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Consensus for multi-agent systems with inherent nonlinear dynamics under directed topologies

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ABSTRACT

This paper considers the consensus problem for multi-agent systems with inherent nonlinear dynamics under directed topologies. A variable transformation method is used to convert the consensus problem to a partial stability problem. Both first-order and second-order systems are investigated under fixed and switching topologies, respectively. It is assumed that the inherent nonlinear terms satisfy the Lipshitz condition. Sufficient conditions on the feedback gains are given based on a Lyapunov function method. For first-order systems under a fixed topology, the consensus is achieved if the feedback gain related to the agents' positions is large enough. For first-order systems under switching topologies, the effect of the minimum dwell time for the switching signal on the consensus achievement is considered. For second-order systems under a fixed topology, the consensus is achieved if the feedback gains related to the agents' positions and velocities, respectively, are both large enough. For second-order systems under switching topologies, a switching variable transformation is given. Then, the consensus problem is investigated when all the digraphs are strongly connected and weighted balanced with a common weighted vector. Finally, numerical simulations are provided to illustrate the effectiveness of the obtained theoretical results.

1. Introduction

The consensus problem for multi-agent systems has drawn much attention from researchers in recent years [1,2], due to its broad range of applications in cooperative control of unmanned air vehicles, formation control of mobile robots and flocking of multiple agents. In the study of the consensus problem, the final convergence state is an important factor. In [3–5], the consensus problem was investigated for first-order and second-order systems, respectively, where the agents' final position was a constant. In [6], the consensus problem was investigated for second-order systems with relative damping introduced. It was proved that using the algorithm with relative damping introduced the agents' final position was time-varying while their final velocity was a constant. Whereas, the scenario for networks of agents with a time-varying asymptotic velocity exists ubiquitously in the study of synchronization [7]. Based on the theory of synchronization, a nonlinear term describing the intrinsic dynamics of each agent was incorporated in the consensus algorithms in [8,9]. The authors defined the generalized algebraic connectivity for the strongly connected networks and used the concept to derive sufficient conditions for the consensus of networks of agents with a time-varying asymptotic velocity. But the relationship between the generalized algebraic connectivity and the eigenvalues of the Laplacian matrix was not direct. The directed graph containing a directed spanning tree had to be divided into the strongly connected components and the generalized algebraic connectivity of each strongly connected components should be calculated to give sufficient conditions to ensure consensus. This obviously weakens the effectiveness of the obtained results when the number of the agents was large and the directed graph was complex. Moreover, the case of switching topologies was not investigated in [8]. The work of [8] was extended to the leader-following case via pinning control in [10]. In [11], the finite-time consensus problem of multi-agent networks with inherent nonlinear dynamics was considered by the comparison method. But only the undirected topologies were considered. Moreover, the convergence time could not be obtained. In [12], the consensus problem for high-order multi-agent systems with inherent nonlinear dynamics was investigated. The linear matrix inequalities were used to give sufficient conditions to ensure consensus. There, it was also assumed that the topology was undirected.

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In this paper, partly based on the ideas in [13,14], we utilize the idea of variable transformation to investigate the consensus problem for multi-agent systems with inherent nonlinear dynamics under directed topologies. By introducing the *star transformation*, the consensus problem is converted to a partial stability problem for the corresponding systems. Both first-order and second-order systems are investigated under fixed and switching topologies, respectively. The inherent nonlinear terms satisfy the Lipshitz condition is assumed. Sufficient conditions on the feedback gains are given based on a Lyapunov function method. For first-order systems under a fixed topology, the consensus is achieved if the feedback gain related to the agents' positions is large enough. For first-order systems under switching topologies, the effect of the minimum dwell time for the switching signal on the consensus achievement is considered. For second-order systems under a fixed topology, the consensus is achieved if the feedback gains related to the agents' positions and velocities, respectively, are both large enough. For second-order systems under switching topologies, we define the switching star transformation. Then, the consensus problem is investigated when all the digraphs are strongly connected and weighted balanced. Here, the given sufficient conditions are related to the matrices which are associated with the Laplacian matrices and are defined in the given theorems and the digraphs do not need to be divided into the strongly connected components as in [8,9].

The paper is organized as follows. In Section 2, we give some basic concepts in graph theory. Then, the models are described and the consensus algorithms are given. In Section 3, the consensus problem for first-order systems under fixed and switching topologies, respectively, is investigated. The star transformation is introduced in this section. The consensus problem for second-order systems under fixed and switching topologies, respectively, is investigated in Section 4. The switching star transformation is defined and analyzed in detail. Numerical simulations and a conclusion are given in Sections 5 and 6, respectively.

The following notations are used throughout this paper. $\underline{n} = \{1, \ldots, n\}$ is an index set. Let I_n be the identity matrix of dimension n, $\mathbf{1}_n = [1, \ldots, 1]^T \in R^n$, and $\mathbf{0}_n = [0, \ldots, 0]^T \in R^n$. We say $X > 0$ (resp., $X < 0$) if the matrix $X \in R^{n\times n}$ is positive definite (resp., negative definite). Given a positive definite matrix $P \in R^{n\times n}$, we denote $\lambda_{\max}(P)$ the maximum of the eigenvalues of P and $\lambda_{\min}(P)$ the minimum of the eigenvalues of P. $\mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ defines a diagonal matrix with diagonal elements being $\lambda_1, \ldots, \lambda_n$. Given $\omega = [\omega_1, \ldots, \omega_n]^T \in R^n$, denote the diagonal matrix with ω_i being the (i, i) element as $\mathrm{diag}\{\omega\}$. $\otimes$ denotes the Kronecker product.

2. Preliminaries

In this section, we introduce the graph theory and formulate the models with inherent nonlinear dynamics and the consensus algorithms.

2.1. Graph theory

A *weighted digraph* $G = (\mathscr{V}, \mathscr{E}, A)$ consists of a node set $\mathscr{V} = \underline{n}$, an edge set $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and a weighted adjacency matrix $A = [a_{ij}] \in R^{n\times n}$ satisfying $a_{ij} > 0$ if $(j, i) \in \mathscr{E}$, while $a_{ij} = 0$, otherwise. Here, we assume that $(i, i) \notin \mathscr{E}$ and hence $a_{ii} = 0$ for all $i \in \underline{n}$. The set of neighbors of node i is denoted by $N_i = \{j \in \mathscr{V} : (j, i) \in \mathscr{E}\}$. The Laplacian matrix $L = [l_{ij}]_{n\times n}$ of a weighted digraph G is defined as $l_{ii} = \sum_{j=1}^n a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. Obviously, L satisfies $L\mathbf{1}_n = 0$. Denote $L_{sym} = \frac{1}{2}(L + L^T)$.

A *directed path* between two distinct nodes i and j is a finite ordered sequence of distinct edges of G with the form $(i, k_1), (k_1, k_2), \ldots, (k_l, j)$. A digraph has a *directed spanning tree* if there exists a node called *the root* such that there exist directed paths from this node to every other node. A digraph is *strongly connected* if there exists a directed path from every node to every other node. A digraph G is called *weighted balanced* if there exists a positive vector $\omega = [\omega_1, \ldots, \omega_n] \in R^n$ satisfying $\sum_{j=1}^n \omega_i a_{ij} = \sum_{j=1}^n \omega_j a_{ji}$ for all $i \in \underline{n}$. Here, the vector ω is called a weighted vector. Note that if ω is a weighted vector, so is $\alpha\omega$, where $\alpha \in R_+$. In the remainder of this paper, without loss of generality, we assume that $\sum_{i=1}^n \omega_i = 1$.

Remark 1. The definition of the weighted balanced digraph will be used to discuss the consensus problem under switching strongly connected topologies. There are two important concepts in the literature to discuss such a problem. One is the balanced digraph [3], which demands $\sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji}$ for all $i \in \underline{n}$. The other is the detailed balanced digraph [15], which demands $\omega_i a_{ij} = \omega_j a_{ji}$ with $\omega_i, \omega_j > 0$ for all $i, j \in \underline{n}$. Obviously, the definition of the weighted balanced digraph includes these two definitions as special cases. Suppose that a strongly connected digraph G is weighted balanced with the weighted vector ω and L is its Laplacian matrix. It is not hard to see that ω^T is the left eigenvector of L associated with the zero eigenvalue. Since the sum of each row of L is zero, it can be verified that $L_{ki} = L_{kj}$ with L_{ki} and L_{kj}, respectively, being the algebraic cofactor of (k, i) and (k, j) elements of L, $k, i, j \in \underline{n}$. This, together with $\det(L) = 0$, implies that $[L_{11}, \ldots, L_{nn}]L = \mathbf{0}_n^T$. By Theorem 1 in [3] or Lemma 3.3 in [4], we know that the rank of L is $n - 1$. So, all the left eigenvectors of L associated with the zero eigenvalue form one dimension space. Then, we have $\omega_i = L_{ii}/\sum_{i=1}^n L_{ii}$ for all $i \in \underline{n}$ since we assume that $\sum_{i=1}^n \omega_i = 1$ for convenience.

The following lemma is needed in the following sections.

Lemma 1 (*[16]*). *For any two real vectors a and b with the same dimension, we have*

$$2a^T b \leq a^T \Phi a + b^T \Phi^{-1} b,$$

where Φ is any positive definite matrix with an appropriate dimension.

2.2. Models and consensus algorithms

Assume that the multi-agent system under consideration consists of n agents each of which can be regarded as a node of the information exchange digraph. Suppose that agent i, $i \in \underline{n}$, is modeled by first-order system with inherent nonlinear dynamics as

$$\dot{x}_i = f(x_i, t) + u_i, \tag{1}$$

where x_i, $u_i \in R^m$ are the position and control input vectors of agent i, respectively, and $f(x_i, t)$ is the inherent nonlinear dynamics of agent i.

