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# WEAKER CONNECTED HAUSDORFF TOPOLOGIES ON SPACES WITH A COUNTABLE NETWORK

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Abstract. We prove that a disconnected Hausdorff space $X$ with a countable network has a weaker connected Hausdorff topology if and only if it is not $H$-closed. This solves in a strong form both problems 3.2 and 3.3 from [10].

## 1. Introduction

If a space $X$ has a weaker topology with certain "nice" properties, then useful information about $X$ can often be obtained by looking for properties common to both the original and the weaker topology; in this sense it might be said that the weaker topologies constitute approximations to the original one. It is natural to look for approximating topologies that possess any of a number of classical properties, of importance not only in topology but in other areas of mathematics and while only a few properties might be deemed to satisfy this criterion, clearly, compactness, metrizability and connectedness are among them.

The existence of weaker compact topologies has been a topic of study for almost 70 years. Among the most important results in this field, that we might mention, are: Pytkeev's theorem [8] that any Borel non-$\sigma$-compact set has a weaker compact topology and Belugin's theorem [3] that if a countable set is removed from a dyadic compact space, then the resulting space has a weaker compact topology. Parkhomenko [7] gave a general method of constructing examples of

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second countable spaces with no weaker compact topology; Smirnov [9] used condensations (that is, continuous one-to-one maps) onto compact spaces to obtain applications in dimension theory.

The existence of a weaker metrizable topology has also been studied intensively. Spaces with a weaker metrizable topology are called submetrizable and many results concerning this class form a part of any serious handbook of general topology. For more on this topic we refer the reader to the excellent survey of Gruenhage [6].

The above mentioned research papers were the motivation for the authors of [10] for studying when a topological space has a weaker connected topology. It was shown, in particular, that any non-compact $T_3$-space with a countable network, as well as every countable Hausdorff space which is not $H$-closed, has a weaker connected Hausdorff topology. A natural simultaneous strengthening of these results could be the existence of a weaker connected Hausdorff topology for every non-$H$-closed (Hausdorff) space with a countable network. The authors of [10] formulated this hypothesis as an open question (see Problems 3.2 and 3.3 of [10]).

In this note we prove (Theorem 3.4) that if $X$ is a disconnected Hausdorff space with a countable network, then $X$ can be condensed onto a connected Hausdorff space if and only if $X$ is not $H$-closed. This answers positively Problems 3.2 and 3.3 of the paper [10]. A number of corollaries are given and we conclude this paper with some open questions.

## 2. Notation and terminology

All spaces under consideration are Hausdorff. A continuous function $f : X \to Y$ is called a *condensation* if it is a bijection (we then say that $f$ *condenses* $X$ onto $Y$). A space $X$ is $H$-closed if $X$ is a $T_2$-space and it is closed in any $T_2$-space containing $X$; the space $X$ is said to be *feebly compact* if each locally finite family of open subsets of $X$ is finite. It is well known that each $H$-closed space and each countably compact space is feebly compact and that a feebly compact (Hausdorff) space which is Lindelöf (in particular, if it has a countable network) is $H$-closed.

If $(X, \nu)$ is a space and $A \subset X$, then $\mathrm{cl}_\nu(A)$, $\mathrm{cl}_X(A)$ (or simply $\mathrm{cl}(A)$ if it does not lead to a misunderstanding) is the closure of $A$ in $(X, \nu)$. Similarly, $\mathrm{int}_\nu(A)$, $\mathrm{int}_X(A)$ or $\mathrm{int}(A)$ will denote the interior of $A$ in the topology $\nu$ on $X$. An open filter $\xi$ on the space $(X, \nu)$ is *free* if $\bigcap\{\mathrm{cl}(U) : U \in \xi\} = \emptyset$. The notation $X \oplus Y$ will denote the disjoint topological union of the spaces $X$ and $Y$. All other notions are standard and can be found in [5].

## 3. Weaker connected Hausdorff topologies

The purpose of this paper is to find weaker connected Hausdorff topologies on Hausdorff spaces with countable networks. Our immediate aim is to reduce the task to finding those for second countable spaces. For this purpose, we will use a result of Arhangel'skiĭ (see for example Lemma 3.1.18 of [5] or Problem 148 of Chapter 2 of [2]), which states that every Hausdorff space with a countable network can be condensed onto a Hausdorff space with a countable base. For the sequel, we need a little more.

