Analog Network Coding in General SNR Regime: Performance of A Greedy Scheme

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***Abstract*—The problem of maximum rate achievable with analog network coding for a unicast communication over a layered relay network with directed links is considered. A relay node performing analog network coding scales and forwards the signals received at its input. Recently this problem has been considered under certain assumptions on per node scaling factor and received SNR. Previously, we established a result that allows us to characterize the optimal performance of analog network coding in network scenarios beyond those that can be analyzed using the approaches based on such assumptions.**

The key contribution of this work is a scheme to greedily compute a lower bound to the optimal rate achievable with analog network coding in general layered networks. This scheme allows for exact computation of the optimal achievable rates in a wider class of layered networks than those that can be addressed using existing approaches. For the specific case of the Gaussian N-relay diamond network, to the best of our knowledge, the proposed scheme provides the first exact characterization of the optimal rate achievable with analog network coding. For general layered networks, our scheme allows us to compute optimal rates within a "small" gap from the cut-set upper bound asymptotically in the source power.

I. Introduction

Analog network coding (ANC) extends to multihop wireless networks the idea of linear network coding [1], where an intermediate node sends out a linear combination of its incoming packets. In a wireless network, signals transmitted simultaneously by multiple sources add in the air. Each node receives a *noisy sum* of these signals, *i.e.* a linear combination of the received signals and noise. A communication scheme wherein each relay node merely amplifies and forwards this noisy sum is referred to as analog network coding [2], [3].

The rates achievable with ANC in layered relay networks is analyzed in [3], [4]. In [3], the achievable rate is computed under two assumptions: (A) each relay node scales the received signal to the maximum extent subject to its transmit power constraint, (B) the nodes in all L layers operate in the high-SNR regime. It is shown that the rate achieved under these two assumptions approaches network capacity as the source power increases. The authors in [4] extend this result to the scenarios where the nodes in at most one layer do not satisfy these assumptions and show that achievable rates still approach the network capacity as the source power increases.

However, requiring each relay node to amplify its received signal to the upper bound of its transmit power constraint results in suboptimal end-to-end performance of analog network coding in general, as we show in this paper and was also previously indicated in [5], [6]. Further, even in low-SNR regimes amplify-and-forward relaying can be capacity-achieving relay strategy in some scenarios, [7]. Therefore, we are concerned with characterizing the performance of analog network coding in general layered networks, without the above two assumptions on input signal scaling factors and received SNRs. However, such a characterization results in a computationally intractable problem in general, [4], [6].

In [6], we establish that a globally optimal set of scaling factors for each node, *i.e.*, a choice of relaying strategies that optimizes end-to-end throughput over all ANC strategies, can be computed in a layer-by-layer manner. This result allows us to computationally efficiently characterize exactly the optimal ANC rate in a large class of layered networks that cannot be addressed using existing approaches under assumptions (A) and (B). Further, for general layered relay networks, this result significantly reduces the computational complexity of computing a set of non-trivial achievable rates.

However, even layer-by-layer computation of a network-wide scaling vector that maximizes the end-to-end ANC rate for general layered networks is a computationally hard problem [6]. In this paper, we propose a greedy scheme to bound from below the optimal rate achievable with analog network coding in general layered networks. The proposed scheme allows us to exactly compute the optimal ANC rate in a much wider class of layered networks than those that can be so addressed using existing approaches, including our approach in [6]. In particular, for the Gaussian N-relay diamond network [9], the proposed scheme allows us to exactly compute the optimal rate achievable with analog network coding. To the best of our knowledge, this is the first characterization of the optimal ANC rate for the Gaussian diamond network. Further, for general layered networks, our scheme allows us to compute the optimal rates within a "small" gap from the cut-set upper bound asymptotically in the source power.

In this paper, we provide the summary of our work. We have omitted most proofs or give only brief outlines. The details can be found in our arXiv submission [10].

Organization: In Section II we introduce a general wireless layered relay network model and formulate the problem of maximum rate achievable with ANC in such a network. Section III addresses the problem of maximum ANC rate achievable in the Gaussian N-relay diamond network and shows that a greedy scheme optimally solves this problem. In Section IV we first generalize this greedy scheme to characterize the optimal performance for a specific subnetwork of general layered network. We then construct a scheme to bound from

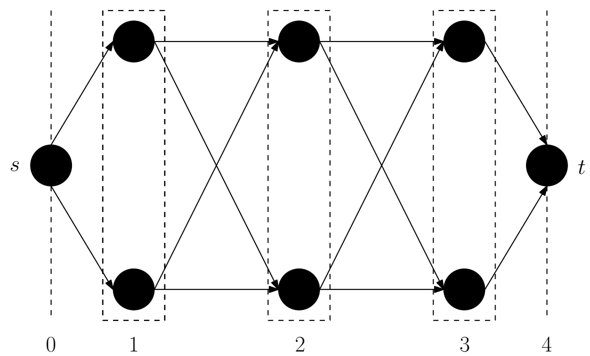

Fig. 1. Layered network with 3 relay layers between source s and destination t. Each layer contains two relay nodes.

below the optimal performance of ANC in general layered networks. Section V illustrates that this scheme leads to exact computation of the optimal ANC rate in a specific class of symmetric layered networks and tight characterization of the optimal rate in the general layered networks asymptotically in the source power. Section VI concludes the paper.