Assume the digraph at time t is $G(t)$. A neighbor-based consensus algorithm for system (1) is given by

$$u_i(t) = k \sum_{j\in N_i(t)} a_{ij}(t)[x_j(t) - x_i(t)], \quad i \in \underline{n}, \tag{2}$$

where $k > 0$ is the feedback gain, $N_i(t)$ is the set of neighbors of agent i at time t, and $a_{ij}(t)$, $i, j \in \underline{n}$, is the (i, j) element of the weighted adjacency matrix and denotes the weight on information link (j, i) at time t. We say that the algorithm (2) asymptotically solves the consensus problem for the system (1) if

$$\lim_{t\to\infty} (x_i - x_j) = 0, \quad \forall i, j \in \underline{n}.$$

When agent i, $i \in \underline{n}$, is modeled by a second-order system, we have

$$\begin{cases}\dot{x}_i = v_i,\\ \dot{v}_i = f(x_i, v_i, t) + u_i,\end{cases} \quad (3)$$

where $x_i, v_i, u_i \in R^m$ are the position vector, velocity vector and control input vector of agent i, respectively, and $f(x_i, v_i, t)$ is the inherent nonlinear dynamics of agent i.

For the system (3), we consider two neighbor-based consensus algorithms as follows:

$$u_i(t) = \sum_{j\in N_i(t)} a_{ij}(t)[\alpha_1(x_j(t) - x_i(t))] - \beta_1 v_i(t), \quad i \in \underline{n}, \quad (4)$$

and

$$u_i(t) = \sum_{j\in N_i(t)} a_{ij}(t)[\alpha_2(x_j(t) - x_i(t)) + \beta_2(v_j(t) - v_i(t))], \quad i \in \underline{n}, \quad (5)$$

where $\alpha_i > 0, \beta_i > 0, i = 1, 2$, are the feedback gains, $N_i(t)$ and $a_i(t)$ are defined the same as those in the algorithm (2). The consensus for the system (3) using the algorithm (4) or (5) is achieved if

$$\lim_{t\to\infty}(x_i - x_j) = 0, \qquad \lim_{t\to\infty}(v_i - v_j) = 0, \quad \forall\, i, j \in \underline{n}.$$

For notational simplicity in the following analysis, we only consider the case $m = 1$. The analysis is valid for any dimension m with the difference being that the expression should be rewritten in terms of the Kronecker product.

3. First-order system

In this section, we analyze the consensus problem for the first-order system (1). Differently from the existing results without the inherent nonlinear dynamics, the final state of the agents will be time-varying. More specifically, the final state depends on the inherent nonlinear function $f(x, t)$. We need the following assumption for further discussion.

Assumption 1. The function $f(x, t)$ satisfies the Lipschitz condition in x with the Lipschitz constant l, i.e.,

$$|f(x_2, t) - f(x_1, t)| \le l|x_2 - x_1|, \quad \forall\, x_1, x_2 \in R, \forall t \ge 0.$$

With the algorithm (2), the system (1) becomes

$$\dot{x}_i = f(x_i, t) + k \sum_{j\in N_i(t)} a_{ij}(t)[x_j(t) - x_i(t)], \quad i \in \underline{n}. \quad (6)$$

Let $x = [x_1, x_2, \dots, x_n]^T$. The system (6) can be written as

$$\dot{x} = \bar{f}(x, t) - kL(t)x, \quad (7)$$

where $\bar{f}(x, t) = [f(x_1, t), \dots, f(x_n, t)]^T$ and $L(t)$ is the Laplacian matrix of $G(t)$. To investigate the system (7), we introduce a variable transformation called *star transformation* as follows:

$$y = Sx, \quad (8)$$

where $S \in R^{n\times n}$ is the transformation matrix defined by

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0\\ 1 & -1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}.$$

It is easy to verify that $S = S^{-1}$. Denote $y = [y_1, y_e^T]^T$, where $y_e = [y_2, \dots, y_n]^T$. By (8), we have $y_1 = x_1$ and

$$y_e = [x_1 - x_2, x_1 - x_3, \dots, x_1 - x_n]^T.$$

Rewriting (7) with respect to y, we have

$$\dot{y} = S\bar{f}(y, t) - kSL(t)S^{-1}y, \quad (9)$$

where $\bar{f}(y, t) = [f(y_1, t), f(y_1 - y_2, t), \dots, f(y_1 - y_n, t)]^T$. Define $E = [\mathbf{1}_{n-1}\ -I_{n-1}]$ and $F = [\mathbf{0}_{n-1}\ -I_{n-1}]^T$. Note that $L(t)\mathbf{1}_n = 0$. It follows that (9) can be rewritten as the following two subsystems

$$\dot{y}_1 = f(y_1, t) - kl_1(t)Fy_e, \quad (10)$$

$$\dot{y}_e = f_e(y, t) - kEL(t)Fy_e, \quad (11)$$

where $l_1(t)$ is the first row of $L(t)$, $f_e(y, t) = [f(y_1, t) - f(y_1 - y_2, t), \dots, f(y_1, t) - f(y_1 - y_n, t)]^T$.

Remark 2. Actually, we can also use another variable transformation called line transformation as follows:

$$z = Tx,$$

where $T \in R^{n\times n}$ is the transformation matrix defined by

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ 1 & -1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

It is easy to see that the two kinds of transformations are similar. Actually, they have the same effect on the study of consensus problem. We will depend on the transformation (8) in the following.

3.1. First-order system under a fixed topology

In this subsection, we study the case where the digraph is fixed, i.e., $L(t) \equiv L$. Note that, for the existence of nonlinear term, the subsystems (10) and (11) are coupled. We will resort to the concept of *partial stability* for which the reader may refer to [17] to analyze the system (7). We have the following proposition to show the relationship between the system (7) and the system given by (10) and (11).

Proposition 1. *The consensus is achieved for the system (7) if and only if the system given by (10) and (11) is asymptotically stable with respect to y_e.*

The proof is trivial. So, we omit it. Moreover, in the following discussion, similar propositions to Proposition 1 also hold for each part and will not be stated anymore. Now, we establish our first theorem.

Theorem 1. *Suppose the fixed digraph G has a directed spanning tree and Assumption 1 holds. The algorithm (2) asymptotically solves the consensus problem for the system (1) if the feedback gain k satisfies the following condition*

$$-k[(ELF)^TP + P(ELF)] + l^2\lambda_{\max}(P)I_{n-1} + P < 0, \quad (12)$$

where $P \in R^{(n-1)\times(n-1)}$ is positive definite with $(ELF)^TP + P(ELF)$ being positive definite. To simplify the condition (12), we can choose P such that $(ELF)^TP + P(ELF) = I_{n-1}$. Then, (12) is equivalent to $l^2\lambda_{\max}(P)I_{n-1} + P < kI_{n-1}$.

Proof. By Proposition 1, we only need to prove that the system given by (10) and (11) is asymptotically stable with respect to y_e. Choose a Lyapunov function candidate as

$$V(t) = y_e^T(t)Py_e(t).$$

Differentiating $V(t)$ along the trajectories of (10) and (11) yields

$$\dot{V}(t) = -ky_e^T[(ELF)^TP + P(ELF)]y_e + 2y_e^T(t)Pf_e(y, t).$$

Using Lemma 1 with $a^T = y_e^T(t), b = Pf_e(y, t)$, and $\Phi = P$, together with Assumption 1, it follows that

$$\begin{aligned}\dot{V}(t) &\le -ky_e^T[(ELF)^TP + P(ELF)]y_e + f_e^T(y_e, t)Pf_e(y_e, t) + y_e^TPy_e\\ &\le -ky_e^T[(ELF)^TP + P(ELF)]y_e + l^2\lambda_{\max}(P)y_e^Ty_e + y_e^TPy_e.\end{aligned}$$

By (12), we have $\dot{V}(t) < 0$ for $y_e \neq 0$, which implies that the system given by (10) and (11) is asymptotically stable with respect to y_e.

Next, we verify the feasibility of (12). By Theorem 3.12 in [4], we know that the algorithm $u_i = \sum_{j\in N_i} a_{ij}(x_j - x_i)$ can solve the consensus problem for the system (1) with no inherent nonlinear dynamics if and only if the digraph has a directed spanning tree. Thus, it is not difficult to get that $-ELF$ is Hurwitz stable. The rest of the proof is trivial and is omitted. □

Remark 3. In Theorem 1, once the consensus is achieved, the final state of all agents is a very important thing which we are interested in. From (10), it is easy to see that the final state as $t \to \infty$ will satisfy the following dynamics

$$\frac{dx}{dt} = f(x, t), \quad \text{where } x \in R.$$

That is, the final state as $t \to \infty$ is time-varying. Moreover, if the function $f(x, t)$ is linear in x, that is, $\alpha f(x_1, t) + \beta f(x_2, t) = f(\alpha x_1 + \beta x_2, t)$, we can prove that the final state as $t \to \infty$ is related to the initial values of all agents, which is defined as the $\mathcal{X}$-consensus problem [3]. Specifically, choose $p^T = [p_1, \ldots, p_n]$ satisfying $\sum_{i=1}^n p_i = 1$ to be the nonnegative left eigenvector of L associated with the zero eigenvalue. Using p^T to pre-multiply the system (7), it is easy to get

$$\frac{d}{dt}\sum_{i=1}^n p_i x_i = f\left(\sum_{i=1}^n p_i x_i, t\right). \tag{13}$$

That is, the weighted sum of the states of all agents satisfies the dynamics (13). Since $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0,\ \forall i, j \in \underline{n}$, we see that the final state as $t \to \infty$ satisfies the following dynamics with initial value

$$\begin{cases} \dfrac{dx}{dt} = f(x, t), \\ x(0) = \displaystyle\sum_{i=1}^n p_i x_i(0). \end{cases}$$

3.2. First-order system under switching topologies

In this subsection, we study the case where the digraph is time-varying. We assume that the digraphs are taken from a finite set $\Gamma = \{G_1, \ldots, G_M\}$ and use a switching signal $\sigma : [0, +\infty) \to \underline{M} \triangleq \{1, \ldots, M\}$ to describe which digraph is active at time t. Moreover, suppose the digraph G_σ switches at $t_r,\ r = 0, 1, \ldots$, with $t_0 = 0$ and remains unchanged during each interval $[t_r, t_{r+1}),\ r = 0, 1, \ldots$. There exists a positive constant T such that $t_{r+1} - t_r \geq T,\ r = 0, 1, \ldots$. The number T can be arbitrarily small and is called the minimum dwell time for the switching signal σ. Let $L_i, i \in \underline{M}$, denote the Laplacian matrix associated with G_i.

We have the following theorem which gives sufficient conditions for the consensus problem of the system (1) under switching topologies.

