$

Note that the degree of a vertex in the small world graph could not be infinite for small choices of  $\beta$  (indeed for  $\beta \leq d$ ). The maximal progress navigation from  $X \in N^0$  to 0 is defined as:

$$\mathcal{A}(X) = \arg \min\{|Y| : (X, Y) \in E\}.$$

As such, the small world graph has isolated points and navigation is ill-defined on non-connected graphs. To circumvent this difficulty three possibilities arise:

- 1) We enlarge slightly  $E$  to ensure a positive progress for  $X \in N$ . This is the approach followed by Ganesh et al. in [12], [8].

- 2) The marks  $V$  are not anymore independent of  $N$ , they are conditioned on the event that a positive progress is feasible at any point  $X$  of  $N$ .
- 3) Loops are allowed and the model is unchanged but if  $\mathcal{A}(X) = X$  then a new set of neighbors for  $X = \mathcal{A}(X)$  is drawn independently of everything else.

We will focus on Model 2, on Models 1 and 3, equivalent results may be proved.

The directed navigation with direction  $e_1$  is defined similarly,

$$\mathcal{A}_{e_1}(X) = \arg \max\{\langle Y, e_1 \rangle : (X, Y) \in E^0\}.$$

Let  $\mathcal{H}_{e_1}(x) = \{y \in \mathbb{R}^d : \langle y, e_1 \rangle > \langle x, e_1 \rangle\}$ , the directed navigation to be properly defined if the set of neighbors of  $X$  in  $\mathcal{H}_{e_1}$  are a.s. finite (that is  $\beta > d$ ).

### B. Compass Routing on Delaunay Graph

*Compass Routing* and its numerous variants is a popular navigation in computer science. It was introduced by Kranakis et al. in [17], see also Morin [20]. Let  $\mathcal{G} = (N^0, E)$  denote a locally finite connected graph. Compass routing on  $\mathcal{G}$  to 0 is a navigation defined by

$$\mathcal{A}(X) = \arg \max\left\{\left\langle \frac{X}{|X|}, \frac{X - Y}{|X - Y|} \right\rangle : (X, Y) \in E\right\},$$

In words:  $\mathcal{A}(X)$  is the neighboring point of  $X$  in  $\mathcal{G}$  which is the closest in direction to the straight line  $\overline{0X}$ . Compass routing is not a proper navigation on any graph: it is possible to cook up examples such that we may have  $\mathcal{A}(X) = Y$  and  $\mathcal{A}(Y) = X$ . However, as it is pointed by Liebeherr et al. in [19], on a Delaunay Graph Compass Routing is a proper navigation. Note also that a variant of this routing called Face Routing is a proper navigation.

The associated directed navigation is naturally:

$$\mathcal{A}_{e_1}(X) = \arg \max\left\{\langle e_1, \frac{X - Y}{|X - Y|} \rangle : (X, Y) \in E\right\},$$

i.e. the direction of  $(X, \mathcal{A}(X, e_1))$  is the closest from  $e_1$ . The algorithm in Baccelli et al. [5] is closely related (but not equivalent).

### C. Radial Navigation

For  $X, Y \in N^0$ ,  $X \neq 0$ ,  $|Y| < |X|$  it is defined as:

$$\mathcal{A}(X) = |Y| \quad \text{if} \quad N(B(X, |X - Y|) \cap B(0, |X|)) = \emptyset.$$

$\mathcal{A}(X)$  is the closest point from  $X$  which is closer from the origin. Radial navigation has an a.s. positive progress and  $\mathcal{A}(X)$  is a.s. uniquely defined. The directed navigation associated to radial navigation is: if  $X, Y \in N$  and  $\langle Y - X, e_1 \rangle > 0$

$$\mathcal{A}_{e_1}(X) = Y \quad \text{if} \quad N(B(X, |X - Y|) \cap \mathcal{H}_{e_1}(X)) = \emptyset.$$

That is  $\mathcal{A}_{e_1}(X)$  is the closest point from  $X$  which has a larger  $e_1$ -coordinate.

The corresponding navigation tree is the radial spanning tree. The directed spanning forest is the directed navigation

forest associated with  $\mathcal{A}_{e_1}$ . This model is examined in [13], [21] and [3].

An instance of the radial path is given in Figure 1 and an instance of the radial spanning tree is given in Figure 2.