II. System Model

Consider a $(L+2)$-layer wireless network with directed links[1]. Source s is at layer '0', destination t is at layer '$L+1$', and the relay nodes from set R are arranged in L layers between them. The l^{th} layer contains n_l relay nodes, $\sum_{l=1}^{L} n_l = |R|$. An instance of such a network is given in Figure 1. Each node is assumed to have a single antenna and operate in full-duplex mode.

At instant n, the channel output at node $i, i \in R \cup \{t\}$, is

$$y_i[n] = \sum_{j \in \mathcal{N}(i)} h_{ji} x_j[n] + z_i[n], \quad -\infty < n < \infty, \tag{1}$$

where $x_j[n]$ is the channel input of node j in neighbor set $\mathcal{N}(i)$ of node i. In (1), h_{ji} is a real number representing the channel gain along the link from node j to node i. It is assumed to be fixed (for example, as in a single realization of a fading process) and known throughout the network. Source symbols $x_s[n], -\infty < n < \infty$, are i.i.d. Gaussian random variables with zero mean and variance P_s that satisfy an average source power constraint, $x_s[n] \sim \mathcal{N}(0, P_s)$. Further, $\{z_i[n]\}$ is a sequence (in n) of i.i.d. Gaussian random variables with $z_i[n] \sim \mathcal{N}(0, \sigma^2)$. We assume that z_i are independent of the input signal and of each other. We also assume that the i^{th} relay's transmit power is constrained as:

$$E[x_i^2[n]] \le P_i, \quad -\infty < n < \infty \tag{2}$$

In analog network coding each relay node amplifies and forwards the noisy signal sum received at its input. More precisely, relay node i at instant $n+1$ transmits the scaled version of $y_i[n]$, its input at time instant n, as follows

$$x_i[n+1] = \beta_i y_i[n], \quad 0 \le \beta_i^2 \le \beta_{i,max}^2 = P_i / P_{R,i}, \tag{3}$$

[1] The layered networks with bidirectional links can be addressed with the *signal subtraction* notion we introduced in [8]. However, for the ease of presentation we do not discuss such networks in this paper.

where $P_{R,i}$ is the received power at node i and choice of scaling factor β_i satisfies the power constraint (2).

In the class of layered networks shown in Figure 1 where the nodes in a layer communicate only with the nodes in the next immediate layer, all copies of a source signal and a noise symbol introduced at a node, traveling along different paths, arrive at the destination with the same respective time delays. Therefore, the outputs of the source-destination channel are free of intersymbol interference. This simplifies the relation between input and output of the channel and allows us to omit the time-index while denoting the input and output signals.

Using (1) and (3), the input-output channel between the source and destination can be written as

$$y_t = \Bigg[\sum_{(i_1,\ldots,i_L) \in K_s} h_{s,i_1} \beta_{i_1} h_{i_1,i_2} \ldots \beta_{i_L} h_{i_L,t} \Bigg] x_s \tag{4}$$
$$+ \sum_{l=1}^{L} \sum_{j=1}^{n_l} \Bigg[\sum_{(i_1,\ldots,i_{L-l+1}) \in K_{lj}} \beta_{i_1} h_{i_1,i_2} \ldots \beta_{i_{L-l+1}} h_{i_{L-l+1},t} \Bigg] z_{lj} + z_t,$$

where K_s is the set of L-tuples of node indices corresponding to all paths from source s to destination t with path delay L. Similarly, K_{lj} is the set of $L-l+1$-tuples of node indices corresponding to all paths from the j^{th} relay of the l^{th} layer to destination t with path delay $L-l+1$.

We introduce *modified* channel gains as follows. For all the paths between source s and destination t define:

$$h_s = \sum_{(i_1,\ldots,i_L) \in K_s} h_{s,i_1} \beta_{i_1} h_{i_1,i_2} \ldots \beta_{i_L} h_{i_L,t} \tag{5}$$

Similarly for all the paths between the j^{th} relay of the l^{th} layer to destination t with path delay $L-l+1$ define:

$$h_{lj} = \sum_{(i_1,\ldots,i_{L-l+1}) \in K_{lj}} \beta_{i_1} h_{i_1,i_2} \ldots \beta_{i_{L-l+1}} h_{i_{L-l+1},t} \tag{6}$$

In terms of these modified channel gains the source-destination channel in (4) can be written as:

$$y_t = h_s x_s + \sum_{l=1}^{L} \sum_{j=1}^{n_l} h_{lj} z_{lj} + z_t \tag{7}$$

In [6] we illustrate the derivation of the source-destination channel expression in (7) for a specific layered network in terms of the modified channel gains introduced above.