**Lemma 3.1.** *If $(X,\tau)$ has a countable network and is not feebly compact then it can be condensed onto a second countable space $(X,\mu)$ which is not feebly compact.*

Proof. Suppose $\mathcal{U}=\{U_n : n\in\omega\}$ is a locally finite family of open sets in $X$. Let $\sigma$ be any second countable topology on $X$ weaker than $\tau$ (such a topology exists by the above-mentioned result of Arhangel'skiĭ). For each $x\in X$, let $V_x\in\tau$ be a neighbourhood of $x$ which meets only finitely many elements of $\mathcal{U}$. Since $(X,\tau)$ is Lindelöf (see 3.8.12 of [5]), the open cover $\{V_x : x\in X\}$ of $X$ has a countable subcover $\{V_{x_n} : n\in\omega\}$. Let $\mu$ be the topology on $X$ generated by the family

$$\sigma\cup\{V_{x_n} : n\in\omega\}\cup\{U_n : n\in\omega\}.$$

It is clear that $\mu$ is second countable, $\mu\subset\tau$, and the family $\{U_n : n\in\omega\}$ is a $\mu$-locally finite family of $\mu$-open sets, indicating that the space $(X,\mu)$ is not feebly compact. □

Given open filters $\{\mathcal{F}_n : n\in\omega\}$ of a topological space $X$, we say that they have *mutually disjoint bases* $\{\mathcal{B}_n : n\in\omega\}$ if for every $n\in\omega$ the family $\mathcal{B}_n$ is a base of the filter $\mathcal{F}_n$ such that $(\bigcup\mathcal{B}_n)\cap(\bigcup\mathcal{B}_m)=\emptyset$ for any $m\neq n$.

**Proposition 3.2.** *Suppose that $\mathcal{U}=\{U_n : n\in\omega\}$ is a locally finite disjoint family of non-empty open subsets of a space $X$. Then there is a family $\{\mathcal{F}_n : n\in\omega\}$ of open filters in $X$ with mutually disjoint bases $\{\mathcal{B}_n : n\in\omega\}$ such that the set $\bigcup\{\bigcup\mathcal{B}_n : n\in\omega\}$ does not meet $\bigcup\mathcal{W}$ for some infinite $\mathcal{W}\subset\mathcal{U}$.*

Proof. Consider a partition $\bigcup\{A_n : n\in\omega\}$ of $\omega$ in which each set $A_n$ is infinite and $A_m\cap A_n=\emptyset$ if $m\neq n$. For each $n\in\omega$ let $\mathcal{C}_n$ be the cofinite filter on $A_{n+1}$ and let $\mathcal{F}_n$ be the open filter on $X$ defined by

$$G\in\mathcal{F}_n \text{ if and only if } G\supset\bigcup\{U_k : k\in F\} \text{ for some } F\in\mathcal{C}_n.$$

It is routine to check that for each $n\in\omega$, the family $\mathcal{F}_n$ is a free open filter and $\mathcal{B}_n=\{\bigcup\{U_k : k\in F\} : F\in\mathcal{C}_n\}$ is a base of $\mathcal{F}_n$. It follows from $A_n\cap A_m=\emptyset$ that $(\bigcup\mathcal{B}_n)\cap(\bigcup\mathcal{B}_m)=\emptyset$ whenever $n\neq m$. Finally, $(\bigcup\{\bigcup\mathcal{B}_n : n\in\omega\})\cap(\bigcup\mathcal{W})=\emptyset$, where $\mathcal{W}=\{U_n : n\in A_0\}$ is an infinite locally finite subfamily of $\mathcal{U}$. □

**Lemma 3.3.** *If $(X, \tau)$ is a second countable Hausdorff space which is not $H$-closed, then there is a second countable dense-in-itself Hausdorff topology $\sigma \subset \tau$ such that either $(X, \sigma)$ is connected or it is not $H$-closed.*