Theorem 2. *Suppose each of the digraphs contains a directed spanning tree and Assumption 1 holds. The algorithm (2) asymptotically solves the consensus problem for the system (1) under arbitrary switching signal if the feedback gain k is large enough such that*

$$\frac{\overline{\lambda}}{\underline{\lambda}} exp\left[\left(-\frac{k}{\overline{\lambda}} + l^2\frac{\overline{\lambda}}{\underline{\lambda}} + 1\right)T\right] < 1, \tag{14}$$

where $\overline{\lambda} = \max_{i\in\underline{M}}\{\lambda_{\max}(P_i)\}$ *and* $\underline{\lambda} = \min_{i\in\underline{M}}\{\lambda_{\min}(P_i)\}$ *with* P_i *being a positive definite matrix satisfying* $(EL_iF)^T P_i + P_i(EL_iF) = I_{n-1},\ i \in \underline{M}$.

Proof. Also, we only need to prove that the system given by (10) and (11) is asymptotically stable with respect to y_e. The proof for the existence of $P_i, i \in \underline{M}$, is the same as that of P in Theorem 1. Choose the following Lyapunov function candidate

$$V(t) = y_e^T(t) P_\sigma y_e(t).$$

Differentiating $V(t)$ along the trajectories of (10) and (11), we have

$$\dot{V}(t) = -k y_e^T y_e + 2 y_e^T P_\sigma f_e.$$

Note that we need to use the right derivative of $V(t)$ at the switching time to fit for the switching case. By virtue of Lemma 1 with $a^T = y_e^T$, $b = P_\sigma f_e$, and $\Phi = P_\sigma$, together with Assumption 1, we have

$$\begin{aligned}\dot{V}(t) &\leq -k y_e^T y_e + f_e^T P_\sigma f_e + y_e^T P_\sigma y_e \leq -k y_e^T y_e + l^2 \lambda_{\max}(P_\sigma) y_e^T y_e \\ &\quad + y_e^T P_\sigma y_e \leq \left(-\frac{k}{\overline{\lambda}} + l^2\frac{\overline{\lambda}}{\underline{\lambda}} + 1\right) V(t).\end{aligned} \tag{15}$$

Note (14) implies that $-\frac{k}{\overline{\lambda}} + l^2\frac{\overline{\lambda}}{\underline{\lambda}} + 1 < 0$. So, we have $\dot{V}(t) < 0$ for $y_e \neq 0$. Multiplying both sides of the above inequality with $exp\left[-\left(-\frac{k}{\overline{\lambda}} + l^2\frac{\overline{\lambda}}{\underline{\lambda}} + 1\right)t\right]$, we get that

$$\left(V(t) exp\left[-\left(-\frac{k}{\overline{\lambda}} + l^2\frac{\overline{\lambda}}{\underline{\lambda}} + 1\right)t\right]\right)' \leq 0.$$

Integrating the above inequality from t_r to t_{r+1}, with a mild simplification, we have

$$V(t_{r+1}^-) \leq V(t_r) exp\left[\left(-\frac{k}{\overline{\lambda}} + l^2\frac{\overline{\lambda}}{\underline{\lambda}} + 1\right)(t_{r+1} - t_r)\right].$$

It follows that

$$\begin{aligned}V(t_{r+1}) &\leq \overline{\lambda} y_e^T(t_{r+1}) y_e(t_{r+1}) \leq \frac{\overline{\lambda}}{\underline{\lambda}} V(t_{r+1}^-) \\ &\leq V(t_r)\frac{\overline{\lambda}}{\underline{\lambda}} exp\left[\left(-\frac{k}{\overline{\lambda}} + l^2\frac{\overline{\lambda}}{\underline{\lambda}} + 1\right)T\right].\end{aligned} \tag{16}$$

Because (14) holds, by the stability theory for switched systems [18], (15) and (16) imply that the asymptotic stability of the system given by (10) and (11) with respect to y_e holds. □

Remark 4. Actually, from the above proof, we can get that all the agents converge to the final consensus state in the following rate expressed by

$$V(t) \leq V(0)\gamma^m exp\left[\left(-\frac{k}{\overline{\lambda}} + l^2\frac{\overline{\lambda}}{\underline{\lambda}} + 1\right)(t - mT)\right] \tag{17}$$

with $\gamma = \frac{\overline{\lambda}}{\underline{\lambda}} exp\left[\left(-\frac{k}{\overline{\lambda}} + l^2\frac{\overline{\lambda}}{\underline{\lambda}} + 1\right)T\right]$ and $m = \lfloor \frac{t}{T} \rfloor$ being the maximum integer no larger than $\frac{t}{T}$. Note that, in Theorem 2, we emphasize the effect of the feedback gain k on the consensus achievement but do not care about how small the minimum dwell time T is. From (17), it is not hard to see that the convergence rate shown by $V(t)$ will increase once the minimum dwell time increases.

Remark 5. Comparing with [9], we focus on the global consensus of first-order multi-agent systems in this section. For the general network, we do not require to divide the digraph with a directed spanning tree into the strongly connected components to give sufficient conditions for the consensus achievement. This facilitates the application of the obtained results. The consensus problem under switching topologies was discussed in Remark 1 of [9] where each of the digraphs was strongly connected. Whereas, we study the case that each of the digraphs contains a directed spanning tree and the minimum dwell time for the switching signal which can be arbitrarily small exists.

4. Second-order system

In many practical applications, the acceleration rather than the velocity is controlled. We hence discuss the consensus problem for agents with second-order dynamics in this section. We will discuss two kinds of algorithms (4) and (5). Before moving on, we also assume that the function f in (3) satisfies the Lipschitz condition in x_i and v_i as follows:

Assumption 2. There exists a nonnegative constant ρ such that

$$|f(x_2, v_2, t) - f(x_1, v_1, t)| \le \rho^{1/2}\sqrt{(x_2 - x_1)^2 + (v_2 - v_1)^2},\quad \forall x_i, v_i \in R, i = 1, 2, \forall t \ge 0.$$

Next, we first convert the consensus problem using the algorithms (4) and (5), respectively, to a partial stability problem for the corresponding systems by virtue of the star transformation.

The dynamics for agents described by (3) using the algorithm (4) can be written as

$$\dot{\xi} = \begin{bmatrix} \mathbf{0}_{n\times n} & I_n \\ -\alpha_1 L(t) & -\beta_1 I_n \end{bmatrix} \xi + \begin{bmatrix} \mathbf{0}_n \\ \bar{f}(\xi, t) \end{bmatrix}, \tag{18}$$

where $\xi = [x^T, v^T]^T$ with $x = [x_1, \ldots, x_n]^T$, $v = [v_1, \ldots, v_n]^T$, and $\bar{f}(\xi, t) = [f(x_1, v_1, t), \ldots, f(x_n, v_n, t)]^T$. Let $y = Sx$ and $\hat{v} = Sv$, where S is defined after (8). Let y_i and $\hat{v}_i$ be, respectively, the ith component of y and $\hat{v}$. We have

$$\dot{\eta} = \begin{bmatrix} \mathbf{0}_{n\times n} & I_n \\ -\alpha_1 SL(t)S^{-1} & -\beta_1 I_n \end{bmatrix} \eta + \begin{bmatrix} \mathbf{0}_n \\ S\bar{f}(\eta, t) \end{bmatrix}, \tag{19}$$

where $\eta = [y^T, \hat{v}^T]^T$ and $\bar{f}(\eta, t) = [f(y_1, \hat{v}_1, t), f(y_1 - y_2, \hat{v}_1 - \hat{v}_2, t), \ldots, f(y_1 - y_n, \hat{v}_1 - \hat{v}_n, t)]^T$. Define $\eta_1 = [y_1, \hat{v}_1]^T$, $\eta_e = [y_2, \ldots, y_n, \hat{v}_2, \ldots, \hat{v}_n]^T$. We can rewrite the system (19) as the following two subsystems

$$\dot{\eta}_1 = \begin{bmatrix} \mathbf{0}_{n-1}^T & 1 \\ -\alpha_1 l_1(t) F & -\beta_1 \end{bmatrix} [\eta_{e1}^T \ \hat{v}_1]^T + \begin{bmatrix} 0 \\ f(y_1, \hat{v}_1, t) \end{bmatrix}, \tag{20}$$

$$\dot{\eta}_e = \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & I_{n-1} \\ -\alpha_1 EL(t)F & -\beta_1 I_{n-1} \end{bmatrix} \eta_e + \begin{bmatrix} \mathbf{0}_{n-1} \\ f_e(\eta, t) \end{bmatrix}, \tag{21}$$

where $y_1 = x_1, \hat{v}_1 = v_1, \eta_{e1} = [y_2, \ldots, y_n], E = [\mathbf{1}_{n-1} - I_{n-1}], F = [\mathbf{0}_{n-1} - I_{n-1}]^T$, $l_1(t)$ is the first row of $L(t)$, and $f_e(\eta, t) = [f(y_1, \hat{v}_1, t) - f(y_1 - y_2, \hat{v}_1 - \hat{v}_2, t), \ldots, f(y_1, \hat{v}_1, t) - f(y_1 - y_n, \hat{v}_1 - \hat{v}_n, t)]$. Also note that the subsystems (20) and (21) are not completely decoupled.

Next, we consider the system (3) using the algorithm (5). The dynamics of the closed-loop system can be written as

$$\dot{\xi} = \begin{bmatrix} \mathbf{0}_{n\times n} & I_n \\ -\alpha_2 L(t) & -\beta_2 L(t) \end{bmatrix} \xi + \begin{bmatrix} \mathbf{0}_n \\ \bar{f}(\xi, t) \end{bmatrix}, \tag{22}$$

where ξ and $\bar{f}(\xi, t)$ are defined the same as those in (18). For simplicity, we directly give the corresponding subsystems through the star transformation (8) as

$$\dot{\eta}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \eta_1 + \begin{bmatrix} \mathbf{0}_{n-1}^T & \mathbf{0}_{n-1}^T \\ -\alpha_2 l_1(t) F & -\alpha_2 l_1(t) F \end{bmatrix} \eta_e + \begin{bmatrix} 0 \\ f(y_1, \hat{v}_1, t) \end{bmatrix}, \tag{23}$$

$$\dot{\eta}_e = \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & I_{n-1} \\ -\alpha_2 EL(t)F & -\beta_2 EL(t)F \end{bmatrix} \eta_e + \begin{bmatrix} \mathbf{0}_{n-1} \\ f_e(\eta, t) \end{bmatrix}, \tag{24}$$

where the notations are the same as those in (20) and (21).