Problem Formulation: For a given network-wide scaling vector $\boldsymbol{\beta} = (\beta_{li})_{1 \le l \le L, 1 \le i \le n_l}$, the achievable rate for the channel in (7) with i.i.d. Gaussian input is ([3], [4], [6]):

$$I(P_s, \boldsymbol{\beta}) = (1/2) \log\big(1 + SNR_t\big), \tag{8}$$

where SNR_t, the signal-to-noise ratio at destination t is:

$$SNR_t = \frac{P_s}{\sigma^2} \frac{h_s^2}{1 + \sum_{l=1}^{L} \sum_{j=1}^{n_l} h_{lj}^2} \tag{9}$$

The maximum information-rate $I_{ANC}(P_s)$ achievable in a given layered network with i.i.d. Gaussian input is defined as the maximum of $I(P_s, \boldsymbol{\beta})$ over all feasible $\boldsymbol{\beta}$ subject to per relay transmit power constraint (3). In other words:

$$I_{ANC}(P_s) \stackrel{def}{=} \max_{\boldsymbol{\beta}:0\le\beta_{li}^2\le\beta_{li,max}^2} I(P_s,\boldsymbol{\beta}) \quad (10)$$

Given the monotonicity of the $\log(\cdot)$ function, we have

$$\boldsymbol{\beta}_{opt} = \operatorname*{argmax}_{\boldsymbol{\beta}:0\le\beta_{li}^2\le\beta_{li,max}^2} I(P_s,\boldsymbol{\beta}) = \operatorname*{argmax}_{\boldsymbol{\beta}:0\le\beta_{li}^2\le\beta_{li,max}^2} SNR_t \quad (11)$$

Therefore in the rest of the paper, we concern ourselves mostly with maximizing the received SNRs.

In [6], we discussed the computational complexity of exactly solving the problem (10) or equivalently the problem (11). Further, we also introduced a key result [6, Lemma 2] that reduces the computational complexity of the problem of computing $\boldsymbol{\beta}_{opt}$ by computing it layer-by-layer as a solution of a cascade of subproblems. This result allows us to characterize the optimal end-to-end rate achievable with analog network coding in communication scenarios that cannot be so addressed using previous approaches. However, each of these subproblems itself is computationally hard for general network scenarios as it involves maximizing the ratio of *posynomials* [11], [12], which is known to be computationally intractable in general [12]. Therefore, in this paper, we introduce a greedy scheme that optimally solves these subproblems and consequently the problem (11) for a large class of symmetric layered networks that cannot be addressed with existing schemes. For general layered networks, the proposed scheme allows us to tightly bound from below the optimal ANC performance. However before discussing this scheme, we motivate it by computing the maximum achievable ANC rate over the diamond network with N relay nodes.

III. Diamond Network: The optimal rate achievable with analog network coding

Consider the diamond network of Figure 2. A diamond network can be considered as a layered network with only one layer of relay nodes. Then using (5), (6), and (9), the SNR at destination t for any scaling vector $\boldsymbol{\beta}$ is

$$SNR_t = \frac{P_s}{\sigma^2}\frac{(\sum_{i=1}^N h_{si}\beta_i h_{it})^2}{1+\sum_{i=1}^N \beta_i^2 h_{it}^2} \quad (12)$$

Using (10), the problem of computing the maximum ANC rate for this network thus can be formulated as

$$\max_{0\le\boldsymbol{\beta}^2\le\boldsymbol{\beta}^2_{max}} SNR_t, \quad (13)$$

where $\boldsymbol{\beta}_{max} = (\beta_{1,max}\ldots,\beta_{N,max})$ with $\beta_{i,max}^2 = P_i/(h_{si}^2 P_s+\sigma^2), i\in\mathcal{N}, \mathcal{N}=\{1,\ldots,N\}$.

Equating the first-order partial derivatives of the objective function with respect to $\beta_i, i\in\mathcal{N}$, to zero, we get the following $N+1$ conditions for local extrema:

$$\sum_{i\in\mathcal{N}} h_{si}\beta_i h_{it} = 0 \quad (14)$$

$$\beta_i = \frac{h_{si}/h_{it}}{\sum_{j\in\mathcal{N}\setminus\{i\}} h_{sj}\beta_j h_{jt}}\left(1+\sum_{j\in\mathcal{N}\setminus\{i\}}\beta_j^2 h_{it}^2\right), i\in\mathcal{N} \quad (15)$$

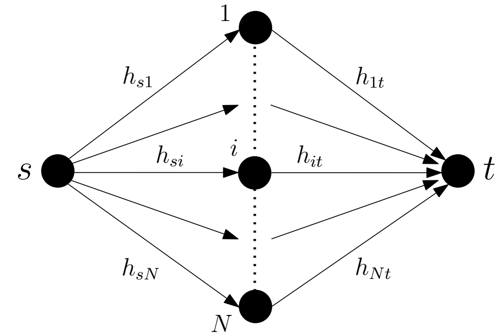

Fig. 2. A diamond network with N relay nodes.