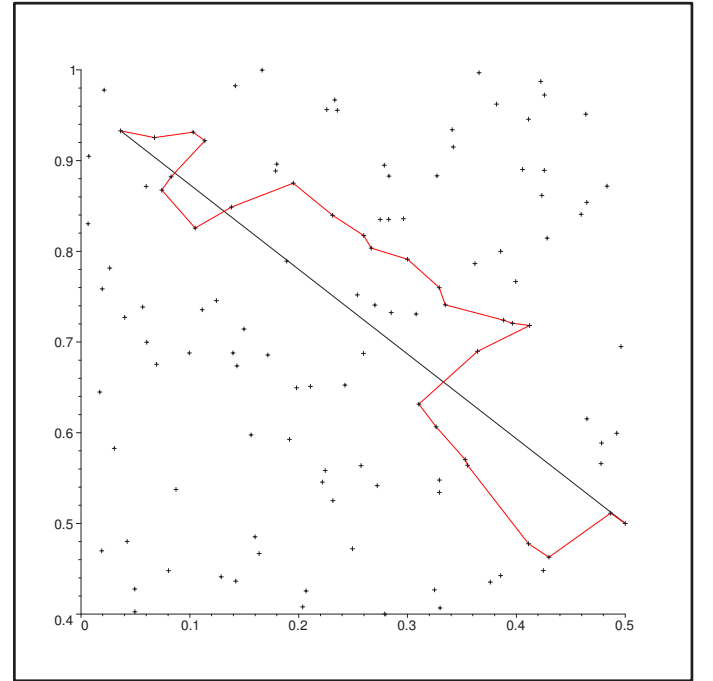


Fig. 1. An example of radial path (with its origin in  $(1/2, 1/2)$ ). The initial point is of generation 26.

### D. Road Navigation

*Road navigation* models a car on  $\mathbb{R}^d$  starting at a point  $X$  and driving to a destination point 0. A road  $R(X, e)$  is the straight line passing through  $X$  with direction  $e \in S^{d-1}$ . The following model has been introduced by Baccelli (private communication).

We consider a family of probability distributions on  $S^{d-1}$ ,  $\{\Pi_X, X \in \mathbb{R}^d\}$ . The starting point  $X$  is on a road  $R_0$  with random direction  $e(X)$  with distribution  $\Pi_X$ . It drives to the closest point on  $R_0$  of 0: the orthogonal projection of 0 on  $R_0$ . From this new point, say  $X_1$ , a new road  $R_1$  starts with direction independently drawn and distribution  $\Pi_{X_1}$ . The driver goes to  $X_2$ , the closest point on  $R_1$  of 0 and so on until it finally reaches its destination (if he ever does).

Note that if  $\Pi_X(X^\perp) = 0$ , where  $X^\perp = \{e \in S^{d-1} : \langle e, X \rangle = 0\}$  then the road navigation has an a.s. positive progress. To be sure that the driver will finally manage to reach its destination we have to assume at least that there exists  $x_0$  such that  $\Pi_X(X/|X|) > 0$  for  $|X| \leq x_0$ .

Our work covers the particular case when the distribution  $|\langle e(X), X/|X| \rangle|$  converges weakly as  $|X|$  tends to infinity.

Generalizations of this model include higher dimensional roads (as hyperplanes) or even successive projections of the origin on more complex sets than straight lines. Note that adding more roads at each point and choosing the road with

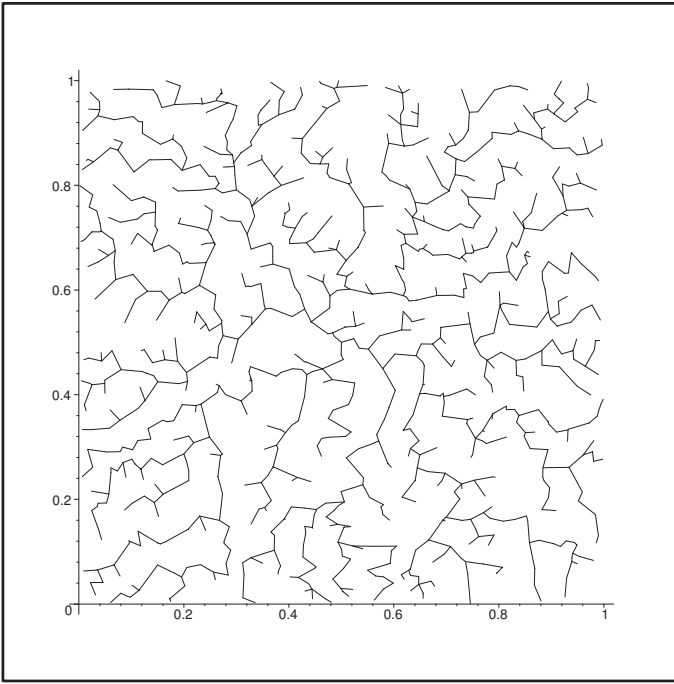


Fig. 2. Radial spanning tree of 1000 points uniformly and independently distributed in the unit square with its origin in  $(1/2, 1/2)$ .

the best possible direction is already included in the original model. Road navigation is not really a navigation since it maps a point in  $\mathbb{R}^d$  to another point in  $\mathbb{R}^d$ . All the results presented for regular navigation also apply to road navigation. Road navigation is clearly memoryless.

### III. OVERVIEW OF THE RESULTS

In this paragraph, we illustrate some of the results with the small world navigation, Model 2. We will denote by

$$\pi_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \quad \text{and} \quad \omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

the  $d$ -dimensional volume of  $B(0, 1)$  and the  $d$ -dimensional surface area measure of  $S^{d-1}$ . The results are proved in [7], [6].

