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# LQG control over lossy TCP-like networks with probabilistic packet acknowledgements

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**Abstract:** This paper focuses on control applications over lossy data networks. Sensor data is transmitted to an estimation-control unit over a network, and control commands are issued to subsystems over the same network. Sensor, control and acknowledgement packets may be randomly lost according to a Bernoulli process. In this context, the discrete-time Linear Quadratic Gaussian (LQG) optimal control problem is considered. We can show how the partial loss of acknowledgements makes the optimal control law a nonlinear function of the information set. For the special case of complete state observation we can compute the optimal controller and show that the stability range increases monotonically with the arrival rate of the acknowledgement packets.

**Keywords:** networked control; optimal control; LQG; linear quadratic gaussian; control.

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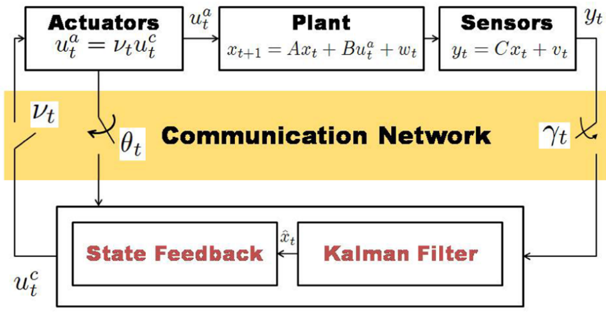
## 1 Introduction

This paper is concerned with the design and analysis of control systems when components are connected via packet based communication networks. This requires a generalisation of classical control techniques that explicitly takes into account the stochastic nature of the communication channel.

In particular we consider a generalised formulation of the LQG optimal control problem where the arrival of both observations and control packets are modelled as random processes whose parameters are related to the characteristics of the communication channel. Accordingly, independent Bernoulli processes are considered, with parameters $\bar{\gamma}$ and $\bar{\nu}$, that govern packet losses between the sensors and the estimation-control unit, and between the latter and the actuation points. The key issue to the proper design of networked control systems is a clear understanding of the information available to the controller at each time instant. For this reason it is usual to distinguish between two information sets depending on two classes of protocols. Such a distinction resides simply in the availability of packet acknowledgements. Adopting the framework proposed by Imer et al. (2004), we will refer to TCP-like protocols if packet acknowledgements are guaranteed at each time instant and to UDP-like protocols otherwise. In many real cases, due to the unreliability of the communication medium, such a distinction is too simplistic. In fact often, while it is impossible to guarantee a perfectly deterministic acknowledgement through a packet dropping channel, it is possible to provide a stochastic one. This 'quasi-TCP-like' protocol will provide the controller with more information than the UDP-like ones, and it could therefore be useful in many applications.

The goal of this paper is to provide some partial answers to the question of how control loop performance is affected by communication constraints and what are the basic system-theoretic implications of using unreliable networks for control in the case of 'quasi-TCP-like' protocols. For such a reason we introduce a third Bernoulli process (see Figure 1) of parameter, $\bar{\theta}$, which models the loss of the acknowledgement packet. Previous work (see Sinopoli et al., 2005a, 2005b, 2005c) showed the existence of a critical domain of values for the parameters of the Bernoulli arrival processes, $\bar{\nu}$ and $\bar{\gamma}$, outside which a transition to instability occurs and the optimal controller fails to stabilise the system. In particular, under TCP-like protocols, the critical arrival probabilities for the control and observation channels are independent of each other. It was shown that in the TCP-like case the classic separation principle holds, and consequently the controller and estimator can be designed independently. Moreover, the optimal controller is a linear function of the state. A more involved situation regards UDP-like protocols. In this case the absence of an acknowledgement structure generates a nonclassical information pattern (Witsenhausen, 1968). A consequence of that the optimal controller is in general a non-linear function of the state. Because of the importance of UDP protocols for wireless sensor networks, the special case when the arrival of a sensor packet provides complete knowledge of the state has been analysed. In this case, despite the lack of acknowledgements, the optimal control design problem yields a linear controller (see Sinopoli et al., 2005b) and the critical arrival probabilities for the control and observation channels are coupled. Also the stability domain and the performance of the optimal controller degrade considerably as compared with TCP-like protocols as shown in Figure 2.