PROOF. Let $D$ denote the set of isolated points of $(X, \tau)$; clearly, $D$ is countable. There are two cases to consider.
1) If $\mathrm{cl}(D)$ is not $H$-closed, then it is not feebly compact, and hence there is an infinite locally finite family of non-empty open sets contained in $D$, which in its turn implies that there is a closed (in $X$) infinite subset $E \subset D$ which we identify with $\omega$. Thus $X$ is homeomorphic to $X^* \oplus \omega$, where $X^* = X \setminus E$. Let $\rho$ be any connected second countable Hausdorff topology on $\omega$. Clearly $X$ condenses onto $X^* \oplus (\omega, \rho)$. Since $(\omega, \rho)$ has no isolated points, it follows from Fact 2.6 of [10], that it is not $H$-closed and hence not feebly compact. Thus, by Proposition 3.2 there is a family $\{\mathcal{F}_n : n \in \omega\}$ of free open filters on $(\omega, \rho)$ with mutually disjoint bases. Choose a dense subset $S = \{s_n : n \in \omega\}$ of $X^*$ and construct a topology $\mu$ on the set $Y = X^* \oplus \omega$ as follows:

$$\mu = \{U : U \cap X^* \in \tau,\ U \cap \omega \in \rho \text{ and for any } n \in \omega \\ \text{if } s_n \in U, \text{ then } F \subset U \text{ for some } F \in \mathcal{F}_n\}.$$

It is now routine to verify that $(Y, \mu)$ is a Hausdorff space which is connected (and hence dense-in-itself) since $(\omega, \rho)$ is connected and dense in $(Y, \mu)$. Furthermore, since $(X, \tau)$ is second countable and condenses onto $(Y, \mu)$, it follows that $\mu$ has a countable network and hence $(Y, \mu)$ can in its turn be condensed onto a second countable Hausdorff space $(Y, \sigma)$ which is necessarily connected.
2) If $\mathrm{cl}(D)$ is $H$-closed, then enumerate $D$ as $\{d_n : n \in \omega\}$. Since $X$ is not $H$-closed, there is an infinite locally finite family of non-empty open subsets of $X$. Only finitely many of them can meet the set $\mathrm{cl}(D)$ which is feebly compact. Thus, there is an infinite locally finite family of open sets $\mathcal{U} = \{U_n : n \in \omega\}$, such that $U_n \cap \mathrm{cl}(D) = \emptyset$ and hence each of the sets $U_n$ is dense-in-itself. By Lemma 2.1 of [11], we may assume the sets $U_n$ to be disjoint. By Proposition 3.2, there are open filters $\{\mathcal{F}_n : n \in \omega\}$ with mutually disjoint bases $\{\mathcal{B}_n : n \in \omega\}$ such that the set $\bigcup\{\bigcup \mathcal{B}_n : n \in \omega\}$ does not meet $\bigcup \mathcal{W}$ for some infinite $\mathcal{W} \subset \mathcal{U}$.

Define a topology $\mu$ as follows:

$$\mu = \{U \in \tau : \text{for any } n \in \omega \text{ if } d_n \in U \text{ then } F \subset U \text{ for some } F \in \mathcal{F}_n\}.$$

For each $n \in \omega$, a $\mu$-neighbourhood of $d_n$ contains an element of $\mathcal{F}_n$ and hence $(X, \mu)$ is dense-in-itself. Suppose that $V, W$ are disjoint $\tau$-open sets and let $P = \{n \in \omega : d_n \in V\}$ and $Q = \{n \in \omega : d_n \in W\}$. It is easy to see that for each $n \in P \cup Q$, we can choose $W_n \in \mathcal{F}_n$ in such a way that the $\mu$-open sets $V' = V \cup \bigcup\{W_n : n \in P\}$ and $W' = W \cup \bigcup\{W_n : n \in Q\}$ are disjoint (note that one or both of the sets $P, Q$ may be empty). To prove that $(X, \mu)$ is Hausdorff, suppose $x, y \in X$ are distinct. There are disjoint $V, W \in \tau$ with $x \in V$ and $y \in W$ and then $V', W'$ constructed above are disjoint $\mu$-neighbourhoods of $x$ and $y$ respectively.

Although the space $(X, \mu)$ may fail to be second countable, the proof can be concluded as follows: observe that the family $\mathcal{W}$ consists of $\mu$-open sets and is locally finite in $\mu$, whence $(X, \mu)$ is not feebly compact. Since $\mu \subset \tau$, the space $(X, \mu)$ has a countable network. Apply Lemma 3.1 to conclude that $(X, \mu)$ can be condensed onto a second countable non-$H$-closed space $(X, \sigma)$. Finally, observe that $(X, \sigma)$ is dense-in-itself because so is $(X, \mu)$. $\square$

Lemmas 3.1 and 3.3 show that in the quest for condensations of Hausdorff spaces with countable networks onto connected Hausdorff spaces, we can restrict attention to those second countable Hausdorff topologies which are dense-in-themselves.