Let $SNR_{\beta_i\beta_j} = \frac{\partial^2 SNR_t}{\partial\beta_i\partial\beta_j}$ denote the second-order partial derivatives of SNR_t with respect to β_i and β_j, $i,j\in\mathcal{N}$, and $H(\boldsymbol{\beta})$ denote the determinant of $N\times N$ Hessian matrix.

First, consider the set of stationary points $S_{\boldsymbol{\beta}} = \{\boldsymbol{\beta} : \boldsymbol{\beta}$ satisfies (14)$\}$. For all points in $S_{\boldsymbol{\beta}}$ we can prove that

$$SNR_{\beta_1\beta_1} > 0$$
$$H(\boldsymbol{\beta}) = 0$$

Therefore, the second partial derivative test to determine if the points in $S_{\boldsymbol{\beta}}$ are local minimum, maximum, or saddle points fails. However, we can establish that for every $\boldsymbol{\beta}\in S_{\boldsymbol{\beta}}$, the following set of conditions holds

$$\left.\frac{\partial SNR_t}{\partial\beta_i}\right|_{\boldsymbol{\beta}+\boldsymbol{\delta}} < 0, \text{ if } \sum_{i\in\mathcal{N}} h_{si}h_{it}\delta_i < 0, \quad (16)$$

$$\left.\frac{\partial SNR_t}{\partial\beta_i}\right|_{\boldsymbol{\beta}+\boldsymbol{\delta}} > 0, \text{ if } \sum_{i\in\mathcal{N}} h_{si}h_{it}\delta_i > 0, \quad (17)$$

$$H(\boldsymbol{\beta}) > 0, \text{ if } \sum_{i\in\mathcal{N}} h_{si}h_{it}\delta_i < 0, \quad (18)$$

$$H(\boldsymbol{\beta}) > 0, \text{ if } \sum_{i\in\mathcal{N}} h_{si}h_{it}\delta_i > 0, \quad (19)$$

for all $\boldsymbol{\delta} = (\delta_1,\ldots,\delta_N)\to\mathbf{0}$. In other words, (16) and (17) imply that the slope of the function changes sign at $\sum_{i\in\mathcal{N}} h_{si}h_{it}\delta_i = 0$, and (18) and (19) imply that the convexity of the function, however, does not change at $\sum_{i\in\mathcal{N}} h_{si}h_{it}\delta_i = 0$. Therefore, together these imply that (14) leads to a local minimum of the objective function.

Next, consider the set of points defined by (15). For all such points we can prove that

$$SNR_{\beta_1\beta_1} < 0$$
$$H(\boldsymbol{\beta}) > 0$$

Therefore, from the second partial derivative test the objective function attains its local maximum at the set of points characterized by (15) above. However, no real solution of the simultaneous system of equations in (15) exists. In other words, no solution of (13) exists where all relay nodes transmit strictly below their respective transmit power constraints.

The above discussion implies that all points satisfying (14) lead to the global minimum of the objective function in (13) and the global maximum of the objective function occurs at one of the N hyperplanes (of dimension $N-1$) defined by $\beta_k = \beta_{k,max}, k \in \mathcal{N}$. Next we identify this hyperplane and characterize the corresponding optimal solution.

Consider the system of simultaneous equations in (15) on the $(N-1)$-dimensional hyperplane defined by $\beta_k = \beta_{k,max}$.

$$\beta_i = \frac{h_{si}}{h_{it}} \frac{1 + \beta_{k,max}^2 h_{kt}^2 + \sum_{j \in \mathcal{N} \setminus \{i,k\}} \beta_j^2 h_{jt}^2}{h_{sk}\beta_{k,max}h_{kt} + \sum_{j \in \mathcal{N} \setminus \{i,k\}} h_{sj}\beta_j h_{jt}}, i \in \mathcal{N} \setminus \{k\} \tag{20}$$

Solving this system of equations results in the following set of optimal solutions for β_i on hyperplane $\beta_k = \beta_{k,max}$:

$$\beta_i^k = \frac{h_{si}}{h_{it}} \frac{1 + \beta_{k,max}^2 h_{kt}^2}{h_{sk}\beta_{k,max}h_{kt}}, i \in \mathcal{N} \setminus \{k\} \tag{21}$$

However, the optimal scaling factors in (21) for $N-1$ nodes are computed without considering the upper bound $\beta_{i,max}$ on each $\beta_i, i \in \mathcal{N} \setminus \{k\}$. Therefore, taking into consideration the upper bound on the scaling factor for each node, the modified solution is computed as per the following lemma.

***Lemma** 1:* The optimal scaling vector $\boldsymbol{\beta}_{opt}^k = (\beta_{1,opt}^k, \ldots, \beta_{N,opt}^k)$ on $\beta_k = \beta_{k,max}$ hyperplane such that each component scaling factor satisfies the corresponding upper bound on its maximum value, is given as

$$\beta_{i,opt}^k = \begin{cases} \beta_{i,max}, i \in S^k \\ \dfrac{h_{si}}{h_{it}} \dfrac{1 + \sum_{j \in S^k} \beta_{j,max}^2 h_{jt}^2}{\sum_{j \in S^k} h_{sj}\beta_{j,max}h_{jt}}, i \notin S^k, \end{cases}$$

where S^k is the set of nodes such that on hyperplane $\beta_k = \beta_{k,max}$, the optimal value of the scaling factor of a node is saturated to its corresponding upper bound, $S^k = \{k\} \cup \{i : \beta_i^k \geq \beta_{i,max}, i \in \mathcal{N} \setminus \{k\}\}$.