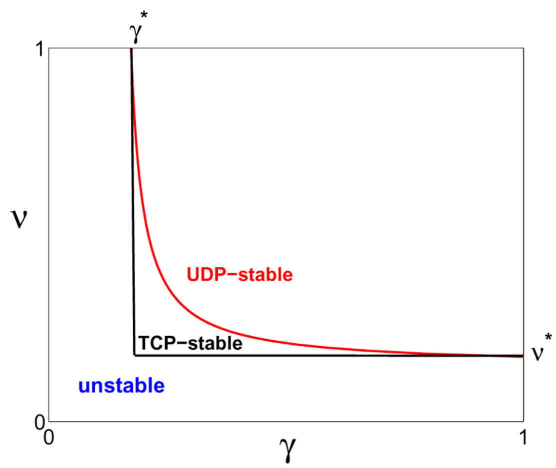
**Figure 1** *Overview of the system*. Architecture of the closed loop system over a communication network. The binary random variables $\nu_t$, $\gamma_t$ and $\theta_t$ indicates whether packets are transmitted successfully (see online version for colours)




For the general UDP-like case, a sub-optimal solution was provided in Sinopoli et al. (2006), by designing the optimal linear static regulator, with constant gains for both the observer and the controller. This is particularly attractive for sensor networks, where simplicity of implementation is highly desirable and complexity issues are a primary concern.

In this paper, we drop the assumption of deterministic and instantaneous availability of acknowledgement. We call this information set Quasi-TCP-like. Loss of acknowledgement leads once again to a nonclassical information pattern, and we are able to prove that in general the optimal control law is a nonlinear function of the information set. By restricting ourselves to the complete observability case, we are able to solve the LQG problem. We show that probabilistic acknowledgements increase the stability range of the system with respect to the UDP-like case. Furthermore, we can show how such range converges to the TCP-like one as the erasure probability for the acknowledgement channel tends to zero. In the past few years networked embedded control systems have drawn considerable attention in the academic world. We will now try to set our work in the context of the existing literature. In Gupta et al. (2007) and Xu and Hespanha (2005), an estimator, i.e., a Kalman filter, is placed at the sensor side of the link and no assumption is made on the statistical model of the data loss process (see Smith and Seiler, 2003) focused on designing a suboptimal yet computationally efficient estimator for Markov Chain arrival processes. In Huang and Dey (2007) the authors study the stability of Kalman Filter under general Markovian packet losses. In Epstein et al. (2006), the authors present a simple estimation scheme that is able to recover the fate of the control packet under UDP-like protocols by constraining the control signal sent to the plant. Drew et al. (2005) analyses the problem of designing a controller over a wireless LAN. Control design has been investigated in the context of Cross Layer Design by Liu and Goldsmith (2006). Finally, in Elia and Eisembeis (2004) and Elia (2005), the plant and the controller are modelled as deterministic time invariant discrete-time systems connected to zero-mean stochastic structured uncertainty, where the variance of the stochastic perturbation is a function of the Bernoulli parameters. Here, the controller design problem is posed an an optimisation problem to maximise mean-square stability of the closed loop system. While this method allows analysis of Multiple Input Multiple Output (MIMO) systems with many different controller and receiver compensation schemes (see Elia and Eisembeis, 2004), it does not include process and observation noise. The resulting controller is restricted to be time-invariant, hence sub-optimal. Finally, within the context of UDP-like control (Epstein et al., 2006) recently proposed to estimate not only the state of the system, but also a binary variable which indicates whether the previous control packet has been received or not. Such strategy improves closed loop performance at the price of a somewhat larger computational complexity.

**Figure 2** Region of stability for UDP-like and TCP-like optimal control relative to measurement packet arrival probability $\bar{\gamma}$, and the control packet arrival probability $\bar{\nu}$ (see online version for colours)



The remainder of this paper is organised as follows. Section 2 provides the problem formulation. In Section 3, the Single Channel problem is studied. In particular, estimator equations are derived and the nonlinearity of the optimal control for the general case is proven. Finally the optimal control for the special case of complete observability is considered. Section 4 generalises the previous section's results to the multi-channel case. Section 5 provides an example showing how the arrival probability of the acknowledgement packet increases the stability region. Section 6 provides conclusions. In order to make the paper more readable most of the proofs have been moved to the Appendix.

