南京大學

研 究 生 毕 业 论 文

（申请博士学位）

论 文 题 目　递归可枚举度的一些代数性质

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学科、专业方向　基础数学

研 究 方 向　数理逻辑及其应用

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二零零六年十一月

学　　　　号: **DG0421009**

论文答辩日期:　　　　年　　月　　日

指 导 教 师:　　　　　　　　　　（签字）

Dissertation for the Degree of Doctor of Philosophy

Some Algebraic Properties of Recursively Enumerable Degrees

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Pure Mathematics

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November, 2006

To my parents and my wife

南京大学研究生毕业论文中文摘要首页用纸

毕业论文题目：递归可枚举度的一些代数性质

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摘 要

长期以来图灵度形成的偏序结构$\mathcal{D} = (\mathbf{D}, \leq)$ 是递归论的一个主要研究对象，对其子结构$\mathcal{R} = (\mathbf{R}, \leq)$ 的研究则是一个重要分支。这里$\mathbf{R}$ 是所有**递归可枚举度**的集合，递归可枚举度是可以由一个**递归可枚举集合**代表的图灵度，而递归可枚举集合在哥德尔不完备定理的证明中扮演了重要角色：给定一个递归的公理系统，如果它蕴含谓词演算，则其所有的定理构成一个非递归的递归可枚举集合。

$\mathcal{R}$ 的研究历史上的第一个著名问题是（Post (1944)）：是否存在除了$\mathbf{0}$（递归集合构成的图灵度））和$\mathbf{0}'$（停机问题代表的图灵度）之外的递归可枚举度？这个问题的肯定答案由Friedberg (1957) 和Mučnik (1956) 独立地发现，他们的证明引入了**优先方法**。接下来的几十年，人们在Friedberg-Muchnik 工作的基础上发展出更复杂的优先方法，并用这些方法发现了$(\mathbf{R}, <)$ 的很多重要性质，比如$\mathcal{R}$ 是稠密的（Sacks (1964)），即$(\forall \mathbf{a}, \mathbf{b} \in \mathbf{R})(\mathbf{a} < \mathbf{b} \rightarrow (\exists \mathbf{c} \in \mathbf{R})(\mathbf{a} < \mathbf{c} < \mathbf{b}))$；$\mathcal{R}$ 是一个上半格，一般来说两个元素的下确界不存在（Lachlan (1966)，Yates (1966)）。

Sacks 稠密性定理的证明导致许多人猜测$\mathcal{R}$是一个简单的结构，比如：Shoenfield (1965) 猜想这个结构是齐次的（即如果一个偏序P可以嵌入$\mathcal{R}$中，而且Q是P的一个扩张，则在$\mathcal{R}$中也可以把P扩张到Q），而Sacks 猜想$\mathcal{R}$ 的一阶理论是递归的。Shoenfield的猜想被Lachlan (1966) 和Yates (1966) 独立地反驳。他们在反驳中构造了称为极小对的递归可枚举度：$\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$（这样的$\mathbf{a}$或$\mathbf{b}$的集合记为$\mathbf{M}$）。Lachlan 和Yates 的工作引起人们对$\mathcal{R}$ 局部性质的兴趣，人们期望通过引入更多的局部性质并且通过对局部性质的研究揭示$\mathcal{R}$ 的整体性质，例如：Sacks 的猜想是否成立？

然而Sacks 猜想的反驳（Harrington and Shelah (1982)）最终借助于在$\mathcal{R}$ 中可

定义地解释其它数学结构这样的模型论手段，而不是引入简单自然的局部性质。事实上模型论的方法早就应用在$\mathcal{D}$的研究中。但是由$\mathcal{R}$的局限性带来的复杂性，意味着应用模型论方法需要更复杂的技巧。这些技巧近年来由Nies, Shore, Slaman 和Woodin 等在Harrington and Shelah (1982) 的基础上系统地发展起来。越来越多整体性质的研究都引入了模型论或者集合论的工具，需要其它逻辑分支的知识和技巧，也需要应用优先方法进行更加复杂的构造。

在本论文中，我们将从格论和模型论的角度出发研究$\mathcal{R}$ 的一些整体性质：可定义理想和滤子的存在性，子结构和$\mathcal{R}$ 的关系，以及同余关系和商结构。为此我们将借助Nies, Shore 和Slaman 等发展的在$\mathcal{R}$ 中解释数论模型$(\mathbb{N},+,0)$ 的工具。

我们将在第二章中研究$\mathcal{R}$ 的一个理想。在这方面的一个重要结果是Ambos-Spies et al. (1984) 发现的：$\mathcal{R}$ 可以分解为一个超滤**NC**和一个素理想**M**，并且它们是可定义的。然而自此以后，人们一直未能找到其它可定义的代数子结构。这方面的突破一直等到Nies (2003) 证明"所有$\mathcal{R}$ 的可定义子集生成的理想也是可定义的"。Yu and Yang (2005) 应用这一有力的结果找到了的更多的理想，其中的一个理想是由**NB** 生成的。Li and Yang (2003) 注意到**NB** 的构造和**PC** 的构造接近，因此问它们是否生成同一个理想。我们将证明上述的集合生成不同的理想，事实上**PC** 生成一个之前未知的理想。另一方面我们证明了：任何非主理想都是$\mathcal{R}$ 的Σ_1-初等子结构。这一结果从模型论的角度说明非主理想在一定程度上反映了$\mathcal{R}$ 的性质。

在本论文的第三章，我们第一次给出定义滤子的一般手段：利用Nies (2003) 的一个定理，我们证明"所有$\mathcal{R}$ 的可定义子集生成的滤子也是可定义的"，这就意味着我们可以通过寻找$\mathcal{R}$ 的可定义子集来寻找可定义的滤子。这是上述Nies 关于可定义理想的结果的对偶。应用上述结论，我们找到两个新的可定义滤子：分别由**Cups**(**M**)和**NSB**生成的滤子。在此之前，**NC**是唯一已知的可定义滤子。

另一方面，$\mathcal{R}$ 的商结构的一些基本性质至今没有被系统地研究，比如稠密性。Schwarz (1984) 证明了$(\mathbf{R}/\mathbf{M},\leq)$ 是向下稠密的。在第四章，我们定义了一个并非由理想诱导的同余关系，并且证明其诱导的商结构并不稠密；另一方面，尽管此同余关系和"模**NCup**"非常相近，我们却能够证明它们并不相同。

关键词: 递归可枚举度; 上半格; 理想; 滤子; 同余关系; 子结构; 可定义.

南京大学研究生毕业论文英文摘要首页用纸

THESIS: Some Algebraic Properties of Recursively Enumerable Degrees

SPECIALITY: Pure Mathematics

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Abstract

For long time the partial order of Turing degrees $\mathcal{D} = (\mathbf{D}, \leq)$ is a major subject of recursion theory, and the study of $\mathcal{R} = (\mathbf{R}, \leq)$ is an important branch, where $\mathbf{R}$ denotes the collection of *recursively enumerable* (*r.e.* for short) degrees, i.e. Turing degrees represented by some r.e. sets. R.e. sets played an important role in Gödel's proof of his Incompleteness Theorem: given a recursive set of axioms implying first order logic, its theorems form a non-recursive r.e. set.

The first famous problem in the history of studies on $\mathcal{R}$ is the existence of a r.e. degree strictly between $\mathbf{0}$ and $\mathbf{0}'$ (Post (1944)). The affirmative answer was given by Friedberg (1957) and Mučnik (1956) independently, and they introduced so called *priority arguments*. During the following decades people developed much more complicated priority arguments, and discovered many important properties of $\mathcal{R}$, e.g. the density of $\mathcal{R}$ by Sacks (1964), and the non-existence of infima in general of two elements in $\mathcal{R}$ independently by Lachlan (1966) and Yates (1966).

Sacks' density theorem led to conjectures implying that $\mathcal{R}$ is simple, e.g. Shoenfield (1965) conjectured that $\mathcal{R}$ is homogenous, and Sacks conjectured that the first order theory of $\mathcal{R}$ is recursive. Shoenfield's conjecture was later refuted. In their refutations Lachlan (1966) and Yates (1966) constructed so called *minimal pairs*, i.e. $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. The collection of halves of minimal pairs is denoted by $\mathbf{M}$. Their proofs inspired interests in *local properties* of $\mathcal{R}$, people kept introducing local properties with expectations to reveal *global properties* of $\mathcal{R}$, like Sacks' conjecture.

But the refutation of Sacks' conjecture by Harrington and Shelah (1982) finally relied on definably interpreting other structures in $\mathcal{R}$ rather than natural local properties.

Actually such model theoretic tools had long appeared in the study of $\mathcal{D}$. But restrictions imposed by $\mathcal{R}$ imply extra difficulty. However recent years see development of these techniques in $\mathcal{R}$ by people like Nies, Shore, Slaman and Woodin based on the work of Harrington and Shelah. Model or set theoretic tools are gradually introduced in study of global properties. And knowledge and skills from other branches of logic are required, as well as complicated priority constructions.

In this thesis we will investigate some global properties of $\mathcal{R}$ from lattice and model theoretic viewpoints: the existence of definable ideals and filters, the relation between substructures and $\mathcal{R}$, congruence relations and quotient structures.

In Chapter 2 we will study an ideal in $\mathcal{R}$. The first important result of this kind is by Ambos-Spies et al. (1984) that $\mathcal{R}$ can be decomposed as a prime ideal $\mathbf{M}$ and an ultra-filter $\mathbf{NC}$. But from then on people found no other definable substructures for a long time. The breakthrough is a theorem by Nies (2003) that ideals generated by definable subsets are also definable. Yu and Yang (2005) applied this to find several definable ideals. Li and Yang (2003) observed that the constructions of $\mathbf{NB}$ and $\mathbf{PC}$ are similar, and thus asked whether they generate a same ideal. We will prove that the ideals generated by $\mathbf{NB}$ and $\mathbf{PC}$ respectively are different. In fact the ideal generated by $\mathbf{PC}$ was unknown. On the other hand we will also show that every non-principal ideal is a Σ_1 element substructure of $\mathcal{R}$. This result from a model theoretic viewpoint indicates that non-principal ideals reflect some properties of $\mathcal{R}$.

In Chapter 3 using a theorem in Nies (2003) we will prove a dual to Nies' result mentioned that filters generated by definable subsets are definable. This gives a general method of finding definable filters. Applying this result we will find some new definable filters: those generated by $\mathbf{Cups}(\mathbf{M})$ and $\mathbf{NSB}$ respectively. Previously $\mathbf{NC}$ was the only known filter.

Finally in Chapter we will study a congruence relation and prove that the induced quotient structure is not dense. There were no known similar results so far, though density is a basic property. (Schwarz (1984) proved the downward density of $\mathcal{R}/\mathbf{M}$.) Despite the analogous between this congruence relation and *modulo* $\mathbf{NCup}$ we will prove that they are actually different.

Keywords: recursively enumerable degrees; upper semi-lattices; ideals; filters; congruence relations; substructures; definability.

Contents

Chapter I

Preliminaries

In this chapter we introduce notions and notations for developing the rest of this thesis. However as limited by time and space we assume that readers have had basic ideas of lattice theory and mathematical logic, in particular some basic recursion theory. For readers not familiar with these, we refer to Grätzer (1998), Nerode and Shore (1997) and Cutland (1980).

I.1 Upper Semi-lattices

A partial order $(P, \prec)$ is an *upper semi-lattice* if and only if for every pair $a, b \in P$ their supreme, denoted as $a \vee b$, always exists. When this pair also has infimum, we denote it by $a \wedge b$. When the ordering is unambiguous we also denote the partial order like P.

A subset I of P is an *ideal* if and only if

1. I is closed downward, i.e. $\forall a \in I, b \in P(b \prec a \to b \in I)$,
2. I is closed by $\vee$, i.e. $\forall a, b \in I(a \vee b \in I)$.

Given $A \subseteq P$, the ideal *generated by* A is the least ideal I containing A, denoted by $(A]$.

Dually, a subset F of P is a *filter* if and only if

1. F is closed upward, i.e. $\forall a \in F, b \in P(a \prec b \to b \in F)$,

2. F is closed by $\wedge$, $\forall a, b \in I(a \wedge b \text{ exists } \rightarrow a \wedge b \in F)$.

If in additional $\forall a, b \in F \exists c \in F(c \preceq a, b)$, then we say that F is a *strong filter*. Given $A \subseteq P$, the filter *generated by* A is the least filter F containing A, denoted by $[A)$.

An ideal I is a *prime ideal* if and only if

$$\forall a, b(a \wedge b \text{ exists and in } I \rightarrow a \in I \text{ or } b \in I).$$

Dually a filter F is *ultra* if and only if

$$\forall a, b(a \vee b \in I \rightarrow a \in F \text{ or } b \in F).$$

I.2 Recursively Enumerable Degrees

In degree theory the objects are subsets of ω and equivalent classes under Turing reducibility. A set A is *Turing reducible* to another set B if and only if there is a Turing machine Φ that using B as an oracle computes the characteristic function of A, denoted by $A \leq_{\mathrm{T}} B$. We denote $A \equiv_{\mathrm{T}} B$ if and only if $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$. $\equiv_{\mathrm{T}}$ is an equivalent relation and equivalent classes induced by $\equiv_{\mathrm{T}}$ are called *Turing degrees*. We denote degrees by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ the set of degrees by $\mathbf{D}$. For $\mathbf{a}, \mathbf{b} \in \mathbf{D}$, $\mathbf{a} \leq \mathbf{b}$ if and only if $A \leq_{\mathrm{T}} B$ for some $A \in \mathbf{a}$ and $B \in \mathbf{b}$. The structure $(\mathbf{D}, \leq)$ is denoted by $\mathcal{D}$ and sometimes also by $\mathbf{D}$ for short.

There is a natural join operation for sets in degree theory. For $A, B \subseteq \omega$, $A \oplus B = \{2n | n \in A\} \cup \{2n+1 | n \in B\}$. It is trivial that $A \oplus B \leq_{\mathrm{T}} C$ for any C with $A, B \leq_{\mathrm{T}} C$. Hence the degree represented by $A \oplus B$ is the supreme of $\mathbf{a}$ and $\mathbf{b}$, i.e. degrees represented by A and B respectively. We denote the degree of $A \oplus B$ by $\mathbf{a} \vee \mathbf{b}$. However infima of pairs do not always exist.

Theorem I.2.1 (Kleene and Post (1954))**.** *There are* $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ *such that* $\mathbf{a} \wedge \mathbf{b}$ *does not exist. Hence* $\mathcal{D}$ *is not a lattice.*

A *recursively enumerable set* is the range of some recursive function mapping ω into ω. A Turing degree containing a recursively enumerable set is called a *recursively enumerable* degree. We use *r.e.* for short of recursively enumerable and denote the set of r.e. degrees by $\mathbf{R}$ and the structure $(\mathbf{R}, \leq)$ by $\mathcal{R}$. From now on without explicit declaration all degrees and sets considered are r.e.

As *recursive function* is defined in an effective way, there are effective enumerations of recursive functions. We may fix an arbitrary one and denote by W_e the range of the e-th function with respect to this specific enumeration. The best known non-recursive r.e. set is so called the *halting problem*, i.e. $K = \{e | e \in W_e\}$. The degree of recursive sets is denoted by $\mathbf{0}$, and that of K by $\mathbf{0}'$. It is immediately from the definition that $A \leq_{\mathrm{T}} K$ for every r.e. A. Thus $\mathbf{0}'$ is also called the complete r.e. degree.

However $\mathbf{0}$ and $\mathbf{0}'$ remained the only known r.e. degrees for a long time, and it was Post's problem whether there exists another r.e. degree, that led to the development of the theory of r.e. degrees. By works of Friedberg, Muchnik and Sacks et al. people learned that there are countably many r.e. degrees. And this non-trivial partial order $(\mathbf{R}, \leq)$ is neither a lattice according to the following.

Theorem I.2.2 (Lachlan (1966) and Yates (1966)). *There are incomparable r.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *with no infimum.*

On the other hand there do exist pairs of r.e. degrees having infima.

Theorem I.2.3 (Lachlan (1966) and Yates (1966)). *There are incomparable r.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

Degrees having $\mathbf{0}$ as infima with other degrees are called *cappable* degrees. The set of cappable degrees is denoted by $\mathbf{M}$, and its complement $\mathbf{NC} = \mathbf{R} \setminus \mathbf{M}$. $\mathbf{M}$ and $\mathbf{NC}$ together form an algebraic decomposition of $\mathbf{R}$.

Theorem I.2.4 (Ambos-Spies et al. (1984)). $\mathbf{M}$ *is a prime ideal and* $\mathbf{NC}$ *is a strong ultrafilter.*

Actually Ambos-Spies et al. (1984) gave more insights about $\mathbf{NC}$. To understand these insights we need more notions.

Given a set X and an enumeration of X-recursive functions we define $K^X = \{e | e \in W_e^X\}$ and denote by $\mathbf{x}'$ the degree of K^X where $\mathbf{x}$ is the degree of X. Then there is a strictly ascending sequence $\mathbf{x} < \mathbf{x}' < \mathbf{x}'' < \ldots$, in particular $\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \ldots$ when $\mathbf{x} = \mathbf{0}$. Inductively we may also define $\mathbf{x}^{(n+1)} = (\mathbf{x}^{(n)})'$. We call K^X and $\mathbf{x}'$ *Turing jumps* or simply jumps of X and $\mathbf{x}$ respectively. Sometimes we also denote K^X

by X'. A degree $\mathbf{a}$ is low_n if and only if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$, and it is $high_n$ if and only if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$. The set of low_n degrees is denoted by $\mathbf{L}_n$ and of $high_n$ degrees by $\mathbf{H}_n$. We write $\mathbf{L}$ and $\mathbf{H}$ respectively when $n = 1$.

A degree $\mathbf{a}$ is *cuppable* if and only if there is a $\mathbf{b} < \mathbf{0}'$ with $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$. If in addition $\mathbf{b}$ can be chosen to be low, then we say that $\mathbf{a}$ is *low cuppable*. Let $\mathbf{NCup}$ denote the complement of the set of cuppable degrees, and $\mathbf{LC}$ the set of low cuppable degrees. It is easy to see that $\mathbf{NCup}$ is also an ideal.

Another notion is related to effective enumeration of r.e. sets. Given an effective enumeration of a set A, we denote by $A[s]$ the finite set of elements enumerated in A up to stage s. Now fix an arbitrary simultaneous enumeration of all r.e. sets, say $\{W_e[s] | e, s \in \omega\}$, we say that a set A is *promptly simple* if and only if A is infinite and co-infinite, and there is a recursive function p and an effective enumeration of A such that

$$W_e \text{ is infinite } \Rightarrow \exists x, s(x \in (W_e[s+1] \setminus W_e[s]) \cap A[p(s)])$$

for all e. We also say that the degree $\mathbf{a}$ represented by A is promptly simple if A is so, and denote by $\mathbf{PS}$ the set of promptly simple degrees.

Now we are ready to introduce the following.

Theorem I.2.5 (Ambos-Spies et al. (1984)). $\mathbf{NC} = \mathbf{PS} = \mathbf{LC}$.

I.3 A Little Model Theory

For basic model theoretic concepts we refer to Marker (2002) or Nerode and Shore (1997). But we will explain notions which are critical to understand definability results in degree theory.

Given a language $\mathcal{L}$ and $\mathcal{L}$-structures $\mathcal{N} \subseteq \mathcal{M}$, $\mathcal{N}$ is a Σ_1*-elementary substructure of* $\mathcal{M}$, denoted by $\mathcal{N} \preceq_1 \mathcal{M}$, if and only if for every Σ_1-formula $\phi(x_0, \ldots, x_n)$ and $a_0, \ldots, a_n \in \mathcal{N}$,

$$\mathcal{N} \models \phi[a_0, \ldots, a_n] \Leftrightarrow \mathcal{M} \models \phi[a_0, \ldots, a_n].$$

If the above holds for all first order formulae then $\mathcal{N}$ is an *elementary substructure of* $\mathcal{M}$, or $\mathcal{N} \preceq \mathcal{M}$ for short.

If $A \subseteq \mathcal{M}^k$ for some k, $\phi(p_1, \ldots, p_m, x_1, \ldots, x_k)$ and $a_1, \ldots, a_m$ are such that for all $b_1, \dots b_k \in \mathcal{M}$,

$$(b_1, \dots, b_k) \in A \Leftrightarrow \mathcal{M} \models \phi[a_1, \dots, a_m, b_1, \dots, b_k]$$

then we say that A is *definable via ϕ in $a_0, \dots, a_m$*.

For another language $\mathcal{L}'$ and an $\mathcal{L}'$-structure $\mathcal{M}'$, $\mathcal{M}'$ is *interpretable in $\mathcal{M}$* if and only if there is a set of parameters $\overline{a} \in \mathcal{M}$ such that there are

1. a subset of $\mathcal{M}^k$ for some k, say A, definable from $\overline{a}$,
2. an equivalent relation E on A definable from $\overline{a}$,
3. for each n-ary relation symbol $R \in \mathcal{L}'$ an n-ary relation ϕ_R on A which is E-invariant and definable from $\overline{a}$,
4. for each n-ary function symbol $F \in \mathcal{L}'$ an n-ary relation ψ_F on A which is actually an n-ary function on A/E and definable from $\overline{a}$, and
5. a bijection $f : \mathcal{M}' \to A/E$ which is an isomorphism if we take A/E as an $\mathcal{L}'$-structure with all symbols interpreted appropriately according to the above.

A, E, ϕ_R's and ψ_F's constitute an *interpretation* of $\mathcal{M}'$ in $\mathcal{M}$. When $\mathcal{L}'$ consists of only finitely many non-logical symbols the parameters can be finite, and the formulae defining A, E etc. form a *scheme*. We also say that $\overline{a}$ *codes* or *defines* a copy of $\mathcal{M}'$ in $\mathcal{M}$.

It is easy to see that interpretations are just generalizations of quotient structures in algebra, except that we require things definable.

Chapter II

Ideals

II.1 Ideals as Substructures

Theorem II.1.1 (Ding et al. (2005))**.** *Every nonprincipal ideal is a Σ_1 elementary substructure of $\mathcal{R}$.*

Proof. Fix $\mathbf{I}$ a nonprincipal ideal of $\mathcal{R}$. Following the analysis in the previous section, it suffices to prove that for any finite partial order P and an embedding $f : P \to \mathcal{R}$, there exists an embedding $g : P \to \mathbf{I}$ with g^{-1} extending $f^{-1} \restriction \mathbf{I}$.

Let $x_1, \ldots, x_n$ be an enumeration of $P^- = f^{-1}(\mathbf{I})$, $\mathbf{a}_i = f(x_i)$ for $i(0 < i \leq n)$ and $\mathbf{a}_0 = \mathbf{0}$. To extend f, define $g(x_i) = f(x_i)$ for $i \leq n$.

Since $\mathbf{I}$ is nonprincipal we can find some $\mathbf{c} \in \mathbf{I}$ such that $\mathbf{c} > \bigvee_{i \leq n} \mathbf{a}_i$. We need a technical lemma.

Lemma II.1.2. *There exist an independent sequence of degrees in $[\mathbf{0}, \mathbf{c}]$, say $\langle \mathbf{b}_{i,k} : i \leq n, k \in \omega \rangle$, such that $\mathbf{a}_i \leq \mathbf{b}_{i,k}$, and for any finite $H \subset \{0, 1, \ldots, n\} \times \omega$,*

$$\mathbf{a}_i \not\leq \bigvee_{j \in H_0} \mathbf{a}_j \Rightarrow \mathbf{a}_i \not\leq \bigvee_{\langle j,k \rangle \in H} \mathbf{b}_{j,k} \tag{II.1}$$

where $H_0 = \{j : \exists k(\langle j, k \rangle \in H)\}$.

Proof. This lemma is essentially a generalization of Sacks Density Theorem.

Assume $A_i \subseteq \omega^{[2i]}$ is a c.e. representative of $\mathbf{a}_i$ for $i \leq n$ and C is a c.e. representative of $\mathbf{c}$. Let $A = \bigcup_{i \leq n} A_i$. We construct pairwise disjoin c.e. sets $\langle B_{i,k} : i \leq n, k \in \omega\rangle$ such that $B_{i,k} \leq_{\mathrm{T}} C$. Let $B = \bigcup_{i \leq n, k \in \omega} B_{i,k}$, we make $A \cap B = \emptyset$.

To make the sequence independent, for $m \in \omega$, $\langle i, k\rangle \in (n+1) \times \omega$ and finite $H \subset (n+1) \times \omega$, we make

$$\mathcal{P}_e : B_{i,k} = \Phi_m(\bigcup_{j \in H_0} A_j \cup \bigcup_{\langle j,l\rangle \in H} B_{j,l}) \Rightarrow C \leq_{\mathrm{T}} A$$

where e is an index of the requirement under some effective encoding of all possible combinations.

To make (II.1), for i and H satisfying the left hand side of (II.1) and m, we make

$$\mathcal{N}_e : A_i = \Psi_m(\bigcup_{j \in H_0} A_j \cup \bigcup_{\langle j,l\rangle \in H} B_{j,l}) \Rightarrow A_i \leq_{\mathrm{T}} \bigcup_{j \in H_0} A_j$$

where e is again an index under analogues settings. Note that there are only finitely many such pairs (i, H_0).

We employee the trick of *true stages computation* where true stages are uniformly defined as true stages of the effective enumeration of $A \cup B$ arisen in the construction.

The strategy for a single $\mathcal{N}_e$ is to preserve agreements between A_i and Ψ_m by imposing restraints on B. Strategies serving less prior requirements are required to respect the restraints.

The strategy for a single $\mathcal{P}_e$ is Sacks Coding with a slight modification. To be precise, at stage $s+1$, we enumerate $\langle x, t, 2e+1\rangle$ in $B_{i,k}$ iff $x \in C_{s+1}$, $\langle x, t, 2e+1\rangle$ is not restrained from entering B by prior requirements and $(\forall v)(t \leq v \leq s \rightarrow x < l^{\Phi}(e, v))$ where l^{Φ} is the length of agreements defined as usual.

In addition, we impose a restraint on B to protect the computation $\Phi_m \upharpoonright l^{\Phi}(e, s)$. This is the only modification comparing to Sacks Coding strategy. Since $C \not\leq_{\mathrm{T}} A$, this restraint imposed for P_e will drop at true stages.

The verifications go essentially in the same way as those for Sacks Density Theorem.

Finally let $\mathbf{b}_{i,k} = \mathbf{deg}(A_i \cup B_{i,k})$. □

Let us return to the proof of Theorem II.1.1. Fix $\langle \mathbf{b}_{i,k} : i \leq n, k \in \omega\rangle$ as in the above lemma. Let $y_1, \ldots, y_m$ enumerate $P \backslash P^-$, define

$$g(y_k) = \bigvee\!\!\bigvee \{\mathbf{b}_{i,l} : x_i <_P y_k \text{ and } y_l \leq_P y_k\}.$$

Immediately, if $u \leq_P y_k$ then $g(u) \leq g(y_k)$.

Assume $u \not\leq_P y_k$. If u is some x_i, then since f is an embedding, $\mathbf{a}_i \not\leq \bigvee\!\!\bigvee_{j \in H_0} \mathbf{a}_j$ where $H_0 = \{j \leq n : x_j < y_k\}$. Hence (II.1) and the definition together imply that $g(u) \not\leq g(y_k)$.

If u is some y_l, then the independence of the sequence $\langle \mathbf{b}_{i,k} : i \leq n, k \in \omega \rangle$ implies that again $g(u) \not\leq g(y_k)$.

Hence g is the desired embedding. □

Remark 1. *Slaman observed that Lemma II.1.2 could be deduced algebraically from the main result in Slaman and Soare (2001). But it might not yield a shorter proof.*

Although no principal ideal could be a Σ_1 elementary substructure since the predicate x *is not maximal* is Σ_1 definable in $\mathcal{R}$, the following follows immediately from the proof above.

Proposition II.1.3. *For* $\mathbf{c}$ *r.e.,* $\varphi \in \Sigma_1$ *and* $\overrightarrow{\mathbf{a}} \in [\mathbf{0}, \mathbf{c}]$ *such that* $\cup \overrightarrow{\mathbf{a}} < \mathbf{c}$*,*

$$\mathcal{R} \models \varphi[\overrightarrow{\mathbf{a}}] \Leftrightarrow [\mathbf{0}, \mathbf{c}] \models \varphi[\overrightarrow{\mathbf{a}}].$$

On the contrary, Σ_2 elementary substructures always contain $\mathbf{0}'$ since the fact *there exists a greatest element* is Σ_2 in $\mathcal{R}$ (plus the remark before the proposition). Hence *no* proper ideal could be a Σ_2 elementary substructure.