Proof: Following the argument similar to the one used to prove the global extrema properties of (14) and (15), we can prove that on the $\beta_k = \beta_{k,max}$ hyperplane, the SNR_t achieves its global minimum at a hyperplane defined by

$$\sum_{i \in \mathcal{N} \setminus \{k\}} h_{si}\beta_i h_{it} = 0$$

and its global maximum at the points defined by β_i^k in (21).

Let M^k denote the set of nodes for which β_i^k computed in (21) is at least equal to the corresponding upper bound $\beta_{i,max}$ on the maximum value of the scaling factor, *i.e.* $M^k = \{i : \beta_i^k \geq \beta_{i,max}, i \in \mathcal{N} \setminus \{k\}\}$. For all such $\beta_i^k, i \in M^k$, after proving that $\frac{\partial SNR_t}{\partial \beta_i}\big|_{\beta_{i,max}} \geq 0$, we set $\beta_i^k = \beta_{i,max}$ and update S^k as follows: $S^k = S^k \cup M^k$. As β_i^k computed in (21) for a node $i \notin S^k$ may no longer be optimal after the above re-assignment of β_i^k, we need to solve the following system of $N - |S^k| = N - |M^k| - 1$ simultaneous equations with $i \in \mathcal{N} \setminus S^k$:

$$\beta_i = \frac{h_{si}}{h_{it}} \frac{1 + \sum\limits_{j \in S^k} \beta_{j,max}^2 h_{jt}^2 + \sum\limits_{j \notin S^k \cup \{i\}} \beta_j^2 h_{jt}^2}{\sum\limits_{j \in S^k} h_{sj}\beta_{j,max}h_{jt} + \sum\limits_{j \notin S^k \cup \{i\}} h_{sj}\beta_j h_{jt}}, \tag{22}$$

Solving this system of equations results in

$$\beta_{i,opt}^k = \frac{h_{si}}{h_{it}} \frac{1 + \sum_{j \in S^k} \beta_{j,max}^2 h_{jt}^2}{\sum_{j \in S^k} h_{sj}\beta_{j,max}h_{jt}}, i \notin S^k \tag{23}$$

Some of the recomputed scaling factors $\beta_{i,opt}^k, i \notin S^k$, may violate the corresponding upper bound on their maximum value. Such nodes are added to set S^k, thus updating it. Then, the system of equations in (22) is solved again for the updated set S^k. This iterative process continues until none of the recomputed β_i in (23) violates its corresponding upper bound. Note that this process always halts with $S^k \subseteq \mathcal{N}$ and $\beta_i^k < \beta_{i,max}, i \in \mathcal{N} \setminus S^k$. ■

Using Lemma 1, for each of the N hyperplanes, defined as $\beta_k = \beta_{k,max}$, $k \in \mathcal{N}$, we can compute $\boldsymbol{\beta}_{opt}^k$, the set of scaling factors for all nodes at which SNR_t attains its maximum on $\beta_k = \beta_{k,max}$ hyperplane. Then the hyperplane at which SNR_t attains its global maximum is identified as follows:

***Proposition** 1:* SNR_t attains its global maximum at the hyperplane corresponding to node $k^\star$, where

$$k^\star = \underset{k \in \mathcal{N}}{\operatorname{argmax}}\, SNR_t(\boldsymbol{\beta}_{opt}^k)$$

Combining Proposition 1 and Lemma 1, we can characterize the scaling vector $\boldsymbol{\beta}_{opt}$ that solves the problem (13) as follows.

***Theorem** 1:* A network-wide scaling vector $\boldsymbol{\beta}_{opt} = (\beta_1^{opt}, \ldots, \beta_N^{opt})$ that maximizes SNR_t for a diamond network with the relay nodes performing ANC is given as

$$\beta_i^{opt} = \begin{cases} \beta_{i,max}, i \in S^{k^\star}, \\ \dfrac{h_{si}}{h_{it}} \dfrac{1 + \sum\limits_{j \in S^{k^\star}} \beta_{j,max}^2 h_{jt}^2}{\sum\limits_{j \in S^{k^\star}} h_{sj}\beta_{j,max}h_{jt}}, i \notin S^{k^\star}, \end{cases}$$

where $k^\star = \operatorname{argmax}_{j \in \mathcal{N}} SNR_t(\boldsymbol{\beta}_{opt}^j)$ and $S^{k^\star} = \{k^\star\} \cup \{i : \beta_i^{k^\star} \geq \beta_{i,max}, i \in \mathcal{N} \setminus \{k^\star\}\}$.

Based on our approach in this section to compute the optimal ANC rate in the Gaussian diamond networks, in the next section we introduce a greedy scheme to bound from below the maximum end-to-end rate achievable with analog network coding in the general layered networks.