Recall that an open set $U$ is *regular open* if $\mathrm{int}(\mathrm{cl}(U)) = U$.

**Theorem 3.4.** *A disconnected second countable dense-in-itself Hausdorff space $(X, \tau)$ can be condensed onto a second countable connected Hausdorff space if and only if it is not $H$-closed.*

Proof. For the necessity, we note that if $(X, \tau)$ is disconnected then there is a non-empty proper clopen set $U \subset X$. If $X$ is $H$-closed then it is easy to see that $U$ will be clopen in any Hausdorff topology $\sigma \subset \tau$. Hence $(X, \sigma)$ is disconnected.

Suppose now that $(X, \tau)$ is not $H$-closed and hence not feebly compact. There is some infinite locally finite family $\mathcal{U} = \{U_n : n \in \omega\}$ of open sets in $X$ which we may assume to be disjoint and regular open. By adding the (regular) open set $X \setminus \mathrm{cl}(\{U_n : n \in \omega\})$ to $\mathcal{U}$ if necessary, we may further assume that$D = \cup\{U_n : n \in \omega\}$ is dense in $X$. Let $E_n$ be a countable dense subset of $U_n$; since $E_n$ is a countable second countable Hausdorff space, it follows from Theorem 2.1 of [1] that $E_n$ has a dense regular subspace, which we denote by $Q_n$. As $Q_n$ is a countable, first countable, dense-in-itself, regular space, it must be homeomorphic to the rationals $\mathbb{Q}$ with the usual metric topology $\nu$. Denote by $h_n$ some homeomorphism from $Q_n$ onto $\mathbb{Q}$. Let $\rho$ be a connected second countable Hausdorff topology on $\omega$. Given $T \in \nu$ and $W \in \rho$, we define

$$O(T, W) = \cup\{\mathrm{int}_\tau(\mathrm{cl}_\tau(h_n^{-1}[T])) : n \in W\}.$$

For each $T \in \nu$ and $W \in \rho$, it is immediate that $O(T, W) \in \tau$. We define a topology $\sigma$ on $X$ as follows:

$$\sigma = \{U \in \tau : \text{for all } x \in U \cap D \text{ there exist } T \in \nu$$

and $W \in \rho$ such that $x \in O(T, W) \subset U\}$.

Obviously, $\sigma \subset \tau$ and we leave to the reader the straightforward verification that $\sigma$ is indeed a topology. We proceed to prove that $(X, \sigma)$ is a Hausdorff space. To this end, suppose $x, y$ are distinct elements of $X$.

1) $x, y \in D$. If $x \in U_n$ and $y \in U_m$, where $m \neq n$, then there are mutually disjoint $W_1, W_2 \in \rho$ such that $n \in W_1$ and $m \in W_2$. The sets $O(\mathbb{Q}, W_1)$ and $O(\mathbb{Q}, W_2)$ are $\sigma$-open disjoint neighbourhoods of $x$ and $y$ respectively. On the other hand, if $x, y \in U_n$ for some $n \in \omega$, then there are disjoint $\tau$-open neighbourhoods $U, V \subset U_n$ of $x, y$ respectively. If we define $T_1 = h_n[U \cap Q_n]$ and $T_2 = h_n[V \cap Q_n]$, then $x \in O(T_1, \omega) \in \sigma$, $y \in O(T_2, \omega) \in \sigma$ and $O(T_1, \omega) \cap O(T_2, \omega) = \emptyset$.

2) $x \in D$, say $x \in U_n$ and $y \in X \backslash D$. There exist disjoint $\tau$-open sets $U, V$ with the following properties: $x \in U \subset U_n$, $y \in V$ and the set $M = \{m \in \omega : V \cap U_n \neq \emptyset\}$ is finite, say $M = \{m_1, \ldots, m_k\}$. Since $(\omega, \rho)$ is Hausdorff, there are disjoint sets $W_1, \ldots, W_k \in \rho$ such that $m_i \in W_i$ for each $i \in 1, \ldots, k$.

If $n \in M$, say $n = m_j$ then the sets $G = O(h_n[U \cap Q_n], W_j)$ and $H = V \cup \bigcup\{O(T_i, W_i) : 1 \leq i \leq k\}$ where $T_i = \mathbb{Q}$ if $i \neq j$ and $T_j = h_n[V \cap Q_n]$ have the property that $H, G \in \sigma$, $H \cap G = \emptyset$ and $x \in G$, $y \in H$.