II.2 The Definable Ideal Generated by Plus-cupping Degrees

A strong version of plus-cupping degrees were introduced by Harrington, later Fejer and Soare isolated a technique in Harrington's construction and introduced a weak version of plus-cupping degrees. Here we follow Fejer and Soare's definition.

Definition II.2.1 (Harrington (1978), Fejer and Soare (1981)). *An r.e. degree* $\mathbf{a}$ *is* plus-cupping *if and only if for every nonrecursive* $\mathbf{b} \leq \mathbf{a}$*, there is an incomplete r.e. degree* $\mathbf{c}$ *such that* $\mathbf{b} \vee \mathbf{c} = \mathbf{0}'$*. We denote the class of plus-cupping degrees by* $\mathbf{PC}$*.*

As remarked by Li and Li (2003), the typical plus cupping constructions resemble those of so called nonbounding degrees to some extent.

Definition II.2.2 (Lachlan (1979)). *An r.e. degree* $\mathbf{a}$ *is* nonbounding *if and only if* $\mathbf{a} > \mathbf{0}$ *and there is no minimal pair below* $\mathbf{a}$*. We denote the class of nonbounding degrees by* $\mathbf{NB}$*.*

However these two notions are different.

Theorem II.2.3 (Li and Li (2003)). $\mathbf{PC} - \mathbf{NB} \neq \emptyset$.

In addition A. Li and Y. Zhao proved the following.

Theorem II.2.4 (Li and Zhao (2004)). *Plus cupping degrees do not form an ideal.*

Based on these facts, A. Li and Yang asked the following question.

Problem II.2.5 (Li and Yang (2003)). *Is* $(\mathbf{PC}]$ *different from* $(\mathbf{NB}]$*?*

We answer this question affirmatively. Actually we will prove a stronger result that $(\mathbf{PC}]$ is a proper subideal of $\mathbf{M}$ and not contained by $(\mathbf{NB} \cup \mathbf{NCup}]$. For this sake, in section 3 we will prove that $\mathbf{NCup}$ is not a subset of $(\mathbf{PC}]$, hence $(\mathbf{PC}]$ is a proper subideal of $\mathbf{M}$; while in section 4, we will prove that $(\mathbf{PC}]$ is not contained by $(\mathbf{NB} \cup \mathbf{NCup}]$. [1]

II.3 $\mathbf{NCup} \not\subseteq [\mathbf{PC}]$

Theorem II.3.1. *There is a noncuppable c.e. degree* $\mathbf{a} \notin [\mathbf{PC}]$.

We prove Theorem II.3.1 by constructing a c.e. set A such that $\mathbf{deg}(A) \in \mathbf{NCup}$ and $\mathbf{deg}(A) \notin [\mathbf{PC}]$.

[1]The results in this section is contained in Wang and Ding (2005)

To make A noncuppable, fix a computable enumeration $(\Phi_e, W_e)_{e\in\omega}$ of c.e. functionals and c.e. sets, we build an additional c.e. set D such that for all e

$$\mathcal{M}_e : D = \Phi_e(A, W_e) \Rightarrow K \leq_{\mathrm{T}} W_e$$

To make $\mathbf{deg}(A) \notin [\mathbf{PC}]$, fix a computable coding of $\bigcup_{1<n<\omega} \omega^n$. For e let $\|e\|$ denote the unique c such that e codes an element, say z, of ω^{c+1}; and let e_i denote the i-th element of z. Fix $(\Psi_{e_{\|e\|}}, B_{e_0}, B_{e_1}, \ldots B_{e_{\|e\|-1}})_{e\in\omega}$, we satisfy the followings requirements for all e

$$\mathcal{P}_e : A = \Psi_{e_{\|e\|}}(B_e) \Rightarrow (\exists i < \|e\|)(B_{e_i} \text{ is not plus cupping})$$

where B_e is the abbreviation of $(B_{e_0}, B_{e_1}, \ldots, B_{e_{\|e\|-1}})$.

We arrange the construction on a tree T of strategies growing upward. Every finite path of the tree is an $\mathcal{X}$-strategy for some requirement $\mathcal{X}$. We will gradually define the set of outcomes and assign a computable linear ordering to this set. Thus we can order strategies on T lexicographically. Denote the order by $<_L$, if $\alpha <_L \beta$ or $\alpha \subset \beta$ are strategies on T then we say $\alpha < \beta$. We also say that α *is to the left of* β or β *is to the right of* α if $\alpha <_L \beta$.

At each stage s we will define a finite approximation TP_s to the *true path* TP of the construction. TP_s will be the union of *accessible* strategies at s, and only accessible strategies are allowed to act at each stage.

II.3.1 $\mathcal{M}$-strategies

Suppose α is an $\mathcal{M}_e$-strategy. We define l^α the length of agreement between D and $\Phi(A, W)$ and α-expansionary stages as usual.

α has two outcomes ∞ (if there are infinitely many α-expansionary stages) and 0 (if there are at most finitely many).

If there are infinitely many expansionary stages, α builds a functional Θ^α such that for all k

$$\mathcal{N}_k^\alpha : D = \Phi_e(A, W_e) \Rightarrow K(k) = \Theta^\alpha(W_e; k).$$

To satisfy $\mathcal{N}_k^\alpha$ and define $\Theta^\alpha(W_e;k)$, we arrange $\mathcal{N}_k^\alpha$-strategies above $\alpha\hat{}\infty$. From now on in this subsection, we occasionally omit α from superscripts.

Suppose $\beta \supseteq \alpha\hat{}\infty$ is an $\mathcal{N}_k^\alpha$-strategy. At the beginning, β picks a *flip point* $d^\beta(k)$ of k and keeps it from entering D. We may write d for $d^\beta(k)$.

If the computation $\Phi_e(A, W_e; d)$ changes infinitely often, β will have $\perp$ as outcome indicating that $\Phi_e(A, W_e; d)$ diverges. In this case, we arrange no more $\mathcal{N}^\alpha$-strategies above $\beta\hat{}\perp$ since $D \neq \Phi_e(A, W_e)$.

Otherwise β has $\top$ as outcome and defines $\Theta(W_e;k) = K(k)$ with $\theta(k) > \phi_e(d)$. In addition, β expects that $A \upharpoonright \phi_e(d)$ changes no longer.

If k is enumerated in K later, β enumerates d in D, then either β establishes a disagreement between D and $\Phi_e(A, W_e)$, or $W_e \upharpoonright \phi_e(d)$ eventually changes and β can safely change the definition of $\Theta(W_e;k)$ to 1. So the key to β's success is the inequality

$$\theta(k) > \phi_e(d).$$

We say that $\theta(k)$ *is honest* if this inequality holds.

II.3.2 $\mathcal{P}$-strategies

Suppose τ is a $\mathcal{P}_e$-strategy. We define l^τ the length of agreement between D and $\Phi(A, W)$ and τ-expansionary stages as usual.

τ has two outcomes ∞ (if there are infinitely many τ-expansionary stages) and 0 (if there are at most finitely many). If there are infinitely many expansionary stages, τ builds $\|e\|$ many c.e. sets $(C_0^\tau, C_1^\tau, \dots, C_{\|e\|-1}^\tau)$ such that $C_i^\tau \leq_\mathrm{T} B_{e_i}$ for $i < \|e\|$,

$$\mathcal{Q}_n^\tau : (\exists i < \|e\|)(C_i^\tau \neq \overline{W}_{n_i}) \text{ for } \|n\| = \|e\| - 1,$$

and for $(i, j) \in \|e\| \times \omega$

$$\mathcal{R}_{i,j}^\tau : D = \Phi_j(C_i^\tau, W_j) \Rightarrow K \leq_\mathrm{T} W_j.$$

To satisfy $\mathcal{Q}_{i,j}^\tau$ and $\mathcal{R}_{i,j}^\tau$, we arrange $\mathcal{Q}_{i,j}^\tau$- and $\mathcal{R}_{i,j}^\tau$-strategies above $\tau\hat{}\infty$. From now on in this subsection, we may occasionally omit τ from superscripts.

Suppose $\alpha \supseteq \tau\hat{}\infty$ is an $\mathcal{R}^{\tau}_{i,j}$-strategies, α acts in the same way as an $\mathcal{M}_e$-strategy described in the previous subsection. α has two outcome ∞ (indicating there are infinitely many α-expansionary stages) and 0 (indicating there are at most finitely many), and builds a functional Θ^{α} such that for all k

$$\mathcal{S}^{\alpha}_k : K = \Phi_j(C_i, W_j) \Rightarrow K_0(k) = \Theta^{\alpha}(W_j; k).$$

To satisfy $\mathcal{S}^{\alpha}_k$, we arrange $\mathcal{S}^{\alpha}_k$-strategies above $\alpha\hat{}\infty$. $\mathcal{S}^{\alpha}_k$-strategies act in the same way as $\mathcal{N}$-strategies above $\mathcal{M}$-strategies.

To make $\mathcal{Q}_n$, assume $\sigma \supseteq \tau\hat{}\infty$ is a $\mathcal{Q}_n$-strategy, at the beginning σ picks an *agitator* a so that $l^{\tau} > a$, and keeps a from entering A. If $B_e \upharpoonright \psi_e(a)$ changes infinitely often, σ has $\perp$ as its outcome indicating that $\Psi_e(B_e; a)$ diverges. Otherwise, σ will eventually fix a *witness* x. If x is never enumerated in W_{n_i} for some $i < \|e\|$, σ has 0 as its outcome. In this case $\mathcal{Q}_n$ is satisfied since $\overline{W}_{n_i} - C_i$ is not empty.

Otherwise at some stage $x \in W_{n_i}$ for all $i < \|e\|$, σ enumerates a in A. If the assumption $A = \Psi_e(B_e)$ is true, then B_{e_i} changes for some i before $A(a) = \Psi_e(B_e; a)$ is established again. We enumerate x in C_i for the least such i. In this case, σ has 1 as its outcome, $\mathcal{Q}_n$ is also satisfied since $C_i - \overline{W}_{n_i}$ is not empty.

Note that in the above paragraph we use the trick of *permitting* to make $C_i \leq_{\mathrm{T}} B_{e_i}$. But in the presence of other strategies we shall in addition use *permitting at τ-expansionary stages* and *links*. On the one hand τ will build a local version of effective enumeration of B, i.e., $B^{\tau}[s] = B[s_0]$ where $s_0 \leq s$ is the latest stage when τ is accessible and $\{B[s] | s \in \omega\}$ is some standard enumeration. The computation $\Psi_e(B_e)$ is also localized, i.e., (for τ and its substrategies) it could change only if τ is accessible. Hence it suffices to capture B changes as above according to these localizations. From now on we may identify these localizations with the standard ones. On the other hand when σ enumerates its agitator in A, it additionally setups a link (τ, σ). At the next τ-expansionary stage the control is passed to σ immediately so that it can catch permission in time. Then we say that *the link (τ, σ) is traveled* and cancel this link immediately.

We will not arrange any $\mathcal{Q}_{n'}$-strategies above $\sigma\hat{}\perp$.

Before proceeding we summarize and order outcomes defined so far,

$$\infty <_\Lambda 1 <_\Lambda 0 <_\Lambda \bot <_\Lambda \top.$$

II.3.3 Coordinating different strategies

Since $\mathcal{Q}$-strategies may enumerate their agitators in A while $\mathcal{N}_{e,k}$-strategies expect that $A \upharpoonright \phi(d(k))$'s will never change after $\Theta(W;k)$'s are defined, conflicts arise. We say that a *threatens the honesty of* $\theta(k)$ if $a \leq \phi(d(k))$.

The technique to solve these conflicts is originally developed by Li et al. (nfty) and then applied by Yu and Yang (2005). However we will give a slightly different formulation and hope that the behaviors of flip points could be made clearer.

On the one hand, whenever an $\mathcal{N}$-strategy β defines $\Theta(W;k)$, strategies $\geq \beta\hat{\ }\top$ are initialized.

On the other hand, the situation is a little more complicated. Suppose τ is some $\mathcal{P}_{e'}$-strategy, σ is a $\mathcal{Q}^\tau$-strategy and $\alpha_0 \subset \alpha_1 \subset \ldots \subset \alpha_{n-1}$ are $\mathcal{M}$-strategies with $\alpha_i\hat{\ }\infty \subseteq \sigma$ ($i < n$). When σ intends to put its agitator $a = a^\sigma$ in A, it first tries to cancel $\theta^{\alpha_i}(k)$'s whose honesties are threatened by a. σ will do this in descending order, i.e. it first tries to cancel $\theta^{\alpha_{n-1}}(k)$'s, then $\theta^{\alpha_{n-2}}(k)$'s and so on.

Now assume $\alpha = \alpha_{n-1}$ is some $\mathcal{M}_e$-strategy. At stage s_0, $\Theta(W;k)[s]$ becomes defined by some $\mathcal{N}_k^\alpha$-strategy β, and at $s \geq s_0$ σ intends to enumerate a in A.

If $\beta < \sigma$ or a^σ is chosen after s_0 then we can easily make $a^\sigma > \phi_e(d^\beta(k))$. Otherwise, in general σ enumerates $d^\beta(k)$ in D to force $W_e \upharpoonright \phi_e(d^\beta(k))$ change. If $W_e \upharpoonright \phi_e(d^\beta(k))$ never changes, then a disagreement between $\Phi_e(A, W_e)$ and D is established; otherwise $\Theta^\alpha(W_e;k)$ diverges eventually and the enumeration of a^σ in A will not harm the intention to make $\Theta^\alpha(W_e;k) = K(k)$.

But there is a special case. Assume there is another $\mathcal{N}_k^\alpha$-strategy $\gamma \subset \gamma\hat{\ }\top \subseteq \sigma$ (then $\gamma <_L \beta$). In this case a might threaten the honesty of $\theta(k)$ according to β. For example, assume γ chooses $d^\gamma(k)$ at t_0, at $t_1 > t_0$ σ chooses its agitator a and $\theta(k)$ is already defined by β (between t_0 and t_1, thus $d^\gamma(k) < d^\beta(k)$), but at some $t_2 > t_1$ we might have $\theta(k)$ redefined by β and $\phi_e(d^\beta(k))[t_2] > a$ but $\phi_e(d^\gamma(k))$ never moves. If σ acts at $t > t_2$ it might find $\phi_e(d^\gamma(k))[t] = \phi_e(d^\gamma(k))[t_0] < a < \phi_e(d^\beta(k))[t] < \theta(k)[t]$.

If this is the case for infinitely many $\mathcal{Q}$-strategies above $\gamma\hat{\ }\top$ and these $\mathcal{Q}$-strategies cancel $\theta(k)$ as described, then $\Theta^\alpha(W_e; k)$ diverges even though $\gamma\hat{}\top$ might be on the true path.

To overcome this difficulty, first, we will allow σ above $\gamma\hat{}\top$ to change $A \upharpoonright \phi_e(d^\beta(k))$ freely. Second, we will make $a^\sigma > \phi_e(d^\gamma(k))$. Assume this is achieved. If later some other strategy wants to cancel $\Theta^\alpha(W_e; k)$, it can enumerate $d^\gamma(k)$ (instead of $d^\beta(k)$) in D.

To keep track of $d^\gamma(k)$ we introduce a new parameter $d^\alpha(k)$, called *the official flip point of* α *and* k, and assign it to α. From now on we measure the honesty of $\theta(k)$ according to this official flip point, instead of $d^\gamma(k)$ or $d^\beta(k)$. We then call $d^\gamma(k)$ *the personal flip point of* γ. Whenever $\gamma\hat{}\top$ is accessible, the official point is defined to be the personal flip point of γ. Furthermore if later W_e changes below $\theta^\alpha(k)$ but not $\phi_e(d^\alpha(k))$ then $\theta^\alpha(k)$ will not be changed. This guarantees that $\Theta^\alpha(W_e; k)$ converges.

Note that when trying to cancel $\theta(k)$'s, σ expects no new $\theta(k')$'s defined, otherwise it might be trapped in endless loops. To this end σ setups a *link* (α, σ) when it initials the above process. At the next α-expansionary stage no $\theta^\alpha(k)$'s are threatened by a, (α, σ) will be traveled and canceled, then σ will proceed to $\mathcal{M}$-strategies below α.

However $\psi_{e'}(a)$ might become $\geq x^\sigma$ when σ is waiting for the link (α, σ) to be traveled. If this happens σ will discard the current x^σ. If σ chooses infinitely many witnesses then it might try to cancel $\theta(k)$ infinitely often, and that would cause $\Theta^\alpha(W_e; k)$ to diverge. But note that then $\Psi_{e'}(B_{e'}; a)$ also diverges. Moreover, this could not happen if $\tau \supset \alpha$, for τ will be *covered by the link* (α, σ) (i.e. $\alpha \subset \tau \subset \sigma$) when σ is waiting for the link to be traveled, and thus the local computation at of $\Psi^\tau(B_{e'}; a^\sigma)$ will not change until τ is accessible again.

Hence we could just arrange a backup strategy α' for α above $\sigma\hat{}\bot$ if $\tau \subset \alpha \subset \sigma$. We will only arrange $\mathcal{N}^{\alpha'}$-strategies but no $\mathcal{N}^\alpha$-strategies above $\sigma\hat{}\bot$. We also backup those $\mathcal{P}$-strategies between τ and σ to guarantee that eventually this backup operation for $\mathcal{M}_e$ will cease. Moreover strategies above $\sigma\hat{}\bot$ will consider α *injured* by σ.

Note that there are similar conflicts between $\mathcal{S}^\tau$-strategies and σ. We apply the same technique to solve these conflicts, and remark that if eventually σ setups a link (τ, σ) then $\mathcal{S}^\tau$-strategies will not act before this link is traveled and x^σ enters some C.

Now we formally describe procedures for $\mathcal{N}$- and $\mathcal{Q}$-strategies.

Let

$$s_0 = \max\{s' < s : d^\beta[s] = d^\beta[s'] \text{ and } \beta \text{ is accessible at } s'\}$$

and

$$s_1 = \max\{s' < s : \theta(k)[s'] \text{ is defined }\}.$$

Procedure II.3.2. *Suppose that β is an $\mathcal{N}_{e,k}$-strategy and $\alpha = top(\beta)$. At stage s, β cancels $d^\beta(k)$ if $d^\beta(k) \in D$ and then acts step by step as followings.*

1. *If $d^\beta(k)$ is undefined, define it to be fresh.*
2. *If $d^\beta(k) > l$, do nothing and stop.*
3. *If $d^\beta(k) \leq l$, take the following actions*
   (a) *If s_0 is defined and the computation $\Phi_e(A, W_e; d^\beta(k))[s]$ is different from that at s_0, let $\perp$ be the outcome.*
   (b) *From now on assume (a) fails. If $\Theta(W_e; k)$ diverges, define $\Theta(W_e; k) = K(k)$ with $\theta(k) = \theta(k)[s_1]$ if $d^\alpha(k)$ is defined, or $\theta(k)$ fresh otherwise.*
   (c) *Let $d^\alpha(k) = d^\beta(k)$ if either $d^\alpha(k)$ is undefined or $d^\beta(k) < d^\alpha(k)$.*
   (d) *If $\Theta(W_e; k) \neq K(k)$ then enumerate $d^\alpha(k)$ in D, cancel $d^\beta(k)$ and stop; otherwise let $\top$ be the outcome.*

We assign states $\{1, c, w, \perp, 0\}$ and a parameter $state(\sigma)$ for σ.

Let $s_0 = \max\{s' < s : a^\sigma[s] = a^\sigma[s_0]$ and σ is accessible at $s'\}$.

Procedure II.3.3. *At the beginning of stage s, σ picks a fresh agitator a^σ if a^σ is undefined, and takes actions according to the following cases.*

1. *$state(\sigma) = \perp$. If $a^\sigma \leq l^\tau$, pick x^σ fresh, and let $state(\sigma) = 0$.*
2. *$state(\sigma) = 0$.*
   (a) *If $B \upharpoonright \psi(a^\sigma)[s] \neq B \upharpoonright \psi(a^\sigma)[s_0]$, let $state(\sigma) = \perp$ and cancel x^σ.*
   (b) *If $B \upharpoonright \psi(a^\sigma)[s] = B \upharpoonright \psi(a^\sigma)[s_0]$ and $x^\sigma \notin \bigcap_{i \leq \|n\|} W_{n_i}$, do nothing.*
   (c) *Both (a) and (c) fail, let $state(\sigma) = w$ and take the actions in (3) immediately.*
3. *$state(\sigma) = w$ (w* for waiting*).*
   (a) *If $\psi(a^\sigma) \geq x^\sigma$, cancel x^σ, let $state(\sigma) = \perp$;*

(b) *If (a) fails and there exist* α *and* k *such that* α *is some* $\mathcal{M}$*- or* $\mathcal{R}$*-strategy not injured so far,* $\alpha\hat{\ }\infty \subseteq \sigma$, $\min\{a^\sigma, x^\sigma\} < \phi^\alpha(d^\alpha(k))$ *and* $\Theta^\alpha(W^\alpha; k) = 0$, *choose* $\alpha(\sigma)$ *as the longest such* α *and* $k(\sigma)$ *as the least such* k *with respect to* $\alpha(\sigma)$*, enumerate* $d^{\alpha(\sigma)}(k(\sigma))$ *in* D*, setup a link* $(\alpha(\sigma), \sigma)$*;*

(c) *If both (a) and (b) fail, enumerate* a^σ *in* A *and setup a link* (τ, σ) *and let* $state(\sigma) = c$.

4. $state(\sigma) = c$. *Let* i_0 *be the least* $i < \|e\|$ *such that* $B_{e_i} \upharpoonright \psi(a^\sigma)[s] = B_{e_i} \upharpoonright \psi(a^\sigma)[s_0]$*, enumerate* x^σ *in* $C^\tau_{i_0}$ *and let* $state(\sigma) = 1$.
5. $state(\sigma) = 1$. *Do nothing.*

If $state(\sigma) \in \{w, c\}$ or (4) happens, then σ has no outcome; otherwise σ has $state(\sigma)$ as outcome.

II.3.4 The tree of strategies

We may consider $\mathcal{N}^\alpha_k$ as subrequirement $\mathcal{N}_{e,k}$ of $\mathcal{M}_e$ where α is an $\mathcal{M}_e$-strategy, $\mathcal{Q}^\tau_n$ and $\mathcal{R}^\tau_{i,j}$ as subrequirements $\mathcal{Q}_{e,n}$ and $\mathcal{R}_{e,i,j}$ of $\mathcal{P}_e$ where τ is a $\mathcal{P}_e$-strategy, and $\mathcal{S}^\eta_k$ as subrequirement $\mathcal{S}_{e,i,j,k}$ of $\mathcal{R}_{e,i,j}$ where η is an $\mathcal{R}^\tau_{i,j}$ strategy and τ is as above. Hence we may regard Θ^α's, C^τ_i's and Θ^η's as local versions of Θ_e's, $C_{e,i}$'s and $\Theta_{e,i,j}$'s.

Fix a computable bijection f mapping ω onto the collection of all requirements such that

1. $f^{-1}(\mathcal{M}_e) < f^{-1}(\mathcal{N}_{e,k})$;
2. $f^{-1}(\mathcal{P}_e) < f^{-1}(\mathcal{Q}_{e,n}), f^{-1}(\mathcal{R}_{e,i,j})$;
3. $f^{-1}(\mathcal{R}_{e,i,j}) < f^{-1}(\mathcal{S}_{e,i,j,k})$.

Let Λ denote the ordered alphabet set $\{\infty <_\Lambda 1 <_\Lambda 0 <_\Lambda \bot <_\Lambda \top\}$.

We define the tree of strategies $T \subset \Lambda^{<\omega}$ inductively.

Suppose $\xi \in T$. If ξ is an $\mathcal{N}_{e,k}$- ($\mathcal{Q}_{e,n}$-, $\mathcal{R}_{e,i,j}$- or $\mathcal{S}_{e,i,j,k}$-) strategy, let $top(\xi)$ be the longest $\eta \subset \xi$ which is an $\mathcal{M}_e$- ($\mathcal{P}_e$-, $\mathcal{P}_e$-, or $\mathcal{R}_{i,j}$-) strategy. We say η is *injured* at ξ if $\eta \subset \xi$ and either

1. η is an $\mathcal{M}_e$- or $\mathcal{P}_e$-strategy and there are μ and ν such that $\mu\hat{\ }\infty \subseteq \eta \subset \nu\hat{\ }\bot \subseteq \xi$, ν is some $\mathcal{Q}$-strategy and $\mu = top(\nu)$; or

2. $top(\eta)$ is defined and injured at ξ.

Suppose $\mathcal{X}$ is a requirement, let $\mathcal{X}(\xi)$ be the longest $\mathcal{X}$-strategy $\zeta \subseteq \xi$ not injured at ξ, or undefined if there is no such strategy. $\mathcal{X}$ is *finished* at ξ if one of the following cases applies

1. $\mathcal{X}$ is an $\mathcal{M}_e$ or $\mathcal{R}_{e,i,j}$, and either $\alpha = \mathcal{X}(\xi)$ is defined and $\alpha\hat{\ }0 \subseteq \xi$, or there is some $\mathcal{Y} = \mathcal{N}_{e,k}$ or $\mathcal{S}_{e,i,j,k}$ such that $\beta = \mathcal{Y}(\xi)$ is defined and $\beta\hat{\ }\bot \subseteq \xi$;
2. $\mathcal{X}$ is a $\mathcal{P}_e$, and either $\tau = \mathcal{X}(\xi)$ is defined and $\tau\hat{\ }0 \subseteq \xi$, or there is some $\mathcal{Q}_{e,n}$ such that $\sigma = \mathcal{Q}_{e,n}(\xi)$ is defined and $\sigma\hat{\ }\bot \subseteq \xi$;
3. $\mathcal{X}$ is an $\mathcal{N}_{e,k}$ or $\mathcal{S}_{e,i,j,k}$ and $\mathcal{M}_e$ or $\mathcal{R}_{e,i,j}$ is finished at ξ;
4. $\mathcal{X}$ is a $\mathcal{Q}_{e,n}$ or $\mathcal{R}_{e,i,j}$, and $\mathcal{P}_e$ is finished at ξ.

Otherwise $\mathcal{X}$ is *unfinished* at ξ. Furthermore, $\mathcal{X}$ is *satisfied* at ξ if either $\mathcal{X}(\xi)$ is defined or $\mathcal{X}$ is finished at ξ. Otherwise $\mathcal{X}$ is *unsatisfied* at ξ.

Label ξ with the $\mathcal{X}$ such that $f^{-1}(\mathcal{X})$ is the least among the unsatisfied ones, and

1. If $\mathcal{X}$ is some $\mathcal{M}$, $\mathcal{P}$ or $\mathcal{R}$, let $\xi\hat{\ }\infty, \xi\hat{\ }0 \in T$;
2. If $\mathcal{X}$ is some $\mathcal{N}$ or $\mathcal{S}$, let $\xi\hat{\ }\bot$ and $\xi\hat{\ }\top \in T$;
3. If $\mathcal{X}$ is some $\mathcal{Q}_{e,n}$, let $\xi\hat{\ }\bot, \xi\hat{\ }0, \xi\hat{\ }1 \in T$.

The following properties of T follow immediately from above.

Lemma II.3.4. *Suppose P is an infinite path of T, $\mathcal{X}$ an requirement. Then there is a finite $\xi \subset P$ such that $\mathcal{X}$ is satisfied at η for any finite η such that $\xi \subseteq \eta \subset P$.*

Let ξ_0 be the shortest ξ as in the last lemma. Then either $\mathcal{X}(\xi_0)$ is defined or $\mathcal{X}$ is finished at ξ_0. In the former case, let $\mathcal{X}(P) = \mathcal{X}(\xi_0)$. Moreover, it is obvious that $\mathcal{M}_e(P)$ and $\mathcal{P}_e(P)$ are always defined for any e and P.

We say that a parameter p (or $p[s']$) *becomes defined at stage s* if p is undefined at stage $s-1$ and is defined at stage s (and is never canceled between s and $s' \geq s$), or *becomes undefined* if the reverse happens. And we say that $p[s']$ *becomes defined by ξ at stage s*, if $s < s'$, ξ is accessible at stage s, p becomes defined at the moment that ξ acts and does not become undefined between s and s'. Or we say that $p[s']$ *becomes undefined by ξ at stage s* if the reverse happens.

II.3.5 Parameters and Conventions

We sum up parameters assigned to strategies.