## 2 Problem and formulation

Consider the following linear stochastic system with intermittent observation and control packets arrival:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k^a + \omega_k, \\ u_k^a &= N_k u_k + [I_{m\times m} - N_k]\, u_k^l, \\ y_k &= \Gamma_k (Cx_k + v_k), \end{aligned} \tag{1}$$

where $x_k \in \mathcal{R}^n$ is the state vector, $y_k \in \mathcal{R}^p$ is the output vector and vectors $(x_0 \in \mathcal{R}^n, w_k \in \mathcal{R}^n, v_k \in \mathcal{R}^p)$ are Gaussian, uncorrelated, white, with mean $(\bar{x}_0, 0, 0)$ and covariance $(P_0, Q, R)$ respectively. The packet arrivals at time $k$ are modelled by means of the following diagonal matrices:

$$N_k = \begin{bmatrix} \nu_{1,k} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \nu_{m,k} \end{bmatrix}, \tag{2}$$

$$\Gamma_k = \begin{bmatrix} \gamma_{1,k} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \gamma_{p,k} \end{bmatrix}, \tag{3}$$

where $(\gamma_{i,k})$, $i = 1, \dots, p$ and $(\nu_{j,k})\, j = 1, \dots, m, \forall k \in \mathbb{Z}$, are binary variables modelling the successful transmission of the information from the $i$th sensor and to the $j$th actuator. $u_k^a \in \mathcal{R}^m$ is the effective control input applied to the actuators while $u_k \in \mathcal{R}^m$ denotes the desired control input computed by the controller. Finally $u_k^l \in \mathcal{R}^m$ is the signal locally provided to the actuators in the case $N_k = 0_{m\times m}$ (all packets to the actuators are lost). While it is possible to choose $u^l(k)$ in several ways, the most common strategies are the following:

1 zero-input scheme: $u_k^l = 0$

2 hold-input scheme: $u_k^l = u_{k-1}^a$.

In this paper we will provide analytical results for the zero-input scheme.

Provided that groups of sensors/actuators could send/receive their data in the same packet, we will suppose that the information transmission is organised in $p'$

sensor and $m'$ actuator clusters independent one of the other. This means we can rewrite $\Gamma_k$ and $N_k$ as follows:

$$\Gamma_k = I_{p\times p} - \prod_{i=1}^{p'}(I_{p\times p} - \gamma'_{i,k}\mathrm{diag}\{g_i\}) \tag{4}$$

$$N_k = I_{m\times m} - \prod_{i=1}^{m'}(I_{m\times m} - \nu'_{i,k}\mathrm{diag}\{\eta_i\}) \tag{5}$$

where $\gamma'_{i,k}$ and $\nu'_{j,k}$ are i.d.d. Bernoulli processes with probabilities of successful transmission $\overline{\gamma'_i} = P(\gamma'_{i,k} = 1), i = 1, \dots, p'$ and $\overline{\nu'_j} = P(\nu'_{j,k} = 1), i = 1, \dots, m'$. $g_i, i = 1, \dots, m'$ and $\eta_i, i = 1, \dots, p'$ are vectors of length $p$ and $m$ respectively such that:

- $(g_i)_j = 1$ ($(\eta_i)_j = 1$) if the $j$th sensor (actuator) belongs to the $i$th cluster
- $(g_i)_j = 0$ ($(\eta_i)_j = 0$) if the $j$th sensor (actuator) does not belong to the $i$th cluster.

A key point toward the design of any control strategy is the definition of the *Information Set*, i.e., the information available to the controller at each time instant. In the literature, it is usual (see Imer et al., 2004) to refer to the following two information sets:

$$I_k = \begin{cases} F_k = \{\Gamma_t y_t, \Gamma_t, N_{t-1} \,|\, t = 0, \dots, k\} & \text{TCP-like} \\ G_k = \{\Gamma_t y_t, \Gamma_t \,|\, t = 0, \dots, k\} & \text{UDP-like} \end{cases}. \tag{6}$$

The difference between the two Information Sets is the acknowledgement of the actually arrived packets to the actuators i.e., the matrix $N_{k-1}$.