#### A. Local Weak Convergence of the Navigation Tree

Under some general conditions the navigation tree converges to the directed navigation forest for the local weak convergence on graphs as it defined by Aldous and Steele in [2].

We consider a navigation  $\mathcal{A}$  with non-negative progress on a PPP  $N$  of intensity 1. Proving the convergence of the navigation tree is not a difficult task, provided that we use the right concepts. We introduce an important class of functional, the *stable functionals*. This class was first introduced by Lee [18] and it was further developed by Penrose and Yukich (see for example [22], [23]); it is slightly modified here to suit to our framework.

*Definition 4:* Let  $F(X, N)$  be a measurable function valued in a complete separable metric space.  $F$  is *stable* on  $N$  if for all  $X \in \mathbb{R}^d$  there exists a random variable  $R(X) > 0$  such that  $F(X, N)$  is  $\mathcal{F}_{B(X, R(X))}^N$ -measurable and  $R(X)$  is stochastically upper bounded uniformly in  $X$ .

A graph  $\mathcal{G} = (N, E)$  is a *stable graph* if for all  $X \in N$ ,  $V(X, N) = \{Y \in N : (X, Y) \in E\}$  (i.e. the set of vertices adjacent to  $X$  in  $\mathcal{G}$ ) is a *stable functional*.

We assume that:

$\mathcal{A}$  is the maximal progress navigation on a stable graph  $\mathcal{G} = (N, E)$ . (4)

This condition is still quite general since a navigation with a positive progress is always a maximal progress navigation on its associated navigation tree. We defined the maximal directed progress navigation with direction  $e_1 \in S^{d-1}$  as

$$\mathcal{A}_{e_1}(X) = \arg \max\{(Y - X, e_1) : Y \in N, (X, Y) \in E\}.$$

Let  $\mathcal{G}^0$  the graph built on  $N^0$  and  $\mathcal{G}$  the graph built on  $N$ . The navigation  $\mathcal{A}$  is defined on  $\mathcal{G}^0$  and the directed navigation  $\mathcal{A}_{e_1}$  on  $\mathcal{G}$ . Let  $\mathcal{T}_{e_1}$  denote the directed navigation forest associated to  $\mathcal{A}_{e_1}$  and  $\mathcal{T}_0$  the navigation tree associated to  $\mathcal{A}$ . A functional is stable on a graph  $\mathcal{G}$  if it is stable on its vertex set.

*Theorem 1:* Let  $F$  be a stable functional on  $\mathcal{T}_{-e_1}$ . If Assumption eq:ass1 holds then as  $x$  tends to  $+\infty$ , the distribution of  $F(xe_1, \mathcal{T}_0)$  converges in total variation toward the distribution of  $F(0, \mathcal{T}_{-e_1})$ .

In [2], the authors define the set of rooted geometric graphs as a metric space and they define a convergence on this space. The local weak convergence is implied by the convergence of all stable functionals. For a graph  $\mathcal{G} = (N, E)$ , we define  $S_x \circ \mathcal{G} = (S_x N, E)$  as the graph obtained by translating all vertices  $N$  by  $x$  and keeping the same edges, let  $\mathcal{T}_{e_1}(N)$  denote the directed navigation forest built on the point set  $N$  and  $\mathcal{T}_0(N)$  the navigation tree built on the point set  $N$ . As a corollary of Theorem 1:

*Corollary 1:* If Assumption eq:ass1 holds and if  $|X_n|$  tends to  $+\infty$  and  $X_n/|X_n|$  to  $e_1$  then  $S_{-X_n} \circ \mathcal{T}_0(N^0, X_n)$  converges to  $\mathcal{T}_{-e_1}(N^0)$  for the local weak convergence.

As an example, on the small world graph:

*Proposition 1:* Assume  $\beta > d$  in the small world graph. If  $|X_n|$  tends to  $+\infty$  and  $X_n/|X_n|$  to  $e_1$  then  $S_{-X_n} \circ \mathcal{T}_0(N^0, X_n)$  converges to  $\mathcal{T}_{-e_1}(N^0)$  for the local weak convergence.

#### B. Local Functional Distribution

Let  $F_X(t) = \mathbb{P}(P(X) \leq t)$  be the distribution function of the progress at  $X$ , and for  $\beta > d$ , let  $F(t) = \mathbb{P}(P_{e_1}(0) \leq t)$  denote the distribution function of the directed progress. It is possible to compute these distributions using some basic stochastic geometry tools, we obtain the following proposition.

*Proposition 2:* For the Model 3, assume  $d \geq 2$ , the following properties hold:

- 1) If  $\beta > d$ , as  $t$  goes to infinity:

$$\overline{F}(t) \sim \frac{2c\omega_{d-2}}{\beta - d} t^{d-\beta} \int_0^{\pi/2} \cos^{\beta-d} \theta d\theta.$$

- 2) If  $\beta > d$ ,  $F_X$  converges weakly to  $F$ .
- 3) If  $\beta = d$ , let  $\tilde{F}_X$  be the distribution of  $\tilde{P}(X) = -\ln(1 - P(X)/X) \in [0, +\infty]$ ,  $\tilde{F}_X$  converges weakly to  $\tilde{F}$  with  $\int \tilde{F}(s)ds = \tilde{\mu} \in (0, +\infty)$ .
- 4) If  $d - 2 < \beta < d$ , the distribution of  $|\mathcal{A}(X)|/|X|^{1-\frac{d-\beta}{2}}$  converges weakly to a non degenerated distribution.