For α an $\mathcal{M}$- or $\mathcal{R}$-strategy, there are

1. The length of agreement l^α;
2. A c.e. functional Θ^α to be built;
3. An official flip point $d^\alpha(k)$ for each k.

For β an $\mathcal{N}_k^\alpha$- or $\mathcal{S}_k^\alpha$-strategy, there is a personal flip point $d^\beta(k)$.

For τ a $\mathcal{P}_e$-strategy, there are

1. The length of agreement l^τ;
2. $\|e\|$ many c.e. sets to be built, namely $C_0^\tau, C_1^\tau, \ldots, C_{\|e\|-1}^\tau$.

For σ a $\mathcal{Q}^\tau$-strategy, there are an agitator a^σ, a witness x^σ and $state(\sigma)$.

Given an arbitrary strategy ξ, if it is initialized then all of its parameters and links with one end being ξ are canceled, i.e. become undefined. But there is an exception, that if ξ is a $\mathcal{Q}$-strategy then $state(\xi)$ is set to be $\perp$.

II.3.6 Construction

Stage 0. Let all c.e. sets and functionals to be constructed be empty, all parameters be undefined and initial states of all $\mathcal{Q}$-strategies are $\perp$.

Stage $s > 0$. Let $\emptyset$ be accessible. Suppose ξ is accessible let $s_0 < s$ be the latest stage such that ξ is accessible at s_0 and never initialized between s_0 and s. We take actions according to the following cases.

Case 1, ξ is an $\mathcal{M}$- or $\mathcal{R}$-strategy.

Subcase 1.1, s is ξ-expansionary.

For each k such that $W \upharpoonright \phi(d^\xi(k))[s] \neq W \upharpoonright \phi(d^\xi(k))[s_1]$ where $s_1 < s$ is the last ξ-expansionary stage, cancel $d^\xi(k)$.

If there is a link (ξ, σ), let σ be accessible and cancel the link. Otherwise let $\xi\hat{\ }\infty$ be accessible.

Subcase 1.2, s is not ξ-expansionary. Let $\xi\hat{\ }0$ be accessible.

Case 2, ξ is a $\mathcal{P}_e$-strategy.

Subcase 2.1, s is ξ-expansionary.

For any $\mathcal{Q}$-strategy σ such that $top(\sigma) = \xi$, $state(\sigma) = w$ and $x^\sigma \le \psi_e(a^\sigma)$, cancel any link with one end being σ.

If there is a link (ξ, σ), let σ be accessible and cancel the link. Otherwise let $\xi\hat{}\infty$ be accessible.

Subcase 2.1, s is not ξ-expansionary. Let $\xi\hat{}0$ be accessible.

Case 3, ξ is a $\mathcal{Q}_{e,n}$-strategy. Let $\tau = top(\xi)$.

If $\bigcup_{i\le\|n\|} C_i^\tau \cap W_{n_i} \neq \emptyset$, let $sate(\xi) = 1$ and $\xi\hat{}1$ be accessible.

Otherwise run Procedure II.3.3. If ξ has no outcome, let $TP_s = \xi$; otherwise let $\xi\hat{}o$ be accessible where o is the outcome.

Case 4, ξ is an $\mathcal{N}$- or $\mathcal{S}$-strategy. Run Procedure II.3.2. If (2)(a), (2)(d) or (3) of Procedure II.3.2 happens, let $TP_s = \xi$; otherwise let $\xi\hat{}o$ be accessible where o is the outcome.

If an outcome o is determined and $\xi\hat{}o = s$, let $TP_s = \xi$. If TP_s is defined, we end stage s immediately by taking the following actions.

(I) If TP_s is some $\mathcal{Q}$-strategy and $state(TP_s) = w$, then initialize all strategies to the right of TP_s.

(II) Otherwise initialize all strategies $> TP_s$.

II.3.7 Verifications

First of all, we study behaviors of flip points.

Lemma II.3.5. *α is an $\mathcal{M}_e$–strategy, β is an $\mathcal{N}_k^\alpha$–strategy extending $\alpha\hat{}\infty$.*
(i) If $\Theta^\alpha(A, W_e; k)[s]$ is defined then $\phi_e(d^\alpha(k))[s] < \theta^\alpha(k)[s]$.
(ii) If σ is some $\mathcal{Q}$-strategy extending $\beta\hat{}\top$, $\beta\hat{}\top$ is accessible at s and $a^\sigma[s]$ (or $x^\sigma[s]$) is defined, then $a^\sigma[s]$ (or $x^\sigma[s]$) $> \phi_e(d^\alpha(k))[s]$.
(iii) If σ is some $\mathcal{Q}$-strategy extending $\beta\hat{}\bot$, $\beta\hat{}\bot$ is accessible at s, $\Theta^\alpha(W_e; k)[s]$ converges and $a^\sigma[s]$ ($x^\sigma[s]$) is defined, then either $a^\sigma[s]$ ($x^\sigma[s]$) $> \phi_e(d^\alpha(k))[s]$ or $d^\alpha(k)[s] > d^\beta(k)[s]$.
(iv) Suppose σ is some $\mathcal{Q}$-strategy $> \beta$. If σ enumerates some d in D at s and $d^\beta(k)[s]$ is defined, then $d > d^\beta[s]$.

Proof. During the proof, we occasionally omit α and β from the superscripts

(i) Let $s_0 \leq s$ be the earliest stage such that $d^\alpha(k)[s_0]$ is defined and never canceled between s_0 and s. Then (i) holds at s_0 by (3)(b) of Procedure II.3.2.

Let $s_0 < s_1 < \ldots < s_n(\leq s)$ be all α-expansionary stages. Assume (i) holds at s_i and let $u_i = \phi_e(d^\alpha(k))[s_i]$.

If $s_i + m < s_{i+1}$ or s and $\Theta(W_e; k)[s_i + m]$ converges then

$$(W_e[s_i + m] - W_e[s_i]) \upharpoonright u_i \subseteq (W_e[s_i + m] - W_e[s_i]) \upharpoonright \theta(k)[s_i] = \emptyset.$$

Moreover $(A[s_i + m] - A[s_i]) \upharpoonright u_i = \emptyset$ because elements in $A[s_i + m] - A[s_i]$ are contributed by strategies $> \alpha\hat{}0$. Hence $\phi_e(d^\alpha(k))[s_i + m] = u_i < \theta(k)[s_i] = \theta(k)[s_i + m]$.

Since $d^\alpha(k)$ is not canceled at s_{i+1}, $(W_e[s_i + m] - W_e[s_i]) \upharpoonright u_i = \emptyset$. Moreover, nothing $\leq u_i$ could be enumerated in A at s_{i+1} and $d^\alpha(k)[s_{i+1}] \leq d^\alpha(k)[s_i]$. Hence (i) holds.

(ii) Let $s_0 \leq s$ be the earliest stage such that $\beta\hat{}\top$ is accessible at s_0 and never initialized between s_0 and s. Then $d^\beta(k)[s] = d^\beta(k)[s_0]$ and $(W_e[s] - W_e[s_0]) \upharpoonright \phi_e(d^\beta(k))[s_0] = \emptyset$. Let $d_0 = d^\beta(k)[s_0]$.

All elements of $A[s] - A[s_0]$ are chosen as agitators of $\mathcal{Q}$-strategies at stages not earlier than s_0 and thus greater than $\phi_e(d_0)[s_0]$. Hence $\phi_e(d_0)[s] = \phi_e(d_0)[s_0] \leq s_0$. Since $a^\sigma[s]$ (or $x^\sigma[s]$) is also chosen at some stage not earlier than s_0 and $d^\alpha(k)[s] \leq d_0$, $a^\sigma[s]$ (or $x^\sigma[s]$) $> s_0 \geq \phi_e(d^\alpha(k))[s]$.

(iii) Let $d = d^\alpha(k)[s]$, s_0 be the earliest stage such that $d^\alpha(k)[s_0] = d$ and $d^\alpha(k)$ is never canceled between s_0 and s, β_0 be an $\mathcal{N}^\alpha_k$-strategy such that $d = d^{\beta_0}(k)[s_0]$ and let $u_0 = \phi_e(d)[s_0]$.

By the choice of s_0 and an argument similar to (i), $\phi_e(d)[s] = u_0$ and $(A, W_e)[s] \upharpoonright u_0 = (A, W_e)[s_0] \upharpoonright u_0$.

If $d^\beta(k)[s] = d$ then $\beta = \beta_0$ and $\beta\hat{}\top$ is accessible at s. This contradicts the assumption of (iii).

If $d^\beta(k)[s] > d$ then $d^\beta[s]$ becomes defined after s_0, and so do $a^\sigma[s]$ (or $x^\sigma[s]$). Hence $a^\sigma[s]$ (or $x^\sigma[s]$) $> u_0 = \phi_e(d)[s]$.

(iv) Let $s_0 \leq s$ be the latest stage at which β is accessible, then $d^\beta(k)[s] =$

$d^\beta(k)[s_0]$. Let α' be some $\mathcal{M}_{e'}$-strategy and k' be such that $d = d^{\alpha'}(k')[s]$. Assume $d^{\alpha'}(k')[s]$ becomes defined at stage $s_1 \le s$ by some $\mathcal{N}^{\alpha'}_{k'}$-strategy β' and is never canceled between s_1 and s, then $d = d^{\beta'}(k')[s_1]$. By an argument similar to (i), $\phi_{e'}(d)[s] = \phi_{e'}(d)[s_1]$.

Suppose $d = d^{\beta'}(k')[s_1] \le d^\beta(k)[s] = d^\beta(k)[s_0]$.

If $\sigma < \beta'$, then $\beta < \beta'$. Thus $d^\beta(k)[s_0]$ and $a^\sigma[s]$ become defined after s_1, and $a^\sigma[s] > \phi_{e'}(d)[s_1] = \phi_{e'}(d)[s]$. Hence σ will not enumerate d in D at s, a contradiction.

If $\beta' <_L \sigma$, then we get a contradiction similar to the previous one.

If $\sigma \supseteq \beta'$, then a contradiction follows from (ii) and (iii). □

By (i) of Lemma II.3.5, if $d^\alpha(k)$ is enumerated in D at s then at $s' > s$, the next α-expansionary stage, either $d^\alpha(k)$ is canceled by α or α is initialized before s'. Moreover the above lemma also holds with $\mathcal{R}$ and $\mathcal{S}$ replacing $\mathcal{M}$ and $\mathcal{N}$ respectively.

Lemma II.3.6. *Suppose σ is some $\mathcal{Q}_{e,n}$-strategy accessible at s_0, and $s_1 > s_0$ is the earliest stage at which σ is accessible again. Let $\tau = top(\sigma)$.*
(i) If $state(\sigma)[s_0] = w$ and σ is not initialized between s_0 and s_1, then either $state(\sigma)[s_1] = \bot$, or $state(\sigma)[s_1] = w$ and $\alpha(\sigma)[s_1] \subset \alpha(\sigma)[s_0]$, or $state(\sigma)[s_1] = c$.
(ii) If $state(\sigma)[s_0] = c$ then σ is initialized between s_0 and s, or $state(\sigma)[s_1] = 1$ and $\bigcup_{i \le \|n\|} C^\tau_{e,i} \cap W_{n_i} \ne \emptyset$.

Proof. (i) Suppose σ is not initialized between s_0 and s_1 and $state(\sigma)[s_1] \ne \bot$, then there is a link (α, σ) at stage s_0, s_0 is α-expansionary and $d^\alpha(k(\sigma))[s_0] \in D[s_0] - D[s_0 - 1]$. By the construction, $s_1 > s_0$ is the earliest α-expansionary stage and α is not initialized between s_0 and s_1.

By (i) of Lemma II.3.5 and the remark after Lemma II.3.5, for each k either $d^\alpha(k)$ is canceled by α at s or $\phi_e(d^\alpha(k))$ does not increase.

Hence (i) holds by (3) of Procedure II.3.3.

(ii) By Procedure II.3.3, $l^\tau[s_0] > a^\sigma[s_0]$,

$$\Psi_e(B_e; a^\sigma)[s_0] = 0 \ne 1 = A(a^\sigma)[s_0],$$

and σ setups a link (τ, σ) at stage s_0. By Case 2 of the construction, $s_1 > s_0$ is the earliest τ-expansionary stage and thus

$$\Psi_e(B_e; a^\sigma)[s_1] = 1 \neq 0 = \Psi_e(B_e; a^\sigma)[s_0].$$

Hence for some $i < \|e\|$, $B_{e_i} \upharpoonright \psi_e(a^\sigma)[s_1] \neq B_{e_i} \upharpoonright \psi_e(a^\sigma)[s_0]$ and (ii) holds. □

Let $TP = \liminf_s TP_s$.

Lemma II.3.7. *For each m,*
(i) $|TP| \geq m$;
(ii) $TP \upharpoonright m$ is accessible infinitely often;
(iii) $TP \upharpoonright m$ is initialized at most finitely often.

Proof. We prove (i)(ii) and (iii) simultaneously by induction of m.

For $m = 0$, (i)(ii) and (iii) hold trivially.

Suppose (i)(ii) and (iii) hold for m. Let $\xi = TP \upharpoonright m$ and fix $s_0 > m$ such that ξ is never initialized after stage s_0. We argue by cases.

Case 1, ξ is some $\mathcal{M}_e$- or $\mathcal{R}_{e,i,j}$-strategy.

It suffices to prove that if there are infinitely many ξ-expansionary stages then $\xi\hat{\ }\infty$ is accessible infinitely often.

Suppose $s_1 > s_0$ is ξ-expansionary and $\xi\hat{\ }\infty$ is inaccessible at stage s_1. Then there exists a link (ξ, σ). Since ξ will no longer be initialized, σ will not be initialized before next ξ-expansionary stage $s_2 > s_1$.

If the link is canceled before s_2 (because of subcase 2.1 of the construction), then $\xi\hat{\ }\infty$ is accessible at s_2.

Otherwise, by Lemma II.3.6, either $\alpha(\sigma)[s_2] \subset \xi$ or $state(\sigma)[s_2] = c$.

By induction hypothesis and Lemma II.3.6, there is $s > s_2$ such that $state(\sigma)[s] = 1$. Let s_3 be the least such s, then $TP_{s_3} = \sigma$ and there is no link along TP_{s_3}. Let s_4 be the earliest ξ-expansionary stage after s_3, then $\xi\hat{\ }\infty$ is accessible.

Case 2, ξ is some $\mathcal{P}_e$-strategy.

It suffices to prove that if there are infinitely many ξ-expansionary stages then $\xi\hat{\ }\infty$ is accessible infinitely often.

Suppose $s_1 > s_0$ is ξ-expansionary and $\xi\hat{\ }\infty$ is inaccessible at stage s_1. Then there exists a link (ξ, σ) and $state(\sigma) = c$. By Lemma II.3.6, the link is canceled at s_1 and $\xi\hat{\ }\infty$ is accessible at next ξ-expansionary stage.

Case 3, ξ is some $\mathcal{Q}_{e,n}$-strategy.

By induction hypothesis, we may assume that $a^\xi[s] = a^\xi[s_0]$ for $s > s_0$.

If ξ has $\perp$ as outcome for infinitely often, then by CASE 3 of the construction, Procedure II.3.3 and Lemma II.3.6, $state(\xi)[s] \neq 1$ for $s > s_0$. The lemma holds because by (I) of the construction, $\xi\hat{\ }\perp$ will not be initialized when $TP_s = \xi$ and $state(\xi)[s] = w$.

If ξ has 1 as outcome at some stage $s > s_0$, then by CASE 3 of the construction and (5) of Procedure II.3.3, ξ eventually has 1 as outcome. Otherwise, ξ eventually has 0 as outcome. In either case the lemma holds obviously.

Case 4, ξ is some $\mathcal{N}_{e,k}$- or $\mathcal{S}_{e,i,j,k}$-strategy. Let $\alpha = top(\xi)$. We only prove the case for $\mathcal{N}_{e,k}$ since the other case is similar.

If $TP_{s_1} = \xi$ at $s_1 > s_0$, then the first clause of (3)(d) of Procedure II.3.2 happens at s_1. Let $s_2 > s_1$ be the next α-expansionary stage, by the remark after Lemma II.3.5 $d^\alpha(k)$ is canceled by α at this stage. Let $s_3 \geq s_2$ be the earliest stage at which ξ is accessible again, then either $\xi\hat{\ }\perp$ is accessible or $\Theta^\alpha(W_e; k)[s_3] = 1$ and $\xi\hat{\ }\top$ is accessible. □

Lemma II.3.8. *If β is an $\mathcal{N}_{e,k}$- or $\mathcal{S}_{e,i,j,k}$-strategy on TP, then d^β is fixed eventually.*

Proof. Let $\alpha = top(\beta)$ and s_0 be the stage such that β is never initialized after s_0. We will only prove the case that β is $\mathcal{N}_{e,k}$-strategy since the other is similar and easier.

By the construction, d^β could be canceled only if it were enumerated in D previously. Moreover, d^β could be enumerated in D after s_0 only if $K(k) = 1 \neq 0 = \Theta^\alpha(A, W_e; k)$ or by some σ such that $\alpha\hat{\ }\infty \subseteq \sigma \subset \beta$.

Note that the former situation could happen at most once. For the latter, if $\sigma\hat{\ }\perp$ is not on TP then σ could enumerate d^β in D at most finitely often.

Assume $\sigma\hat{\ }\perp \subset TP$. If $\tau \subset \alpha\hat{\ }\infty \subseteq \sigma$, then by the definition of T, $\beta \subset \sigma$. By Lemma II.3.5 (iv), σ will never enumerate d^β in D.

If $\alpha\hat{\ }\infty \subseteq \tau$ and σ enumerates d^β in D at $s_1 > s_0$, then σ setups a link (α, σ) at s_1. From then on τ is skipped and the enumeration B^τ will never change until later $state(\sigma) = c$ and a link (τ, σ) is setup. By Lemma II.3.6 and the choice of s_0, $\sigma\hat{}1 \subset TP$. This contradicts the assumption that $\sigma\hat{}\bot \subset TP$. □

Let $\alpha = \mathcal{M}_e(TP)$ and assume α is never initialized after s_0, then

$$\Theta^\alpha = \bigcup_{s>s_0} \Theta^\alpha[s]$$

is a consistent p.r. functional.

If β is an $\mathcal{N}_k^\alpha$-strategy on TP, then by the lemma above, $d^\beta(k)$ is fixed eventually. If in addition $\beta\hat{}\top \subset TP$, then $d^\alpha(k)$ is eventually fixed too by (ii) of Lemma II.3.5, and $\Theta^\alpha(W_e; k) = K(k)$ by Case 4 in the proof of Lemma II.3.7 and (3)(d) of Procedure II.3.2.

Thus we get the following.

Lemma II.3.9. *$\mathcal{M}_e$ is satisfied for every e.*

Now we turn to $\mathcal{P}_e$.

Let $\tau = \mathcal{P}_e(TP)$ and assume it is never initialized after s_0, then

$$C_i^\tau = \bigcup_{s>s_0} C_i^\tau[s]$$

is c.e. for $i < \|e\|$.

If $\tau\hat{}0 \subset TP$ then C_i^τ is finite for $i < \|e\|$. Otherwise, to determine whether $x \in C_i^\tau$ for an arbitrary x and $i < \|e\|$, let $s > s_0$ be the earliest τ-expansionary stage such that $B_{e_i} \upharpoonright x = B_{e_i}^\tau[s] \upharpoonright x$, then $x \in C_i^\tau$ iff $x \in C_i^\tau[s]$. Hence we establish $C_i^\tau \leq_{\mathrm{T}} B_{e_i}$ for $i < \|e\|$.

Suppose $A = \Psi_e(B_e)$, and let σ be a $\mathcal{Q}_n^\tau$-strategy on TP. Then the satisfaction of $\mathcal{Q}_n^\tau$ follows from Lemma II.3.6. The argument for $\mathcal{R}_{e,i,j}$'s is similar to that for Lemma II.3.9. Hence we get the next lemma and finish the proof of Theorem II.3.1.

Lemma II.3.10. *$\mathcal{P}_e$ is satisfied for every e.*

II.4 [PC] $\not\subseteq$ [NB $\cup$ NCup]

Yu and Yang (2005) showed that $I = [\mathbf{NB} \cup \mathbf{NCup}] \subset \mathbf{M}$. In this section, we will prove the following.

Theorem II.4.1. *There is a plus cupping degree* $\mathbf{a} \notin I$.

We construct a c.e. set A satisfying the plus cupping requirements

$$\mathcal{M}_e : W_e = \Phi_e(A) \Rightarrow W_e \leq_{\mathrm{T}} \emptyset \text{ or } W_e \text{ is cupping},$$

and the requirements guaranteeing $\mathbf{deg}(A) \notin [\mathbf{NB} \cup \mathbf{NCup}]$

$$\mathcal{P}_e : A = \Psi_{e_c}(X_e, Y_{e_{c-1}}) \Rightarrow (\exists i < c-1)(X_{e_i} \text{ is bounding}) \text{ or } Y_{e_{c-1}} \text{ is cupping}$$

where X_e is the abbreviation of the tuple $(X_{e_0}, \ldots, X_{e_{c-2}})$ and $c = \|e\|$.

We will arrange the construction on a tree of strategies as in the previous section, and will follow conventions described there.

During the construction, we will in addition build a c.e. set D for some diagonalization purposes which will be clear.

II.4.1 $\mathcal{M}$-strategies

We follow the technique originally developed by Harrington (1978) and refined by Fejer and Soare (1981).

Suppose α is an $\mathcal{M}_e$-strategy, let l^α the length of agreement between W_e and $\Phi_e(A)$ and α-expansionary stages be defined as usual. If there are at most finitely many α-expansionary stages, α has 0 as outcome; otherwise α has ∞ as outcome.

In the latter case, α will build a c.e. set C^α and a p.r. functional Δ^α such that $K = \Delta^\alpha(W_e, C^\alpha)$, and

$$\mathcal{N}_i^\alpha : D \neq \Gamma_i(C^\alpha) \text{ or } W_e \leq_{\mathrm{T}} \emptyset.$$

From now on we will omit the superscript α in this section.

To define $\Delta(W_e, C; k)$, at the beginning α defines $\Delta(W_e, C; k) = K(k)$ with an arbitrary use. If later k is enumerated in K, α enumerates $\delta(k)$ in C and redefines $\Delta(W_e, C; k) = 1$ with a fresh use.

To make $\mathcal{N}_i^\alpha$, we arrange $\mathcal{N}_i^\alpha$-strategies above $\alpha\hat{\ }\infty$. If β is an $\mathcal{N}_i^\alpha$-strategy, β picks a fresh *diagonalizer* d^β and a *lifting point* k^β at the beginning and keeps d^β from entering D. From now on we will omit the superscript β in this section.

β intends to make $\delta(k) > \gamma_i(d)$. If this is achieved, d will be enumerated in D. Suppose β has many chances to do this, eventually the inequality $D(d) \neq \Gamma_i(C; d)$ will be established.

To this end, whenever β finds $\Gamma_i(C; d) = 0$ it will *open a gap* by creating a *shortcut* (α, β) (whose purpose will be clear later), having g as outcome and allowing strategies extending $\beta\hat{\ }g$ to contribute anything to A. That is, β hopes that $W_e \upharpoonright \delta(k)$ will be changed by changes to A.

At the next α-expansionary stage $s_1 > s_0$, we will have α *close the gap for* β.

If α finds that $W_e \upharpoonright \delta(k)[s_1] \neq W_e \upharpoonright \delta(k)[s_0]$, then $\Delta(W_e, C; k)$ diverges. α will *closes the gap successfully* by defining $\delta(k) > \gamma_i(d)$ and enumerating d in D. In this case, β's intention will be achieved.

If α finds that $W_e \upharpoonright \delta(k)[s_1] = W_e \upharpoonright \delta(k)[s_0]$, it will try to preserve the computation $\Phi_e(A) \upharpoonright \delta(k)[s_1]$ by initializing strategies to the right of $\beta\hat{\ }g$. Then α will enumerate $\delta(k)[s_0]$ in C and thus canceling $\Delta(W_e, C; k')$ for $k' \geq k$. We say that α *closes the gap unsuccessfully*.

In either cases above, α will cancel (α, β). The purpose of using shortcuts is to guarantee validity of the argument below.

If there are infinitely many gaps opened and closed (unsuccessfully), let $(s_m : m \in \omega)$ increasingly enumerate the stages at which β opens a gap. For each m let t_m be the earliest α-expansionary stage after s_m, then the gap opened at s_m is closed by α at t_m. Since $\delta(k)[s_{m+1}] > \delta(k)[s_m]$, $W_e \upharpoonright \delta(k)[s_m]$ is fixed between s_m and t_m while $\Phi_e(A) \upharpoonright \delta(k)[s_m]$ is fixed between t_m and s_{m+1}, W_e is computable if $W_e = \Phi_e(A)$.

Thus we will arrange no $\mathcal{N}^\alpha$-strategies above $\beta\hat{\ }g$.

However, to guarantee that $\Delta(W_e, C; k)$ converges, we must arrange the distribution of lifting points so that there are at most finitely many $\mathcal{N}^\alpha$-strategies having lifting point less than k' for each k'.

We formally describe the behavior of α at stage s as below. Let

$$s_0 = \max\{s' < s : \alpha \text{ is accessible at } s' \text{ and not initialized between } s' \text{ and } s\}.$$

Procedure II.4.2. *There are two cases.*

(i) Case 1, s is not α-expansionary. *Just have 0 as outcome.*

(ii) Case 2, s is α-expansionary. *If in additional there is a shortcut (α, β), then the shortcut is setup at stage s_0, let $k_0 = k^\beta$; otherwise let $k_0 = s$. Let $k_1 = \min\{k < k_0 : \Delta(W_e, C; k) = 0 \neq 1 = K(k)\}$. Whatever α does, let ∞ be the outcome, and if there is a shortcut then it will be canceled.*

1. *If k_1 is defined, enumerate $\delta(k_1)$ in C. Redefine $\Delta(W_e, C; k') = K(k')$ for $k' \geq k$ with $\delta(k')$ fresh.*
2. *From now on assume k_1 is undefined. For $k' < k_0$, if $\Delta(W_e, C; k') \uparrow$, define $\Delta(W_e, C; k') = K(k')$ with $\delta(k') = \delta(k')[s_0]$ if s_0 is defined and $\Delta(W_e, C; k')[s_0]$ converges, or with $\delta(k')$ fresh.*
3. *If $k_0 = k^\beta$ and $(W_e[s] - W_e[s_0]) \upharpoonright \delta(k_0)[s_0] \neq \emptyset$, then define $\Delta(W_e, C; k') = K(k')$ with $\delta(k')$ fresh for $k' \geq k_0$ and enumerate d^β in D.*
4. *If $k_0 = k^\beta$ and $(W_e[s] - W_e[s_0]) \upharpoonright \delta(k_0)[s_0] = \emptyset$, enumerate $\delta(k_0)[s_0]$ in C if $\delta(k_0)[s_0]$ is defined, define $\Delta(W_e, C; k') = K(k')$ with $\delta(k')$ fresh for $k' \geq k_0$ and initialize strategies $\geq \beta\hat{\ }0$.*

We formally describe the behavior of β at stage s as below. Once the outcome is determined, β stops immediately.

Procedure II.4.3. *Define k to be fresh if it is undefined. Whenever β finds $\Delta(W_e, C; k)$ is undefined or $\delta(k) > l$, it simply stops. Otherwise, β acts as below.*

1. *If $\Gamma_i(C; d) = 1 = D(d)$, cancel d.*
2. *If d is undefined, define it to be fresh.*
3. *If $\Gamma_i(C; d) \neq 0$, let 0 be the outcome.*
4. *If $\Gamma_i(C; d) = 0 \neq 1 = D(d)$, let 1 be the outcome.*
5. *Otherwise $\Gamma_i(C; d) = 0 = D(d)$, setup a shortcut (α, β) and let g be the outcome.*

II.4.2 $\mathcal{P}$-strategies

We follow the proof of Theorem 1.6 in Yu and Yang (2005).

Suppose τ is a $\mathcal{P}_e$-strategy, the length of agreement l^τ and the τ-expansionary stages are defined as usual. If there are at most finitely many τ-expansionary stages, τ has 0 as outcome; otherwise τ has ∞ as outcome.