IV. General layered networks: a greedy scheme to lower bound the maximum ANC rate

In a general layered network with L layers of relay nodes, consider layer $l, 1 \leq l \leq L$, and a node in the next $l+1^{\text{st}}$ layer, denoted as t_{l+1} or with a little abuse of notation as t. This scenario is depicted in Figure 3. For this subnetwork we have for any scaling vector $\boldsymbol{\beta}$

$$SNR_t = \frac{P_s(\sum_{i=1}^N s_i\beta_i h_{it})^2}{\mathbb{E}(z_t + \sum_{i=1}^N z_i\beta_i h_{it})^2} \tag{24}$$

Using (10) the problem of computing the maximum ANC rate for this subnetwork can be formulated as

$$\max_{0 \leq \boldsymbol{\beta}^2 \leq \boldsymbol{\beta}_{max}^2} SNR_t, \tag{25}$$

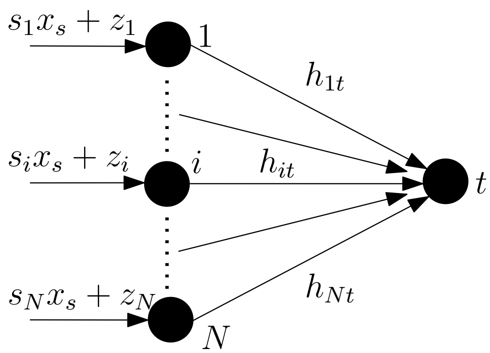

Fig. 3. A subnetwork of general layered network with L relay layers, depicting l^{th} layer with N relay nodes and a node in the $l+1^{\text{st}}$ layer. The received signal component at node $i, 1 \le i \le N$, in the l^{th} layer is denoted as $s_i x_s$, where x_s is the source symbol and the corresponding noise component is denoted as z_i.

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$ and $\boldsymbol{\beta}_{max} = (\beta_{1,max} \cdots, \beta_{N,max})$ with $\beta_{i,max}^2 = P_i / \mathbb{E}(s_i x_s + z_i)^2, i \in \mathcal{N}, \mathcal{N} = \{1, \ldots, N\}$.

Following a sequence of arguments similar to those used to establish Theorem 1 for diamond networks, we can characterize the scaling vector for the nodes in the l^{th} layer that optimally solve problem (25) for the subnetwork under consideration. Note that in this subnetwork the noises at different nodes in a relay layer are correlated unlike the independent noises at relay nodes in a diamond network as in Figure 2. This explains the difference between the SNR expression in (24) and the one in (12) for the diamond network, and results in more complex analysis in the present case.

***Lemma** 2:* A scaling vector $\boldsymbol{\beta}_{opt} = (\beta_1^{opt}, \ldots, \beta_N^{opt})$ that maximizes SNR_t for any subnetwork, as in Figure 3, of the general layered network with the relay nodes in l^{th} layer performing analog network coding is given as

$$\beta_i^{opt} = \begin{cases} \beta_{i,max}, i \in S^{k^\star}, \\ \dfrac{s_i + s_i \mathbb{E}\big(\sum_{j \in S^{k^\star}} z_j \beta_{j,max} h_{jt}\big)^2 - \alpha_j \gamma_j}{h_{it}(\alpha_j \mathbb{E} z_i^2/\sigma^2 - s_i \gamma_j)}, i \notin S^{k^\star}, \end{cases}$$

where $k^\star = \text{argmax}_{\{\boldsymbol{\beta}^j : j \in \mathcal{N}\}} SNR_t(\boldsymbol{\beta}^j)$, $S^{k^\star} = \{k^\star\} \cup \{i : \beta_i^{k^\star} \ge \beta_{i,max}, i \in \mathcal{N} \setminus \{k^\star\}\}$, and $\boldsymbol{\beta}^j = (\beta_1^j, \ldots, \beta_N^j)$ with

$$\beta_i^j = \begin{cases} \beta_{j,max}, i = j \\ \dfrac{s_i + \{s_i \mathbb{E} z_j^2/\sigma^2 - s_j \mathbb{E}(z_i z_j)/\sigma^2\} \beta_{j,max}^2 h_{jt}^2}{h_{it}\{s_j \mathbb{E} z_i^2/\sigma^2 - s_i \mathbb{E}(z_i z_j)/\sigma^2\} \beta_{j,max} h_{jt}}, i \neq j, \end{cases}$$

$$\alpha_j = \sum_{j \in S^{k^\star}} s_j \beta_{j,max} h_{jt}, \qquad \text{(signal component at } t\text{)}$$

$$\gamma_j = \sum_{j \in S^{k^\star}} \beta_{j,max} h_{jt} \mathbb{E}(z_i z_j)/\sigma^2, \quad \text{(noise component at } t\text{)}$$

Note that Lemma 2 reduces to Theorem 1 when the noise components at the relay nodes are independent.