The computation for  $d = 1$  is simpler and the same result holds with different constants. Note that this lemma implies a similar result on Model 2, in statement 1, it suffices to rescale by  $\mathbb{P}(P(0) = 0) = \exp(-\int_{\mathcal{H}(0)} f(y)dy)$  and statements 2, 3, and 4 hold without change.

The limit distribution in statements 3 and 4 can be computed explicitly. In Appendix, the distribution  $\tilde{F}$  in statement 3 is given by Equation (10) and the weak limit of  $|\mathcal{A}(X)|/|X|^{1-\frac{d-\beta}{2}}$  has a distribution obtained in Equation (9). For  $d \geq 3$  and  $0 < \beta < d - 2$ , the same method can be used to prove a convergence of the properly scaled progress.

Exact computations is not limited to the progress. For example, in [3], we compute on the radial navigation the distribution of  $(X, \mathcal{A}(X))$  and the mean degree of a vertex in the radial spanning tree.

A proof of Proposition 2 is given in Appendix.

### C. Path Average

The path from  $X$  to 0 in the navigation tree  $\mathcal{T}_0$  is given by a sequence of vertices  $\pi(X) = (X_0 = X, \dots, X_{H(X)} = 0)$  where  $H(X)$  is the generation of  $X$  in  $\mathcal{T}_0$ :

$$H(X) = \inf\{k : \mathcal{A}^k(X) = 0\}.$$

Let  $g$  be a measurable function from  $\mathbb{R}^d \times \mathbb{R}^d$  to  $\mathbb{R}$ ,  $G(0) = 0$  and

$$G(X) = \sum_{k=0}^{H(X)-1} g(X_k, X_{k+1}) = g(X, \mathcal{A}(X)) + G(\mathcal{A}(X)). \quad (5)$$

In [7], we state the various convergence results that can be expected for Equation (5) for a memoryless navigation. This amounts to analyze a non-homogeneous random walks. Analogous results for regenerative navigation will be obtained as corollaries.

For example, assume that  $\mathcal{A}$  is a regenerative navigation and  $\theta$  its associated regenerative time. We define  $P^\theta(X) = |X| - |X_\theta| = |X| - |\mathcal{A}^\theta(X)|$  with distribution  $F_X^\theta$ , the assumptions are as follows

- (i)  $\mathcal{A}$  is a regenerative navigation with non-negative progress.
- (ii)  $\mathbb{E}\theta < \infty$ .
- (iii)  $F_X^\theta$  converges weakly to  $F$  with  $\mu = \frac{1}{\mathbb{E}\theta} \int rF(dr) < \infty$  and  $F_X^\theta$  is uniformly integrable.

*Theorem 2:* Under the foregoing assumption, a.s.

$$\frac{H(X)}{|X|} \rightarrow \frac{1}{\mu}$$

Moreover if  $\sum_{k=0}^{\theta-1} g(X_k, X_{k+1})$  converges weakly as  $|X|$  tends to  $\infty$  and is uniformly integrable then a.s.

$$\frac{G(X)}{|X|} \rightarrow \frac{\nu(g)}{\mu},$$

where  $\nu(g) = \lim_{|X| \rightarrow \infty} \frac{1}{\mathbb{E}\theta} \mathbb{E} \sum_{k=0}^{\theta-1} g(X_k, X_{k+1})$ .

In the PWIT model on the small world graph, a strengthening of Proposition 2 will imply a result on the convergence of  $H(X)$  for all  $\beta > d - 2$ .

*Proposition 3:* For the maximal progress navigation in the small world graph of the PWIT model,

- If  $\beta > d + 1$  and  $\mu = \int rF(dr)$ , a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{|X|} = \frac{1}{\mu}.$$

- If  $\beta = d + 1$  a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X) \ln |X|}{|X|} = \frac{1}{c}.$$

- If  $\beta \in (d, d + 1)$

$$\liminf_{|X| \rightarrow \infty} \mathbb{E} \frac{H(X)}{|X|^{\beta-d}} > 0 \quad \text{and} \quad \limsup_{|X| \rightarrow \infty} \mathbb{E} \frac{H(X)}{|X|^{\beta-d}} < \infty.$$

- If  $\beta = d$  and  $\tilde{\mu}$  as in Proposition 2 a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln |X|} = \frac{1}{\tilde{\mu}}.$$

- If  $d - 2 < \beta < d$ , a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln \ln |X|} = -\frac{1}{\ln(1 - \frac{d-\beta}{2})}.$$

### D. How to prove that a navigation is regenerative ?

In [7], we explain a general method to prove that a navigation algorithm is regenerative. This original method relies on geometric properties of the navigation and tail bounds in the GI/GI/ $\infty$  queue.