4.1. Second-order System under a Fixed Topology

In this subsection, we consider the consensus problem for second-order systems under a fixed topology, i.e., $L(t) \equiv L$.

Theorem 3. *Suppose the fixed digraph G has a directed spanning tree and Assumption 2 holds. Consensus of the system (3) using the algorithm (4) is achieved if the feedback gains α_1 and β_1 satisfy the following condition*

$$-\begin{bmatrix} \alpha_1 I_{n-1} & \alpha_1 (ELF)^T P \\ \alpha_1 P(ELF) & 2(\beta_1 - 1)P \end{bmatrix} + I_2 \otimes (2\rho\bar{\lambda} I_{n-1} + P) < 0, \tag{25}$$

where $P \in R^{(n-1)\times(n-1)}$ is the positive definite matrix satisfying $P(ELF) + (ELF)^T P = I_{n-1}$, $\beta_1 > 1$, and $\bar{\lambda} = \lambda_{\max}(P)$. To simplify the choice of α_1 and β_1, we can let $\beta_1 = \frac{1}{2}k\alpha_1 + 1$ with $k > 0$ satisfying $kP - P(ELF)(ELF)^T P > 0$. Then, once α_1 is large enough, the condition (25) will hold.

Proof. First, we see that the positive definite matrix P always exists since it is easy to see that $-ELF$ is Hurwitz stable under the assumption that G has a directed spanning tree. Choose a Lyapunov function candidate as

$$V(t) = \eta_e^T \begin{bmatrix} \mu P & \nu P \\ \nu P & \gamma P \end{bmatrix} \eta_e \triangleq \eta_e^T \bar{P} \eta_e,$$

where μ, ν, γ are positive constants satisfying $\mu\gamma > \nu^2$ to guarantee that V is positive definite. The values of these three numbers will be given below. Differentiating $V(t)$ along the trajectories of the system given by (20) and (21), we have

$$\begin{aligned} \dot{V}(t) &= \eta_e^T \left(\bar{P} \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & I_{n-1} \\ -\alpha_1 ELF & -\beta_1 I_{n-1} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & I_{n-1} \\ -\alpha_1 ELF & -\beta_1 I_{n-1} \end{bmatrix}^T \bar{P} \right) \eta_e \\ &\quad + 2\eta_e^T \bar{P} \begin{bmatrix} \mathbf{0}_{n-1} \\ f_e(\eta, t) \end{bmatrix} \\ &= \eta_e^T \begin{bmatrix} -\alpha_1 \nu I_{n-1} & (\mu - \beta_1\nu)P - \alpha_1\gamma (ELF)^T P \\ (\mu - \beta_1\nu)P - \alpha_1\gamma P(ELF) & 2(\nu - \beta_1\gamma)P \end{bmatrix} \eta_e \\ &\quad + 2\eta_e^T \mathrm{diag}\{\nu P, \gamma P\} \begin{bmatrix} f_e(\eta, t) \\ f_e(\eta, t) \end{bmatrix}. \end{aligned}$$

To meet our need, we choose $\mu = \beta_1 \nu$. Meanwhile, for simplicity, we choose $\nu = \gamma = 1$. So, $\mu\gamma > \nu^2$ since $\beta_1 > 1$, which implies that V is positive definite. Then, we get

$$\begin{aligned} \dot{V}(t) &= -\eta_e^T \begin{bmatrix} \alpha_1 I_{n-1} & \alpha_1 (ELF)^T P \\ \alpha_1 P(ELF) & 2(\beta_1 - 1)P \end{bmatrix} \eta_e + 2\eta_e^T \mathrm{diag}\{P, P\} \begin{bmatrix} f_e(\eta, t) \\ f_e(\eta, t) \end{bmatrix} \\ &\le -\eta_e^T \begin{bmatrix} \alpha_1 I_{n-1} & \alpha_1 (ELF)^T P \\ \alpha_1 P(ELF) & 2(\beta_1 - 1)P \end{bmatrix} \eta_e \\ &\quad + \eta_e^T \mathrm{diag}\{P, P\} \eta_e + \left[f_e^T(\eta, t) \quad f_e^T(\eta, t) \right] \mathrm{diag}\{P, P\} \begin{bmatrix} f_e(\eta, t) \\ f_e(\eta, t) \end{bmatrix} \\ &\le -\eta_e^T \begin{bmatrix} \alpha_1 I_{n-1} & \alpha_1 (ELF)^T P \\ \alpha_1 P(ELF) & 2(\beta_1 - 1)P \end{bmatrix} \eta_e \\ &\quad + \eta_e^T \left[I_2 \otimes (2\rho\bar{\lambda} I_{n-1} + P) \right] \eta_e. \end{aligned}$$

Here, Lemma 1 has been applied by choosing $a = \eta_e$ and $b = \mathrm{diag}\{P, P\} \left[f_e^T(\eta, t), f_e^T(\eta, t) \right]^T$ to get the first inequality and Assumption 2 has been used to obtain the second inequality.

Next, we show that the condition (25) is feasible for large enough α_1 and β_1. The condition (25) is equivalent to the following condition

$$\begin{aligned} &-\begin{bmatrix} I_{n-1} & \mathbf{0}_{(n-1)\times(n-1)} \\ -P(ELF) & I_{n-1} \end{bmatrix} \begin{bmatrix} \alpha_1 I_{n-1} & \alpha_1 (ELF)^T P \\ \alpha_1 P(ELF) & 2(\beta_1 - 1)P \end{bmatrix} \\ &\times \begin{bmatrix} I_{n-1} & \mathbf{0}_{(n-1)\times(n-1)} \\ -P(ELF) & I_{n-1} \end{bmatrix}^T + \begin{bmatrix} I_{n-1} & \mathbf{0}_{(n-1)\times(n-1)} \\ -P(ELF) & I_{n-1} \end{bmatrix} \end{aligned}$$

$$\times \left[I_2 \otimes (2\rho\bar{\lambda} I_{n-1} + P)\right] \begin{bmatrix} I_{n-1} & \mathbf{0}_{(n-1)\times(n-1)} \\ -P(ELF) & I_{n-1} \end{bmatrix}^T < 0.$$

By direct calculation, the above condition can be simplified as shown in Eq. (26) which is in Box I.

Obviously, (26) will hold only if α_1 and β_1 are large enough. By letting $\beta_1 = \frac{1}{2}k\alpha_1 + 1$ with $k > 0$ satisfying $kP - P(ELF)(ELF)^T P > 0$, we can guarantee (25) holds by only changing one parameter. Now that (25) holds for large α_1 and β_1, then $\dot{V}(t) < 0$ for $\eta_e \neq 0$. So, the system given by (20) and (21) is asymptotically stable with respect to η_e, which means the consensus of (3) using the algorithm (4) is achieved. □

Next, we consider the consensus problem for the system (3) using the algorithm (5).

Theorem 4. *Suppose the fixed digraph G has a directed spanning tree and Assumption 2 holds. Consensus of the system (3) using the algorithm (5) is achieved if the feedback gains α_2 and β_2 satisfy the following conditions*

$$\alpha_2 > (3 + 2\rho)\bar{\lambda}, \tag{27}$$

$$\beta_2 > \max\left\{\sqrt{\alpha_2(\bar{\lambda}+1)},\quad \bar{\lambda}(1+2\rho)\right\}, \tag{28}$$

and

$$[\alpha_2^2 - \alpha_2\lambda_i - (\alpha_2+\beta_2)\bar{\lambda}\rho] \times [\beta_2^2 - (2\alpha_2+\beta_2)\lambda_i - (\alpha_2+\beta_2)\bar{\lambda}\rho] - \alpha_2^2\beta_2^2\left(\frac{\lambda_i}{\bar{\lambda}+1} - 1\right)^2 > 0, \quad 1 \le i \le n-1, \tag{29}$$

where λ_i, $1 \le i \le n-1$, is the eigenvalue of P and $\bar{\lambda} = \lambda_{\max}(P)$ with $P \in R^{(n-1)\times(n-1)}$ being the positive definite matrix satisfying $P(ELF) + (ELF)^T P = I_{n-1}$.

Denote $\underline{\lambda} = \lambda_{min}(P)$. To simplify the calculation for (29), we can replace (27) and (29) with the following conditions, respectively,

$$\alpha_2 > \max\{(3+2\rho)\bar{\lambda},\ (2+4\rho)\bar{\lambda}\}, \tag{30}$$

and

$$[\alpha_2^2 - \alpha_2\bar{\lambda} - (\alpha_2+\beta_2)\bar{\lambda}\rho] \times [\beta_2^2 - (2\alpha_2+\beta_2)\bar{\lambda} - (\alpha_2+\beta_2)\bar{\lambda}\rho] - \alpha_2^2\beta_2^2\left(\frac{\underline{\lambda}}{\bar{\lambda}+1} - 1\right)^2 > 0. \tag{31}$$

If $1 \le \frac{\bar{\lambda}}{\underline{\lambda}} < 2$, besides (30), we can replace (28) and (29), respectively, with the following conditions

$$\beta_2 > \max\{\sqrt{\alpha_2\underline{\lambda}},\ \bar{\lambda}(1+2\rho)\}, \tag{32}$$

and

$$[\alpha_2^2 - \alpha_2\bar{\lambda} - (\alpha_2+\beta_2)\bar{\lambda}\rho] \times [\beta_2^2 - (2\alpha_2+\beta_2)\bar{\lambda} - (\alpha_2+\beta_2)\bar{\lambda}\rho] - \alpha_2^2\beta_2^2\left(\frac{\bar{\lambda}}{\underline{\lambda}} - 1\right)^2 > 0. \tag{33}$$

Furthermore, if we let $\alpha_2 = \beta_2$, the consensus of the system (3) using the algorithm (5) is achieved under the following condition

$$\alpha_2 > \max\left\{2(1+\rho)\bar{\lambda},\ \frac{[(2\theta^2-6)-8\rho]\bar{\lambda}}{2\theta-4},\ \frac{\theta^2\underline{\lambda}^2 - (\underline{\lambda}+2\rho\bar{\lambda})(3\underline{\lambda}+2\rho\bar{\lambda})}{(2\theta-4)\underline{\lambda} - 4\rho\bar{\lambda}}\right\}, \tag{34}$$

where $\theta > 2(1+\rho\frac{\bar{\lambda}}{\underline{\lambda}})$.