In the latter case, τ will construct $2c-1$ ($c = \|e\|$) c.e. sets

$$M^\tau_{0,0}, M^\tau_{0,1}, \ldots, M^\tau_{c-2,0}, M^\tau_{c-2,1}, Z^\tau$$

and one p.r. functional Θ^τ so that $M^\tau_{i,0}, M^\tau_{i,1} \leq_{\mathrm{T}} X_{e_i}$ for $i < c-1$, $K = \Theta^\tau(Y_{c-1}, Z^\tau)$, for $\|n\| = c-1$ and $(i,j) \in (c-1) \times 2$

$$\mathcal{Q}^\tau_{n,j} : D \neq \Phi_{n_{c-1}}(Z^\tau) \text{ or } (\exists i < c-1)(M^\tau_{i,j} \neq \overline{W}_{n_i}), \text{ and}$$

$$\mathcal{R}^\tau_{i,j} : \Phi_j(M^\tau_{i,0}) = \Phi_j(M^\tau_{i,1}) \text{ is total} \Rightarrow \Phi_j(M^\tau_{i,0}) \leq_{\mathrm{T}} \emptyset, \text{ for } (i,j) \in \omega^2.$$

From now on in this subsection, we will drop the superscript τ and occasionally also drop the subscripts such as e and e_i.

To define $\Theta(Y, Z; k)$, at the beginning τ defines $\Theta(Y, Z; k) = K(k)$ with an arbitrary use. If k is enumerated in K later, τ enumerates $\theta(k)$ in Z and redefines $\Theta(Y, Z; k) = 1$ with a fresh use.

To satisfy $\mathcal{Q}^\tau$'s and $\mathcal{R}^\tau$'s, we arrange ζ's for $\mathcal{Q}^\tau$'s and η's for $\mathcal{R}^\tau$s above $\tau\hat{\ }\infty$.

Suppose ζ is a $\mathcal{Q}^\tau_{n,0}$-strategy. At the beginning ζ picks a fresh *lifting point* k, a fresh *diagonalizer* d and a fresh *agitator* a, and keeps d and a from entering D or A respectively. ζ makes $\theta(k) > \psi(a)$ by lifting $\theta(k)$ whenever $\psi(a)$ grows.

If ζ finds that the computation $\Psi(X, Y; a)$ diverges, then it has $\perp$ as outcome indicating that $\Psi(X, Y; a)$ diverges. We will have neither $\mathcal{Q}$- nor $\mathcal{R}$-strategies above $\zeta\hat{\ }\perp$.

If $\Psi(X, Y; a)$ converges eventually, ζ defines a *witness* $x > \psi(a)$ and waits for $\Phi(Z; d) = 0$ and $x \in \bigcap_{i<c-1} W_{n_i}$. If ζ keeps waiting for ever, it will have 0 as outcome.

If at some stage s_0, $\Phi(Z; d) = 0$ and $x \in \bigcap_{i<c-1} W_{n_i}$, ζ will try to make either $\theta(k) > \phi(d)$ while preserving the computation $\Phi(Z; d) = 0$ or to enumerate x in some $M_{i,0}$ with permission from X_{e_i}. If the former is achieved, ζ will enumerate d in D and establish $D(d) = 1 \neq \Phi(Z; d)$. In both cases, ζ will have a local win.

To this end, ζ will enumerate a in A and setup a link (τ, ζ). At next τ-expansionary stage $s_1 > s_0$, one of $X_{e_0}, \ldots, X_{e_{c-2}}$ and Y must have been changed below $\psi(a)[s_0]$.

The control will be passed immediately from τ to ζ and the link will be canceled, i.e., the link will be *traveled*.

If Y changes, then ζ will redefine $\Theta(Y, Z; k)$ with $\theta(k)$ fresh and enumerates d in D. If some X_{e_i} does, ζ will enumerate x in $M_{i,0}$. In both cases ζ will have 1 as outcome.

The purpose of using links is to either make the lifting of $\theta(k)$ in time or make the enumeration of x in $M_{i,0}$ permitted by X_{e_i}.

We formally describe the actions of τ at stage s as below. Let s_0 be defined as before Procedure II.4.2 (with τ in place of α).

Procedure II.4.4. *There are two cases.*

(i) Case 1, s is not τ-expansionary. *Just let 0 be the outcome.*

(ii) Case 2, s is τ-expansionary. *If there is a link (τ, ζ), then it is setup by ζ at stage s_0, let $k_0 = k^{\zeta}$; otherwise let $k_0 = s$. Let $k_1 = \min\{k < k_0 : \Theta(Y, Z; k) = 0 \neq 1 = K(k)\}$.*

1. *If k_1 is defined, enumerate $\theta(k_1)$ in Z and redefine $\Theta(Y, Z; k') = K(k')$ with $\theta(k')$ fresh for $k' \geq k_1$; if there is a link (τ, ζ), travel and cancel it.*
2. *From now on, assume k_1 is undefined. For $k' < k_0$, if $\Theta(Y, Z; k')$ diverges define $\Theta(Y, Z; k') = K(k')$ with $\theta(k') = \theta(k')[s_0]$ if s_0 is defined and $\Theta(Y, Z; k')[s_0]$ converges, or with $\theta(k')$ fresh.*
3. *If there is no link, let ∞ be the outcome and stop. Otherwise assume that there is a link (τ, ζ), travel and cancel the link.*

We formally describe the actions of ζ at stage s as below.

Procedure II.4.5. *There are two cases.*

(i) Case 1, a link (τ, ζ) is traveled. *Suppose the link is setup at stage $s_0 < s$. Take actions according to the following subcases.*

1. *If $K[s] \upharpoonright k \neq K[s_0] \upharpoonright k$ then cancel a, d and x.*
2. *If $Y[s] \upharpoonright \psi(a)[s_0] \neq (Y \upharpoonright \psi(a))[s_0]$, then $\Theta(Y, Z; k)[s-1]$ diverges, define $\Theta(Y, Z; k') = K(k')$ with $\theta(k')$ fresh for $k' \geq k$ and enumerate d in D.*
3. *Otherwise, there is some $i < c-1$ such that $X_{e_i}[s] \upharpoonright \psi(a)[s_0] \neq (X_{e_i} \upharpoonright \psi(a))[s_0]$. Let i_0 be the least such i, enumerate x in $M_{i_0,j}$.*

(ii) Case 2, otherwise. *Check the followings one by one. Once an outcome is determined, ζ stops immediately.*

1. *If k is undefined, define it to be fresh.*
2. *If $D(d) = 1 = \Phi_{n_{c-1}}(Z; d)$, cancel a, d and x.*
3. *If $D(d) = 1 \neq \Phi_{n_{c-1}}(Z; d)$, or $\bigcup_{i<c} M_{i,j} \cap W_{n_i} \neq \emptyset$, let 1 be the outcome.*
4. *If a is undefined, define it to be fresh. If $l^\tau < a$, stop.*
5. *Otherwise if $\psi(a) \geq \theta(k)$, enumerate $\theta(k)$ in Z and redefine $\Theta(Y, Z; k')$ with $\theta(k')$ fresh for $k' \geq k$ (if $\Theta(Y, Z; k)[s-1]$ is defined), cancel d and x, let $\perp$ be the outcome.*
6. *If d and x are undefined, define them to be fresh. If $D(d) = 0 \neq \Phi_{n_{c-1}}(Z; d)$ or $x \notin \bigcap_{i<c} W_{n_i}$, let 0 be the outcome.*
7. *Otherwise enumerate a in A and setup a link (τ, ζ).*

The $\mathcal{R}$-strategies η's act in the same way as typical minimal pair constructions. We define l^η the length of agreement between $\Phi_j(M_{i,0})$ and $\Phi_j(M_{i,1})$ and η-expansionary stages as usual. Each η has two outcomes, namely ∞ indicating there are infinitely many η-expansionary stages, and 0 indicating there are at most finitely many such stages. We refer the readers to in (Soare, 1987, XIV.3.2) for details.

II.4.3 Conflicts

Different $\mathcal{M}$-strategies do not injure each other, because they never intend to change A and they build local Δ's and C's. Neither do different $\mathcal{N}^\alpha$-strategies above a certain $\mathcal{M}_e$-strategy α injure each other, because none of them intend to change C^α.

If β is some $\mathcal{N}_i^\alpha$-strategy, then the intention of β to preserve $C^\alpha \upharpoonright \gamma_i(d^\beta)$ may be injured by the intention of α to define $\Delta^\alpha(W_e, C^\alpha; k) = K(k)$ for $k < k^\beta$, and the intention of β to lift $\delta^\alpha(k^\beta)$ may injure the intention of α to make $\Delta^\alpha(W_e, C^\alpha; k)$ converge. The first conflict is solved by guaranteeing that k^β is eventually fixed, hence it could happen at most finitely often (this is also the solution of similar conflicts between $\mathcal{P}$-strategies and $\mathcal{Q}$-strategies). To solve the second conflict, note that β intends to lift $\delta^\alpha(k^\beta)$ infinitely often only if it opens infinitely many gaps. In this case we will make $W_e \leq_{\mathrm{T}} \emptyset$ hence will not worry about the definition of Δ^α. Otherwise we arrange the distribution of lifting points so that each k is used as a lifting point by at most one $\mathcal{N}^\alpha$-

strategy. This is achieved by the first sentence of Procedure II.4.3. Hence $\delta^\alpha(k)$ will not be lifted for ever if every $\mathcal{N}^\alpha$-strategy lifts its lifting point at most finitely often.

Now the intention of α to preserve $\Phi_e(A;k^\beta)$ when unsuccessfully closing a gap opened by β could be injured by some $\mathcal{Q}^\tau$-strategy ζ where τ is some $\mathcal{P}$-strategy since ζ may enumerate its agitator in A. The solution is to initialize ζ if $\zeta \geq \beta\hat{}0$. Hence α will succeeded in preserving $\Phi_e(A;k^\beta)$ if $\mathcal{Q}$-strategies $< \beta\hat{}g$ are never accessible later, since A can be freely changed above $\beta\hat{}g$. This is already incorporated by (ii)(4) of Procedure II.4.2.

The last kind of conflicts is between $\mathcal{R}^\tau_{i,j}$-strategies η's and $\mathcal{Q}^\tau_{n,j'}$-strategies ζ's. The solution is to allow at most one side of $\Phi_j(M^\tau_{i,0}) = \Phi_j(M^\tau_{i,1})$ be destroyed between η-expansionary stages. To this end we will run no more strategies at a stage once (i)(3) of Procedure II.4.5 happens.

II.4.4 Parameters

We sum up parameters associated with strategies.

For α an $\mathcal{M}$-strategy, there are the length of agreement l^α, a c.e. set C^α to be built and a p.r. functional Δ^α.

For β an $\mathcal{N}^\alpha$-strategy, there are a diagonalizer d^β and a lifting point k^β.

For τ a $\mathcal{P}_e$-strategy, there are

1. The length of agreement l^τ;
2. $2\|e\|-1$ many c.e. sets $M^\tau_{0,0}, M^\tau_{0,1}, \dots, M^\tau_{\|e\|-2,0}, M^\tau_{\|e\|-2,1}$ and Z^τ;
3. A p.r. functional Θ^τ.

For ζ a $\mathcal{Q}^\tau$-strategy, there are a lifting point k^ζ, a diagonalizer d^ζ, an agitator a^ζ and a witness x^ζ.

For η an $\mathcal{R}^\tau$-strategy, there is the length of agreement l^η.

Assume ξ is an arbitrary strategy. If it is initialized then all of its parameters, and shortcuts or links with one end being ξ are canceled.

II.4.5 The tree of strategies

We may consider $\mathcal{N}_i^\alpha$ as a subrequirement $\mathcal{N}_{e,i}$ of $\mathcal{M}_e$ where α is a $\mathcal{M}_e$-strategy, and $\mathcal{Q}_{n,j}^\tau$ and $\mathcal{R}_{i,j}^\tau$ as subrequirements $\mathcal{Q}_{e,n,j}$ and $\mathcal{R}_{e,i,j}$ of $\mathcal{P}_e$ where τ is a $\mathcal{P}_e$-strategy. Hence $C^\alpha, \Delta^\alpha, M_{i,j}^\tau$ and Z^τ, Θ^τ may be taken as local versions of $C_e, \Delta_e, M_{e,i,j}$ and Z_e, Θ_e respectively.

Let Λ be the set of outcomes $\infty <_\Lambda 1 <_\Lambda g <_\Lambda \perp <_\Lambda 0$.

Fix a computable bijection f mapping ω onto the collection of all requirements and subrequirements such that $f^{-1}(\mathcal{M}_e) < f^{-1}(\mathcal{N}_{e,k})$, and $f^{-1}(\mathcal{P}_e) < f^{-1}(\mathcal{Q}_{e,n,j}), f^{-1}(\mathcal{R}_{e,i,j})$. We inductively define T the tree of strategies as a computable subset of $\Lambda^{<\omega}$.

Let $\emptyset \in T$. If $\xi \in T$, we say that a requirement $\mathcal{O}$ is *finished at* ξ if and only if one of the followings applies

1. $\mathcal{O}$ is $\mathcal{M}_e$ and either there is an $\mathcal{M}_e$-strategy $\alpha \subset \alpha\hat{\ }0 \subseteq \xi$ or there is an $\mathcal{N}_{e,i}$-strategy $\beta \subset \beta\hat{\ }g \subseteq \xi$.
2. $\mathcal{O}$ is $\mathcal{P}_e$ and either there is a $\mathcal{P}_e$-strategy $\tau \subset \tau\hat{\ }0 \subseteq \xi$ or there is a $\mathcal{Q}_{e,n,j}$-strategy $\zeta \subset \zeta\hat{\ }\perp \subseteq \xi$.
3. $\mathcal{O}$ is $\mathcal{N}_{e,i}$ ($\mathcal{Q}_{e,n,j}$ or $\mathcal{R}_{e,i,j}$) and $\mathcal{M}_e$ ($\mathcal{P}_e$) is finished at ξ.

We say that $\mathcal{O}$ is *satisfied at* ξ if either $\mathcal{O}$ is finished at ξ or there is an $\mathcal{O}$-strategy $\xi' \subset \xi$; otherwise we say that $\mathcal{O}$ is *unsatisfied at* ξ.

We assign the unique $\mathcal{O}$ to ξ such that $f^{-1}(\mathcal{O})$ is the least among the requirements unsatisfied at ξ.

If ξ is some $\mathcal{M}$-, $\mathcal{P}$- or $\mathcal{R}$-strategy, let $\xi\hat{\ }\infty$ and $\xi\hat{\ }0 \in T$; if ξ is an $\mathcal{N}$-strategy, let $\xi\hat{\ }1, \xi\hat{\ }g$ and $\xi\hat{\ }0 \in T$; if ξ is a $\mathcal{Q}_{e,n,j}$-strategy, let $\xi\hat{\ }1, \xi\hat{\ }\perp$ and $\xi\hat{\ }0 \in T$.

Furthermore, if ξ is an $\mathcal{N}_{e,i}$-strategy, let $top(\xi)$ be the unique $\mathcal{M}_e$-strategy $\alpha \subset \xi$; if ξ is a $\mathcal{Q}_{e,n,j}$- or $\mathcal{R}_{e,i,j}$-strategy, let $top(\xi)$ be the unique $\mathcal{P}_e$-strategy $\tau \subset \xi$.

We will use some terminologies defined in subsection II.3.4.

II.4.6 Construction

Stage 0. Let all parameters associated with all strategies be undefined, and all c.e. sets and p.r. functionals to be built be empty.

Stage $s > 0$. Let $\emptyset$ be accessible. If ξ is accessible and $|\xi| = s$, let $TP_s = \xi$. Otherwise we take actions according to the following cases.

Case 1, ξ is an $\mathcal{M}_e$-strategy. Run Procedure II.4.2. Let $\xi\hat{\ }o$ be accessible where o is the outcome.

Case 2, ξ is an $\mathcal{N}_{e,i}$-strategy. Run Procedure II.4.3. If there is no outcome, let $TP_s = \xi$; otherwise let $\xi\hat{\ }o$ be accessible where o is the outcome.

Case 3, ξ is a $\mathcal{P}_e$-strategy. Run Procedure II.4.4. If there is an outcome o let o be accessible; otherwise there is a link (ξ, ζ) at the beginning of s and the last clause of (ii)(3) of Procedure II.4.4 happens, let ζ be accessible.

Case 4, ξ is a $\mathcal{Q}_{e,n,j}$-strategy. Let $\tau = top(\xi)$. Run Procedure II.4.5. If there is no outcome, let $TP_s = \xi$; otherwise let $\xi\hat{\ }o$ be accessible where o is the outcome.

Case 5, ξ is an $\mathcal{R}_{e,i,j}$-strategy. If s is ξ-expansionary, let $\xi\hat{\ }\infty$ be accessible; otherwise let $\xi\hat{\ }0$ be accessible.

In addition, once TP_s is defined, we end stage s immediately by initializing all strategies $> TP_s$.

II.4.7 Verification

First we study an important behavior of $\mathcal{N}$-strategies.

Lemma II.4.6. *If α is an $\mathcal{M}_e$-strategy and β is an $\mathcal{N}_i^\alpha$-strategy above $\alpha\hat{\ }\infty$, then either β is initialized infinitely often or d^β is eventually fixed.*

Proof. During the proof, we occasionally omit α and β from the superscripts.

If β is accessible at most finitely often, then it is trivial that d^β is eventually fixed (including the possibility that it is canceled at some stage and never becomes defined from then on).

From now on we assume that β is accessible infinitely often and initialized at most finitely often. We may assume in addition that every proper initial segment of β being also some $\mathcal{N}^\alpha$-strategy has its diagonalizer eventually fixed. Let s_0 be such that

1. β is not initialized after s_0 and $k^\beta = k^\beta[s_0]$;
2. For all $k < k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined, and if $k \in K$ then $\Delta(W_e, C; k)[s_0] = 1 = K(k)[s_0]$;

3. For $\mathcal{N}^\alpha$-strategy $\beta' \subset \beta$, $d^{\beta'} = d^{\beta'}[s_0]$.

If at stage $s > s_0$, d is canceled by β then $\Gamma_i(C;d) = 1 = D(d)$. Let $s_2 < s$ be the stage at which $d^\beta[s_2 - 1]$ is enumerated in D by α. Then there is a shortcut (α, β) at the beginning of s_2, suppose it is setup by β at stage $s_1 < s_2$. We may assume $s_1 > s_0$ otherwise d could be canceled at most s_0 many times.

At stage s_1, $\Gamma_i(C;d) = 0 = D(d)$. At the beginning of stage s_2 the computation $\Gamma_i(C;d)$ is same as that at s_1 since C can only be changed by α, and $\delta(k)[s_2] > \gamma_i(d)$ for $k \geq k^\beta$ by (3) of Procedure II.4.2.

Since d is canceled at $s > s_2$, there is some $\delta(k^{\beta'})$ enumerated in C with $k^{\beta'} < k^\beta$, at some stage $s'(s_2 \leq s' < s)$. Then $\beta' < \beta$ and setups a shortcut (α, β') at some stage $s''(s_2 \leq s'' < s')$. Hence $\beta'\hat{\ }g \subseteq TP_{s''}$.

By the definition of the tree, $\beta \not\supseteq \beta'\hat{\ }g$. Since in addition β is not initialized at s'', $\beta'\hat{\ }1 \subseteq \beta$. Hence $\beta'\hat{\ }1$ is accessible at s_1 and $\Gamma_{i'}(C; d^{\beta'})[s_1] \neq 1 = D(d^{\beta'}[s_1])$ (assume β' is an $\mathcal{N}^\alpha_{i'}$-strategy).

But $\Gamma_{i'}(C; d^{\beta'})[s''] = 0 = D(d^{\beta'}[s''])$. Hence $d^{\beta'}[s_1] \neq d^{\beta'}[s'']$. This contradicts with the choice of s_0. □

Next we study some important behaviors of $\mathcal{Q}$-strategies.

Lemma II.4.7. *Let τ be a $\mathcal{P}_e$-strategy, $c = \|e\|$ and $\zeta \supseteq \tau\hat{\ }\infty$ be a $\mathcal{Q}^\tau_{n,j}$-strategy.*
(i) Either ζ is initialized infinitely often or a^ζ is eventually fixed;
(ii) If ζ is initialized at most finitely often and accessible infinitely often, then there is a stage s_0 at which both k^ζ and a^ζ are defined and fixed for ever, and for no $k < k^\zeta$ and $s > s_0$, $\theta^\tau(k)[s-1]$ is enumerated in Z^τ;
(iii) Let s_0 be as in (ii) and moreover ζ setups a link (τ, ζ) at stage s_0. Let $s_1 > s_0$ be the earliest τ-expansionary stage. Then (τ, ζ) is traveled at stage s_1, $d^\zeta = d^\zeta[s_0]$ is fixed and either

1. *$\Phi_{n_{c-1}}(Z^\tau; d^\zeta) = 0 \neq 1 = D(d^\zeta)$ and the computation $\Phi_{n_{c-1}}(Z^\tau; d^\zeta)$ is exactly $\Phi_{n_{c-1}}(Z^\tau; d^\zeta)[s_0]$, or*
2. *$M^\tau_{i,j} \cap W_{n_i} \neq \emptyset$ for some $i < c-1$.*

Proof. During the proof we occasionally omit τ and ζ from the superscripts, and we may write X for X_e, etc.. Let $c = \|e\|$.

(i) As in the proof of Lemma II.4.6, let s_0 be such that

1. ζ is not initialized after s_0 and k^ζ is defined at s_0 and fixed for ever;
2. For all $k < k^\zeta$, $\Theta(Y, Z; k)[s_0]$ is defined, and if $k \in K$ then $\Theta(Y, Z; k)[s_0] = 1 = K(k)[s_0]$;
3. For $\mathcal{Q}^\tau$-strategy $\zeta' \subset \zeta$, $a^{\zeta'}$ is defined at s_0 and fixed for ever.

If a is canceled at $s > s_0$ by ζ, then $\Phi(Z; d) = 1 = D(d)$ at s. Suppose d is enumerated in D by ζ at $s_2 < s$, then there is a link (τ, ζ) at the beginning of s_2. Suppose the link is setup by ζ at $s_1 < s_2$. As in the proof of Lemma II.4.6, we assume $s_1 > s_0$.

Then at s_1, $\Phi(Z; d) = 0 = D(d)$ and $\theta(k^\zeta) > \psi(a)$. The computation $\Phi(Z; d)[s_2]$ is same as that at s_1 by the choice of s_0, and $\theta(k^\zeta)[s_2] > \phi(d)$ by (i)(2) of Procedure II.4.5.

Hence at some stage s' between s_2 and s, some $\mathcal{Q}^\tau$-strategy $\zeta' < \zeta$ enumerates $\theta(k^{\zeta'})$ in Z. Then $\zeta'\hat{}\bot \subseteq TP_{s'}$. By the definition of the tree, $\zeta'\hat{}1 \subseteq \zeta$. Hence $\zeta'\hat{}1$ is accessible at s_1 and $a^{\zeta'}$ is changed after s_1. This contradicts with the choice of s_0.

(ii) follows immediately from the proof of (i).

(iii) It is obvious that ζ is never initialized after stage s_0. Hence the link is traveled at stage s_1.

By Procedure II.4.5, at stage s_0

1. $\Psi(X, Y; a) = 0 \neq 1 = A(a)$;
2. $\Phi(Z; d) = 0 = D(d)$;
3. $x \in W_{n_i}$ for $i < c - 1$;
4. $\psi(a) < \theta(k^\zeta)$.

Since s_1 is τ-expansionary, $\Psi(X, Y; a)[s_1] = 1$ and

$$((X, Y) \upharpoonright \psi(a))[s_1] \neq ((X, Y) \upharpoonright \psi(a))[s_0].$$

If $(Y \upharpoonright \psi(a))[s_1] \neq (Y \upharpoonright \psi(a))[s_0]$, then $\Theta(Y, Z; k^\zeta)[s_1 - 1]$ diverges and $\theta(k')[s_1] > \phi(d)[s_0]$ for $k' \geq k^\zeta$ by (i)(2) of Procedure II.4.5. By (ii) the computation $\Phi(Z; d)[s_0]$ no longer changes.

If $(X_{e_i} \upharpoonright \psi(a))[s_1] \neq (X \upharpoonright \psi(a))[s_0]$ for some $i < c - 1$ then $x \in M_{i_0, j}$ for i_0 being the least such i. □

Let $TP = \liminf_s TP_s$, the next lemma states that TP is infinite and every strategy on TP is eligible to win.

Lemma II.4.8. *For every m*
(i) $|TP| \geq m$;
(ii) $TP \upharpoonright m$ is initialized at most finitely often;
(iii) $TP \upharpoonright m$ is accessible infinitely often.

Proof. It is trivial for $m = 0$.

Assume (i)(ii) and (iii) hold for m. Let $\xi = TP \upharpoonright m$. Assume ξ is accessible at $s_0 > m$ and never initialized after s_0.

Case 1, ξ is some $\mathcal{M}$- or $\mathcal{R}$-strategy. (i)(ii) and (iii) hold trivially.

Case 2, ξ is an $\mathcal{N}_i^\alpha$-strategy where $\alpha = top(\xi)$. By Procedure II.4.3, ξ always has outcome when it is accessible. Let o be the $<_\Lambda$-least outcome which ξ has infinitely often, then $\xi\hat{\ }o \subseteq TP$.

Hence we may assume that either $TP_s \geq \xi\hat{\ }o$ for $s > s_0$. At stage $s > s_0$, if $\xi\hat{\ }o$ is initialized, then the initialization could only be launched by α and $o = 0$. If this happens then ξ setups a shortcut (α, ξ) at some stage $s_1 < s$ and $\xi\hat{\ }g \subseteq TP_{s_1}$. But this could happen at most finitely often by the choice of o.

Case 3, ξ is a $\mathcal{P}_e$-strategy. If there are at most finitely many ξ-expansionary stages, then $\xi\hat{\ }0 \subseteq TP$. Assume there are infinitely many ξ-expansionary stages.

If $s_1 > s_0$ is ξ-expansionary but $\xi\hat{\ }\infty$ is not accessible at stage s_1, then there is a link (ξ, ζ) at the beginning of stage s_1, it is traveled and no new link is setup at stage s_1 by (ii)(3) of Procedure II.4.4 and (i) of Procedure II.4.5. Let $s_2 > s_1$ be the next ξ-expansionary stage, then $\xi\hat{\ }\infty$ is accessible.

Case 4, ξ is a $\mathcal{Q}_{n,j}^\tau$-strategy where $\tau = top(\xi)$. Let s_0 be as in (ii) of Lemma II.4.7.

If $TP_{s_0} = \xi$ then at stage s_0, either ξ setups a link (τ, ξ) or Procedure II.4.5(i) happens. In either case by (iii) of Lemma II.4.7, there is some $s_1 > s_0$ such that either $D \neq \Phi_{n_{c-1}}(Z^\tau)$ or $M_{i,j}^\tau \neq \overline{W}_{n_i}$ for some $i < c - 1$ is established for ever at stage s_1. Hence whenever ξ is accessible after stage s_1, $\xi\hat{\ }1$ is also accessible. $\square$

Now we are ready to prove the satisfactions of plus cupping requirements.

Lemma II.4.9. *Let α be the unique $\mathcal{M}_e$-strategy on TP, and β be the unique $\mathcal{N}_i^\alpha$-strategy on TP.*

(i) C^α is c.e. and Δ^α is consistent;

(ii) If $\beta\hat{}g \not\subset TP$, then $\Delta^\alpha(W_e, C^\alpha; k^\beta)$ converges eventually;

(iii) If $\beta\hat{}g \subset TP$, then W_e is computable;

(iv) $\mathcal{M}_e$ is eventually satisfied.

Proof. During the proof, we occasionally omit α and β from the superscripts.

(i) By Lemma II.4.8, assume α is not initialized after s. Then $C^\alpha = \bigcup_{t>s} C^\alpha[t]$ and $\Delta^\alpha = \bigcup_{t>s} \Delta^\alpha[t]$, and (i) follows from the construction.

(ii) Let $o = TP(|\beta|)$, then $o = 1$ or 0. By Lemma II.4.6, $d = d^\beta$ is eventually fixed.

If $o = 1$ then $\beta\hat{}g \subseteq TP_s$ for at most finitely many stages, otherwise d could not be fixed. If $o = 0$, then $\beta\hat{}g \subseteq TP_s$ for at most finitely many stages too by the definition of TP.

Assume for every $\mathcal{N}^\alpha$-strategy $\beta' \subset \beta$, $\Delta(W_e, C; k^{\beta'})$ eventually converges. By the assumption above, let s_0 be such that

1. $k^\beta = k^\beta[s_0]$;
2. For $k \le k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined, and $k \in K$ iff $\Delta(W_e, C; k)[s_0] = 1$ and $k \in K[s_0]$;
3. For $k < k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined and fixed for ever;
4. β is not initialized and opens no gap after s_0.