As an example, we prove that the small world navigation on a PPP has good regenerative properties for  $\beta \leq d$  and  $\beta > d + 2$ . Our method fails in the case  $d < \beta \leq d + 2$ .

*Proposition 4:* - If  $\beta > d + 1$ ,  $\mathcal{A}$  is regenerative.

- If  $\beta > d + 2$  there exists  $\mu > 0$  such that a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{|X|} = \frac{1}{\mu}.$$

- If  $\beta = d$  and  $\tilde{\mu}$  as in Proposition 2 a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln |X|} = \frac{1}{\tilde{\mu}}.$$

- If  $d - 2 < \beta < d$ , a.s.

$$\lim_{|X| \rightarrow \infty} \frac{H(X)}{\ln \ln |X|} = -\frac{1}{\ln(1 - \frac{d-\beta}{2})}.$$

This proposition implies that the *PWIT model gives the exact order of magnitude for  $H(X)$* . It is also worth to mention that our method has enabled us to determine the exact asymptotic limit for  $\beta \in (d - 2, d]$ .

### E. Path Deviation and Tree Topology

We examine the path from  $X$  to  $O$  in the navigation tree. For regenerative navigation algorithms, we establish an upper bound on the maximal deviation of this path with respect to the straight line  $\overline{OX}$ :

$$\Delta(X) = \max_{0 \leq k \leq H(X)} |X_k - \overline{X}_k|, \quad (6)$$

with  $\overline{X}_k = \langle X_k, X/|X| \rangle X/|X|$  is the projection of  $X_k$  on the straight line  $\overline{OX}$ .

To understand the intrinsic structure of  $\mathcal{T}_0$  we need to characterize its *ends*. An end is a semi-infinite self-avoiding path in  $\mathcal{T}_0$ , starting from the origin:  $(0 = X_0, X_1, \dots)$ . The set of ends of a tree is the set of distinct semi-infinite, self-avoiding paths (two semi-infinite paths are not distinct if they share an infinite sub-path). A semi-infinite path  $(0 = X_0, X_1, \dots)$  has an asymptotic direction if  $X_n/|X_n|$  has a limit in the unit sphere  $S^{d-1}$ . Following Howard and Newman in [14], some properties of the semi-infinite self-avoiding paths in  $\mathcal{T}_0$  follows from tail bounds on  $\Delta(X)$ .

For  $X \in N$ , let  $\Pi_{out}(X)$  be the set of offspring of  $X$  in the  $\mathcal{T}_0$ , namely the set  $X' \in N$  such that for some  $k$ ,  $X = \mathcal{A}^k(X')$ . We now state a definition introduced in [14].

*Definition 5:* Let  $f$  be a measurable positive function tending to 0 at  $+\infty$ , a tree is said to be  $f$ -straight at the origin, if for all but finitely many vertices :

$$\Pi_{out}(X) \subset C(X, f(|X|)),$$

where for all  $X \in \mathbb{R}^d$  and  $\epsilon \in \mathbb{R}^+$ ,  $C(X, \epsilon) = \{Y \in \mathbb{R}^d : \theta(X, Y) \leq \epsilon\}$  and  $\theta(X, Y)$  is the angle (in  $[0, \pi]$ ) between  $X$  and  $Y$ .

A bound on  $\mathbb{P}(\Delta(X) \geq |X|^\gamma)$  will enable us to find conditions under which  $\mathcal{T}_0$  is an  $f$ -straight tree, with  $f(x) = |x|^{\gamma-1}$ . In particular, it will characterize the semi-infinite paths of the navigation tree. Indeed  $f$ -straight trees have a simple topology described by Proposition 2.8 of [14] and restated in the following proposition.

*Proposition 5 (Howard and Newman):* Let  $\mathcal{T}$  be an  $f$ -straight spanning tree on a PPP. The following set of properties holds almost surely:

- every semi-infinite path has an asymptotic direction,
- for every  $u \in S^{d-1}$ , there exists at least one semi-infinite path with asymptotic direction  $u$ ,
- the set of  $u$ 's of  $S^{d-1}$  such that there is more than one semi-infinite path with asymptotic direction  $u$  is dense in  $S^{d-1}$ .

On the small world navigation, we obtain the following proposition.

*Proposition 6:* There exists  $C \geq 1$ , such that if  $\gamma > C(d+1)/(\beta-d)$ , then for some  $\eta > 0$ , there exists  $C_1 > 0$  such that

$$\mathbb{P}(\Delta(X) \geq |X|^\gamma) \leq C_1 |X|^{-d-\eta}.$$

and  $\mathcal{T}_0$  is  $f$ -straight with  $f(x) = |x|^{\gamma-1}$ .

A bound for the constant  $C$  could be explicitly computed. We only point out that for a small world navigation on a PWIT,  $C = 1$ .