Proof. As in the proof of Theorem 3, we start with a Lyapunov function candidate of the form

$$V(t) = \eta_e^T \begin{bmatrix} \mu P & \nu P \\ \nu P & \gamma P \end{bmatrix} \eta_e \triangleq \eta_e^T \bar{P} \eta_e,$$

where μ, ν, γ are positive constants satisfying $\mu\gamma > \nu^2$ to guarantee that V is positive definite. The values of these three numbers will be given below. Differentiating $V(t)$ along the trajectories of the system given by (23) and (24), we have

$$\begin{aligned}
\dot{V}(t) &= \eta_e^T \left(\bar{P} \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & I_{n-1} \\ -\alpha_2 ELF & -\beta_2 ELF \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & I_{n-1} \\ -\alpha_2 ELF & -\beta_2 ELF \end{bmatrix}^T \bar{P} \right) \eta_e + 2\eta_e^T \bar{P} \begin{bmatrix} \mathbf{0}_{n-1} \\ f_e(\eta,t) \end{bmatrix} \\
&= \eta_e^T \left(\begin{bmatrix} -\alpha_2\nu P(ELF) & \mu P - \beta_2\nu P(ELF) \\ -\alpha_2\gamma P(ELF) & \nu P - \beta_2\gamma P(ELF) \end{bmatrix} + \begin{bmatrix} -\alpha_2\nu P(ELF) & \mu P - \beta_2\nu P(ELF) \\ -\alpha_2\gamma P(ELF) & \nu P - \beta_2\gamma P(ELF) \end{bmatrix}^T \right) \eta_e \\
&\quad + 2\eta_e^T \operatorname{diag}\{\nu P, \gamma P\} \begin{bmatrix} f_e(\eta,t) \\ f_e(\eta,t) \end{bmatrix}.
\end{aligned}$$

We let $\nu = \alpha_2$ and $\gamma = \beta_2$ first. Then

$$\begin{aligned}
\dot{V}(t) &\le \eta_e^T \begin{bmatrix} -\alpha_2^2 I_{n-1} & \mu P - \alpha_2\beta_2 I_{n-1} \\ \mu P - \alpha_2\beta_2 I_{n-1} & 2\alpha_2 P - \beta_2^2 I_{n-1} \end{bmatrix} \eta_e + \eta_e^T \begin{bmatrix} \alpha_2 P & \mathbf{0}_{(n-1)\times(n-1)} \\ \mathbf{0}_{(n-1)\times(n-1)} & \beta_2 P \end{bmatrix} \eta_e \\
&\quad + \begin{bmatrix} f_e^T & f_e^T \end{bmatrix} \begin{bmatrix} \alpha_2 P & \mathbf{0}_{(n-1)\times(n-1)} \\ \mathbf{0}_{(n-1)\times(n-1)} & \beta_2 P \end{bmatrix} \begin{bmatrix} f_e \\ f_e \end{bmatrix} \\
&\le \eta_e^T \begin{bmatrix} -\alpha_2^2 I_{n-1} & \mu P - \alpha_2\beta_2 I_{n-1} \\ \mu P - \alpha_2\beta_2 I_{n-1} & 2\alpha_2 P - \beta_2^2 I_{n-1} \end{bmatrix} \eta_e \\
&\quad + \eta_e^T \operatorname{diag}\{\alpha_2 P, \beta_2 P\}\eta_e + (\alpha_2+\beta_2)\bar{\lambda}\rho\eta_e^T\eta_e.
\end{aligned}$$

Next, we investigate how to guarantee that $\dot{V}(t) < 0$ if $\eta_e \neq 0$. It is equivalent to the positive definiteness of the matrix shown in Box II. Since P is positive definite, there exists an orthogonal matrix U such that $U^T PU = \operatorname{diag}\{\lambda_1, \ldots, \lambda_{n-1}\} \triangleq \Lambda$. Using $\operatorname{diag}\{U^T,\ U^T\}$ and $\operatorname{diag}\{U,\ U\}$ to pre- and post-multiply Ω, respectively, we get the similar matrix of Ω as given in Box III:

Then, Ω is positive definite is equivalent to all the following matrices are positive definite

$$\begin{bmatrix} \alpha_2^2 - \alpha_2\lambda_i - (\alpha_2+\beta_2)\bar{\lambda}\rho & -\mu\lambda_i + \alpha_2\beta_2 \\ -\mu\lambda_i + \alpha_2\beta_2 & \beta_2^2 - (2\alpha_2+\beta_2)\lambda_i - (\alpha_2+\beta_2)\bar{\lambda}\rho \end{bmatrix},$$

$i = 1, \ldots, n-1.$

By Vieta's Theorem, we need to test that

$$[\alpha_2^2 - \alpha_2\lambda_i - (\alpha_2+\beta_2)\bar{\lambda}\rho] + [\beta_2^2 - (2\alpha_2+\beta_2)\lambda_i - (\alpha_2+\beta_2)\bar{\lambda}\rho] > 0, \tag{35}$$

$$[\alpha_2^2 - \alpha_2\lambda_i - (\alpha_2+\beta_2)\bar{\lambda}\rho] \times [\beta_2^2 - (2\alpha_2+\beta_2)\lambda_i - (\alpha_2+\beta_2)\bar{\lambda}\rho] - (\mu\lambda_i - \alpha_2\beta_2)^2 > 0. \tag{36}$$

It is not difficult to get that (27) and (28) guarantee that (35) holds.

Next, we analyze how to meet (36). Choosing $\mu = \frac{\alpha_2\beta_2}{\bar{\lambda}+1}$, then (36) becomes (29). Noting that $-1 < \frac{\lambda_i}{\bar{\lambda}+1} - 1 < 0$, it is obvious that (29) or (36) will hold once α_2 and β_2 are large enough. Noting that we require $\mu\gamma > \nu^2$ to guarantee V being positive definite at the beginning of the proof, we have $\beta_2 > \sqrt{\alpha_2(\bar{\lambda}+1)}$, which is shown in (28).

$$-\begin{bmatrix} \alpha_1 I_{n-1} & \mathbf{0}_{(n-1)\times(n-1)} \\ \mathbf{0}_{(n-1)\times(n-1)} & 2(\beta_1-1)P-\alpha_1 P(ELF)(ELF)^T P \end{bmatrix} + \begin{bmatrix} 2\rho\bar{\lambda} I_{n-1}+P & -[2\rho\bar{\lambda}I_{n-1}+P][(ELF)^T P] \\ -[P(ELF)][2\rho\bar{\lambda}I_{n-1}+P] & [P(ELF)](2\rho\bar{\lambda}I_{n-1}+P)[(ELF)^T P]+(2\rho\bar{\lambda}I_{n-1}+P) \end{bmatrix} < 0. \quad (26)$$

Box I.

$$\Omega \triangleq \begin{bmatrix} \alpha_2^2 I_{n-1}-\alpha_2 P-(\alpha_2+\beta_2)\bar{\lambda}\rho I_{n-1} & -\mu P+\alpha_2\beta_2 I_{n-1} \\ -\mu P+\alpha_2\beta_2 I_{n-1} & \beta_2^2 I_{n-1}-(2\alpha_2+\beta_2)P-(\alpha_2+\beta_2)\bar{\lambda}\rho I_{n-1} \end{bmatrix}.$$

Box II.

$$\begin{bmatrix} \alpha_2^2 I_{n-1}-\alpha_2 \Lambda-(\alpha_2+\beta_2)\bar{\lambda}\rho I_{n-1} & -\mu \Lambda+\alpha_2\beta_2 I_{n-1} \\ -\mu \Lambda+\alpha_2\beta_2 I_{n-1} & \beta_2^2 I_{n-1}-(2\alpha_2+\beta_2)\Lambda-(\alpha_2+\beta_2)\bar{\lambda}\rho I_{n-1} \end{bmatrix}.$$

Box III.

Obviously, once n is large, the verification for (29) is boring. Denote

$$h(\lambda_i) = [\alpha_2^2-\alpha_2\lambda_i-(\alpha_2+\beta_2)\bar{\lambda}\rho] \times [\beta_2^2-(2\alpha_2+\beta_2)\lambda_i-(\alpha_2+\beta_2)\bar{\lambda}\rho].$$

We have

$$\begin{aligned} h'(\lambda_i) &= -\alpha_2[\beta_2^2-(2\alpha_2+\beta_2)\lambda_i-(\alpha_2+\beta_2)\bar{\lambda}\rho] \\ &\quad -(2\alpha_2+\beta_2)([\alpha_2^2-\alpha_2\lambda_i-(\alpha_2+\beta_2)\bar{\lambda}\rho]) \\ &= -\alpha_2^2[2\alpha_2-(4\lambda_i+3\bar{\lambda}\rho)]-\alpha_2\beta_2[\alpha_2-(2\lambda_i+4\bar{\lambda}\rho)] \\ &\quad -\beta_2^2(\alpha_2-\bar{\lambda}\rho). \end{aligned}$$

It is easy to see that $h'(\lambda_i) < 0$ if $\alpha_2 > 2\lambda_i + 4\bar{\lambda}\rho$, which means $h(\lambda_i)$ decreases in λ_i. Note that $\alpha_2^2\beta_2^2(\frac{\lambda_i}{\bar{\lambda}+1}-1)^2$ decreases in λ_i for $0<\lambda_i\le\bar{\lambda}$. (31), together with (30), implies that (36) holds.

If $1 \le \frac{\bar{\lambda}}{\underline{\lambda}} < 2$, denote the left polynomial about λ_i of (36) as $f(\lambda_i)$. Then,

$$f'(\lambda_i) = h'(\lambda_i) - 2\mu(\mu\lambda_i - \alpha_2\beta_2).$$

Choosing $\mu = \frac{\alpha_2\beta_2}{\underline{\lambda}}$, we have

$$0 \le \mu\lambda_i - \alpha_2\beta_2 = \alpha_2\beta_2\left(\frac{\lambda_i}{\underline{\lambda}}-1\right) < 1.$$

So, $f'(\lambda_i) < 0$ if $\alpha_2 > 2\lambda_i + 4\bar{\lambda}\rho$, which means that $f(\lambda_i)$ decreases in λ_i. Then, we only need to show that (33) holds to guarantee that $f(\lambda_i) > 0$ for all $1 \le i \le n-1$. Since $0 \le \frac{\bar{\lambda}}{\underline{\lambda}} - 1 < 1$, (33) holds only if α_2 and β_2 are large enough. Also, from $\mu\gamma > \nu^2$, we have $\beta_2 > \sqrt{\alpha_2\underline{\lambda}}$.