Let $s_1 > s_0$ be the earliest α-expansionary stage, then $\Delta(W_e, C; k^\beta)[s_1]$ is defined and fixed for ever.

(iii) By the definition of the tree, for $\mathcal{N}^\alpha$-strategy $\beta' \subset \beta$, $\beta'\hat{}g \not\subset TP$. By (ii) above, Lemma II.4.6 and II.4.8, let s_0 be such that

1. $k^\beta = k^\beta[s_0]$ and $d^\beta = d^\beta[s_0]$ are defined and fixed for ever;
2. For $k \le k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined, and $k \in K$ iff $\Delta(W_e, C; k)[s_0] = 1$ and $k \in K[s_0]$;
3. For all $k < k^\beta$, $\Delta(W_e, C; k)[s_0]$ is defined and fixed for ever;
4. β and $\beta\hat{}g$ are accessible at s_0 and not initialized after s_0.

Let $(s_m : m \in \omega)$ increasingly enumerate all stages such that $s_m \ge s_0$, both β and $\beta\hat{}g$ are accessible at s_m. For each m, let t_m be the first α-expansionary stage after s_m, then $t_m \le s_{m+1}$. Hence β open a gap at s_m while α closes this gap at t_m.

If α closes a gap successfully at t_m, by (4) of Procedure II.4.2, $D(d)[t_m] = 1$. By the choice of s_0, d is fixed for ever, and β opens no gap after t_m.

Hence α always closes gaps unsuccessfully. Thus

$$W_e[t_m] \upharpoonright \delta(k)[s_m] = (W_e \upharpoonright \delta(k))[s_m] \text{ and } \delta(k)[s_m] < \delta(k)[t_m] \qquad (*).$$

We claim that

$$W_e[s_{m+1}] \upharpoonright \delta(k)[t_m] = (W_e \upharpoonright \delta(k))[t_m] \qquad (**).$$

If $s_{m+1} = t_m$ then $(**)$ holds trivially.

Assume $s_{m+1} > t_m$. If there exists some stage $t(t_m \leq t < s_{m+1})$ at which some ζ enumerates a^ζ in A, then $\zeta > \beta$.

ζ could not be above $\beta\hat{\ }1$ otherwise $\beta\hat{\ }g$ is initialized. If ζ is above $\beta\hat{\ }g$ then $\beta\hat{\ }g$ is accessible at t and $s_{m+1} \leq t$ by the definition of s_{m+1}. This contradicts with the choice of t.

Now it could only be the case that $\zeta \geq \beta\hat{\ }0$. Thus ζ is initialized by α when the gap opened at s_m is closed. Hence

$$a^\zeta > t_m > \phi_e(l^\alpha)[t_m] = \phi_e(\delta(k))[t_m]$$

and $(**)$ holds since s_{m+1} is α-expansionary.

(iii) follows from $(*)$ and $(**)$.

(iv) By (i)(ii) and the construction, $K = \Delta(W_e, C)$ if $\beta\hat{\ }g \not\subset TP$ for every $\mathcal{N}^\alpha$-strategy β. (iv) follows from this and (iii). $\square$

Finally we prove the satisfactions of $\mathcal{P}$-strategies.

Lemma II.4.10. *τ is the unique $\mathcal{P}_e$-strategy on TP, ζ is the unique $\mathcal{Q}^\tau_{n,j}$-strategy on TP and η is the unique $\mathcal{R}^\tau_{i,j}$-strategy on TP. Let $c = \|e\|$.*

(i) $M^\tau_{i',0}$, $M^\tau_{i',1}$ and Z^τ are c.e. sets ($i' < c-1$), and Θ^τ is a p.r. functional;

(ii) If $\zeta\hat{\ }\bot \not\subset TP$ then $\Theta^\tau(Y_{e_{c-1}}, Z^\tau; k^\zeta)$ converges;

(iii) $\mathcal{Q}^\tau_{n,j}$ is eventually satisfied;

(iv) $\mathcal{R}^\tau_{i,j}$ is eventually satisfied;

(v) $\mathcal{P}_e$ is eventually satisfied.

Proof. During the proof, we occasionally omit τ, ζ and η from the superscripts, and write X for X_e, etc..

(i) follows from an argument similar to that for (i) of Lemma II.4.9.

(ii) Let $o = TP(|\zeta|)$. By Lemma II.4.7, we may choose s_0 as in the proof for (ii) of Lemma II.4.9 such that

1. a^ζ is defined at stage s_0 and fixed for ever;
2. For all $k \leq k^\zeta$, $\Theta(Y, Z; k)[s_0]$ is defined, and if $k \in K$ then $K(k)[s_0] = 1 = \Theta(Y, Z; k)[s_0]$;
3. For all $k < k^\zeta$, $\Theta(Y, Z; k)[s_0]$ is defined and fixed for ever;
4. $\zeta\hat{\ }o$ is accessible at s_0 and never initialized after s_0.

If $o = 0$ then $\Theta^\tau(Y, Z; k)$ converges since $\zeta\hat{\ }\bot$ is never accessible after s_0.

If $o = 1$ then $\Theta^\tau(Y, Z; k)$ converges by (iii) of Lemma II.4.7 and (i) of Procedure II.4.5.

(iii) Let $o = TP(|\zeta|)$, and a denote the final value of a^ζ.

If $o = \bot$ then $\Psi(X, Y; a)$ diverges and $\mathcal{Q}^\tau_{n,j}$ is satisfied trivially.

Otherwise $d = d^\zeta$ and $x = x^\zeta$ are eventually fixed. The satisfaction follows easily from the definitions of outcomes 0 and 1.

(iv) If there are at most finitely many η-expansionary stages, then $\mathcal{R}^\tau_{i,j}$ is satisfied trivially.

Otherwise, assume η is never initialized after s_0 and s_0 is η-expansionary. Let $(s_m : m \in \omega)$ increasingly enumerate all η-expansionary stages $\geq s_0$.

It suffices to prove that

$$|\{x < \phi_j(l^\eta)[s_m] : x \in (M_{i,0} \cup M_{i,1})[s_{m+1} - 1] - (M_{i,0} \cup M_{i,1})[s_m]\}| \leq 1.$$

Assume $s_{m+1} > s_m$ and some ζ enumerate x^ζ in $M_{i,0}$ at some stage $t (s_m \leq t < s_{m+1})$.

If $t > s_m$ then $\zeta = \eta\hat{\ }0$ or $> \eta\hat{\ }0$. Hence ζ is initialized at s_m and $x_\zeta > s_m \geq \phi_j(l^\eta)[s_m]$.

If $t = s_m$ then $TP_{s_m} = \zeta$ and no more strategies act at s_m. This is the only case that a number less than $\phi_j(l^\eta)[s_m]$ enters $M_{i,0} \cup M_{i,1}$.

(v) By (ii) and Procedure II.4.4, if $A = \Psi(X, Y)$ then $K = \Theta(Y, Z)$. Now the satisfaction of $\mathcal{P}_e$ follows immediately from (iii) and (iv). $\square$

This end the proof of Theorem II.4.1.

Chapter III

Filters

III.1 Filters Generated by Sets

Yu and Yang (2005) and Chapter 2 demonstrate several examples of definable ideals in $\mathcal{R}$. However $\mathbf{NC}$ remains the only definable filter so far. In this chapter we will show that $\mathbf{NC}$ is not the only one. [1]

Theorem III.1.1 (Nies (2003)). *There is a scheme S_M for coding a standard model of PA^- and a scheme S_h for coding functions such that for each $\mathbf{d} > \mathbf{0}$ there are an $M \subseteq \mathbf{R}$ and a map h coded by S_M and S_h respectively and $h : M \to [\mathbf{d}, \mathbf{0}']$ is onto.*

Since the meet operation is somehow ill behaved, there are several versions of filters. For $n \leq \omega$, an *n-filter* F is a upward closed set such that if $\mathbf{a}_0, \ldots, \mathbf{a}_m \in F (m < n)$ and $\mathbf{a} = \bigwedge_{0 \leq i \leq m} \mathbf{a}_i$ exists then $\mathbf{a} \in F$.

First of all, let us formally define filters generated by sets.

Definition III.1.2. *Given $n \leq \omega$ and $\mathbf{C} \subseteq \mathbf{R}$, let $[\mathbf{C})_n$ denote the n-filter generated by $\mathbf{C}$, i.e., the set of degrees $\mathbf{x}$ such that there exists a finite sequence of degrees $\mathbf{a}_0, \ldots, \mathbf{a}_l = \mathbf{x}$ and for each $i \leq l$ either*

(i) $\mathbf{a}_i \in \mathbf{C}$*, or*

(ii) *there are $m < n$ and $j_0, \ldots, j_m < i$ such that $\bigwedge_{0 \leq m \leq k} \mathbf{a}_{j_k}$ exists and $\leq \mathbf{a}_i$.*

[1]The results in this chapter is contained in Wang and Ding (2006b)

It is obvious that $[\mathbf{C})_n$ is the least n-filter containing $\mathbf{C}$. We denote $[\mathbf{C})_\omega$ by $[\mathbf{C})$.

From now on we assume $\mathbf{0} \notin [\mathbf{C})$. An easy observation follows immediately from the definition.

Proposition III.1.3. *If $\mathbf{C} = \{\mathbf{c}_0, \ldots, \mathbf{c}_k\} \subset [\mathbf{y}, \mathbf{0}']$ then $[\mathbf{C})_n \subseteq [\mathbf{y}, \mathbf{0}']$ for $n \leq \omega$. Hence $\bigcap_{i=0}^k [\mathbf{0}, \mathbf{c}_i] \subseteq [\mathbf{0}, \mathbf{x}]$ for any $\mathbf{x} \in [\mathbf{C})_n$.*

Proof. The first part follows from an easy induction on length of finite sequences in Definition III.1.2 and directly implies the second part. □

Note that $[\mathbf{C})_n = \bigcup\{[\mathbf{F})_n : \mathbf{F}$ is a finite subset of $\mathbf{C}\}$.

Next we fix some definable (in arithmetic) coding of finite sequences of natural numbers. Let $lth(n)$ denote the length of the sequence coded by n and $(n)_i$ denote the i-th element of the sequence ($i < lth(n)$).

Theorem III.1.4. *Given $\mathbf{C} \subseteq \mathbf{R}$ definable, $[\mathbf{C})_n$ is also definable.*

Proof. Let $F = [\mathbf{C})$. For any $\mathbf{x} \in F$ and a sequence of degrees $\mathbf{a}_0, \ldots, \mathbf{a}_l$ as in Definition III.1.2. We may assume that $(\forall i \leq p)(\mathbf{a}_i \in \mathbf{C})$ for some $p \leq l$.

Fix $\mathbf{d}$ such that $\mathbf{d} > \mathbf{0}$ and $\mathbf{a}_j \in [\mathbf{d}, \mathbf{0}']$ for $j \leq l$. Let M and h be as in Theorem III.1.1, and let e code a finite sequence such that $lth(e) = l + 1$ and $h(((e)_j)^M) = \mathbf{a}_j$ for $j \leq l$. The predicate stating that the conditions of Definition III.1.2 hold for $h(((e)_0)^M), h(((e)_1)^M), \ldots, h(((e)_l)^M)$, can be expressed by a formula in the language of partial ordering and satisfactions in M. By the fact that M is interpreted in $\mathbf{R}$ and the choice of e, this predicate is definable (say by $\varphi(\mathbf{x}, \mathbf{d}, M, h, e^M)$) and holds in $\mathbf{R}$. So we have

$$\mathbf{R} \models (\exists \mathbf{d} > \mathbf{0})(\exists M)(\exists h)(\exists e^M)\varphi(\mathbf{x}, \mathbf{d}, M, h, e^M).$$

On the other hand, if we have the satisfaction above then the image under h of the finite sequence coded by e^M is a finite sequence, say $\overrightarrow{\mathbf{a}}$, satisfying the conditions of Definition III.1.2. Hence we have $\mathbf{x} \in [\overrightarrow{\mathbf{a}}) \subseteq [\mathbf{C})$. □

III.2 Degrees Cupping Cappables

The following is a well known result.

Theorem III.2.1 (Ambos-Spies et al. (1984)). *The class of capping degrees,* $\mathbf{M}$*, and that of non-capping degrees,* $\mathbf{NC}$*, form an algebraic decomposition of* $\mathbf{R}$ *into a definable prime ideal and a definable strong ultra filter.*

In Ambos-Spies et al. (1984), it is also proved that $\mathbf{NC}$, $\mathbf{ENC}$ (effectively non-capping degrees) and $\mathbf{PS}$ (promptly simple degrees) are all coincident with $\mathbf{LC}$ (low cuppable degrees).

Let $\mathbf{Cups}(\mathbf{a})$ denote the collection of degrees which cup $\mathbf{a}$ to $\mathbf{0}'$ and $\mathbf{Cups}(\mathbf{C}) = \bigcup_{\mathbf{a}\in\mathbf{C}} \mathbf{Cups}(\mathbf{a})$ where $\mathbf{C} \subseteq \mathbf{R}$. It is obvious that $\mathbf{Cups}(\mathbf{C})$ is closed upward. Note that $\mathbf{LC} = \mathbf{Cups}(\text{Ł})$. Furthermore, let $\mathbf{NCups}(\mathbf{a}) = \mathbf{R} - \mathbf{Cups}(\mathbf{a})$ and $\mathbf{NCups}(\mathbf{C}) = \mathbf{R} - \mathbf{Cups}(\mathbf{C})$.

$\mathcal{F}_1 = [\mathbf{Cups}(\mathbf{M}))$ is definable by Theorem III.1.4. We prove an interesting property for $\mathcal{F}_1$.

Theorem III.2.2 (Low Non-Diamond, Ambos-Spies (1984)). *If* $\mathbf{a}_0, \ldots, \mathbf{a}_n$ *and* $\mathbf{b}_0, \ldots, \mathbf{b}_n$ *are such that* $\bigvee\!\!\!\bigvee_{i\leq n}\mathbf{a}_i = \mathbf{0}'$ *and* $\bigvee\!\!\!\bigvee_{i\leq n}\mathbf{b}_i \in \mathbf{L}$*, then there is some* $i \leq n$ *such that* $[\mathbf{0}, \mathbf{a}_i] \cap [\mathbf{0}, \mathbf{c}] \not\subseteq [\mathbf{0}, \mathbf{b}_i]$ *for any* $\mathbf{c} \not\leq \mathbf{b}_i$*.*

Theorem III.2.3 (Harrington and Soare (1992)). *If* $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$ *then there is some* $\mathbf{c} > \mathbf{a}$ *such that* $\mathbf{c} \wedge \mathbf{b} = \mathbf{0}$*.*

The proof of Theorem III.2.3 in Harrington and Soare (1992) can be easily extended to yield a little stronger result.

Corollary III.2.4. *If* $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$ *and* $\mathbf{d} < \mathbf{0}'$ *then there is some* $\mathbf{c} > \mathbf{a}$ *such that* $\mathbf{c} \not\leq \mathbf{d}$ *and* $\mathbf{c} \wedge \mathbf{b} = \mathbf{0}$*.*

Theorem III.2.5. $\mathcal{F}_1 \cap \mathbf{L} = \emptyset$ *and hence* $\subset \mathbf{NC}$*.*

Proof. For contradiction, assume in $\mathcal{F}_1$ there is a sequence $\mathbf{a}_0, \ldots, \mathbf{a}_p, \mathbf{a}_{p+1}, \ldots, \mathbf{a}_l = \mathbf{a} \in \mathbf{L}$ as in Definition III.1.2 where $\mathbf{a}_0, \ldots, \mathbf{a}_p \in \mathbf{Cups}(\mathbf{M})$ and $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_l \notin \mathbf{Cups}(\mathbf{M})$. Then $\bigcap_{0\leq i\leq p}[\mathbf{0}, \mathbf{a}_i] \subseteq [\mathbf{0}, \mathbf{a}]$.

Let $q \leq p$ be the least such that $\bigcap_{0\leq i\leq q}[\mathbf{0}, \mathbf{a}_i] \subseteq [\mathbf{0}, \mathbf{a}]$, then $q > 0$ since $a_0 \notin \mathbf{L}$. Fix $\mathbf{b} \in \bigcap_{0\leq i<q}[\mathbf{0}, \mathbf{a}_i]$ and $\mathbf{b} \not\leq \mathbf{a}$, then $[\mathbf{0}, \mathbf{b}] \cap [\mathbf{0}, \mathbf{a}_q] \subseteq \bigcap_{0\leq i\leq q}[\mathbf{0}, \mathbf{a}_i] \subseteq [\mathbf{0}, \mathbf{a}]$.

By Corollary III.2.4, choose $\mathbf{x}, \mathbf{y} \in \mathbf{M}$ such that $\mathbf{a}_q \vee \mathbf{x} = \mathbf{0}'$, $\mathbf{x} \wedge \mathbf{y} = \mathbf{0}$ and $\mathbf{y} \not\leq \mathbf{a}$. Then we get a contradiction to Theorem III.2.2. □

It is obvious that the above argument works for any joint-closed $\mathbf{C} \subseteq \mathbf{M}$. Moreover such $[\mathbf{Cups}(\mathbf{C}))$'s are always non-ultra since every c.e. degree could be split into two low ones.

Proposition III.2.6. *For any joint-closed* $\mathbf{C} \subseteq \mathbf{M}$, $[\mathbf{Cups}(\mathbf{C}))$ *is non-ultra and disjoint with* $\mathbf{L}$, *hence* $\subset \mathbf{NC}$. *In particular,* $\mathcal{F}_1$ *is non-ultra.*

A natural question arises that whether $\mathcal{F}_1$ is strong. Or more aggressively,

Problem III.2.7. *Given* $\bar{\mathbf{a}} \in \mathbf{Cups}(\mathbf{M})$, *is there always a* $\mathbf{b} \in \bigcap_{\mathbf{a} \in \bar{\mathbf{a}}} [\mathbf{0}, \mathbf{a}]$ *which is the infimum of some* $\bar{\mathbf{c}} \in \mathbf{Cups}(\mathbf{M})$*?*

III.3 Non-splitting Bases

In this section we will prove that the so called Lachlan-Harrington's nonsplitting bases generate a filter different from $\mathbf{NC}$ and $\mathcal{F}_1$. Lachlan (1976) introduced the $\mathbf{0}'''$ injury argument when constructing a pair of degrees $\mathbf{a} < \mathbf{b}$ with $\mathbf{b}$ non-splittable above $\mathbf{a}$. Later Harrington improved this monster result by showing that $\mathbf{b}$ could be $\mathbf{0}'$. We call those $\mathbf{a}$'s in Harrington's result as Lachlan-Harrington's nonsplitting bases and denote them by $\mathbf{NSB}$.

Proposition III.3.1. $\mathbf{NSB} \subset \mathbf{PS}$. *Hence* $[\mathbf{NSB}) \subset \mathcal{R}$.

Proof. To see $\mathbf{NSB} \subseteq \mathbf{PS}$, let $\mathbf{a} \in \mathbf{NSB}$ and choose a pair of low c. e. degrees joining to $\mathbf{0}'$. Then $\mathbf{a}$ cups one of the two to $\mathbf{0}'$.

On the other hand $\mathbf{NSB} \cap \mathbf{L} = \emptyset$ while $\mathbf{PS} \cap \mathbf{L} \neq \emptyset$. □

We denote $[\mathbf{NSB})$ by $\mathcal{F}_2$ and prove the following.

Theorem III.3.2. $\mathcal{F}_1 - \mathcal{F}_2 \supseteq \mathbf{Cups}(\mathbf{M}) - \mathcal{F}_2 \neq \emptyset$.

To prove the above theorem, we construct a c. e. set A such that $\mathbf{deg}(A) \in \mathbf{Cups}(\mathbf{M}) - \mathcal{F}_2$ using a tree of strategies, say T.

III.3.1 Making A cupping a cappable

To make $\mathbf{deg}(A) \in \mathbf{Cups}(\mathbf{M})$, we simultaneously build c. e. sets B, C and a c. e. functional Δ so that $K = \Delta(A, B)$,

$$\mathcal{M}_e : f = \Gamma_e(B) = \Gamma_e(C) \text{ is total} \Rightarrow f \text{ is computable}$$

and

$$\mathcal{P}_e : C \neq \overline{W}_e$$

where $(\Gamma_e : e < \omega)$ is an effectively enumeration of all functionals and $(W_e : e < \omega)$ sets.

Note that in general we should also make $\mathcal{P}$'s for B. But combining these requirements with those in the next subsection automatically guarantees B non-computable.

To make $K = \Delta(A, B)$ we have one global strategy which is also identified as Δ. At any stage of the construction Δ either extends the domain of Δ or repairs a $\Delta(A, B; k)$ disagreeing with $K(k)$ by enumerating $\delta(k)$ in A and redefines it.

As classical minimal pair constructions we arrange many strategies on T for an $\mathcal{M}$. Assume α is a such one. Its length of agreement l^α and α-expansionary stages are defined as usual. α has two possible outcomes, say ∞ indicating infinitely many expansionary stages and 0 for finite. If ∞ appears true then only one of $\Gamma(B) \upharpoonright l^\alpha$ and $\Gamma(B) \upharpoonright l^\alpha$ is allowed to be destroyed.

We also arrange many strategies for a $\mathcal{P}$. Let β be one. It chooses a fresh witness, say c^β, at the beginning, and waits for this witness to enter W. If β keeps waiting then it has 0 as outcome, or it puts c in C and has 1 as outcome.

The possible outcomes defined so far are ordered reversely, i. e. $\infty < 1 < 0$.

III.3.2 Keeping $\mathbf{deg}(A)$ away from $\mathcal{F}_2$

To make $\mathbf{deg}(A) \notin \mathcal{F}_2$, for each finite tuples of c. e. sets, we should either prove that one of them is not in $\mathbf{NSB}$, or find a c. e. set computable in every element of the tuple but not in A. For notational simplifications we do the above only for tuples of two elements. The general case would be an easy variance.

Hence for each pair (X_e, Y_e) of c. e. sets, we either refute that $\mathbf{deg}(A)$ is in the filter generated by $\mathbf{deg}(X_e)$ and $\mathbf{deg}(Y_e)$, or find at least two pairs (E_0, E_1) and (F_0, F_1) of c. e. sets and two c. e. functionals Θ and Ξ so that one of the pairs witnesses (via Θ or Ξ) that X_e or Y_e is not a non-splitting base.

Thus we should have a strategy, say $\tau \in T$ building the sets and functionals mentioned, and make

$$\mathcal{Q}_e : K = \Theta(E_0, E_1) = \Xi(F_0, F_1).$$

τ has exactly one outcome denoted by 0. When τ is accessible during the construction, it either extends the domain of Θ and Ξ, or repairs the definitions on some k by enumerating $\theta(k)$ and $\xi(k)$ into (E_0, E_1) and (F_0, F_1) respectively. However it depends on τ's substrategies whether $\theta(k)$ ($\xi(k)$) should go to E_0 or E_1 (F_0 or F_1).

τ has many children extending $\tau\hat{\ }0$ for all tuples (n, i, j) (where $n < \omega$ and $i, j < 2$) making

$$\mathcal{N}^{\tau}_{n,i,j} : D = \Phi_n(E_i, X) = \Psi_n(F_j, Y) \Rightarrow \exists G \leq_{\mathrm{T}} X, Y (G \not\leq_{\mathrm{T}} A)$$

for some effective enumeration of all possible combinations of the form (Φ_n, Ψ_n, i, j).

Let $\pi \supseteq \tau\hat{\ }0$ be an $\mathcal{N}^{\tau}_{n,i,j}$-strategy and $top(\pi) = \tau$. During the construction π builds a c. e. set $G \leq_{\mathrm{T}} X, Y$. It has a parameter l^{π} to measure the length of agreement between D, $\Phi(E_i, X)$ and $\Psi(F_j, X)$. Thus we may define π-expansionary stages as usual. In addition, π has two outcomes, say ∞ for infinitely many expansionary stages, and 0 for finitely many. Above $\pi\hat{\ }\infty$, we arrange children of π to make

$$\mathcal{O}^{\pi}_k : G \neq \Lambda_k(A), \text{ for each functional } \Lambda_k.$$

Now assume $\sigma \supseteq \pi\hat{\ }\infty$ is a child strategy of π, let $top(\sigma) = \pi$ and $toop(\sigma) = top(\pi)$. In isolation σ acts as below

1. Pick an agitator d^{σ} and wait for $l^{\pi} > d$. We say d *realized* if the inequality realized.
2. Choose a witness $g^{\sigma} > \phi(d), \psi(d)$ and keep it from entering G. If later $\phi(d)$ or $\psi(d)$ grows then cancel g and go back to (1) for waiting.
3. Wait for $\Lambda(A; g) \downarrow = 0$, i. e. g to be *realized.*

4. Enumerate d in D, impose restraints $r^\sigma(E) = \phi(d)$ and $r^\sigma(F) = \psi(d)$ on E_i and F_j respectively, and $r^\sigma(A) = \lambda(g)$ on A.
5. At next π-expansionary stage, enumerate g in G.

If σ waits at (1) for ever or goes back to (1) infinitely often, then it has ∞ as outcome indicating Φ or Ψ diverges at d and hence a failure of the premise of $\mathcal{N}$. We arrange no more children for π above $\sigma\hat{}\infty$. If g is never realized then σ has 0 as its outcome. If σ eventually reaches (5) then it has 1 as final outcome.

The intuition of the above procedure is that when g is realized, σ diagonalizes against the premise of π for obtaining permissions from X and Y of enumerating g in G. As there may exist many strategies between π and σ, σ in addition creates a link (π, σ) when it finishes (4), for catching permissions in time. At next π-expansionary stage the construction should jump via this link from π to σ and let σ finish (5) promptly.

The outcomes are ordered as before.

III.3.3 Coordinating strategies

The first kind of conflicts between strategies arises when $\mathcal{O}$-strategies on T try to protect Λ-computations at their witnesses. They do this by imposing a restraint on A, but Δ may need to enumerate some $\delta(k)$ in A for repairing $\Delta(A, B; k)$.

To solve these conflicts, we assign a new parameter k^σ for each $\mathcal{O}$-strategy σ on T. σ defines k fresh at the beginning. When it is ready to change $D(d)$, besides setting $r^\sigma(A)$, σ lifts $\delta(k')$ for $k' \geq k$ by enumerating $\delta(k')$ in B and redefining all such $\delta(k')$'s. Thus Δ would not violate $r^\sigma(A)$ if it were to repair $\Delta(A, B; k')$ for $k' \geq k$.

But the restraint may still be violated if Δ needs to repair $\Delta(A, B; k')$ for $k' < k$. If this happens, we let Δ *reset* σ by canceling $g(\sigma)$, $r^\sigma(Z)$ for $Z = A, X$ or Y, and any link $(top(\sigma), \sigma)$. Note that Δ does not cancel k.

Moreover to make $\mathcal{M}$'s at the same time, when σ lifts $\delta(k)$ the stage should be ended immediately as in typical minimal pair constructions.

There are similar conflicts between $\mathcal{Q}$- and $\mathcal{O}$-strategies, but the solution is different. We apply the trick in the proof of Sacks Splitting Theorem. Suppose an $\mathcal{Q}$-strategy τ needs to repair some $\theta(k)$ and $\xi(k)$. First it finds the most prior grandchild $\mathcal{O}$-strategy, say σ, having restraints $r^\sigma(E) \geq \theta(k)$ or $r^\sigma(F) \geq \xi(k)$. Then τ enumerates $\theta(k)$ in E_{1-i^σ} and $\xi(k)$ in F_{1-j^σ}. Thus $r^\sigma(E)$ and $r^\sigma(F)$ are eventually never violated.

III.3.4 The construction

Stage 0. Initialize all parameters. Let $TP_0 = \emptyset$.

Stage $s > 0$. First deal with Δ. Let x be the least number such that either $\Delta(A, B; x)$ is undefined or $K(x) \neq \Delta(A, B; x)$. If $\Delta(A, B; x)$ is undefined, simply define it to be $K(x)$ with $\delta(x) = s$. Otherwise, let $\sigma \in T$ be the most prior $\mathcal{O}$-strategy with $r^\sigma(A) \geq \delta(x)$, do the followings to reset σ

1. cancel g^σ, $r^\sigma(Z)$ for $Z = A, E$ and F, and
2. cancel any link of the form $(top(\sigma), \sigma)$.

Then enumerate $\delta(x)$ in A and redefine $\Delta(A, B; x) = K(x)$ with $\delta(x)[s] = s$, initialize all $\zeta > \sigma$ on T.

Then deal with strategies on T. We will define an approximation TP_s to the true path. Once TP_s is determined, initialize all strategies $> TP_s$ and goto stage $s + 1$.