### F. Shape of the Navigation Tree

Finally, we state a shape theorem for regenerative navigation algorithms.

Another interesting feature is the set of points at tree-distance less than  $k$  from the origin

$$\mathcal{T}_0(k) = \{X \in N : \mathcal{A}^k(X) = 0\}.$$

Let  $\mathcal{A}$  be a regenerative navigation and  $\theta$  its associated regenerative time. We define  $P^\theta(X) = |X| - |X_\theta| = |X| - |\mathcal{A}^\theta(X)|$  with distribution  $F_X^\theta$ , the assumptions are as follows

- (i)  $\mathcal{A}$  is a regenerative navigation with non-negative progress.
- (ii)  $\sup_{X \in \mathbb{R}^d} \mathbb{E}P^\theta(X)^r < \infty$  and  $\mathbb{E}\theta^r < \infty$  for some  $r > d+2$ .
- (iii)  $F_X^\theta$  converges weakly to  $F$  with  $\mu = \frac{1}{\mathbb{E}\theta} \int rF(dr) > 0$ .

*Theorem 3:* Under the foregoing assumption, for all  $\epsilon > 0$  there exists a.s.  $K$  such that if  $k \geq K$ :

$$N \cap B(0, (1-\epsilon)k\mu) \subset \mathcal{T}_0(k) \subset B(0, (1+\epsilon)k\mu). \quad (7)$$

Moreover a.s. and in  $L^1$  :

$$\frac{|\mathcal{T}_0(k)|}{\pi_d k^d} \rightarrow \mu^d, \quad (8)$$

In other words, the navigation tree generated by a PPP inside a ball grows linearly with the number of points. In the literature, this constant is known as the *volume growth*. The intuition behind Theorem 3 is as follows: by Theorem 2 a point  $k$  hops away from the origin is asymptotically at Euclidean distance  $D_k \sim k\mu$  from the origin. The ball of radius  $D_k$  contains  $\pi_d D_k^d$  points in  $N$  asymptotically. In order to prove Theorem 3, we need an estimate of the tail of the fluctuations of  $H(X)$  around its mean.

On the small world graph, we have the following proposition.

*Proposition 7:* Let  $\mu$  (resp.  $\tilde{\mu}$ ) as in Proposition 4 (resp. Proposition 2).

- There exists  $C \geq 1$  such that if  $\beta > (C+1)d + 2C$ , for all  $\epsilon > 0$  there exists a.s.  $K$  such that if  $k \geq K$ :

$$N \cap B(0, (1-\epsilon)k\mu) \subset \mathcal{T}_0(k) \subset B(0, (1+\epsilon)k\mu).$$

Moreover a.s. and in  $L^1$ :

$$\frac{|\mathcal{T}_0(k)|}{\pi_d k^d} \rightarrow \mu^d,$$

- If  $\beta = d$ , for all  $\epsilon > 0$  there exists a.s.  $K$  such that if  $k \geq K$ :

$$N \cap B(0, e^{(1-\epsilon)k\tilde{\mu}}) \subset \mathcal{T}_0(k) \subset B(0, e^{(1+\epsilon)k\tilde{\mu}}).$$

Moreover a.s. and in  $L^1$ :

$$\frac{\ln |\mathcal{T}_0(k)|}{k} \rightarrow d\tilde{\mu}.$$

- For  $d-2 < \beta < d$ , let  $\alpha = 1 - (d-\beta)/2$ , for all  $\epsilon > 0$  there exists a.s.  $K$  such that if  $k \geq K$ :

$$N \cap B(0, \exp(\alpha^{(1-\epsilon)k})) \subset \mathcal{T}_0(k) \subset B(0, \exp(\alpha^{(1+\epsilon)k})).$$

Moreover a.s. and in  $L^1$  :

$$\frac{\ln \ln |\mathcal{T}_0(k)|}{k} \rightarrow \ln \alpha.$$

Again, a bound for the constant  $C$  could be computed. In the PWIT model  $C = 1$ .

#### IV. FINAL REMARKS

Our analysis has focused on three different features of navigation algorithms. Firstly, we have computed the distribution of local functionals, such as the progress, and proved the convergence of the navigation tree to the directed navigation forest. The main ingredients for this analysis are stochastic geometric and geometric probability tools. Secondly, important properties of the path leading a point to the root have been shown: path average and deviation. There, we have introduced the notion of memoryless and regenerative navigation as a convenient way to classify the different behaviors that can be expected. Finally, some global properties of the navigation trees have been obtained from the analysis of the path properties: shape of the navigation tree and characterization of the semi-infinite paths.

Some important problems remain unsolved. The method used to prove that a navigation is regenerative should be improved in order to derive results on the small world navigation for  $\beta \in (d, d+2)$ . The deviation properties and the semi-infinite paths of navigation trees with super-polynomial growth rate are not well understood, for the small world navigation it corresponds the case  $\beta \leq d$ . Another direction of research is to state a weak convergence theorem for the path from a point far away to the origin, or also for the navigation tree.