Summarizing the above analysis, we conclude that (27)–(29) or (28) and (30) and (31) or (30), (32) and (33) guarantee that $\dot{V}(t) < 0$ for $\eta_e \ne 0$, which means the consensus of the system (3) using the algorithm (5) is achieved.

When the feedback gains satisfy $\alpha_2 = \beta_2$, we can choose a Lyapunov function candidate as

$$V(t) = \eta_e^T \begin{bmatrix} \theta P & P \\ P & P \end{bmatrix} \eta_e.$$

For simplicity, we directly give the two inequalities corresponding to (35) and (36), respectively, to guarantee that $\dot{V}(t) < 0$ for $\eta_e \ne 0$ as follows:

$$(\alpha_2-\lambda_i-2\rho\bar{\lambda})+(\alpha_2-3\lambda_i-2\rho\bar{\lambda}) > 0, \quad i=1,\dots,n-1, \quad (37)$$

$$(\alpha_2-\lambda_i-2\rho\bar{\lambda})(\alpha_2-3\lambda_i-2\rho\bar{\lambda})-(\theta\lambda_i-\alpha_2)^2 > 0, \quad i=1,\dots,n-1. \quad (38)$$

Obviously, (34) guarantees that (37) holds. Denote the left polynomial about λ_i of (38) as $g(\lambda_i)$. We have, through simplification,

$$\begin{aligned} g'(\lambda_i) &= -4\alpha_2+6\lambda_i+8\rho\bar{\lambda}-2\theta(\theta\lambda_i-\alpha_2) \\ &= (2\theta-4)\alpha_2+6\lambda_i+8\rho\bar{\lambda}-2\theta^2\lambda_i, \end{aligned}$$

$$g''(\lambda_i) = 6-2\theta^2 < 0 \quad \text{for } \theta > \sqrt{3}.$$

Then, we only need to demand $g'(\bar{\lambda}) = (2\theta-4)\alpha_2+6\bar{\lambda}+8\rho\bar{\lambda}-2\theta^2\bar{\lambda} > 0$ to guarantee that $g'(\lambda_i) > 0$ for all $i=1,\dots,n-1$, which can be guaranteed by (34). By direct calculation, we have that (38) is equivalent to

$$[(2\theta-4)\lambda_i-4\rho\bar{\lambda}]\alpha_2+(\lambda_i+2\rho\bar{\lambda})(3\lambda_i+2\rho\bar{\lambda})-\theta^2\lambda_i^2 > 0.$$

Noting $g'(\lambda_i) > 0$ under (34), it follows that (34) implies that (38) holds for all $i=1,\dots,n-1$. □

4.2. Second-order system under switching topologies

In this subsection, we consider the consensus problem under switching topologies using algorithms (4) and (5), respectively. We also assume that the digraphs $G(t)$ are taken from a finite set $\Gamma = \{G_1,\dots,G_M\}$ and use a switching signal $\sigma : [0,+\infty) \to \underline{M} \triangleq \{1,\dots,M\}$ to describe which digraph is active at time t. We impose a stronger restriction on the switching digraphs, that is, every digraph G_i, $i \in \underline{M}$, is strongly connected.

To further our discussion, we modify the *star transformation* (8) as the *switching star transformation*, that is, the transformation is related to the switching signal. Define the switching star transformation as

$$y = S_\sigma x \quad (39)$$

with

$$S_\sigma = \begin{bmatrix} \omega_{\sigma 1} & \omega_{\sigma 2} & \omega_{\sigma 3} & \cdots & \omega_{\sigma(n-1)} & \omega_{\sigma n} \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix},$$

where $[\omega_{\sigma 1},\dots,\omega_{\sigma n}]$ is the left eigenvector associated with the zero eigenvalue of the Laplacian matrix L_σ for a strongly connected digraph G_σ. For convenience, we require $\sum_{i=1}^n \omega_{\sigma i} = 1$. Then, by direct calculation, we have the following lemma.

Lemma 2. *For the matrix S_σ defined above, we have*

$$S_\sigma^{-1} = \begin{bmatrix} 1 & \omega_{\sigma 2} & \omega_{\sigma 3} & \cdots & \omega_{\sigma n} \\ 1 & \omega_{\sigma 2}-1 & \omega_{\sigma 3} & \cdots & \omega_{\sigma n} \\ 1 & \omega_{\sigma 2} & \omega_{\sigma 3}-1 & \cdots & \omega_{\sigma n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_{\sigma 2} & \omega_{\sigma 3} & \cdots & \omega_{\sigma n}-1 \end{bmatrix}, \tag{40}$$

and

$$S_\sigma^{-T} \mathrm{diag}\{\omega_\sigma\} S_\sigma^{-1} = \mathrm{diag}\{1, \bar{S}_\sigma\}, \tag{41}$$

$$S_\sigma^{-T} \mathrm{diag}\{\omega_\sigma\} L_\sigma S_\sigma^{-1} = \begin{bmatrix} 0 & \mathbf{0}_{(n-1)}^T \\ \mathbf{0}_{(n-1)} & F^T \mathrm{diag}\{\omega_\sigma\} L_\sigma F \end{bmatrix}, \tag{42}$$

where $\omega_\sigma = [\omega_{\sigma_1}, \ldots, \omega_{\sigma_n}]^T$ and $\bar{S}_\sigma = [\bar{s}_{\sigma ij}]_{(n-1)\times(n-1)}$ is a positive definite matrix with $\bar{s}_{\sigma ij} = -\omega_{\sigma(i+1)}\omega_{\sigma(j+1)}$ for $i > j$ and $\bar{s}_{\sigma ii} = (1-\omega_{\sigma(i+1)})\omega_{\sigma(i+1)}$, and $F = [\mathbf{0}_{n-1} \ -I_{n-1}]^T$.

By [3], it is not difficult to show that the matrix $\mathrm{diag}\{\omega\}L + L^T \mathrm{diag}\{\omega\}$ has positive eigenvalues except for one simple zero eigenvalue, where L is the Laplacian matrix associated with a strongly connected digraph and ω^T is its nonnegative left eigenvector associated with the zero eigenvalue. So the following lemma is direct using the form of (42).

Lemma 3. *Suppose that the digraph G is strongly connected and its Laplacian matrix is L. Then, the matrix $F^T(\mathrm{diag}\{\omega\}L + L^T \mathrm{diag}\{\omega\})F$ is positive definite, where ω^T is a nonnegative left eigenvector of L associated with the zero eigenvalue and $F = [\mathbf{0}_{n-1} \ -I_{n-1}]^T$.*

Before moving on, we use the switching star transformation (39) to convert the system (18) to the following two subsystems

$$\dot{\eta}_1 = \begin{bmatrix} 0 & 1 \\ 0 & -\beta_1 \end{bmatrix} \eta_1 + \begin{bmatrix} 0 \\ f_{\sigma 1}(\eta, t) \end{bmatrix}, \tag{43}$$

$$\begin{aligned} \mathrm{diag}\{\bar{S}_\sigma, \bar{S}_\sigma\}\dot{\eta}_e &= \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & \bar{S}_\sigma \\ -\alpha_1 F^T \mathrm{diag}\{\omega_\sigma\} L_\sigma F & -\beta_1 \bar{S}_\sigma \end{bmatrix} \eta_e \\ &\quad + \begin{bmatrix} \mathbf{0}_{n-1} \\ \bar{S}_\sigma f_e(\eta, t) \end{bmatrix}, \end{aligned} \tag{44}$$

where $f_{\sigma 1}(\eta, t) = \sum_{i=1}^n \omega_{\sigma i} f(x_i, v_i, t)$, $\bar{S}_\sigma$ is defined as in (41) and other notations are the same as the ones in (20) and (21). We are now ready to establish the theorem which gives sufficient conditions for the consensus problem of the system (3) under switching topologies.

Theorem 5. *Under Assumption 2, assume that all the digraphs are strongly connected and weighted balanced with a common weighted vector ω. Then, the consensus of the system (3) using the algorithm (4) is achieved under arbitrary switching signal if the feedback gains α_1 and β_1 satisfy, for all $i \in \underline{M}$,*

$$\begin{aligned} &-\begin{bmatrix} \alpha_1 F^T(\mathrm{diag}\{\omega\}L_i)_{sym} F & \frac{1}{2}\alpha_1 F^T L_i^T \mathrm{diag}\{\omega\} F \\ \frac{1}{2}\alpha_1 F^T \mathrm{diag}\{\omega\} L_i F & (\beta_1 - 1)\bar{S}_\omega \end{bmatrix} \\ &\quad + I_2 \otimes \left(\rho\bar{\mu} I_{n-1} + \frac{1}{2}\bar{S}_\omega\right) < 0, \end{aligned} \tag{45}$$

where $\beta_1 > 1$ and $\bar{\mu} = \lambda_{\max}(\bar{S}_\omega)$ with $\bar{S}_\omega$ playing the role of $\bar{S}_\sigma$ in (41) because ω^T is the common nonnegative left eigenvector associated with the zero eigenvalue for all L_i, $i \in \underline{M}$, by Remark 1. To simplify the choices of α_1 and β_1, we can let $\beta_1 = \frac{1}{2}k\alpha_1 + 1$ with $k > 0$ satisfying $k\bar{S}_\omega - \frac{1}{\mu_{min}} F^T \mathrm{diag}\{\omega\} L_i F F^T L_i^T \mathrm{diag}\{\omega\} F > 0$, where $\mu_{min} = \min_{i \in \underline{M}}\{\lambda_{min}(F^T(\mathrm{diag}\{\omega\}L_i)_{sym} F)\}$. Then, once α_1 is large enough, the condition (45) will hold.