If, on the other hand, $n \notin M$, then we choose disjoint sets $W_0, W_1, \ldots, W_k \in \rho$ such that $n \in W_0$ and $m_i \in W_i$ for $i = 1, \ldots, k$. The sets $G = O(\mathbb{Q}, W_0)$ and $H = V \cup \bigcup\{O(\mathbb{Q}, W_i) : 1 \leq i \leq k\}$ are disjoint $\sigma$-open neighbourhoods of $x$ and $y$ respectively.

3) If $x, y \in X \backslash D$, then choose disjoint $\tau$-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and the sets $A = \{n \in \omega : U \cap U_n \neq \emptyset\}$ and $B = \{n \in \omega : V \cap U_n \neq \emptyset\}$ are both finite. For each $n \in A \cup B$, choose a $\rho$-open set $W_n$ in such a way that if $m \neq n$, then $W_m \cap W_n = \emptyset$. The sets $G = U \cup \bigcup\{O(h_m[U \cap Q_m], W_m) : m \in A\}$ and $H = V \cup \bigcup\{O(h_m[V \cap Q_m], W_m) : m \in B\}$ are disjoint $\tau$-open neighbourhoods of $x$ and $y$ respectively.

For each $q \in \mathbb{Q}$, consider the set $Y_q = \{h_n^{-1}[q] : n \in \omega\}$. It is clear that $(Y_q, \sigma|Y_q)$ is homeomorphic to $(\omega, \rho)$ and hence is connected. The set $Y = \cup\{Y_q : q \in \mathbb{Q}\}$ is $\tau$-dense and hence $\sigma$-dense in $X$. Since the space $(\omega, \rho)$ is countable, connected and Hausdorff, it is not $H$-closed (see, for example, Lemma 3.3 of [11]); being second countable it is not feebly compact. Thus there is an infinite locally finite family $\mathcal{W} = \{W_n : n \in \omega\}$, of open subsets of $(\omega, \rho)$, which we may assume to be disjoint. We claim that the infinite family of mutually disjoint $\sigma$-open sets $\mathcal{V} = \{O(\mathbb{Q}, W_n) : n \in \omega\}$ is locally finite.

To prove our claim, suppose $x \in X$. If $x \in D$, say $x \in U_n$, then since $\mathcal{W}$ is locally finite, there is some open $\rho$-neighbourhood $V$ of $n$ which meets only finitely many elements of $\mathcal{W}$. Now $O(\mathbb{Q}, V)$ is a $\sigma$-neighbourhood of $x$ which meets only finitely many elements of $\mathcal{V}$. If, on the other hand, $x \in X \setminus D$, then $x$ has a $\tau$-neighbourhood $V$ which meets only finitely many elements of $\mathcal{U}$, say $\{U_{n_1}, \ldots, U_{n_k}\}$. Since $\mathcal{W}$ is locally finite there are disjoint $\rho$-neighbourhoods $\{V_{n_1}, \ldots, V_{n_k}\}$ of $n_1, \ldots, n_k$ respectively each meeting only a finite number of elements of $\mathcal{W}$. Now $V \cup \bigcup\{O(\mathbb{Q}, V_{n_j}) : 1 \le j \le k\}$ is a $\sigma$-neighbourhood of $x$ meeting only finitely many elements of $\mathcal{V}$, and our claim is proved.

Let $\mathcal{F}$ be the $\sigma$-open filter generated by the family of sets

$$\{\{O(\mathbb{Q}, W_n) : n \ge k\} : k \in \omega\};$$

then $\mathcal{F}$ is a free open filter on $(X, \sigma)$. Choose an $x \in X$ and define a new topology $\mu = \{U \in \sigma : x \notin U \text{ or } U \in \mathcal{F}\}$.