#### APPENDIX: PROOF OF PROPOSITION 2

The proof relies on explicit computations and does not involve any subtle argument, we skip most details.

*Statement 1.*

Let  $\mathcal{G} = (N, E)$  denote the Small World graph and  $V(X) = \{Y : (X, Y) \in E\}$  the set of neighbors of  $X$  in the graph  $G$ ,  $V(X)$  is a non-homogenous Poisson point process of intensity  $f(|X - x|)dx$ . We have

$$\begin{aligned} \mathbb{P}(P(0) > t) &= \mathbb{P}(V(0) \cap \mathcal{H}(te_1) \neq \emptyset) \\ &= 1 - \exp\left(-\int_{\mathcal{H}(te_1)} f(y)dy\right) \\ &\sim \int_{\mathcal{H}(te_1)} f(y)dy, \end{aligned}$$

as  $t$  tends to infinity. Let  $\Lambda_t = \int_{\mathcal{H}(te_1)} f(y)dy$ , writing  $y = r \cos \theta e_1 + r \sin \theta e_2$  with  $\langle e_1, e_2 \rangle = 0$  and  $e_2 \in S^{d-1}$ , we

obtain

$$\begin{aligned} \Lambda_t &= 2\omega_{d-2} \int_0^{\pi/2} \int_{t/\cos\theta}^{\infty} f(r)r^{d-1}drd\theta \\ &\sim 2\omega_{d-2} \int_0^{\pi/2} \int_{t/\cos\theta}^{\infty} cr^{d-\beta-1}drd\theta \\ &\sim 2\frac{2c\omega_{d-2}}{\beta-d} \int_0^{\pi/2} \left(\frac{t}{\cos\theta}\right)^{d-\beta}d\theta \\ &\sim \frac{2c\omega_{d-2}}{\beta-d} t^{d-\beta} \int_0^{\pi/2} \cos^{\beta-d}\theta d\theta. \end{aligned}$$

*Statement 2.*

We can suppose without loss of generality that  $X = -xe_1$ , with  $x > 0$ . By definition, for  $t < x$ :

$$\begin{aligned} \mathbb{P}(P(X) > t) &= \mathbb{P}(V(X) \cap B(0, x-t)(t) \neq \emptyset) \\ &= 1 - \mathbb{P}(0 \in V(X))e^{-\int_{B(0, x-t)} f(|X-y|)dy} \\ &= 1 - (1 - f(x))e^{-\int_{B(0, x-t)} f(|X-y|)dy}. \end{aligned}$$

In  $\mathbb{R}^2$  for  $u \in (0, 1)$  and  $0 \leq \theta < \arcsin(1-u)$ , the straight line with equation  $y = \tan \theta$  intersects the sphere of radius  $u$  and center  $(1, 0)$  at two points of respective norms  $A(\theta, u)$  and  $B(\theta, u)$ . A direct computation leads to

$$\begin{aligned} A(\theta, u) &= \cos\theta \left(1 - \sqrt{1 - \frac{u(2-u)}{\cos^2\theta}}\right) \\ &= \frac{u}{\cos\theta} + o\left(\frac{u}{\cos\theta}\right) \\ B(\theta, u) &= \cos\theta \left(1 + \sqrt{1 - \frac{u(2-u)}{\cos^2\theta}}\right) \\ &= 2\cos\theta - \frac{u}{\cos\theta} + o\left(\frac{u}{\cos\theta}\right). \end{aligned}$$

Let  $\Lambda_t(x) = \int_{B(0, x-t)} f(|X-y|)dy$ , we get as  $t, x$  tend to infinity and  $t/x$  tends to 0:

$$\begin{aligned} \Lambda_t(x) &= 2\omega_{d-2} \int_0^{\arcsin(1-t/x)} \int_{xA(\theta, t/x)}^{xB(\theta, t/x)} f(r)r^{d-1}drd\theta \\ &\sim 2\omega_{d-2} \int_0^{\arcsin(1-t/x)} \int_{xA(\theta, t/x)}^{xB(\theta, t/x)} cr^{d-\beta-1}drd\theta \\ &\sim \frac{2c\omega_{d-2}}{\beta-d} \int_0^{\arcsin(1-t/x)} (xA(\theta, t/x))^{d-\beta} \\ &\quad - (xB(\theta, t/x))^{d-\beta} d\theta \\ &\sim \bar{F}(t). \end{aligned}$$

It follows also

$$\begin{aligned} |\Lambda_t(x) - \Lambda(t)| &\leq \int_{\arcsin(1-t/x)}^{\pi/2} f(r)r^{d-1}drd\theta \\ &\quad + \int_0^{\arcsin(1-t/x)} \int_{t/\cos\theta}^{xA(\theta, t/x)} f(r)r^{d-1}drd\theta \\ &\quad + \int_0^{\arcsin(1-t/x)} \int_{xB(\theta, t/x)}^{\infty} f(r)r^{d-1}drd\theta. \end{aligned}$$

If  $t = x^{d-\beta}\epsilon(x)$ , with  $\epsilon \in \ell^0$ , we easily get that  $t^{\beta-d}|\Lambda_t(x) - \Lambda(t)|$  tends to 0.