Let $\emptyset$ be accessible. Assume $\eta \in T$ is accessible.

Case 1. η is an $\mathcal{M}_e$-strategy. If s is η-expansionary, let $o = \infty$. Otherwise let $o = 0$.

If $|\eta| = s$, then let $TP_s = \eta$. If $|\eta| < s$ then let $\eta\hat{\ }o$ be accessible.

The above routine of checking the length of an accessible strategy is also called in other cases when an outcome is determined. We will take this for granted and mention no more.

Case 2. η is an $\mathcal{P}_e$-strategy. If $C \cap W \neq \emptyset$, let 1 be the current outcome.

Otherwise define c^η fresh if it is undefined. If $c \in W$ then enumerate c in C, let $TP_s = \eta$ and goto stage $s + 1$. If $c \notin W$ then let 0 be the current outcome.

Case 3. η is an $\mathcal{Q}_e$-strategy. Let x be the least number such that

1. $\Theta(E_0, E_1; x)$ and $\Xi(F_0, F_1; x)$ are undefined, or
2. $K(x) \neq \Theta(E_0, E_1; x)$ and $K(x) \neq \Xi(F_0, F_1; x)$.

If (1) applies, define $\Theta(E_0, E_1; x) = \Xi(F_0, F_1; x) = K(x)$ with $\theta(x) = \xi(x) = s$.

If (2) applies, let σ be the most prior $\mathcal{O}$-strategy on T with $toop(\sigma) = \eta$ and $\min\{\theta(x), \xi(x)\} \leq \max\{r^\sigma(E), r^\sigma(F)\}$, enumerate $\theta(x)$ in E_{1-i^σ} and $\xi(x)$ in F_{1-j^σ}. Moreover, initialize $\zeta > \sigma$ on T.

Either one of the above two happens, let 0 be the current outcome.

Case 4. η is an $\mathcal{N}$-strategy. If s is η-expansionary and there is a link (η, σ) then let σ be accessible.

If s is expansionary and there is no link of the form (η, σ) then let ∞ be the current outcome. Otherwise let 0 be the current outcome.

Case 5. η is an $\mathcal{O}$-strategy. Let $\pi = top(\eta)$.

Subcase 5.1. There is a link (π, η). Cancel this link, enumerate g^η in G^π and let $TP_s = \eta$.

Subcase 5.2. There is no link (π, η). If g is defined and in G, then let 1 be the current outcome.

If k^η (d^η) is undefined, let it be fresh. If either $d > l^\pi$ or g is defined but $\leq \max\{\phi(d), \psi(d)\}$, then cancel g, $r^\eta(Z)$ (for $Z = A, E$ and F) and let ∞ be the current outcome.

Otherwise, let g be fresh if it is undefined. If $\Lambda(A; g) \neq 0$ then let 0 be the current outcome. If $\Lambda(A; g) = 0$, put d in D, $r^\eta(A) = \lambda(g)$, $r^\eta(E) = \phi(d)$ and $r^\eta(F) = \psi(d)$, enumerate $\delta(k)$ in B and redefine $\Delta(A, B; k) = K(k)$ with $\delta(k) = s$. Finally create a link (π, η) and let $TP_s = \eta$.

III.3.5 The verification

It follows immediately from the construction that all sets and functionals built are c. e..

Lemma III.3.3 (True Path)**.** *Let $TP = \liminf_s TP_s$, for each $n \in \omega$,*

1. *$|TP| \geq n$,*
2. *$TP \restriction n$ is accessible infinitely often,*
3. *$TP \restriction n$ is initialized and reset at most finitely many times, and*
4. *if $TP \restriction n$ is an $\mathcal{O}$-strategy then $r^{TP\restriction n}(Z)$ converges for $Z = A, E$ and F.*

Proof. We prove (1)-(4) simultaneously by induction on n. Let $\eta = TP \restriction n$. The cases where $n = 0$ or η is an $\mathcal{M}$-, $\mathcal{P}$- or $\mathcal{Q}$-strategy are trivial.

η is an $\mathcal{N}_{n,i,j}$-strategy. It suffices to prove that $\eta\hat{\ }\infty$ is accessible infinitely often if there are infinitely many η-expansionary stages. Let s be an η-expansionary stage

and assume η is never initialized after s. If $\eta\hat{}\infty$ is not accessible at s, then there is a link (η, σ) at the beginning of stage s. By Subcase 5.1 in the construction, this link is canceled and no link is created at s. Hence $\eta\hat{}\infty$ is accessible at next η-expansionary stage.

η is an $\mathcal{O}_k$-strategy. Let $\pi = top(\eta)$ and $\tau = toop(\eta)$. By induction hypothesis, $d = \lim_s d^\eta[s]$, $k = \lim_s k^\eta[s]$ and

$$r = \lim_s \max\{r^\sigma(Z) : \sigma < \eta, Z = A, E, F\}[s]$$

exist. Let s_0 be a stage when d, k and r reach their final values. In addition we may assume that

1. η is never initialized after s_0,
2. $K(x) = \Delta(A, B; x)[s_0] \downarrow$ for $x < k$ and
3. $K(y) = \Theta(E_0, E_1; y)[s_0] \downarrow = \Xi(F_0, F_1; y)[s_0] \downarrow$ if $\theta(x)[s_0]$ or $\xi(x)[s_0] \leq r$

where Θ, Ξ, E_0, E_1, F_0 and F_1 are c. e. functionals and sets built at τ.

If η changes $D(d)$ at some $s_1 > s_0$ then $\Lambda(A; g) \downarrow = 0 = G(g)$ and

$$D(d) = \Phi(E_{i(\eta)}, X; d) \downarrow = \Psi(F_{j(\eta)}, Y; d) \downarrow$$

at the beginning of s_1. While at the end of s_1, we have

1. $D(d) \neq \Phi(E_{i(\eta)}, X; d) \downarrow = \Psi(F_{j(\eta)}, Y; d) \downarrow$,
2. $\delta(k) > \lambda(g) = r(\eta, A)$, and
3. $r(\eta, E), r(\eta, F)$ are defined.

By the choice of s_0, the computation $\Gamma(A; g)$ and g are fixed for ever. Let $s_2 > s_1$ be the next π-expansionary stage, then $g \in G[s_2]$. Hence $\eta\hat{}1 \subseteq TP$.

The lemma follows immediately. $\square$

From the second part of the above proof, $\Delta(A, B)$ is total and equals K. Moreover each $\mathcal{P}$ is satisfied from (2) and (3) of the Lemma.

Lemma III.3.4 (Minimal Pair)**.** *Each $\mathcal{M}_e$ is satisfied.*

Proof. Let $\alpha \subset \alpha\hat{}\infty \subset TP$ be an $\mathcal{M}_e$-strategy. By the True Path Lemma we may assume that α is neither initialized nor reset after s. Let $s_0 < s_1 < \ldots < s_n < \ldots$ enumerate all α-expansionary stages after s, and let $l_n = l^\alpha[s_n]$.

If at the end of s_n there is no link of the form (π, σ) with $\pi \subset \alpha \subset \sigma$, then a typical minimal pair argument shows that $f[s_n] \upharpoonright l_n = f[s_{n+1}] \upharpoonright l_n$.

If there does exist a link as described then σ changes B but no one changes C at s_n. Moreover no strategy could change $C \upharpoonright \gamma(l_n)[s_n]$ before s_{n+1}, by initializations and the choice of s. Hence we have

$$f[s_n] \upharpoonright l_n = \Gamma(C)[s_n] \upharpoonright l_n = \Gamma(C)[s_{n+1}] \upharpoonright l_n = f[s_{n+1}] \upharpoonright l_n.$$

So we can compute f. □

For each $\mathcal{Q}$-strategy $\tau \subset TP$, the equality $K = \Theta^\tau(E_0, E_1) = \Xi^\tau(F_0, F_1)$ is automatically guaranteed by the construction. We have the last lemma.

Lemma III.3.5. *Let $\pi \subset TP$ be an $\mathcal{N}^\tau_{n,i,j}$-strategy and top$(\pi) = \tau$, then $G^\pi \leq_{\mathrm{T}} X^\tau, Y^\tau$ and each $\mathcal{O}^\pi_k$ is satisfied.*

Proof. The lemma holds trivially if there are at most finitely many π-expansionary stages. So we may assume that $\pi\hat{}\infty \subset TP$ and π is neither initialized nor reset after some stage s_0.

To compute $G(x)$ from X, find a stage $t > s_0$ at which $\pi\hat{}\infty$ is accessible and $X \upharpoonright x = X[t] \upharpoonright x$. Then $G(x) = G(x)[t]$. The algorithm of computing G from Y is similar.

Now let σ be an $\mathcal{O}^\pi_k$-strategy on T with $top(\sigma) = \pi$. By the True Path Lemma we may assume that $d = \lim_s d^\sigma[s] = d^\sigma[s_0]$. If $\sigma\hat{}\infty \subset TP$ then the premise of $\mathcal{N}$ is obviously false and the lemma holds.

Otherwise we may assume $g = \lim_s g^\sigma[s] = g^\sigma[s_0]$. By the construction, once π enumerates g^σ in G, $\Lambda(A; g^\sigma)$ could be changed only if some strategy reset π. But then g^σ were canceled. Hence after s_0 whenever an inequality $G(g) = 1 \neq \Lambda(A; g)$ is setup, it lasts for ever. If such an inequality never appears then g is never realized and $\Lambda(A; g) \neq 0 = G(g)$. □

III.4 The Supreme of $\mathcal{F}_1$ and $\mathcal{F}_2$

In this section we prove that the supreme of $\mathcal{F}_1$ and $\mathcal{F}_2$ is a proper subset of $\mathbf{NC}$.

Theorem III.4.1. $[\mathbf{Cups}(\mathbf{M}) \cup \mathbf{NSB})$ *is a proper definable subfilter of* $\mathbf{NC}$.

The definability is obvious. It suffices to construct a promptly simple set A with its degree not in $[\mathbf{Cups}(\mathbf{M}) \cup \mathbf{NSB})$.

III.4.1 Making A promptly simple

To make A promptly simple, we will make A coinfinite and

$$\mathcal{P}_e : W_e \text{ is infinite} \Rightarrow \exists x, s(x \in W_{e,\text{at } s} \cap A_s).$$

We will have exactly one strategy for each $\mathcal{P}_e$, denoted by P_e. These P's will be ordered naturally by their indices. If at some stage s, P_e finds that $\mathcal{P}_e$ is not satisfied so far and there is an element greater than $2e$ showed up in W_e then P_e puts the least x as above in A and declares $\mathcal{P}_e$ satisfied.

III.4.2 Avoiding the filter

As in the last section, we will only make $\mathbf{deg}(A) \notin [\{\mathbf{u}, \mathbf{v}, \mathbf{x}^0, \mathbf{x}^1\})$ for any tuple $(\mathbf{u}, \mathbf{v}, \mathbf{x}^0, \mathbf{x}^1) \in \mathbf{NSB}^2 \times \mathbf{Cups}(\mathbf{M})^2$. The theorem should follow from a simple generalization.

The strategies described in this subsection will be arranged on a tree of strategies, say T.

Given $(U_e, V_e, X_e^0, X_e^1, Y_e, Z_e)$ a tuple of c. e. sets. We have a $\tau \in T$ assuming that (U_e, V_e, X_e^0, X_e^1) is a tuple of representatives of degrees as above, $\mathbf{y}$ witnesses $\mathbf{x}^0, \mathbf{x}^1 \in \mathbf{Cups}(\mathbf{M})$ (i. e. Y_e cups both X_e^0 and X_e^1 to $\emptyset'$) and $\mathbf{z}$ witnesses $\mathbf{y} \in \mathbf{M}$ (i. e. they form a minimal pair). τ builds c. e. sets E_0, E_1, F_0, F_1, functionals Γ, Λ and an additional c. e. set D for diagonalization so that

$$\mathcal{Q}^\tau : K = \Gamma(E_0, E_1) = \Lambda(F_0, F_1)$$

and

$$\mathcal{M}^{\tau}_{m,i,j} : D = \Phi_m(E_i, U) = \Psi_m(F_j, V) = \Delta^0_m(X^0_e, Y_e) = \Delta^1_m(X^1_e, Y_e) \Rightarrow$$
$$\exists B \leq_{\mathrm{T}} U_e, V_e, X^0_e, X^1_e (B \not\leq_{\mathrm{T}} A) \text{ or } \exists C \leq_{\mathrm{T}} Y_e, Z_e (C \not\leq_{\mathrm{T}} \emptyset)$$

where $(\Phi_m, \Psi_m, \Delta^0_m, \Delta^1_m, i, j)$ effectively ranges over all possible combinations of functionals when (m, i, j) effectively ranges over $\omega \times 2 \times 2$.

The only direct responsibility of τ is to meet $\mathcal{Q}$, hence it will be referred as $\mathcal{Q}$-strategy. τ has exactly one immediate successor on T, denoted by $\tau\hat{\ }0$. To define Γ, Λ, τ extends the definitions of the functionals by stage using big uses. Once it finds Γ or Λ is wrong at some parameter, τ enumerates the corresponding use in E_0, E_1 or F_0, F_1 and corrects the definition.

To meet $\mathcal{M}$'s, τ has many substrategies for each $\mathcal{M}$ as its successors. Suppose π is such one, let $top(\pi) = \tau$. We then define the length of agreement l^{π} and expansionary stages for π as usual. π has two possible outcomes, say ∞ indicating that there are infinitely many expansionary stages, and 0 for finitely many. Now π has l^{π} to measure the premise of $\mathcal{M}$. For the consequence of $\mathcal{M}$, π builds B and C subjecting to proper permissions and has its substrategies above $\pi\hat{\ }\infty$ to make for some proper effective indexing,

$$\mathcal{N}^{\pi}_n : B = \Theta_n(A) \text{ and } C = \overline{W}_n \Rightarrow Z_e \text{ is computable}$$

where (Θ_n, W_n) effectively ranges over all combinations of functional and set.

Let σ be a substrategy of π, $top(\sigma) = \pi$ and $toop(\sigma) = top(\pi)$. At the beginning, σ picks a fresh *agitator* d and waits for the premise of $\mathcal{M}$ established at d. Secondly it picks a fresh *witness* b, and waits for $B(b) = 0 = \Theta(A)$, say *realized*. Next σ picks a fresh follower c_0 and wait for c_0 realized, i. e. $c_0 \in W_n$. If b and c are realized then σ waits for changes of $Z \upharpoonright c_0$ (i. e. permission by Z) and simultaneously picks another follower c_1 and so on.

If $Z \upharpoonright c_k$ never changes after c_k realized then we can compute Z. Thus we refute the assumption of τ and have a global win for τ. If $Z \upharpoonright c_k$ changes for some c_k realized, then σ enumerates d in D *immediately* (even if σ does not appear to be on the true path) and creates a link (π, σ) for catching further permissions needed promptly. At next π-expansionary stage, the flow of control follows the link and jumps from π to σ (say the link is *traveled*). Now either U, V, X^0 and X^1 change below $u(d) = \max\{\phi(d), \psi, \delta^0(d), \delta^1(d)\}$ or Y changes below $u(d)$. If the first case happens, σ enumerates b in B and the second enumerates c_k in C. Either one σ obtains a local win, cancels the link and terminates.

Note that $u(d)$ could move after σ picks b and c_0. If this happens b or c_0 might be too small to be permitted, then σ cancels this witness and all defined followers. Similarly, $\theta(b)$ could move after σ picks c_0, then σ cancels all defined followers.

σ has four possible outcomes listed below.

- ∞ - $u(d)$ moves infinitely often;
- b - $\theta(b)$ moves infinitely often, b is never realized or $b \in B$;
- z - There are infinitely many defined followers but none is ever permitted by Z, thus Z is computable;
- c - Some follower is either never realized or enumerated in C.

Both ∞ and z mean global wins for $\tau = toop(\sigma)$, thus we arrange no descendance of τ above $\sigma\hat{}\infty$ and $\sigma\hat{}z$. While b and c indicate local wins for σ, descendance of τ and π are arranged above to make $\mathcal{M}^\tau$'s and $\mathcal{N}^\pi$'s.

All possible outcomes in this subsection are ordered as $\infty < b < z < c < 0$. The tree of strategies, say T, is defined in a usual way.

III.4.3 Coordinating strategies

Note that once an $\mathcal{N}$-strategy, say σ, puts its agitator in D, it expects no move of $\theta(b)$. Hence it should impose a restraint on A. On the other hand, to force changes of $U \upharpoonright u(d)$ and $V \upharpoonright u(d)$, it should also restrain E_i and F_j. Hence there are two kinds of conflicts, the first kind between $\mathcal{N}$-strategies and P's and the second between $\mathcal{N}$-strategies and their grandparents $\mathcal{Q}$-strategies.

To solve the first kind, we have σ choose a fresh *killing point*, say k, at the very beginning. When it expects no change of $A \upharpoonright \theta(b)$, σ imposes a restraint $r(A) = \theta(b)$ on A. This $r(A)$ will be respected by every P_e with $e \geq k$. But the other k many P's are free to violate $r(A)$. If such a violation happens, we cancel all parameters (including possible links) associated with σ *except the killing point*, and say that σ *is reset*. Since each P needs only one chance for winning, σ can be reset by P's at most finitely often.

On the other hand we will ensure that there are only finitely many strategies having killing point below one e and their restraints on A are bounded.

The second kind of conflicts is solved in essentially the same way as in the last section.

III.4.4 The construction

During the construction, the even stages are devoted to P's and the odd stages are devoted to T. At the beginning we have all parameters undefined and all c. e. sets and functionals to be built empty.

At an even stage s, we say a P_i *requiring attention* if and only if $i < s$, $\mathcal{P}_i$ is not satisfied so far and there is an x such that

$$x \in W_{i,\text{at } s} - A \text{ and } x > 2i, \max\{r^\sigma(A) : \sigma \in T, k^\sigma < i\}$$

If there is no P requiring attention, proceed to stage $s+1$. Otherwise let i be the least index with P_i requiring attention, x be the corresponding least element and σ be the most prior strategy on T with $r^\sigma(A) > x$, put x in A, reset σ and initialize all strategies on T and less prior than σ. Finally declare $\mathcal{P}_i$ *satisfied* and proceed to next stage.

At an odd stage s, we will define a finite sequence of accessible strategies on T and an approximation of the true path, say TP_s. A strategy, say η, acts immediately after it becomes accessible. When it finishes its jobs, there are three possible cases, i. e. an outcome o is determined, TP_s is defined, or the next accessible strategy is determined. If the first case happens and $|\eta| \geq s$ then define $TP_s = \eta$. If the first case happens but $|\eta| < s$ then let $\eta\hat{\ }o$ be accessible. In whichever case, if TP_s is determined, initialize strategies less prior than TP_s and proceed to next stage. Initially let $\emptyset$ be accessible. Given η accessible, the jobs of η are defined by cases.

Case 1. η is a $\mathcal{Q}_e$-strategy. Let x be the least number such that

$$\Gamma(E_0, E_1; x) \uparrow \text{ and } \Lambda(F_0, F_1; x) \uparrow \text{ or } K(x) \neq \Gamma(E_0, E_1; x) = \Lambda(F_0, F_1; x).$$

If $\Gamma(E_0, E_1; x) \uparrow$ and $\Lambda(F_0, F_1; x) \uparrow$, let $\Gamma(E_0, E_1; x) = \Lambda(F_0, F_1; x) = K(x)$ with $\gamma(x)$ and $\lambda(x)$ fresh.

Otherwise, let $\sigma \in T$ be the most prior $\mathcal{N}$-strategy such that $toop(\sigma) = \eta$ and $\max\{r^\sigma(E), r^\sigma(F)\} > \min\{\gamma(x), \lambda(x)\}$. Put $\gamma(x)$ in E_{1-i^σ} and $\lambda(x)$ in F_{1-j^σ}, redefine $\Gamma(E_0, E_1; x) = \Lambda(F_0, F_1; x) = K(x)$ with $\gamma(x)$ and $\lambda(x)$ fresh, initialize strategies less prior than σ.

In both cases, let 0 be the outcome.

Case 2. η is an $\mathcal{M}^\tau_{m,i,j}$-strategy where $\tau = top(\eta)$.

If s is not η-expansionary, let 0 be the outcome.

If s is expansionary and there is a link of the form (η, σ), let σ be accessible and cancel this link.

Otherwise, let $s_0 < s$ be the last η-expansionary stage. Say an $\mathcal{N}^\eta$-strategy $\sigma \supseteq \eta\hat{\ }\infty$ *requiring attention* if and only if

1. d^σ, b^σ are defined and $b^\sigma > u(d) = \min\{\phi(d), \psi(d), \delta^0(d), \delta^1(d)\}$,
2. $\Theta(A; b^\sigma) \downarrow= 0 = B(b^\sigma)$,
3. $C \cap W_{n^\sigma} = \emptyset$,
4. c^σ_k is realized at s_0 and $Z[s_0] \upharpoonright c^\sigma_k \neq Z \upharpoonright c^\sigma_k$ for some k.

If s_0 is undefined or there is no $\mathcal{N}$-strategies requiring attention, let ∞ be the outcome. Otherwise let the most prior one requiring attention, say σ, be accessible.

Case 3. η is an $\mathcal{N}^\pi_n$-strategy where $\pi = top(\eta)$. If b is defined and in B, let b be the outcome. If $C \cap W_n \neq \emptyset$, let c be the outcome. Otherwise act according to the applicable subcase below.

Subcase 3.1. η becomes accessible because a link (π, η) is just traveled. Let $s_0 < s$ be the stage when this link was created. If $Y \upharpoonright u(d)[s_0] \neq (Y \upharpoonright u(d))[s_0]$ then put the greatest defined follower in C. Otherwise put b in B. Let $TP_s = \eta$.

Subcase 3.2. η becomes accessible because π finds it requiring attention. Put d in D, let $r(E) = r(F) = u(d)$ and $r(A) = \theta(b)$, create a link (π, η) and let $TP_s = \eta$.

Subcase 3.3. None of the above subcases applicable. Define k and d fresh if they are undefined. If $d > l^\pi$ then let $TP_s = \eta$ and stop.

Let $u(d)$ be as in Case 2. If b is defined and less than $u(d)$ then cancel b and all defined followers, let ∞ be the outcome.

Otherwise define b fresh if it is not defined. If b is not realized, let b be the outcome. If b is realized and c_0 is defined, let s_0 be the stage when c_0 became defined. If $(A \upharpoonright \theta(b))[s_0] \neq A \upharpoonright \theta(b)$, then cancel all followers and let b be the outcome.

Otherwise let k be the least number such that c_k is undefined. If $k = 0$ or c_{k-1} is realized, define c_k fresh and let z be the outcome. If $k > 0$ and c_{k-1} is not realized then let c be the outcome.

III.4.5 The verification

It is obvious from the construction that all sets and functionals constructed are c. e..

Lemma III.4.2 (True Path)**.** *Let $TP = \liminf_s TP_s$. For each n we have the following.*

1. *$|TP| \geq n$,*
2. *$TP \upharpoonright n$ is accessible infinitely often,*
3. *$TP \upharpoonright n$ is initialized at most finitely often,*
4. *If $TP \upharpoonright n$ is an $\mathcal{N}$-strategy then the limits of $r^{TP\upharpoonright n}(A), r^{TP\upharpoonright n}(E)$ and $r^{TP\upharpoonright n}(F)$ exist, and $TP \upharpoonright n$ is reset at most finitely often.*

Proof. We prove (1)-(4) simultaneously by induction. It is obvious for $n = 0$. Assume the lemma holds for n and $\eta = TP \upharpoonright n$.

Case 1. η is a $\mathcal{Q}_e$-strategy. (1)-(3) hold trivially as η has only one immediate successor on T. (4) is included in Case 3.

Case 2. η is a $\mathcal{M}^{\tau}_{n,i,j}$-strategy. Firstly we prove (1)-(3) for $n + 1$. To this end it suffices to prove that $|TP| \geq n + 1$.

The situation that there are at most finitely many η-expansionary stages is trivial. Now assume there are infinitely many η-expansionary stage. Let $s_0 > n$ be a stage from when on η is never initialized. Assume $s > s_0$ is η-expansionary but $\eta\hat{\ }\infty$ is not accessible at stage s. Then either there is a $\mathcal{N}^{\eta}$-strategy requiring attention or there is a link (η, σ).

In the first case, a link like (η, σ) is created at stage s. Thus this case is reduced to the second one.

In the second case this link is canceled and no new link is created. Note that only those $\mathcal{N}^{\eta}$-strategies prior than σ could require attention before $\eta\hat{\ }\infty$ becomes accessible again. Hence $\eta\hat{\ }\infty$ is eventually accessible again.

(4) for this case is included in Case 3.

Case 3. η is an $\mathcal{N}_n^\pi$-strategy where $\pi = top(\eta)$. By induction hypothesis, let s_0 be a stage from when on k^η is fixed. Hence η is neither initialized nor reset after s_0. Suppose $r^\eta(A), r^\eta(E)$ and $r^\eta(F)$ become defined at some stage $s > s_0$. Then at stage s, we have for $d = d^\eta$,

$$D(d)[s] = 1 \neq 0 = D(d)[s-1] = \Xi(H; d)[s]$$

where $\Xi(H) = \Phi(E_i, U), \Psi(F_j, V), \Delta^0(X^0, Y)$ or $\Delta^1(X^1, Y)$.

Let $s' > s$ be the next π-expansionary stage, then

$$D(d)[s'] = \Xi(H; d)[s'] = 1 \neq 0 = \Xi(H; d)[s]$$

where $\Xi(H)$ is as above. As by induction hypothesis the restraints are never initialized after s_0, either U, V, X^0 and X^1 change below $u(d)$ or $Y \upharpoonright u(d)$ changes, and either $b^\eta \in B[s']$ or $C[s'] \cap W$ is not empty. By the construction, the restraints are fixed for ever.

(1) and (2) are obvious. For (3), since the restraints imposed by η are eventually fixed, $toop(\eta)$ could initialized $TP \upharpoonright n+1$ for correcting Γ and Λ at most finitely often. □

The next lemma follows immediately from (4) of the True Path Lemma and the construction.

Lemma III.4.3. *All $\mathcal{P}$'s and $\mathcal{Q}$'s are satisfied.*

Finally let $\pi \subset TP$ be an $\mathcal{M}_{m,i,j}^\tau$-strategy where $\tau = top(\pi) \subset TP$ is a $\mathcal{Q}_e$-strategy, we prove the satisfaction of this $\mathcal{M}$ and end the proof of the theorem.

Lemma III.4.4. *$\mathcal{M}_{m,i,j}^\tau$ is eventually satisfied.*

Proof. If $\pi\hat{\ }0 \subset TP$, then the lemma holds trivially. Assume $\pi\hat{\ }\infty \subset TP$, then there are infinitely many π-expansionary stages. Let σ be an $\mathcal{N}_n^\pi$-strategy on TP.

By the True Path Lemma, $d = \lim_s d^\sigma[s]$ exists. Thus if $\sigma\hat{\ }\infty \subset TP$ then some $\Xi(H; d)$ diverges for $\Xi(H) = \Phi(E_i, U), \Psi(F_j, V), \Delta^0(X^0, Y)$ or $\Delta^1(X^1, Y)$ and $\mathcal{M}$ is satisfied since its premise is refuted.

If $\sigma\hat{}\infty \not\subset TP$ then $b = \lim_s b^\sigma[s]$ exists. Suppose $\sigma\hat{}b \subset TP$, then either $\Theta(A; b)$ changes infinitely often, b is never realized or $b \in B$. The first two cases imply $B(b) = 0 \neq \Theta(A; b)$. For the last case, assume b is enumerated in B at stage s then $B(b) = 1 \neq 0 = \Theta(A; b)[s]$ and $(r^\sigma(A) = \theta(b))[s]$ and is never violated. Hence $B(b) = 1 \neq 0 = \Theta(A; b)$.

Next assume $\sigma\hat{}z \subset TP$, then σ has infinitely many realized permanent followers. By the True Path Lemma and Case 2 in the construction, $Z \upharpoonright c_k^\sigma = Z[s_k] \upharpoonright c_k^\sigma$ where c_k^σ is a permanent follower, s_k is a stage when c_k^σ has been realized and $\pi\hat{}\infty$ is accessible.

Finally assume $\sigma\hat{}c \subset TP$, then either σ has finitely many permanent followers and the greatest one is never realized, or $C \cap W_n \neq \emptyset$. In the first case $\overline{W}_n - C \neq \emptyset$.