Proof. We only need to prove the system given by (43) and (44) is asymptotically stable with respect to η_e. Choose a Lyapunov function candidate as

$$V(t) = \frac{1}{2}\eta_e^T \begin{bmatrix} \beta_1 \bar{S}_\omega & \bar{S}_\omega \\ \bar{S}_\omega & \bar{S}_\omega \end{bmatrix} \eta_e.$$

Differentiating $V(t)$ along the trajectories of (43) and (44), we have

$$\begin{aligned} \dot{V}(t) &= -\eta_e^T \begin{bmatrix} \alpha_1 F^T(\mathrm{diag}\{\omega\}L_\sigma)_{sym} F & \frac{1}{2}\alpha_1 F^T L_\sigma^T \mathrm{diag}\{\omega\} F \\ \frac{1}{2}\alpha_1 F^T \mathrm{diag}\{\omega\} L_\sigma F & (\beta_1 - 1)\bar{S}_\omega \end{bmatrix} \eta_e \\ &\quad + \eta_e^T \begin{bmatrix} \bar{S}_\omega f_e(\eta, t) \\ \bar{S}_\omega f_e(\eta, t) \end{bmatrix}. \end{aligned}$$

The rest of the proof is similar to that of Theorem 3. So, we only concentrate on the feasibility of (45). From Lemma 3, we know that each $F^T(\mathrm{diag}\{\omega\}L_\sigma)_{sym}F$ is positive definite. To simplify the proof, we verify a stronger condition

$$\begin{aligned} \Phi &\triangleq -\begin{bmatrix} \alpha_1 \mu_{min} I_{n-1} & \frac{1}{2}\alpha_1 F^T L_\sigma^T \mathrm{diag}\{\omega\} F \\ \frac{1}{2}\alpha_1 F^T \mathrm{diag}\{\omega\} L_\sigma F & (\beta_1 - 1)\bar{S}_\omega \end{bmatrix} \\ &\quad + I_2 \otimes \left(\rho\bar{\mu} I_{n-1} + \frac{1}{2}\bar{S}_\omega\right) < 0. \end{aligned} \tag{46}$$

Pre- and post-multiplying Φ with $\begin{bmatrix} I_{n-1} & 0 \\ -\frac{1}{2\mu_{min}} F^T \mathrm{diag}\{\omega\} L_\sigma F & I_{n-1} \end{bmatrix}$ and its transpose, respectively, we get the equivalent condition of (46) as follows:

$$\begin{aligned} &-\begin{bmatrix} \alpha_1 \mu_{min} I_{n-1} & \mathbf{0}_{(n-1)\times(n-1)} \\ \mathbf{0}_{(n-1)\times(n-1)} & (\beta_1 - 1)\bar{S}_\omega - \frac{\alpha_1}{4\mu_{min}} F^T \mathrm{diag}\{\omega\} L_\sigma F F^T L_\sigma^T \mathrm{diag}\{\omega\} F \end{bmatrix} \\ &+ \left[\begin{matrix} \rho\bar{\mu} I_{n-1} + \frac{1}{2}\bar{S}_\omega \\ -\frac{1}{2\mu_{min}} F^T \mathrm{diag}\{\omega\} L_\sigma F \left(\rho\bar{\mu} I_{n-1} + \frac{1}{2}\bar{S}_\omega\right) \end{matrix}\right. \\ &\qquad \left.\begin{matrix} -\frac{1}{2\mu_{min}}\left(\rho\bar{\mu} I_{n-1} + \frac{1}{2}\bar{S}_\omega\right) F^T L_\sigma^T \mathrm{diag}\{\omega\} F \\ \frac{1}{4\mu_{min}^2} F^T \mathrm{diag}\{\omega\} L_\sigma F \left(\rho\bar{\mu} I_{n-1} + \frac{1}{2}\bar{S}_\omega\right) F^T L_\sigma^T \mathrm{diag}\{\omega\} F + \left(\rho\bar{\mu} I_{n-1} + \frac{1}{2}\bar{S}_\omega\right) \end{matrix}\right] \\ &< 0. \end{aligned} \tag{47}$$

Obviously, (47) will hold only if α_1 and β_1 are large enough. Moreover, by (47), we can let $\beta_1 = \frac{1}{4}k\alpha_1 + 1$ with $k > 0$ satisfying $k\bar{S}_\omega - \frac{1}{\mu_{min}} F^T \mathrm{diag}\{\omega\} L_\sigma F F^T L_\sigma^T \mathrm{diag}\{\omega\} F > 0$ to simplify the adjustment of the feedback gains. □

Next, we consider the consensus problem of the system (3) using algorithm (5) under switching topologies. Also, we use the switching star transformation (39) to convert the system (22) to the following two subsystems

$$\dot{\eta}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \eta_1 + \begin{bmatrix} 0 \\ f_{\sigma 1}(\eta, t) \end{bmatrix}, \tag{48}$$

$$\begin{aligned} \mathrm{diag}\{\bar{S}_\sigma, \bar{S}_\sigma\}\dot{\eta}_e &= \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & \bar{S}_\sigma \\ -\alpha_2 F^T \mathrm{diag}\{\omega_\sigma\} L_\sigma F & -\beta_2 F^T \mathrm{diag}\{\omega_\sigma\} L_\sigma F \end{bmatrix} \eta_e \\ &\quad + \begin{bmatrix} \mathbf{0}_{n-1} \\ \bar{S}_\sigma f_e(\eta, t) \end{bmatrix}. \end{aligned} \tag{49}$$

Then, we have the following theorem.

Theorem 6. *Under Assumption 2, assume all the digraphs are strongly connected and weighted balanced with a common weighted* *vector ω. Then, the consensus of the system (3) using the algorithm (5) is achieved under arbitrary switching signal if the feedback gains α_2 and β_2 satisfy, for all $i \in \underline{M}$,*

$$\frac{\alpha_2^2}{\beta_2}F^T(\mathrm{diag}\{\omega\}L_i)_{sym}F > \frac{\alpha_2}{\beta_2}\bar{S}_\omega + \left(\frac{\alpha_2}{\beta_2}+1\right)\bar{\mu}\rho I_{n-1}, \tag{50}$$

and

$$\beta_2F^T(\mathrm{diag}\{\omega\}L_i)_{sym}F > \left(\frac{\alpha_2}{\beta_2}+1\right)(\bar{S}_\omega + \bar{\mu}\rho I_{n-1}), \tag{51}$$

where $\bar{\mu} = \lambda_{\max}(\bar{S}_\omega)$ with $\bar{S}_\omega$ playing the role of $\bar{S}_\sigma$ in (41).

Proof. Choose a Lyapunov function candidate as

$$V(t) = \frac{1}{2}\eta_e^T\begin{bmatrix}\alpha_2F^T(\mathrm{diag}\{\omega\}L_\sigma + L_\sigma^T\mathrm{diag}\{\omega\})F & \frac{\alpha_2}{\beta_2}\bar{S}_\omega \\ \frac{\alpha_2}{\beta_2}\bar{S}_\omega & \bar{S}_\omega\end{bmatrix}\eta_e.$$

By the Schur complement theorem, $V(t) > 0$ for $\eta_e \neq 0$ if and only if $2\alpha_2F^T(\mathrm{diag}\{\omega\}L_\sigma)_{sym}F > \frac{\alpha_2^2}{\beta_2^2}\bar{S}_\omega$. Obviously, (51) implies that this condition holds. Differentiating $V(t)$ along the trajectories of the system given by (48) and (49), we have

$$\begin{aligned}\dot{V}(t) = &-\eta_e^T\begin{bmatrix}\frac{\alpha_2^2}{\beta_2}F^T(\mathrm{diag}\{\omega\}L_\sigma)_{sym}F & \mathbf{0}_{(n-1)\times(n-1)} \\ \mathbf{0}_{(n-1)\times(n-1)} & \beta_2F^T(\mathrm{diag}\{\omega\}L_\sigma)_{sym}F - \frac{\alpha_2}{\beta_2}\bar{S}_\omega\end{bmatrix}\eta_e \\ &+\eta_e^T\begin{bmatrix}\frac{\alpha_2}{\beta_2}\bar{S}_\omega f_e(\eta,t) \\ \bar{S}_\omega f_e(\eta,t)\end{bmatrix} \\ \leq &-\eta_e^T\begin{bmatrix}\frac{\alpha_2^2}{\beta_2}F^T(\mathrm{diag}\{\omega\}L_\sigma)_{sym}F & \mathbf{0}_{(n-1)\times(n-1)} \\ \mathbf{0}_{(n-1)\times(n-1)} & \beta_2F^T(\mathrm{diag}\{\omega\}L_\sigma)_{sym}F - \frac{\alpha_2}{\beta_2}\bar{S}_\omega\end{bmatrix}\eta_e \\ &+\eta_e^T\mathrm{diag}\left\{\frac{\alpha_2}{\beta_2}\bar{S}_\omega, \bar{S}_\omega\right\}\eta_e + [f_e^T\ f_e^T]\mathrm{diag}\left\{\frac{\alpha_2}{\beta_2}\bar{S}_\omega, \bar{S}_\omega\right\}\begin{bmatrix}f_e(\eta,t)\\ f_e(\eta,t)\end{bmatrix} \\ \leq &-\eta_e^T\begin{bmatrix}\frac{\alpha_2^2}{\beta_2}F^T(\mathrm{diag}\{\omega\}L_\sigma)_{sym}F & \mathbf{0}_{(n-1)\times(n-1)} \\ \mathbf{0}_{(n-1)\times(n-1)} & \beta_2F^T(\mathrm{diag}\{\omega\}L_\sigma)_{sym}F - \frac{\alpha_2}{\beta_2}\bar{S}_\omega\end{bmatrix}\eta_e \\ &+\eta_e^T\begin{bmatrix}\frac{\alpha_2}{\beta_2}\bar{S}_\omega + \left(\frac{\alpha_2}{\beta_2}+1\right)\bar{\mu}\rho I_{n-1} & \mathbf{0}_{(n-1)\times(n-1)} \\ \mathbf{0}_{(n-1)\times(n-1)} & \bar{S}_\omega + \left(\frac{\alpha_2}{\beta_2}+1\right)\bar{\mu}\rho I_{n-1}\end{bmatrix}\eta_e.\end{aligned}$$

By (50) and (51), we have $\dot{V}(t) < 0$ for $\eta_e \neq 0$. Hence the system given by (48) and (49) is asymptotically stable with respect to η_e, which means that the consensus of (3) using (5) is achieved. □

Remark 6. In this subsection, we do not require the minimum dwell time for the switching signal as in Section 3.2 since we can no longer choose a Lyapunov function which is not related to the feedback gains as in the proof of Theorem 2 when the minimum dwell time for the switching signals exists. With our approach, when all the digraphs are strongly connected and weighted balanced with a common weighted vector, we have given the sufficient conditions to ensure consensus for second order systems with no constraints at the switching instants. It will be meaningful to study the general case when each digraph contains a directed spanning tree. We expect that the average dwell time theory for the stability of the switched systems in [19] might be useful to investigate such a case and it will be our future direction.