Using Lemma 2, we can compute $\boldsymbol{\beta}_{l,opt}^{l+1,j}$, the scaling vector for the nodes in the l^{th} layer that maximizes the received SNR for node $j, 1 \le j \le n_{l+1}$, in the $l+1^{\text{st}}$ layer. Among these n_{l+1} scaling vectors for the nodes in the l^{th} layer, let $\boldsymbol{\beta}_l^{low}$ denote the one that solves the following problem

$$\boldsymbol{\beta}_l^{low} = \underset{\substack{\boldsymbol{\beta}_{l,opt}^{l+1,j} \\ 1 \le j \le n_{l+1}}}{\text{argmax}} \prod_{k=1}^{n_{l+1}} (1 + SNR_{l+1,k}(\boldsymbol{\beta}_{l,opt}^{l+1,j})) \tag{26}$$

The following corollary of Lemma 2 in [6] establishes that among n_{l+1} such scaling vectors, the scaling vector characterized by $\boldsymbol{\beta}_l^{low}$ computes the tightest lower bound for the optimal value of the objective function in (11).

***Corollary** 1 ([6], Lemma 2):* Consider two scaling vectors $\boldsymbol{\beta}_l$ and $\hat{\boldsymbol{\beta}}_l$ for the nodes in layer l. If $\prod_{k=1}^{n_{l+1}} (1 + SNR_k)\big|_{\boldsymbol{\beta}_l} > \prod_{k=1}^{n_{l+1}} (1 + SNR_k)\big|_{\hat{\boldsymbol{\beta}}_l}$, then $SNR_t(\boldsymbol{\beta}_l) > SNR_t(\hat{\boldsymbol{\beta}}_l)$.

Computing $\boldsymbol{\beta}_l^{low}$ as above for each layer $l, 1 \le l \le L$, in conjunction with Corollary 1 allows us to construct a network-wide scaling vector $\boldsymbol{\beta}_{low} = (\boldsymbol{\beta}_1^{low}, \ldots, \boldsymbol{\beta}_L^{low})$ to compute a lower bound[2] to the optimal solution of (10). Formally, for a given layered network, $\boldsymbol{\beta}_{low}$ is constructed as follows.

***Proposition** 2:* Consider a layered relay network of $L+2$ layers, with source s in layer '0', destination t in layer '$L+1$', and L layers of relay nodes between them. The l^{th} layer contains n_l nodes, $n_0 = n_{L+1} = 1$. A network-wide scaling vector $\boldsymbol{\beta}_{low} = (\boldsymbol{\beta}_1^{low}, \ldots, \boldsymbol{\beta}_L^{low})$ that provides a lower bound to the optimal solution of (10) for this network, can be computed recursively for $1 \le l \le L$ as

$$\boldsymbol{\beta}_l^{low} = \underset{\substack{\boldsymbol{\beta}_{l,opt}^{l+1,j} \\ 1 \le j \le n_{l+1}}}{\text{argmax}} \prod_{k=1}^{n_{l+1}} (1 + SNR_{l+1,k}(\boldsymbol{\beta}_1^{low}, \ldots, \boldsymbol{\beta}_{l-1}^{low}, \boldsymbol{\beta}_{l,opt}^{l+1,j}))$$

Here $\boldsymbol{\beta}_l^{low} = (\beta_{l1}^{low}, \ldots, \beta_{ln_l}^{low})$ is the vector of scaling factors for the nodes in the l^{th} layer and $\boldsymbol{\beta}_{l,opt}^{l+1,j}$ (computed using Lemma 2) is the scaling vector for the nodes in the l^{th} layer that maximizes the received SNR for node $j, 1 \le j \le n_{l+1}$, in layer $l+1$.

In the next section we analyze the performance of the greedy scheme of Proposition 2 in the context of both a special class of layered networks and the general layered networks.

V. Illustration

We first demonstrate that the greedy scheme of Proposition 2 allows us to exactly compute the optimal ANC rate for a broad class of layered networks. Then, we give an example to show that for the general layered networks, the proposed scheme leads to the optimal rates within a small gap from the cut-set upper bound asymptotically in the source power.

[2]Clearly, choosing $\boldsymbol{\beta}_l^{low}$ as in (26) for each layer l may lead, in general, to some performance loss at each layer as $\boldsymbol{\beta}_l^{low}$ may not be the optimal vector [6, Lemma 2] of the scaling factors for the nodes in layer l that solves

$$\underset{\boldsymbol{\beta}_l^2 \le \boldsymbol{\beta}_{l,max}^2}{\text{argmax}} \prod_{k=1}^{n_{l+1}} (1 + SNR_k)$$

The cumulative effect of this performance loss at each layer is that the end-to-end ANC rate computed at $\boldsymbol{\beta}_{low}$ may not lead to the optimal solution of problem (10). However, our results in the next section show that for a large class of layered networks there is no loss in the optimality and for other layered networks, the loss is *small* asymptotically in the network parameters.