Clearly, $\mu \subset \sigma$ and it is straightforward to check that $(X, \mu)$ is a Hausdorff space. The filter $\mathcal{F}$ converges to $x$ in the topology $\mu$ and for any $q \in \mathbb{Q}$ any element of $\mathcal{F}$ meets $Y_q$. Therefore $x \in \mathrm{cl}_\mu(Y_q)$ for all $q \in \mathbb{Q}$. As a consequence, the subspace $\{x\} \bigcup\{Y_q : q \in \mathbb{Q}\}$ is connected and dense in $(X, \mu)$, whence $(X, \mu)$ is connected. Since $\mu \subset \tau$, the space $(X, \mu)$ has a countable network. If $\lambda \subset \mu$ is any Hausdorff second countable topology, then $(X, \tau)$ condenses onto the space $(X, \lambda)$ which is Hausdorff, second countable and connected. □

Combining the results of Lemmas 3.1 and 3.3 and Theorem 3.4, we obtain:

**Corollary 3.5.** *A disconnected Hausdorff space $X$ with a countable network can be condensed onto a connected (second countable) Hausdorff space if and only if it is not $H$-closed.*

Recall that, given a Tychonoff space $X$, a point $y \in \beta X \setminus X$ is called *remote* if for any nowhere dense subset $D$ of the space $X$ we have $y \notin \mathrm{cl}_{\beta X}(D)$. In the paper [4], van Douwen proved, among other things, that there are remote points in $\beta\mathbb{R} \setminus \mathbb{R}$.

**Example 3.6.** *There exists a Tychonoff non-compact (and hence non-$H$-closed) space $X$ of countable $\pi$-weight, which does not have a weaker Hausdorff connected topology.*

Proof. Let $p \in \beta\mathbb{R} \setminus \mathbb{R}$ be a remote point of $\beta\mathbb{R}$. Take any points $a, b \notin \beta\mathbb{R}$ and let $X = (\beta\mathbb{R} \setminus \{p\}) \oplus \{a\} \oplus \{b\}$. It is clear that $X$ is not compact and has countable $\pi$-weight. We will show that every weaker Hausdorff topology on $X$ has an isolated point. To this end, suppose that $\tau$ is such a topology on $X$. By Corollary 4.3 of [11], $\beta\mathbb{R} \setminus \{p\}$ has only one free open ultrafilter, which we denote by $\mathcal{F}$. If neither $a$ nor $b$ are isolated points of $(X, \tau)$, then it follows that the traces on $\beta\mathbb{R} \setminus \{p\}$ of the $\tau$-open neighbourhood filters of $a$ and of $b$, are free open filter bases on $\beta\mathbb{R} \setminus \{p\}$ which we denote by $\mathcal{V}_a$ and $\mathcal{V}_b$ respectively. Both $\mathcal{V}_a$ and $\mathcal{V}_b$ are contained in free open ultrafilters on $\beta\mathbb{R} \setminus \{p\}$ and since $\mathcal{F}$ is the unique one, we must have $\mathcal{V}_a, \mathcal{V}_b \subset \mathcal{F}$. Thus each $\tau$-neighbourhood of $a$ meets each $\tau$-neighbourhood of $b$, a contradiction, and so one of the points $a$ or $b$ is isolated in $(X, \tau)$. □

**Corollary 3.7.** *Any non-compact metric space $(X, \tau)$ of weight $\leq \mathfrak{c}$ has a weaker second countable connected Hausdorff topology.*

Proof. Since $X$ is not compact, there is a continuous unbounded function $f : X \to \mathbb{R}$. Let $\mathcal{U} = \{f^{-1}((p, q)) : p, q \in \mathbb{Q}\}$. Then $\mathcal{U}$ is a countable family of open subsets of $X$ such that for any topology $\mu$ on $X$, if $\mathcal{U} \subset \mu$ then the function $f$ is $\mu$-continuous.

Any metrizable space of weight $\leq \mathfrak{c}$ embeds into the countable power of the hedgehog with $\mathfrak{c}$ spines. It is clear that such a hedgehog condenses onto a subset of the plane $\mathbb{R}^2$ with the natural topology. Thus, $X$ condenses onto a second countable space $(Y, \sigma)$. The topology $\sigma$ might be compact but the following modification helps us get round this obstacle.