The satisfaction of $\mathcal{M}$ follows from the above argument. □

III.5 Conclusions

III.5.1 A counter example

People might expect that diamond bases, i. e. infima of splittings of $\mathbf{0}'$, also generate a proper filter. However by Ambos-Spies et al. (1994), the distributive lattice shown in Figure 1 can be embedded in $\mathcal{R}$ preserving 0 and 1 (where $a_0 \wedge a_1$ is joint-irreducible and noncappable). Thus there are two diamond bases, namely $\mathbf{c}_0$ and $\mathbf{c}_1$ (the images of c_0 and c_1), forming a minimal pair.

Proposition III.5.1. $[\mathrm{DB})$ *is not a proper filter.*

III.5.2 Remarks

Firstly, observe that the proof of Theorem III.4.1 could be improved to made $\mathbf{deg}(A) \in \mathbf{H}$. Hence by the Jump Interpolation Theorem in Robinson (1971), we have a clear picture of the relation between $\mathcal{F}_1$ and jump hierarchy.

Proposition III.5.2. $\mathrm{NC} - \mathcal{F}_1$ *meets every jump classes.*

Secondly we raise a natural question about $\mathcal{F}_1$ and $\mathcal{F}_2$.

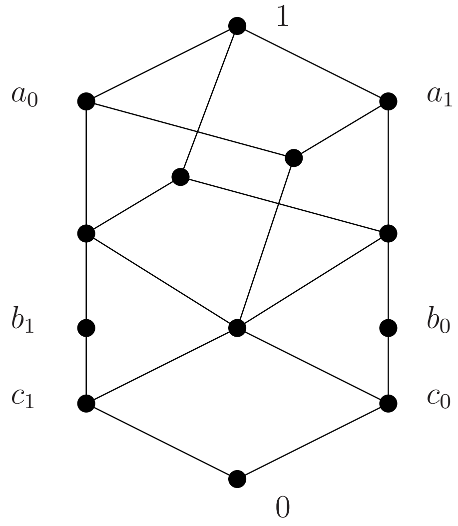

Figure III.1 Two diamond bases forming a minimal pair

Problem III.5.3. *Is $\mathcal{F}_2$ a subset of $\mathcal{F}_1$? Or even* $\mathbf{NSB} \subseteq \mathbf{Cups}(\mathbf{M})$*?*

Thirdly, if there exist no n-sequence of degrees pairwise cupping to $\mathbf{0}'$ above $\mathbf{a}$ then we call $\mathbf{a}$ *n-nonsplitting*. Let $\mathbf{NSB}_n$ denote the class of n-nonsplitting degrees. Obviously $\mathbf{NSB} = \mathbf{NSB}_2$ and $\mathbf{NSB}_n = \{\mathbf{0}'\}$ if $n < 2$. Leonhardi in Leonhardi (1997) proved that $\mathbf{NSB}_{n+1} - \mathbf{NSB}_n \neq \emptyset$ for $n > 0$. Let $\mathbf{NSB}_{\leq n} = \bigcup_{i \leq n} \mathbf{NSB}_i$. Then we have the question.

Problem III.5.4. *Does* $[\mathbf{NSB}_{\leq n+1}) - [\mathbf{NSB}_{\leq n}) \neq \emptyset$ *for each* $n > 0$*?*

Finally, let $\mathbf{NSB}_\omega = \bigcup_{i<\omega} \mathbf{NSB}_i$.

Proposition III.5.5. $\mathbf{NSB}_\omega$ *is definable.*

Proof. It is easy to see that $\mathbf{NSB}_\omega \cap (\mathbf{M} \cup \mathbf{L}) = \emptyset$. Now let $\mathbf{d} > \mathbf{0}$ be an element of $\mathbf{NSB}_\omega$, then we could have a standard model M and a surjection $h : M \to [\mathbf{d}, \mathbf{0}']$ as in Theorem III.1.1. The predicate that $h(((e)_0)^M), \ldots, h(((e)_{l-1})^M)$ *do not form an l-splitting sequence above* $\mathbf{d}$ *where* $(e)_i$ *and* $l = lth(e)$ *are as before*, is obviously definable and uniform in e^M. □

It is then natural to ask the following.

Problem III.5.6. *Does* $\mathbf{NSB}_\omega$ *form a (strong) filter?*

Chapter IV

A Congruence Relation

IV.1 Introduction

Ambos-Spies et al. (1984) suggested that the quotient structure $\mathcal{R}/\mathbf{M}$ might give insights on $\mathcal{R}$. Schwarz (1984) contained a systematic study and found a major property of $\mathcal{R}/\mathbf{M}$ analogous $\mathcal{R}$, namely $\mathcal{R}/\mathbf{M}$ is downward dense. But the density of $\mathcal{R}/\mathbf{M}$ remains open. Another important result on $\mathcal{R}/\mathbf{M}$ is that Shoenfield's homogenous conjecture fails, by Yi (1996). Recently Li et al. (2006) studied another quotient structure $\mathcal{R}/\mathbf{NCup}$ and built a minimal pair in it.

[1] In this chapter we will introduce a new congruence relation, denoted by $\sim$, and deduce from known results that $\mathcal{R}/\sim$ is not dense. This shows that quotient structures of $\mathcal{R}$ could behave very differently from $\mathcal{R}$.

We will also show that $\sim$ is a relation strictly coarser than *modulo* $\mathbf{NCup}$ in the last section. This again suggests great complexity of $\mathcal{R}$.

IV.2 A congruence relation on upper semilattices

We introduce some notions.

Definition IV.2.1. *Fix* $(L, \leq, \vee, 1)$ *an upper semilattice with a greatest element* 1.

[1] The results in this chapter is contained in Wang and Ding (2006a)

1. *For* $a \in L$ *let* $Cups(a) = \{b \in L | a \vee b = 1\}$ *and* $Cups(X) = \bigcup_{a \in X} Cups(a)$ *for* $X \subseteq L$. *We call elements of* $Cups(a)$ cupping partners *of* a.
2. *Let* $NCup(L) = \{a \in L | Cups(a) = \{1\}\}$. *Obviously* $NCup(L)$ *is an ideal and we call it the ideal of noncuppables and its elements* noncuppables.
3. $a \sim b$ *if and only if* $Cups(a) = Cups(b)$, *for* $a, b \in L$.

Fix L be as in the above definition. It is immediate that $\sim$ is an equivalent relation on L. In fact, $\sim$ is a congruence relation.

Proposition IV.2.2. *For* a, b, c *and* d *elements of* L, $a \sim b$ *and* $c \sim d$ *imply* $a \vee c \sim b \vee d$.

Proof. If $a \vee c \vee x = 1$ then $b \vee c \vee x = 1$ as $a \sim b$. Similarly $b \vee d \vee x = 1$ as $c \sim d$. By symmetry $a \vee c \sim b \vee d$. □

We denote $\tilde{a}$ the congruence class represented by a and $\tilde{L} = L/\sim$.

Proposition IV.2.3. $\tilde{a} \leq \tilde{b}$ *if and only if* $Cups(a) \subseteq Cups(b)$.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Assume $Cups(a) \subseteq Cups(b)$. If $x \vee a \vee b = 1$ then $x \vee b \vee b = x \vee b = 1$. Thus $Cups(b) = Cups(a \vee b)$ and $b \sim a \vee b \geq a$. □

Moreover $Cups(\tilde{a}) = \{\tilde{b} | b \in Cups(a)\}$ as $\sim$ is commutative with $\vee$ and $\tilde{1} = \{1\}$.

Corollary IV.2.4. $\tilde{a} \leq \tilde{b} \Leftrightarrow Cups(a) \subseteq Cups(b) \Leftrightarrow Cups(\tilde{a}) \subseteq Cups(\tilde{b})$. *Hence* $\tilde{a} \mapsto Cups(\tilde{a})$ *is a partial order monomorphism from* $\tilde{L}$ *into the powerset of* L *ordered by inclusion.*

Proof. Immediately from the last proposition and the above remarks. □

If $NCup(L)$ is not empty, then we could define another congruence relation *modulo* $NCup(L)$ as $NCup(L)$ is an ideal . If moreover L has a least element 0, then from the definitions, elements in $NCup(L)$ represent the least element in $L/NCup(L)$ as well as in $\tilde{L}$. Thus some local properties of $L/NCup(L)$ might also hold for $L/\sim$. The following is an example for $L = \mathbf{R}$.

Proposition IV.2.5. *There is at least a minimal pair in* $\mathbf{R}/\sim$.

Proof. (Li et al., 2006, Theorem 6) proved that there is a minimal pair in $\mathbf{R}/\mathbf{NCup}$ by constructing $\mathbf{a}$ and $\mathbf{b}$ such that $Cups(\mathbf{a}) \cap Cups(\mathbf{b}) = \{\mathbf{0}'\}$. It is easy to see $\tilde{\mathbf{a}} \wedge \tilde{\mathbf{b}} = \tilde{\mathbf{0}}$. □

However the two relationships could be different. A simple example of this is the natural restriction of the least non-modular lattice N_5. Later we will prove that these two congruence relations are different for $L = \mathbf{R}$.

We introduce another notion motivated by degree theory.

Definition IV.2.6. *Call a pair x, y a* nontrivial splitting *of z if $x \vee y = z$, $x < z$ and $y < z$. $u \in L$ is a* nonsplitting base *if and only if there are no nontrivial splittings of* 1 *above u. Let $NSB(L) = \{u \in L | u$ is a nonsplitting base$\}$.*

Proposition IV.2.7. *The interval $(\tilde{u}, \tilde{1})$ is empty if and only if $u \in NSB(L)$.*

Proof. ($\Rightarrow$) Assume $\tilde{a} \in (\tilde{u}, \tilde{1})$ then $a < 1$ and there is an $x \in Cups(a) - Cups(u)$. Thus $x \vee u < 1$. Hence $a \vee u$ and $x \vee u$ form a nontrivial splitting of 1.

($\Leftarrow$) If there are $a, b \geq u$ forming a nontrivial splitting of 1 then we will have $\tilde{a} > \tilde{u}$ as $b \in Cups(a) - Cups(u)$. Hence $(\tilde{u}, \tilde{1})$ is not empty. □

We call $\tilde{u}$ *maximal* if $\tilde{u} < \tilde{1}$ and $(\tilde{u}, \tilde{1})$ is empty.

Corollary IV.2.8. *If $\tilde{u}$ is maximal then $\tilde{u} = \{v \in NSB(L) | u \vee v < 1\}$.*

Proof. $\tilde{u} \subseteq \{v \in NSB(L) | u \vee v < 1\}$ by the last proposition.

On the other hand, if $v \in NSB(L)$ and $u \vee v < 1$ then $\tilde{v} = (u \vee v)^{\sim} = \tilde{u}$ where the first equality is by the maximality of $\tilde{v}$ and the second by that of $\tilde{u}$. □

Corollary IV.2.9. $\mathbf{R}/\sim$ *is not dense.*

Proof. Lachlan (1976) proved that there are $\mathbf{a}, \mathbf{b} \in \mathbf{R}$ such that $\mathbf{a} < \mathbf{b}$ and there are no nontrivial splittings of $\mathbf{b}$ above $\mathbf{a}$. Harrington in an unpublished manuscript (a published and modern presentation can be found in Leonhardi (1997)) improved this by showing that $\mathbf{b}$ could be $\mathbf{0}'$ (i. e. the greatest element of $\mathbf{R}$). Hence $NSB(\mathbf{R})$ is not empty and $\mathbf{R}/\sim$ is not dense by the above proposition. □

IV.3 Comparing $\sim$ and modulo $NCup$

In this section we prove that $\sim$ and *modulo NCup* are different congruence relations in $\mathbf{R}$. Let $\mathbf{NCup} = NCup(\mathbf{R})$ and $\hat{\mathbf{a}}$ be the element in $\mathbf{R}/\mathbf{NCup}$ represented by $\mathbf{a} \in \mathbf{R}$.

Theorem IV.3.1. *There are c. e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\hat{\mathbf{b}} \not\leq \hat{\mathbf{a}}$ *but* $\tilde{\mathbf{b}} \leq \tilde{\mathbf{a}}$.

We construct two c. e. sets A and B representing $\mathbf{a}$ and $\mathbf{b}$ mentioned above. To make $\hat{\mathbf{b}} \not\leq \hat{\mathbf{a}}$ it suffices to make

$$\mathcal{M}_e : B = \Phi_e(A, X_e) \Rightarrow X_e \text{ is cuppable}$$

for each e, where e is an index of some effective enumeration of pairs of functional and set. To make $\tilde{\mathbf{b}} \leq \tilde{\mathbf{a}}$ we build an additional set D and make

$$\mathcal{P}_e : D = \Psi_e(B, Y_e) \Rightarrow K \leq_{\mathrm{T}} A \oplus Y_e$$

for each e, where e is again an appropriate index.

We arrange strategies for meeting requirements on a tree, say T.

IV.3.1 Meeting $\mathcal{M}$'s

Let $\tau \in T$ be an $\mathcal{M}_e$-strategy, l^τ and τ-expansionary stages are defined as usual. τ has two possible outcomes, say ∞ indicating that there are infinitely many expansionary stages and 0 indicating finitely many.

If ∞ appears to be true, τ builds a set C^τ and a functional Δ^τ such that $K = \Delta(C, X)$ and

$$\mathcal{N}_i^\tau : D \neq \Gamma_i(C)$$

where Γ_i effectively ranges over all functionals when i ranges over ω.

To make $K = \Delta(C, X)$, at an expansionary stage τ either extends $dom(\Delta)$ or repairs a disagreement between K and Δ by putting an appropriate use in C. The uses of Δ obey the prescribed conventions and an additional rule,

(δ-rule) If τ acts at stage s then $\delta(n)[s-1]\downarrow\in C$ implies $\delta(n)[s] > s$.

To make $\mathcal{N}_i$, τ has children above $\tau\hat{}\infty$ assigned to $\mathcal{N}_i$. Let σ be such one, define $top(\sigma) = \tau$. σ acts as below.

1. Pick a *killing point* k^σ, an *agitator* b^σ and a *witness* d^σ fresh, keep b from entering B and d from D.
2. Wait for $b \leq l^\tau$.
3. If later $\phi(b)$ moves, return to (2).
4. Wait for $D(d) = 0 = \Gamma(C; d)\downarrow$.
5. If $\delta(k) < \phi(b)$, put $\delta(k)$ in C and return to (4).
6. Put b in B, impose a restraint $r = \phi(b)$ on A and wait for the next τ-expansionary stage.
7. Put d in D and redefine $\delta(n)$ fresh for $n \geq k$.

If σ returns from (3) to (2), it will have outcome ∞. If this happens infinitely often then we will have $\phi(b)$ diverging and Φ partial. Hence $\sigma\hat{}\infty$ indicates a global win of τ and we arrange no children of τ above it.

If σ keeps waiting at (4) confinitely often then $\Gamma(d) \neq 0 = D(d)$, σ has outcome w indicating a local win of itself.

Note that by (δ-rule), σ will not return from (5) to (4) infinitely often unless $\sigma\hat{}\infty$ is true. Thus we assign no additional outcome for this case.

If σ finally reaches (7) we will have

$$D(d) = 1 \neq 0 = \Gamma(C; d). \qquad (*)$$

In this case σ has s as its outcome. Moreover, $(*)$ will not be destroyed by $\delta(n)(n \geq k)$. If $(*)$ is destroyed by some $\delta(n)(n < k)$ then we let σ pick a fresh agitator and a fresh witness and restart from (2). As k is fixed σ can reach (7) at most $k+1$ times and $\sigma\hat{}s$ indicates a finite local win of σ.

The possible outcome defined so far are ordered in reverse lexicographic, i.e. $\infty < w < s < 0$.

IV.3.2 Meeting $\mathcal{P}$'s

Let $\alpha \in T$ be a $\mathcal{P}_e$-strategy, l^α and α-expansionary stages are defined as usual. α has two possible outcomes, say ∞ and 0, with meanings similar to those of $\mathcal{M}$-strategies. α acts like strategies building nontrivial noncuppable degrees (e.g. see (Wang and Ding, 2005, Theorem 2.1)).

If ∞ appears to be true α builds a functional Θ^α such that $K = \Theta(A, Y)$. To this end α manages an *agitator* $d^\alpha(k)$ for each k so that $\theta(k) \geq \psi(d(k))$ whenever $\Theta(A, Y; k)$ converges (say k *is honest for* α), and has its children above $\alpha\hat{\ }\infty$ to make

$$\mathcal{Q}_k^\alpha : K(k) = \Theta(A, Y; k).$$

Let $\beta \supseteq \alpha\hat{\ }\infty$ be a $\mathcal{Q}_k^\alpha$-strategy and $top(\beta) = \alpha$. β acts as below.

1. Pick a *guard* g^β fresh.
2. Wait for $g \leq l^\alpha$.
3. If later $\psi(g)$ moves, return to (2).
4. If $\Theta(A, Y; k)$ diverges, $d(k)$ is undefined or $d(k) > g$, let $d(k) = g$.
5. If $\Theta(A, Y; k)$ diverges then let $\Theta(A, Y; k) = K(k)$ with $\theta(k) = \psi(d(k))$.
6. If $\Theta(A, Y; k) \downarrow \neq K(k)$ then put $d(k)$ in D and cancel $d(k)$.

If β returns from (3) to (2) it has ∞ as outcome. Otherwise β has 0 as outcome.

It depends on two assumptions for β to win.

($\beta 1$) $g = \lim_s g^\beta[s]$ exists, and

($\beta 2$) If $\theta(k)$ is defined, then either $B \upharpoonright \psi(d(k))$ never changes or $\theta(k)$ could be put in A for the honesty of k.

If these are achieved then $\beta\hat{\ }\infty$ indicates Ψ partial and a global win of α. Hence we arrange no children of α above $\beta\hat{\ }\infty$. On the other hand, if $\beta\hat{\ }0$ is true, then either (6) never happens and $K(k) = 0 = \Theta(A, Y; k)$, or k is always honest and the enumeration of $d(k)$ in D would cause $\Theta(A, Y; k)$ to diverge. Thus eventually $\Theta(A, Y; k) \downarrow = K(k)$ and β achieves a local win.

Let us remark on the intuition behind the above procedure. β picks a guard in order to test the premise of α, i.e. whether $\Psi(B, Y)$ is total. On the other hand all $\mathcal{Q}_k^\alpha$-strategies share a same agitator and manage the honesty of k according to this agitator.

The purpose of agitators is for strategies to force Θ^α diverge on appropriate parameters. To manage the honesty of k, β should prevent numbers below $\psi(d(k))$ from entering B. However β can not always control $\psi(d(k))$ as $d(k)$ might be defined by other $\mathcal{Q}_k^\alpha$-strategies. But to some extent β can control $\psi(g^\beta)$ and g^β would also fit the purpose of agitators under certain circumstances. This explains Step (4) of the above procedure.

In the proof of (Wang and Ding, 2005, Theorem 2.1) we call guards *personal flip points* and agitators *official flip points*.

IV.3.3 Coordinating strategies

An $\mathcal{M}$-strategy τ never interferes with other strategies except its children as it builds its own C^τ and changes nothing other than C^τ. Moreover τ could injure its children on the true path at most finitely often as argued.

A $\mathcal{P}$-strategy changes nothing and imposes no restraints and thus never interferes with other strategies.

The only conflicts lie between $\mathcal{N}$-strategies and $\mathcal{Q}$-strategies. Recall that an $\mathcal{N}$-strategy might need to put its agitator in B and then hope $A \upharpoonright r$ not changed while a $\mathcal{Q}$-strategy assumes ($\beta 2$).

Let σ be an $\mathcal{N}_i^\tau$-strategy and β a $\mathcal{Q}_k^\alpha$-strategy where $\tau = top(\sigma)$ is an $\mathcal{M}_e$-strategy and $\alpha = top(\beta)$ is some $\mathcal{P}$-strategy. To solve the described conflict between σ and β, we would have σ destroy $\Theta^\alpha(k)$ by putting $d^\alpha(k)$ in D before it could put b^σ in B. However to respect α's expectation on totality of Θ, we will forbid σ to do this if $\beta\hat{\ }0 \subseteq \sigma$. Moreover under this situation we will make $b^\sigma > \psi(d(k))$ by initialization thus b^σ would not threat the honesty of k for α.

If $\sigma \supseteq \beta\hat{\ }\infty$ then either $d(k) < g^\beta$ and $\theta(k)$ converges or $\Theta(A, Y; k)$ diverge when σ acts. By initialization again we will either have $b > \theta(k)\downarrow$ or never worry the honesty of k when σ acts.

If $\sigma \not\supset \beta$ then σ might need to worry the honesty of k when it wants to put b in B. As described σ then delays the enumeration of b in B and puts $d(k)$ in D first. Moreover σ brings forward the defining of r and defines it right now. Before σ could act again, an α-expansionary stage should be seen and $\Theta(A, Y; k)$ would diverge by the assumed honesty of k.

Moreover as there might be many $\mathcal{P}$-strategies, say $\alpha_0 \subset \alpha_1 \subset \ldots \subset \alpha_n$, below σ, we make σ destroy θ^{α_j} in decreasing order. When σ puts some $d^{\alpha_j}(k_j)$ in D, it setups a link (α_j, σ). At the next α_j-expansionary stage this link is *traveled*, i.e. the control passes from α_j directly to σ, and σ will attempt to destroy some $\theta^{\alpha_{j-1}}$ if $j > 0$ or put b^σ in B.

We could still make $\theta(k)$ converge if each $\mathcal{N}$-strategy only attempts to destroy $\theta(k)$ finitely often. However σ might want to put its agitator in B infinitely often and thus destroy $\theta(k)$ constantly. For example, assume that σ put $d^\alpha(k)$ in D at stage s_0 which is both τ- and α-expansionary, but τ at the next τ-expansionary stage $s_1 > s_0$ finds a disagreement between K and Δ^τ, and repairs it by putting a use less than $\gamma_i(d^\sigma)$ in C. If this happens infinitely often then σ might want to put its agitator in B infinitely often.

To solve this new problem, we build local enumerations for Φ_e and X_e, i.e.

$$X^\tau[s] = X_e[r] \text{ and } \Phi^\tau(A, X^\tau)[s] = \Phi^\tau(A, X^\tau)[r]$$

for each s, where $r \leq s$ is the latest stage when τ acted. Provided τ acts infinitely often, we will have

$$X^\tau = \bigcup_s X^\tau[s] = X_e \text{ and } \Phi^\tau(A, X^\tau) = \bigcup_s \Phi^\tau(A, X^\tau)[s] = \Phi_e(A, X_e).$$

Intuitively, we will freeze the computation $\Phi_e(A, X_e)$ whenever τ does not act. Children of τ will also deal with these enumerations instead of the standard enumerations. In particular, when the computation is frozen an $\mathcal{N}^\tau$-strategy will not detect any changes of uses. During the construction below we will *not* mention the above again but just take it as granted.

Under this setting the above paragraph will not happen if $\alpha \subset \tau$ as then τ will be covered by a link (α, σ).

Assume $\tau \subset \alpha \subset \sigma$ and $s_2 \geq s_1$ is the earliest α-expansionary stage. We in addition assume that $\phi^\tau(b^\sigma)$ had reached its final value at s_0 (called b-*correct* assumption). Then we will have

$$\theta(k)[s_2] > r^\sigma[s_0] \geq r^\rho[s_0] \qquad (k\text{-safe})$$

if $\Theta(A, Y; k)[s_2] \downarrow$ and $r^\rho[s_0]$ is defined for some $\mathcal{N}$-strategy ρ prior to σ.

This observation might not be critical for $\rho = \sigma$, for after σ makes $\delta(k^\sigma)$ diverge it is safe for σ to honestify k for α by putting $\theta(k)$ in A. But it is critical for $\rho \neq \sigma$. ρ might have attempted to destroy some $\theta^{\mu}(n)$ before s_0 and was waiting for a μ-expansionary stage at s_0. Hence before a μ-expansionary stage were seen the restraint r^{ρ} should be respected. Our observation guarantees that the restraint would be respected even if $\theta(k)$ were put in A.

Thus σ simply accepts $\theta(k)$ at s_2. If later σ will put b in B and hurt the honesty of k for α, we will make σ simultaneously put $\theta(k)$ in A to repair it.

Finally if the b-correct assumption fails infinitely often, then σ might cause $\theta(k) \to \infty$. But we will have a global win for τ since $\Phi(A, X; b)$ then diverges. As in typical $\mathbf{0}'''$-arguments we arrange backup strategies for α and its children above $\sigma\hat{}\infty$. We consider such arrangement as α *injured by* τ. Furthermore we will have a final strategy free of injuries for each $\mathcal{P}$ along every infinite path of T.

IV.3.4 Final behaviors of $\mathcal{N}$- and $\mathcal{Q}$-strategies

In this subsection we summarize behaviors of $\mathcal{N}$- and $\mathcal{Q}$-strategies subjecting to adjustments in the last subsection, so as to help readers building a clear mind picture.

Let σ be an $\mathcal{N}_i^{\tau}$-strategy where $\tau = top(\sigma)$. σ has four parameters: k^{σ}, b^{σ}, d^{σ} and r^{σ}. Assume σ is initialized only finitely often then $k = \lim_s k^{\sigma}[s]$ exists. We may assume $k = k^{\sigma}[s_0]$. If σ waits for $\Gamma_i(C; d) \downarrow = 0$ cofinitely often then it wins.

If σ finds that $\phi(b)$ moves infinitely often then it will never attempt to destroy any $\theta^{\alpha}(x)$ for $\alpha \subset \tau$, otherwise τ would be covered by a link and the computation of Φ would be frozen. σ will cancel r before $\sigma\hat{}\infty$ will act. In this case $b = \lim_s b^{\sigma}[s]$ exists and a global win of τ is achieved. Moreover totality of Θ^{α} for $\alpha \subset \tau$ is not hurt and those $\mathcal{P}$-strategies between τ and σ will be backup.

If $\phi(b)$ converges, and d is realized but $\Gamma_i(C; d)$ is destroyed infinitely often, then σ achieves a win of itself and will stop attempting to destroy θ^{α}'s for $\alpha \subset \sigma$.

Otherwise σ will attempt to destroy some $\theta^{\alpha}(k)$ with $\alpha \subset \tau$ at some stage. We could assume that r and thus the computation $\Phi(A, X; b)$ would be respected before σ would put d in D. This assumption could be verified by induction and (k-safe). Provided σ acts infinitely often it will eventually make $D(d) = 1 \neq 0 = \Gamma_i(C; d)$. Though such disagreement might be destroyed by enumerations of $\delta(n)$ in C for $n < k$, σ needs at most $k + 1$ times to establish a final one.

In all cases $b = \lim_s b^\sigma[s]$ and $d = \lim_s d^\sigma[s]$ exist. And either r^σ is canceled infinitely often or $\lim_s r^\sigma[s]$ exists.

Next consider a $\mathcal{Q}_k^\alpha$-strategy β with $\alpha = top(\beta)$. β has one parameter g^β. Assume β is not initialized after stage s_0 then $g^\beta[s]$ ($s > s_0$) could be canceled only if it enters D, and $g^\beta[s]$ could enter D only if β at some stage set $d^\alpha(k) = g^\beta$. $d^\alpha(k)$ could enter D for two reasons: some $\mathcal{Q}_k^\alpha$-strategy needs to repair a disagreement between K and Θ, or some $\mathcal{N}$-strategy attempts to destroy $\theta(k)$. The first case happens at most once and so might be assumed never do after s_0. To argue that $g = \lim_s g^\beta[s]$ exists we examine $\mathcal{N}$-strategies according to their positions on T. Let σ be as above.

1. σ is to the left of β. Then σ never acts after s_0 by the choice of s_0.
2. $\sigma\hat{}\infty \subseteq \beta$. As in classical nonbounding constructions we will have $\alpha \supseteq \sigma\hat{}\infty$. But σ would never attempt to destroy any $\theta^\alpha(x)$.
3. $\sigma\hat{}w$ or $\sigma\hat{}s \subseteq \beta$. As argued $b = \lim_s b^\sigma[s]$ exists and either $\phi^\tau(b)$ converges or σ eventually stops checking $\phi^\tau(b)$. Hence σ could put at most finitely many $d^\alpha(k)$'s in D.
4. Otherwise, by initialization we may assume that $\Theta(A, Y; k)$ diverges or $b^\sigma > \psi(d(k))$ whenever σ acts. Then σ never puts $d(k)$ in D as b^σ would not threat the honesty of k for α.

If $\beta\hat{}0$ is on the true path we will also have $d(k) = \lim_s d^\alpha(k)[s]$ exist and $\Theta(A, Y; k)$ converge.