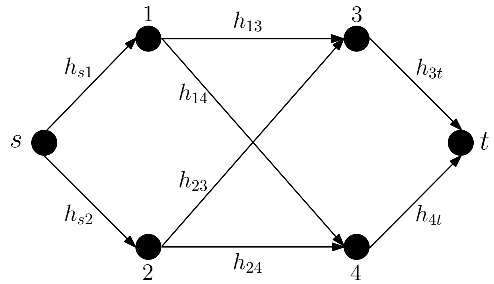

Fig. 4. General layered network with 2 relay layers between source s and destination t. Each layer contains two relay nodes.

Example 1 *(A class of exactly solvable layered networks):* Let us consider a class of symmetric layered networks where the channel gains along all outgoing links from a node are equal. An instance of such a network is obtained from the network in Figure 4 when $h_{s1} = h_{s2}, h_{13} = h_{14}$, and $h_{23} = h_{24}$. An implication of this property of the channel gains is that the received SNRs at every node in a layer are equal: $SNR_{l,j} = SNR_l, 1 \le j \le n_l, 1 \le l \le L$. In this case, for each layer $l, 1 \le l \le L$, $\boldsymbol{\beta}_l^{low}$ computed in Proposition 2 is equal to the optimal $\boldsymbol{\beta}_l^{opt}$ computed in [6, Lemma 2]. Therefore, $\boldsymbol{\beta}_{low}$ is the optimal solution of problem (11) for this class of networks.

Consider an instance of the network in Figure 4 when $h_{s1} = h_{s2} = h_0, h_{13} = h_{14} = h_1$, and $h_{23} = h_{24} = h_2$. Such an instance belongs to the class of symmetric networks we are concerned with in this example. Using Proposition 2, the optimal solution of problem (11) for this instance is:

$$\boldsymbol{\beta}_{opt} = \left(\beta_{1,max}, \frac{1 + \beta_{1,max}^2 h_1^2}{h_2 \beta_{1,max} h_1}, \beta_{3,max}, \beta_{4,max}\right), \text{ where}$$

$$\beta_{1,max}^2 = \frac{P_1}{h_0^2 P_s + \sigma^2}$$

$$\beta_{3,max}^2 = \frac{P_3}{S^2 P_s + Z^2 \sigma^2}, \beta_{4,max}^2 = \frac{P_4}{S^2 P_s + Z^2 \sigma^2}$$

$$S = h_0(\beta_{1,opt} h_1 + \beta_{2,opt} h_2), Z^2 = 1 + \beta_{1,opt}^2 h_1^2 + \beta_{2,opt}^2 h_2^2$$

and we assume that $P_1 h_1^2 > P_2 h_2^2$. ■

Example 2 *(General layered networks):* Let us consider the layered network of Figure 4. We compute a lower bound to the optimal ANC rate for this network using the greedy scheme in Proposition 2 and compare it with the MAC upper bound in Figure 5. We observe that in this case the ANC rate achieved with the greedy scheme of Proposition 2 approaches the capacity within one bit when $P_s > 100$.

VI. Conclusion and Future Work

We consider the problem of maximum rate achievable with analog network coding in general layered networks. Previously, this problem has been considered under certain assumptions on per node scaling factor and received SNR as without these assumptions the problem was presumed to be intractable. The key contribution of this work is a greedy scheme to exactly compute the optimal rates in a wider class of layered networks than those that can be addressed using

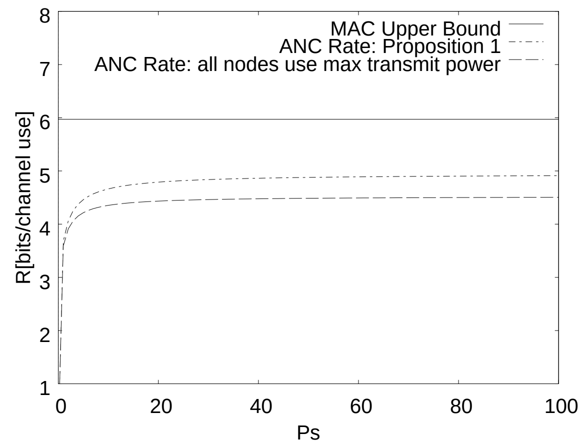

Fig. 5. Comparison of the ANC rate achievable with the scheme in Proposition 2 with the MAC upper bound for the layered network in Figure 4 with $P_1 = P_2 = P_3 = P_4 = 10$, $h_{14} = h_{24} = 2$ and all other channel gains are equal to 10. Also plotted is the ANC rate when the scaling factors for all relay nodes are set to their respective upper-bounds.

prior approaches. In particular, using the proposed scheme for the Gaussian N-relay diamond network, to the best of our knowledge, we provide the first exact characterization of the optimal rate achievable with analog network coding. Further, for general layered networks, our scheme allows us to compute optimal rates within a "small" gap of the cut-set upper bound asymptotically in the source power. In the future, we plan to extend this work to non-layered networks, and to construct the optimal distributed relay schemes.

Acknowledgment

This work is partially supported by a grant from the University Grants Committee of the Hong Kong Special Administrative Region, China (Project No. AoE/E-02/08), the CERG grant 412207, and the Project MMT-p7-11 of the Shun Hing Institute of Advanced Engineering, CUHK.

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