Take a countable base $\mathcal{B}$ for $\sigma$ and let $\mu$ be the topology generated by the family $\mathcal{U} \cup \mathcal{B}$ as a subbase. Having a countable subbase, $\mu$ has a countable base. Clearly, $\mu \subset \tau$ and $\mu$ is Hausdorff being stronger than the Hausdorff topology $\sigma$. Moreover, the function $f$ is unbounded and $\mu$-continuous, whence $(X, \mu)$ is not $H$-closed. The result now follows from Corollary 3.5. □

**Corollary 3.8.** *If either*

*(1) $X$ is a non-compact regular Lindelöf space with a $G_\delta$-diagonal, or*

*(2) $X$ is a non-$H$-closed Hausdorff space with a $G_\delta$-diagonal and $X \times X$ is Lindelöf,*

*then $X$ condenses onto a connected Hausdorff space.*

Proof. It is well-known that a regular Lindelöf space with a $G_\delta$-diagonal condenses onto a regular second countable space (see 210, Chapter V of [2]). However, the result of the condensation might be compact. We will show that under either hypothesis (1) or (2) the space $X$ condenses onto a non-$H$-closed second countable space. Note that, in both cases $X$ is not feebly compact and therefore there exists a locally finite family $\gamma = \{U_n : n \in \omega\}$ of non-empty open subsets of $X$.

Let $\mathcal{U}$ be a countable open cover of $X$ such that each $U \in \mathcal{U}$ intersects only finitely many elements of $\gamma$.

(1) By 210, Chapter V of [2], $X$ condenses onto a regular second countable space $(Y, \sigma)$. Take a countable base $\mathcal{B}$ for $\sigma$ and let $\mu$ be the topology generated by the

family $\gamma \cup \mathcal{U} \cup \mathcal{B}$ as a subbase. Having a countable subbase, $\mu$ has a countable base. Clearly, $\mu \subset \tau$ and $\mu$ is Hausdorff being stronger than the regular topology $\sigma$. Moreover, the family $\gamma$ is locally finite in $\mu$ whence $(X, \mu)$ is not $H$-closed. Apply Corollary 3.5 to finish the proof of (1).

(2) Let $\Delta = \{(x, x) : x \in X\}$ be the diagonal of $X$. Since $X \times X$ is Lindelöf and $(X \times X) \setminus \Delta$ is a countable union of closed subsets of $X \times X$, it is Lindelöf. For any distinct $x, y \in X$ fix $U_{(x,y)}, V_{(x,y)} \in \tau$ with $x \in U_{(x,y)},\ y \in V_{(x,y)}$ and $U_{(x,y)} \cap V_{(x,y)} = \emptyset$. Let $\mathcal{C} = \{U_{(x,y)} \times V_{(x,y)} : (x, y) \in (X \times X) \setminus \Delta\}$. Clearly, $\mathcal{C}$ is an open cover of $(X \times X) \setminus \Delta$; let $\mathcal{C}'$ be a countable subcover of $\mathcal{C}$. Take any countable family $\mathcal{W} \subset \tau$ such that $\mathcal{C}' \subset \{U \times V : U, V \in \mathcal{W}\}$.

Let $\sigma$ be the topology generated by the family $\mathcal{W}$. Observe that it is second countable; to show that it is Hausdorff take distinct $x, y \in X$. There exist $U, V \in \mathcal{W}$ such that $U \times V \in \mathcal{C}'$ and $(x, y) \in U \times V$. Then $x \in U \in \sigma, y \in V \in \sigma$ and $U \cap V = \emptyset$ because $(U \times V) \cap \Delta = \emptyset$. Now let $\mu$ be the topology generated by the family $\gamma \cup \mathcal{U} \cup \mathcal{W}$ as a subbase. Having a countable subbase, $\mu$ has a countable base. Clearly, $\mu \subset \tau$ and $\mu$ is Hausdorff being stronger than the Hausdorff topology $\sigma$. Moreover, the family $\gamma$ is locally finite in $\mu$, whence $(X, \mu)$ is not $H$-closed. Now Corollary 3.5 concludes the proof of (2). $\square$

We end with some open questions:

**Problem 3.9.** *Let $X$ be a non-$H$-closed Hausdorff space with a $\sigma$-locally finite base. Does $X$ have a weaker connected Hausdorff topology?*

The authors and G. Gruenhage proved recently that for metrizable spaces the answer is positive. Note however, that Corollary 3.7 gives more for metrizable spaces of weight $\leq \mathfrak{c}$, because the weaker topology in this case can be chosen to be second countable.

**Problem 3.10.** *Suppose that $X$ is a non-compact regular Lindelöf space. Does $X$ have a weaker connected Hausdorff topology? What if $X$ is first countable?*

**Problem 3.11.** *Suppose that $X$ is a non-$H$-closed Lindelöf Hausdorff space. Does $X$ have a weaker connected Hausdorff topology? What if $X$ is first countable or has a $G_\delta$-diagonal?*

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