*Statement 4.*

Let  $U(X) = |\mathcal{A}(X)|/x^\alpha = (x - P(X))/x^\alpha$  with  $|X| = x$  and  $\alpha = 1 - (d - \beta)/2 \in (0, 1)$ . Let  $0 < s < x^{1-\alpha}$ , we have

$$\begin{aligned}\mathbb{P}(U(X) < s) &= 1 - (1 - f(x))e^{-\int_{B(0, sx^\alpha)} f(|X-y|)dy}, \\ &= 1 - (1 - f(x))e^{-\Lambda_{x-sx^\alpha}(x)},\end{aligned}$$

with as  $x$  tends to  $+\infty$ , uniformly in  $s < x^{1-\alpha'}$ ,  $\alpha' > \alpha$ :

$$\begin{aligned}\Lambda_{x-sx^\alpha}(x) &\sim 2\omega_{d-2} \int_0^{\arcsin(sx^{\alpha-1})} \int_{xA(\theta, 1-sx^{\alpha-1})}^{xB(\theta, 1-sx^{\alpha-1})} cr^{d-\beta-1} dr d\theta \\ &\sim \frac{2c\omega_{d-2}}{d-\beta} \int_0^{\arcsin(sx^{\alpha-1})} (xB(\theta, 1-sx^{\alpha-1}))^{d-\beta} \\ &\quad - (xA(\theta, 1-sx^{\alpha-1}))^{d-\beta} d\theta.\end{aligned}$$

We have  $B(\theta, 1 - sx^{\alpha-1}) = \cos\theta(1 + \sqrt{s^2x^{2(\alpha-1)}/\cos^2\theta - \tan^2\theta})$  and  $A(\theta, 1 - sx^{\alpha-1}) = \cos\theta(1 - \sqrt{s^2x^{2(\alpha-1)}/\cos^2\theta - \tan^2\theta})$ . Hence as  $x$  tends to  $\infty$ ,  $(xB(\theta, 1 - sx^{\alpha-1}))^{d-\beta} - (xA(\theta, 1 - sx^{\alpha-1}))^{d-\beta}$  is equivalent to

$$2(d-\beta)x^{d-\beta} \cos^{d-\beta}\theta \sqrt{s^2x^{\beta-d}/\cos^2\theta - \tan^2\theta},$$

and we obtain:

$$\begin{aligned}\Lambda_{x-sx^\alpha}(x) &\sim 4c\omega_{d-2} \int_0^{\arcsin(sx^{(\beta-d)/2})} x^{d-\beta} \cos^{d-\beta}\theta \sqrt{\frac{s^2x^{\beta-d}}{\cos^2\theta} - \tan^2\theta} d\theta \\ &\sim 4c\omega_{d-2}s^2.\end{aligned}$$

Finally we have proved that uniformly in  $s < x^{(d-\beta)/2-\eta}$  (for some  $\eta > 0$ ):

$$\lim_{|X| \rightarrow \infty} \mathbb{P}(U(X) > s) = \exp(-4c\omega_{d-2}s^2). \quad (9)$$

and this concludes the proof of statement 4.

*Statement 3.* Similarly, we still suppose that  $X = -xe_1$ , with  $x > 0$ , let  $s > 0$  and  $u = 1 - \exp(-s) \in (0, 1)$ :

$$\begin{aligned}\mathbb{P}(\tilde{P}(X) > s) &= \mathbb{P}(P(X) > xu) \\ &= 1 - (1 - f(x))e^{-\int_{B(0, (1-u)x)} f(|X-y|)dy},\end{aligned}$$

as above with  $\Lambda_t(x) = \int_{B(0, x-t)} f(|X-y|)dy$ :

$$\begin{aligned}\Lambda_{ux}(x) &= 2\omega_{d-2} \int_0^{\arcsin(1-u)} \int_{xA(\theta, u)}^{xB(\theta, u)} f(r)r^{d-1} dr d\theta \\ &\sim 2\omega_{d-2} \int_0^{\arcsin(1-u)} \int_{xA(\theta, u)}^{xB(\theta, u)} c/r dr d\theta \\ &\sim 2c\omega_{d-2} \int_0^{\arcsin(1-u)} \ln \frac{B(\theta, u)}{A(\theta, u)} d\theta\end{aligned}$$

We define

$$\begin{aligned}\tilde{F}(s) &= 1 - e^{-2c\omega_{d-2} \int_0^{\arcsin(\exp(-s))} \ln \frac{B(\theta, 1-\exp(-s))}{A(\theta, 1-\exp(-s))} d\theta} \\ &= 1 - \exp(-c \int_{B(0, \exp(-s))} |e_1 - y|^{-d} dy).\end{aligned} \quad (10)$$

A direct analysis shows that, as  $s$  tends to  $+\infty$ :

$$\tilde{F}(s) \sim 4c\omega_{d-2}e^{-2s}. \quad (11)$$

The statement 3 follows.

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