Remark 7. In [8], sufficient conditions to ensure consensus for second-order systems under a fixed topology were given by virtue of a concept of the generalized algebraic connectivity which was defined to the strongly connected digraph and the general digraph needed to be divided into the strongly connected components. Our results utilize the Laplacian matrix of the digraph directly and can be applied more easily. Moreover, we investigate the consensus problem for second-order systems under switching topologies.

5. Simulations

In this section, we give some numerical simulations to demonstrate the effectiveness of the theoretical results. The considered system consists of four agents. Fig. 1 shows three digraphs $G_1 - G_3$ each of which contains a directed spanning tree. For simplicity, assume all the adjacency matrices of $G_1 - G_3$ have 0 or 1 elements. Fig. 2 shows three digraphs $G_4 - G_6$ each of which is strongly connected with the weights shown beside the edges.

Example 1. Consider the consensus of (1) under the topologies represented by $G_1 - G_3$. The inherent nonlinear dynamics is given as $f(x, t) = x\sin(t)$. By Theorem 1, when the feedback gain $k > 1.3954$, the algorithm (2) asymptotically solves the consensus problem for (1) under the fixed topology G_1. Fig. 3(a) shows the states of the closed-loop system with $k = 1.4$. By Theorem 2, when the feedback gain $k > 4.6398$, the algorithm (2) asymptotically solves the consensus problem for (1) under switching topologies with the minimum dwell time $T = 1$ for the switching signal. Fig. 3(b) shows the states of the closed-loop system with the feedback gain $k = 4.7$ and the common dwell time $T = 1$ for the switching signal. Note that the final state can no longer be determined in advance because of the switching topologies.

Example 2. Consider the consensus of (3) under the fixed topology represented by G_4. For the algorithm (4), we consider the system with the inherent nonlinear dynamics $f(x, v, t) = x + \sin(v)$. For simplicity, we let the feedback gains satisfy $\beta_1 = \frac{1}{2}k\alpha_1 + 1$ with $k > 0$. From Theorem 3, when $\alpha_1 > 2.799$ together with $k = 1.4$, the algorithm (4) asymptotically solves the consensus problem for the system (3). Fig. 4 shows the states when $\alpha_1 = 2.9$. Note that because of the existence of the inherent nonlinear dynamics, the velocities do not tend to zero, which is different from the case where there is no inherent nonlinear dynamics. For the algorithm (5), we consider the system with the inherent nonlinear dynamics $f(x, v, t) = -x + v\sin(t)$. From Theorem 4, we can get that $\alpha_2 = \beta_2 > 13.4654$ ensures consensus. Fig. 5 shows the states of the system (3) using the algorithm (5) with the feedback gains $\alpha_2 = \beta_2 = 13.5$.

Example 3. Consider the consensus of (3) using the algorithm (5) under switching topologies represented by $G_4 - G_6$. For simplicity, we also let $\alpha_2 = \beta_2$. By direct calculation, we have that all the digraphs are weighted balanced with the weighted vector $\omega = [\frac{2}{\sqrt{10}}\ \frac{1}{\sqrt{10}}\ \frac{2}{\sqrt{10}}\ \frac{1}{\sqrt{10}}]^T$. When $\alpha_2 = \beta_2 > 1.5247$, the algorithm (5) asymptotically solves the consensus problem for (3) under switching topologies. Fig. 6 shows the states when the inherent nonlinear dynamics is $f(x, v, t) = \frac{1}{\sqrt{2}}(\sin(x) - \cos(v))$ and $\alpha_2 = \beta_2 = 3$.

6. Conclusion

The consensus problem for multi-agent systems with inherent nonlinear dynamics under directed topologies has been investigated. We have introduced a kind of variable transformation, i.e., *star transformation*, to convert the consensus problem to a corresponding partial stability problem. By avoiding to use the concept of the generalized algebraic connectivity for the strongly connected digraph introduced in the existing work, we have given the results which are more effective when the number of the agents

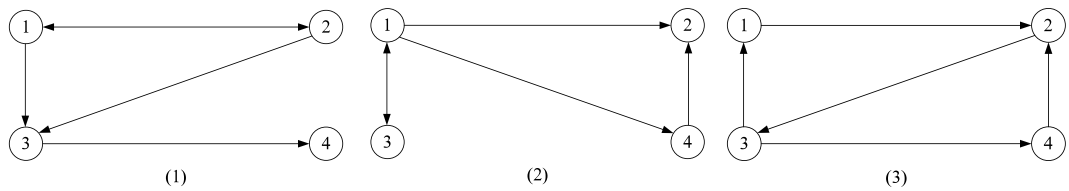

Fig. 1. Three digraphs $G_1 - G_3$ each of which contains a directed spanning tree.

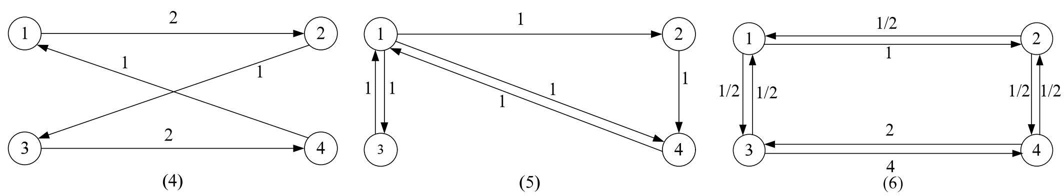

Fig. 2. Three digraphs $G_4 - G_6$ each of which is strongly connected.

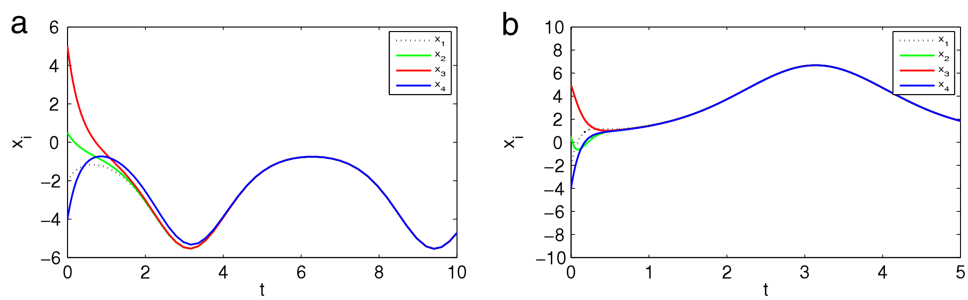

Fig. 3. States of system (1) using the algorithm (2) under G_1 and $G_1 - G_3$, respectively.

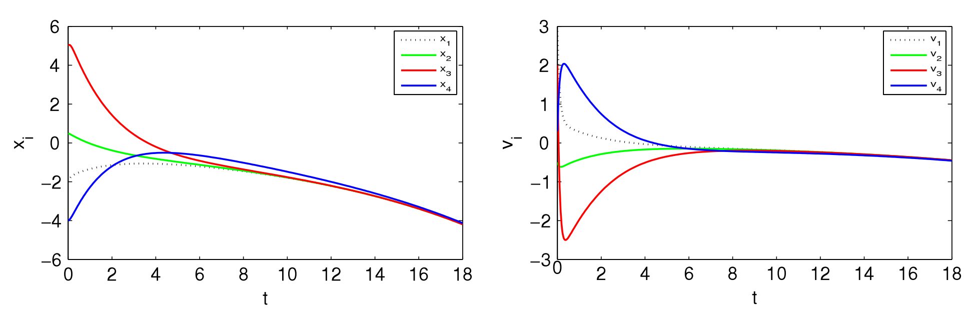

Fig. 4. States of (3) using the algorithm (4) under the fixed topology G_4.

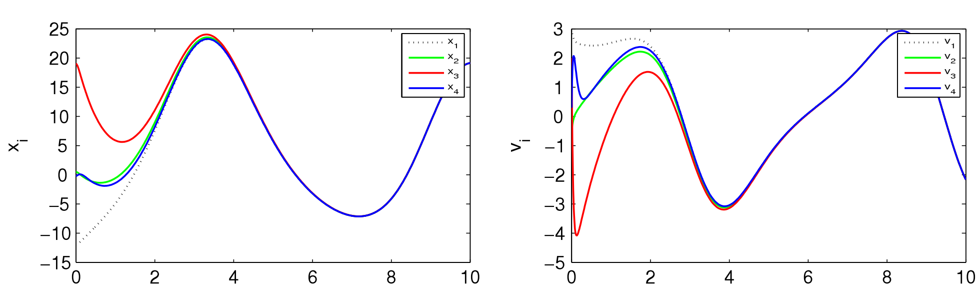

Fig. 5. States of (3) using the algorithm (5) under the fixed topology G_4.

is large and the directed graph is complex. The final consensus state is time-varying. Specifically, it is related to the inherent nonlinear term. So, we can also design the nonlinear term for the purpose of application. In our future work, we will study the consensus problem for the multi-agent systems with communication delays. In addition, we will investigate more general switching topologies (e.g., each digraph containing a spanning tree) for second-order systems.

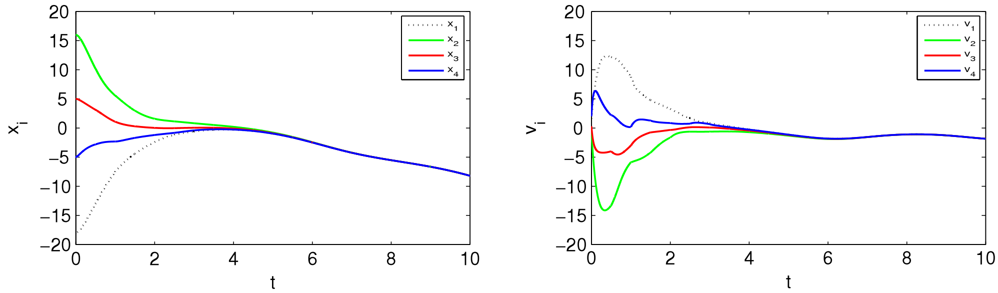

Fig. 6. States of (3) using the algorithm (5) under switching topologies $G_4 - G_6$.

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