IV.3.5 Defining the tree of strategies

Firstly, we may restate all requirements in the following forms:

$$\mathcal{M}_e : B = \Phi_e(A, X_e) \Rightarrow \exists C_e, \Delta_e(C_e \text{ is incomplete } \wedge K = \Delta_e(C_e, X_e)),$$
$$\mathcal{N}_{e,i} : D \neq \Gamma_i(C_e),$$
$$\mathcal{P}_e : D = \Psi_e(B, Y_e) \Rightarrow \exists \Theta_e(K = \Theta_e(A, Y_e),$$
$$\mathcal{Q}_{e,k} : K(k) = \Theta_e(A, Y_e; k).$$

Moreover we may consider $\mathcal{N}^\tau$'s and $\mathcal{Q}^\alpha$'s as local variances of $\mathcal{N}_{e,i}$'s and $\mathcal{Q}_{e,k}$'s, as well as C^τ's, Δ^τ's and Θ^α's.

Secondly fix a computable bijection f from the set of requirements to ω such that $f(\mathcal{M}_e) < f(\mathcal{N}_{e,i})$ and $f(\mathcal{P}_e) < f(\mathcal{Q}_{e,i})$ for any pair (e, i).

Finally we define T, a partial function $top : T \to T$ and for each requirement $\mathcal{X}$, a partial function also denoted as $\mathcal{X} : T \to T$.

At the beginning we put $\emptyset$ in T and assign it to $f^{-1}(0)$. Assume ξ is enumerated in T and assigned to $\mathcal{Z}$, we say that ξ is a $\mathcal{Z}$-strategy. We define the immediate successors of ξ:

1. If ξ is some $\mathcal{M}$-, $\mathcal{P}$- or $\mathcal{Q}$-strategy then put $\xi\hat{\ }\infty$ and $\xi\hat{\ }0$ in T.
2. If ξ is an $\mathcal{N}$-strategy then put $\xi\hat{\ }\infty$, $\xi\hat{\ }w$ and $\xi\hat{\ }s$ in T.

To define the partial functions we need some notions. For $\rho \subset \xi$, ρ *is injured at* ξ if and only if it is assigned to some $\mathcal{M}$, $\mathcal{N}$ or $\mathcal{P}$, and there are τ and σ such that $\tau\hat{\ }\infty \subseteq \rho \subset \sigma\hat{\ }\infty \subseteq \xi$, τ is some $\mathcal{M}$-strategy and $top(\sigma) = \tau$.

For each requirement $\mathcal{X}$, let $\mathcal{X}(\xi)$ be the longest $\rho \subseteq \xi$ assigned to $\mathcal{X}$. $\mathcal{X}$ *is satisfied at* ξ if and only if

1. either $\mathcal{X}(\xi)$ is defined and not injured at ξ, or
2. $\mathcal{X}$ is some $\mathcal{N}_{n,j}$ or $\mathcal{Q}_{m,l}$, $\mu = \mathcal{M}_n(\xi)$ or $\mathcal{P}_m(\xi)$ is defined and not injured at ξ, and there is some $\mathcal{Y} = \mathcal{N}_{n,p}$ or $\mathcal{Q}_{m,q}$ such that $\nu = \mathcal{Y}(\xi)$ is defined, $top(\nu) = \mu$ and $\nu\hat{\ }\infty \subseteq \xi$.

If ξ is some $\mathcal{N}_{e,i}$ or $\mathcal{Q}_{e,k}$-strategy then let $top(\xi) = \mathcal{M}_e(\xi)$ or $\mathcal{P}_e(\xi)$, otherwise $top(\xi)$ is undefined.

If ζ is an immediate successor of ξ in T then assign ζ to the unique $\mathcal{X}$ such that

$$f(\mathcal{X}) = \mu x(f^{-1}(x) \text{ is not satisfied at } \xi).$$

This completes the definition of T.

Some useful notions can be derived. For any infinite path $P \subset T$ and a requirement $\mathcal{X}$, let $\mathcal{X}(P) = \bigcup_{\xi \subset P} \mathcal{X}(P)$ given the righthand side is finitely defined; $\mathcal{X}$ *is satisfied on P* if and only if there is a finite $\xi \subset P$ and $\mathcal{X}$ is satisfied at any ζ on P extending ξ.

The facts below are not difficult.

Lemma IV.3.2. *For each requirement $\mathcal{X}$ and each infinite path P of T,*

1. *if $\mathcal{X}$ is some $\mathcal{M}_e$ or $\mathcal{P}_e$ then $\mathcal{X}(P)$ is defined;*
2. *if $\mathcal{X}$ is some $\mathcal{M}_e$ or $\mathcal{P}_e$ and for some $\mathcal{N}_{e,i}$ or $\mathcal{Q}_{e,k}$, $\rho = \mathcal{N}_{e,i}(P)$ or $\mathcal{Q}_{e,k}(P)$ is defined and $\rho\hat{~}\infty \subset P$, then there is no $\mathcal{Y} = \mathcal{N}_{e,j}$ or $\mathcal{Q}_{e,l}$ with $\mathcal{Y}(P)$ defined and extending ρ;*
3. *again $\mathcal{X}$ is some $\mathcal{M}_e$ or $\mathcal{P}_e$ but the assumption above fails then $\rho = \mathcal{Y}(P)$ is defined and $\rho\hat{~}\infty \not\subset P$ for any $\mathcal{Y} = \mathcal{N}_{e,i}$ or $\mathcal{Q}_{e,k}$;*
4. *$\mathcal{X}$ is satisfied on P.*

IV.3.6 The construction

At the beginning let all sets and functionals empty and other parameters undefined.

At stage s we will define a finite approximation to the true path, denoted by $TP[s]$, and probably an *indicator* $\eta[s] \in T$. As soon as $TP[s]$ is determined, we will initialize strategies less prior than $TP[s]$ *except* those extending $\eta[s]$ given $\eta[s]$ defined, then proceed to stage $s+1$.

To define $TP[s]$ we will define a finite sequence of *accessible* strategies. Initially $\emptyset$ will be accessible. If a strategy ξ becomes accessible, it will acts immediately. When ξ acting it might defined its outcome $o = o(\xi)[s]$, or the next accessible strategy, or $TP[s]$. If one of these happens we end ξ immediately. On defining of o, if $|\xi| < s$ we will declare $\xi\hat{~}o$ accessible, otherwise let $TP[s] = \xi$ and $\eta[s]$ undefined.

The actions of ξ depend on which requirement ξ is serving.

Case 1. ξ is an $\mathcal{M}_e$-strategy.

If s is not ξ-expansionary then let $o(\xi) = 0$ and end.

If s is ξ-expansionary and there is no link of the form (ξ, σ) then let n be the least number such that either $\Delta(C, X; n)$ diverges or $K(n) \neq \Delta(C, X; n)$.

If $\Delta(C, X; n)$ diverges then let $\Delta(C, X; n) = K(n)$ with $\delta(n)$ fresh if $\delta(n)[s-1]$ is undefined or is in C, or $\delta(n) = \delta(n)[s-1]$ otherwise.

If $K(n) \neq \Delta(C, X; n)$ then put $\delta(n)[s-1]$ in C and redefine $\Delta(C, X; n) = K(n)$ with $\delta(n)[s]$ fresh.

In both situations let $o(\xi) = \infty$.

If s is ξ-expansionary and there exists a link of the form (ξ, σ) then let σ be accessible.

Case 2. ξ is an $\mathcal{N}_i^\tau$-strategy where $\tau = top(\xi)$. Let $s_0 < s$ be the last stage when ξ was accessible.

Subcase 2.1. d^ξ is defined and $D(d^\xi) = 1 \neq \Gamma_i(C^\tau; d^\xi)$. Let $o(\xi) = s$.

Subcase 2.2. ξ becomes accessible because a link (α, ξ) was traveled where $\alpha \subset \xi$ is some $\mathcal{P}$-strategy. Then b^ξ and s_0 are defined.

If $\phi^\tau(b)[s] > \phi^\tau(b)[s_0]$ or $C[s] \restriction \gamma_i(d)[s_0] \neq (C \restriction \gamma_i(d))[s_0]$ then cancel r, let $TP[s] = \xi$ and $\eta[s] = \xi\hat{\ }\infty$.

Otherwise check whether there exists $\mathcal{P}$-strategy $\mu \subset \sigma$ such that

$$\exists x[\Theta^\mu(A, Y^\mu; x)[s] \downarrow \wedge b \leq \psi^\mu(d^\mu(x))[s] \wedge (\mu \subset \tau \vee \theta^\mu(x)[s] \downarrow \leq r)].$$

If there exists such $\mathcal{P}$-strategies, let μ be the longest one and x be the least number where the matrix of the above holds, put $d^\mu(x)$ in D and setup a link (μ, ξ).

If there does not exist $\mathcal{P}$-strategies as above, put b in B and setup a link (τ, ξ).

In both situation let $TP[s] = \eta[s] = \xi$.

Subcase 2.3. ξ becomes accessible because a link of the form (τ, ξ) was traveled.

1. Put d in D, redefine $\Delta(C, X; k) = K(k)$ with $\delta(k)$ fresh.
2. For each $\mathcal{P}$-strategy $\alpha \subset \xi$ with $\theta^\alpha(x)$ defined and $\psi^\alpha(d^\alpha(x)) \geq b$ for some x, put $\theta^\alpha(x)$ in A for the least such x.
3. Let $TP[s] = \xi$ and $\eta[s]$ undefined.

Subcase 2.4. The above subcases do not apply. Cancel r^ξ if it is defined. Let $s_1 < s$ be the last stage when ξ was accessible and this subcase applied, act as below.

1. If k^ξ is undefined then define it fresh. If b^ξ is undefined or in B then define or redefine it fresh. If d^ξ is undefined or in D then define or redefine it fresh.
2. If $b > l^\tau$ let $TP[s] = \xi$ and $\eta[s]$ undefined.
3. If $b \leq l^\tau$ but $\phi^\tau(b)[s] > \phi^\tau(b)[s_1]$, let $o(\xi) = \infty$.
4. If $b \leq l^\tau$, $\phi^\tau(b)[s] = \phi^\tau(b)[s_1]$ and $\Gamma_i(C; d) \neq 0$ then let $o(\xi) = w$.
5. If $b \leq l^\tau$, $\phi^\tau(b)[s] = \phi^\tau(b)[s_1]$, $\Gamma_i(C; d) \neq 0$ but $\delta(k) < \phi(b)$, then put $\delta(k)$ in C and let $o(\xi) = w$.
6. Otherwise all the above tests fail, let

$$r = \max(\{\phi(b)\} \cup \{r^\rho | \rho \text{ is prior to } \xi\})$$

and go to subcase 2.2.

Case 3. ξ is a $\mathcal{P}_e$-strategy.

If s is ξ-expansionary and there exists a link (ξ, σ), cancel this link and let σ be accessible.

If s is ξ-expansionary and there is no link as above then let $o(\xi) = \infty$.

Otherwise let $o(\xi) = 0$.

Case 4. ξ is a $\mathcal{Q}_k^\alpha$-strategy where $\alpha = top(\xi)$. Let $s_0 < s$ be the last stage when ξ was accessible, act as below.

1. If g^ξ is undefined or in D, define or redefine it fresh.
2. If $g > l^\alpha$ then let $TP[s] = \xi$ and $\eta[s]$ undefined.
3. If $g \leq l$ but $\psi^\alpha(g)[s] > \psi^\alpha(g)[s_0]$ then let $o(\xi) = \infty$.
4. Otherwise. If $\Theta^\alpha(A, Y^\alpha; k) \uparrow$, $d^\alpha(k) \uparrow$ or $d^\alpha(k) > g$ then let $d(k) = g$.
5. If $\Theta(A, Y; k) \uparrow$, let $\Theta(A, Y; k) = K(k)$ with $\theta(k)$ fresh if $\theta(k)[s-1] \uparrow$ or $\theta(k)[s-1] \in A$, or $\theta(k) = \max\{\theta(k)[s-1], \psi(d(k))\}$ otherwise.
6. If $\Theta(A, Y; k) \downarrow \neq K(k)$ then put $d(k)$ in D, cancel $d(k)$ and let $TP[s] = \xi$ and $\eta[s] = \xi\hat{\ }\infty$; otherwise let $o(\xi) = 0$.

IV.3.7 The verification

Let $TP = \liminf_s TP[s]$. We have the following facts immediately from the construction.

1. A, B and D are enumerable.
2. If $\eta[s]$ is defined then $\eta[s] \supseteq TP[s]$.
3. For any $\mathcal{P}$-strategy α, no one above $\alpha\hat{\ }\infty$ could ever put any $d^\alpha(x)$ in D or $\theta^\alpha(x)$ in A.

Lemma IV.3.3 (Honesty). *Let α be a $\mathcal{P}_e$-strategy. For any k and at any stage s if $d^\alpha(k)[s]$ is defined and $\Theta^\alpha(A, Y_e; k)[s]$ converges then k is honest for α.*

Proof. By Case 4 in the construction when $\Theta^\alpha(A, Y_e; k)$ becomes defined k is always honest. And k keeps honest if nothing enters $B \upharpoonright \theta^\alpha(k)$.

But the only case where B is changed is Subcase 2.3 in the construction. If that happens then either $\theta^\alpha(k)$ is put in A or α is initialized. This would cause $\Theta^\alpha(A, Y_e; k)$ to be redefined unless α is later initialized. □

Lemma IV.3.4 ($\mathcal{N}$-behaviors)**.** *Let τ be an $\mathcal{M}_e$-strategy, $\sigma \supseteq \tau\hat{}\infty$ be an $\mathcal{N}_i^\tau$-strategy and $\alpha \subset \sigma$ be a $\mathcal{P}_n$-strategy. If σ is prior to or on TP, we have the followings:*

1. *There is a stage s_0 such that $k = \lim_s k^\sigma[s] = k^\sigma[s_0]$ exists. (The other assertions will assume this s_0)*
2. *At any stage $s > s_0$ if $r^\sigma[s]$ is defined then nothing enters $A \restriction r^\sigma[s]$.*
3. *If σ created a link (α, σ) at $s_1 > s_0$ and is accessible at $s_2 > s_1$, then at the end of s_2, exactly one of the followings happens:*
   (a) $\phi^\tau(b^\sigma)[s_2] > \phi^\tau(b^\sigma)[s_1]$,
   (b) a $\delta(x) \leq \gamma_i(d)$ is enumerated in C and $\alpha \supset \tau$,
   (c) a link (μ, σ) is created where $\mu \subset \alpha$ is another $\mathcal{P}$-strategy,
   (d) a link (τ, σ) is created.
4. *If σ created a link (τ, σ) at $s_3 > s_0$ and acts at $s_4 > s_3$, then d^σ is put in D, $\Gamma_i(C; d) = 0$ and $\delta(k) > \gamma_i(d)$ at the end of s_4.*
5. *If σ never has ∞ as its outcome after s_0 or $\alpha \subset \tau$ then σ will put no agitators of α in D after some stage. Moreover even without the assumption σ will eventually stop putting anything in A.*

Proof. (1) By the construction we choose s_0 be a stage such that σ is never initialized after s_0. Then either k^σ is undefined cofinitely often or becomes defined at some point and fixes for ever. We may consider the first case a special situation of our assertion. While in the latter case we may assume that k becomes defined at s_0.

(2) Assume some strategy puts something in A at some $s_1 > s_0$. From the construction we have that this strategy, say ρ, is an $\mathcal{N}_j^\pi$-strategy where $\pi = top(\rho)$, and those enumerated in A are uses of functionals built by $\mathcal{P}$-strategies. Fix an arbitrary one which is $\theta^\mu(x)$ of some $\mathcal{P}$-strategy μ. Moreover by the assumption of s_0, we have $\rho = \sigma$ or less prior than σ.

We may in addition assume that $r^\sigma[s_1]$ is defined and became defined at stage $s > s_0$. By Case 2 of the construction we have that $r^\rho[s_1]$ became defined at some $s' \geq s$ and thus $r^\rho[s_1] \geq r^\sigma[s_1]$. By Subcase 2.2 of the construction, we have $\theta^\mu(x)[s_1] > r^\rho[s_1] \geq r^\sigma[s_1]$. This establishes (2).

(3) Assume neither (a) nor (b) happens then $r = r^\sigma[s_2] = r^\sigma[s_1]$. Recall that at s_1 some $d^\alpha(x)[s_1]$ was put in D and note that s_2 is α-expansionary and no children of α acts. By the Honesty Lemma (and induction decreasingly on the length of α) we have

$$\forall x(\Theta^\epsilon(A, Y^\epsilon; x)\downarrow \rightarrow \theta^\epsilon(x) > r)$$

at s_2, for any $\mathcal{P}$-strategy ϵ such that $\alpha \subseteq \epsilon \subset \sigma$. Hence we have (3).

(4) $D(d^\sigma) = 1$ and $\Gamma_i(C; d) = 0$ are immediately from Subcase 2.3 in the construction. By (2) and Subcase 2.2 in the construction we have

$$\Phi^\tau(A, X^\tau; b^\sigma)[s_4] = 1 \neq 0 = \Phi^\tau(A, X^\tau; b^\sigma)[s_3]$$

and $(A \upharpoonright \phi(b))[s_4] = (A \upharpoonright \phi(b))[s_3]$. Hence Step (1) of Subcase 2.3 in the construction is feasible and $\delta(k) > \gamma_i(d)$.

(5) By (1) we may assume that no $\delta(x)$ for $x < k$ enters C after stage s_0. By Subcase 2.1 in the construction, Subcase 2.3 could happen at most finitely often and may be assumed never do after some stage s_0.

If $\alpha \subset \tau$ then no link of the form (α, σ) could be created after s_0, otherwise by (3) and (4) Subcase 2.3 eventually happened. Hence σ never puts any agitators of α in D after s_0.

If $\tau \subset \alpha \subset \sigma$ and σ never has ∞ as its outcome after s_0 then either Subcase 2.1 in the construction happens whenever σ acts after s_0, or we may assume $\phi(b) = \phi(b)[s_0]$. The former case is trivial. In the latter case we may assume $r^\sigma[s] = r^\sigma[s_0]$ whenever $s > s_0$ and $r^\sigma[s]$ is defined. Hence σ will put at most finitely many agitators of α in D after s_0.

Now we have established the first half of (5). For the remaining, note that σ could put uses in A only if Subcase 2.3 in the construction happens. But by (4) it could not happen infinitely often. □

Lemma IV.3.5 (True Path)**.** *For each n we have that $|TP| \geq n$, $TP \upharpoonright n$ is accessible infinitely often and initialized for finitely many times.*

Proof. We prove by induction on n. The case for $n = 0$ is trivial. Let $\xi = TP \upharpoonright n$ and s_0 be a stage such that ξ is never initialized after s_0.

Case 1: ξ is an $\mathcal{M}_e$-strategy. If there are finitely many ξ-expansionary stages then the lemma holds for $TP \upharpoonright (n+1) = \xi\hat{\ }0$.

Assume there are infinitely many ξ-expansionary stages, and $s > s_0$ is one when $\xi\hat{\ }\infty$ is not accessible. Then there is a link (ξ, σ) where σ is a child of ξ. But this link will be canceled and no new links are created at the end of s. Thus $\xi\hat{\ }\infty$ will be accessible at the next ξ-expansionary stage and the lemma holds for $TP \upharpoonright (n+1) = \xi\hat{\ }\infty$.

Case 2: ξ is an $\mathcal{N}_i^\tau$-strategy where $\tau = top(\xi)$. By the $\mathcal{N}$-behaviors Lemma we may assume that

1. $k = \lim_s k^\xi[s]$, $b = \lim_s b^\xi[s]$ and $d = \lim_s d^\xi[s]$ exist and reached their final values before s_0,
2. no $\delta^\tau(n)$ $(n < k)$ enters C after s_0, and
3. $d \notin D$ (otherwise the lemma holds for $TP \upharpoonright (n+1) = \xi\hat{\ }s$).

Then either infinitely often ξ finds that $\phi^\tau(b)$ moves or $\Gamma_i(C; d) \neq 0$. In the former case $TP \upharpoonright (n+1) = \xi\hat{\ }\infty$ and in the latter $TP \upharpoonright (n+1) = \xi\hat{\ }w$. The remaining part is obvious.

Case 3: ξ is a $\mathcal{P}_e$-strategy. By (3) of the $\mathcal{N}$-behaviors Lemma, our assertion holds for $TP \upharpoonright (n+1) = \xi\hat{\ }\infty$ if there are infinitely many ξ-expansionary stages, or for $TP \upharpoonright (n+1) = \xi\hat{\ }0$ if there are only finitely many.

Case 4: ξ is a $\mathcal{Q}_k^\alpha$-strategy where $\alpha = top(\xi)$. Note that Step (6) of Case 4 in the construction is the only one which could stop the process at ξ, at enough large stages. Fortunately this happens at most once by the Honesty Lemma. □

For each $\xi \subset TP$, let s^ξ be an arbitrary stage such that ξ is never initialized after s^ξ.

Lemma IV.3.6 ($\mathcal{P}$-behaviors)**.** *Let $\alpha = \mathcal{P}_e(TP)$.*

1. $\Theta = \bigcup_{s > s^\alpha} \Theta^\alpha[s]$ *is consistent.*
2. *If $\alpha\hat{\ }\infty \subset TP$ and $\beta = \mathcal{Q}_{e,k}(TP)$ is defined then $g = \lim_s g^\beta[s]$ exists.*
3. *If β is as in (2) and $\beta\hat{\ }0 \subset TP$ then $d(k) = \lim_s d^\alpha(k)[s]$ and $\theta(k) = \lim_s \theta^\alpha(k)[s]$ exist, and $\Theta(A, Y_e; k) \downarrow = K(k)$.*

Hence $\Theta(A, Y_e)$ is total and equals K if $\mathcal{Q}_{e,n}(TP)$ is defined and $\mathcal{Q}_{e,n}(TP)\hat{\ }0 \subset TP$ for every n.

Proof. (1) The consistency of Θ is obvious from the construction.

(2) By the definition of T and the $\mathcal{N}$-behaviors Lemma, we may assume that no $\mathcal{N}$-strategies below β put any agitator of α in D after s^β. In addition assume that no $\mathcal{Q}_k^\alpha$-strategies put $d^\alpha(k)$ in D after s^β, as this could happen at most once.

Suppose g^β became defined or was redefined at stage $s_0 > s^\beta$. Then at this stage, β acted, strategies less prior than β were initialized.

Assume some strategy σ put $g^\beta = g^\beta[s_0]$ in D at $s_1 > s_0$. Then by the construction there was a stage s between s_0 and s_1 at which β set $d^\alpha(k) = g^\beta$. Let $s \leq s_1$ be the latest such stage and $u = \psi(g^\beta)[s]$.

Claim 1. *$(B,Y)[s] \upharpoonright u = (B,Y)[s_1] \upharpoonright u$, hence $\psi(g^\beta)[s_1] = u$.*

Proof. For contradiction suppose s' is the least stage between s and s_1 when $B \upharpoonright u$ or $Y \upharpoonright u$ changed. But $B \upharpoonright u$ could not change at s' otherwise $d^\alpha(k)[s] = g^\beta$ were enumerated in D before s' by Subcase 2.2 in the construction. If $Y \upharpoonright u$ changed at s' then as $\theta(k)[s] \geq u$, we have $\Theta(A,Y;k)[s']$ diverged and $d^\alpha(k)$ redefined at some stage after $s' \geq s$. Thus we get a contradiction with the choice of s. □

By Claim 1 we have $\sigma \supseteq \beta\hat{\ }0$ and $\theta(k)[s_1] = \theta(k)[s]$.

Claim 2. *$b^\sigma[s_1] > u$.*

Proof. Assume $b^\sigma[s_1]$ became defined at $t < s_1$. If $t \geq s$, we have $b^\sigma[t] > \theta(k)[s] \geq u$.

If $t < s$ then $b^\sigma[t] > t > \psi(g^\beta)[t]$. As $b^\sigma[t] > t \geq l^\tau[t]$ where $\tau = top(\sigma)$, σ did nothing more at t hence $TP[t] = \sigma$ and no link existed along $TP[t]$. Then if $\psi(g^\beta)[t] < \psi(g^\beta)[s]$, $\beta\hat{\ }\infty$ would have been accessible before s and σ initialized and b^σ canceled. Hence we have $\psi(d(k))[t] = u$ and again $b^\sigma[s_1] > u$. □

Claim 2 refutes our assumption that σ put $d^\alpha(k)$ in D at s_1 and establishes (2).

(3) The existence of the limits follows from (2) and its proof, and the equality follows from the existence and the construction.

The final assertion follows from (3). □

Lemma IV.3.7 ($\mathcal{P}$-satisfaction)**.** *Every $\mathcal{P}_e$ is satisfied.*

Proof. Let $\alpha = \mathcal{P}_e(TP)$. If there is a child of α, say β, such that $\beta\hat{\ }\infty \subset TP$, then by (2) of the $\mathcal{P}$-behaviors Lemma and Case 4 in the construction, $\Psi_e(B, Y_e; g^\beta)$ diverges and $\Psi_e(B, Y_e)$ is partial.

Otherwise by the definition of T, α has a child, say β_k, for each $\mathcal{Q}_k^\alpha$, and $\beta_k\hat{\ }0 \subset TP$. By the final assertion of the $\mathcal{P}$-behaviors Lemma we can compute K from A and Y_e. □

Lemma IV.3.8 ($\mathcal{M}$-satisfaction)**.** *Every $\mathcal{M}_e$ is satisfied.*

Proof. Let $\tau = \mathcal{M}_e(TP)$. Immediately from the construction, $\Delta = \bigcup_{s>s^\tau} \Delta^\tau[s]$ is consistent and $C = \bigcup_{s>s^\tau} C^\tau[s]$ is enumerable. If $\tau\hat{\ }\infty \subset TP$ then $\mathcal{M}_e$ is satisfied trivially. Assume otherwise.

By induction on i, assume $\sigma = \mathcal{N}_{e,i}(TP)$ is defined. We may assume $s^\sigma > s_4$ where s_4 is as in (4) of the $\mathcal{N}$-behaviors Lemma. By (3) and (4) of the $\mathcal{N}$-behaviors Lemma $k = \lim_s k^\sigma[s]$, $b = \lim_s b^\sigma[s]$ and $d = \lim_s d^\sigma[s]$ exist.

Thus if $\sigma\hat{\ }\infty \subset TP$ then $\Phi_e(A, X_e; b)$ diverges. If $\sigma\hat{\ }w \subset TP$ then $\Gamma_i(C^\tau; d) \neq 0 = D(d)$. Otherwise $\Gamma_i(C^\tau; d) = 0 \neq 1 = D(d)$. This establishes the satisfaction of $\mathcal{N}_i^\tau$.

For $\mathcal{M}_e$ we may assume that each $\sigma_i = \mathcal{N}_{e,i}(TP)$ is defined and $\sigma_i\hat{\ }\infty \not\subset TP$. Then by (4) of the $\mathcal{N}$-behaviors Lemma each σ_i will eventually stop changing $\delta(k_i)$ ($k_i = \lim_s k^{\sigma_i}[s]$. As τ could put each $\delta(k)$ in D at most once, we have $\Delta(C, X_e)$ total.

$\Delta(C, X_e) = K$ then follows directly from Case 1 of the construction. □

Theorem IV.3.1 follows immediately from the $\mathcal{P}$-satisfaction Lemma and the $\mathcal{M}$-satisfaction Lemma.

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Acknowledgements

I learned basic concepts of logic when I was an undergraduate student and from a course given by Dr. Qing Zhou. His lectures inspired my interests in this subject.

Most of my knowledge on logic, particularly recursion theory, was obtained under the supervision of Prof. Decheng Ding. In the past years he has spent much time and energy developing me, always shared with me his insights in this subject, encouraged me when I was in confusion and constantly in great patience.

Dr. Hong Lu introduced to me the first concepts of recursion theory and Dr. Liang Yu gave me many helpful advices. And I often enjoyed talking with them.

I have also benefited from visits in the winter of 2003 and the spring of 2004 to Beijing supported by Prof. Angsheng Li, and to National University of Singapore in the summers of 2005 and 2006 supported by Prof. Chitat Chong and Yue Yang. I thank them not only for their generosity and hospitality, but also for invaluable discussions and conversations. During my visits to Singapore I had benefited from many logicians, esp. Steffen Lempp and Ted Slaman. Peter Cholak read parts of results presented in this thesis during his visit to Nanjing in the spring of 2006, and gave many helpful comments.

My friends in Botwave, though have no direct connections with my years in Nanjing, had important influence on my personality, which certainly has direct connections.

Finally but most importantly I thank my parents and my wife, Jinhe, for their love and support.