While for the 'TCP-like' case several useful and important results (separation principles, LQG optimal control, etc.) are known, it is well known from networks and computer science literature that guaranteeing a deterministic 'perfect' acknowledgement is in general a very difficult task and, in the case where the acknowledgement packet uses unreliable channels, a theoretically impossible one as it can be seen as a particular case of the *two-armies problem* (see Tanenbaum, 1981).

On the other hand it is extremely difficult (see Sinopoli et al., 2005b) to design optimal estimators and controllers under the information set $G_k$, since the separation principle does not hold and it can be shown that the optimal control is not linear. Even in the special case where the optimal control is linear performance and stability regions are highly affected as shown in Figure 2, due to the fact that no 'real' information on the actual input is exploited.

In many practical cases, it is reasonable to use communication channels where acknowledgements are provided although they can be dropped with a probability which depends on both the channel reliability and the protocol employed. This means that, during each process, we have a non-zero probability to lose the acknowledgement from the channel $j$. In order to formalise this assumption we introduce the matrix

$$\Theta_k = \begin{bmatrix} \theta_{1,k} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \theta_{m,k} \end{bmatrix}, \tag{7}$$

where $\theta_{i,k}$ is the acknowledgement event from the $i$th actuator at time $k$. It can be rewritten as

$$\Theta_k = I_{m\times m} - \prod_{i=1}^{m'} (I_{m\times m} - \theta'_{i,k}\text{diag}\{\eta_i\}) \tag{8}$$

where $\theta'_{i,k}$ are i.i.d. Bernoulli processes with $\overline{\theta'_i} = P(\theta'_{i,k} = 1), i = 1, \ldots, m'$. The information structure of a networked system with stochastic acknowledgements is the following:

$$E_k = \{\Gamma_t y_t, \Gamma_t, \Theta_{kt-1}, \Theta_{t-1} N_{t-1} \mid t = 0, \ldots, k\}. \tag{9}$$

Let us now define $u^{N-1} = \{u_0, u_1, \ldots, u_{N-1}\}$ as the set of all the input values between 0 and $N-1$. In this paper, we will tackle the LQG control problem, i.e., we will look for a control input sequence $u^{N-1*}$, function of the information set $E_k$, i.e., $u_k = f_k(E_k)$, that solves the following optimisation problem

$$J_N^*(\bar{x}_0, P_0) = \min_{\substack{u_k = f_k(E_k) \\ k = 0, \ldots, N-1}} J_N(u^{N-1}, \bar{x}_0, P_0), \tag{10}$$

where the cost function $J_N(u^{N-1}, \bar{x}_0, P_0)$ is defined as follows:

$$J_N(u^{N-1}, \bar{x}_0, P_0) = E\left[x_N^T W_N x_N + \sum_{k=0}^{N-1} x_k^T W_k x_k + u_k^{aT} U_k u_k^a \middle| u^{N-1}, \bar{x}_0, P_0\right]. \tag{11}$$

The general multichannel formulation involves large matrix calculations. We feel that such technicalities will affect the intuitive nature of the results while not providing any additional insight. For this reason we will first concentrate on single channels case, i.e., $m' = 1$ and $n' = 1$ and provide the major results for it. We will then generalise the results to the multi-channel case. We will defer most of the proofs to the Appendix.

## 3 Single input/output channel case

### 3.1 Estimator design

If $m' = 1$ and $n' = 1$ system (1) becomes

$$\begin{aligned} x_{k+1} &= Ax_k + \nu_k B u_k + \omega_k \\ y_k &= \gamma_k C x_k + v_k \end{aligned} \tag{12}$$

and $\Theta_k = \theta_k$. By the knowledge of the information set equation (9), the one-step prediction can be written as:

$$\hat{x}_{k+1\,|\,k} = A\hat{x}_{k\,|\,k} + \theta_k \nu_k B u_k + (1 - \theta_k)\bar{\nu} B u_k. \tag{13}$$

Using equation (13) it is possible to rewrite the predicted error as follows:

$$\begin{aligned} e_{k+1\,|\,k} &= x_{k+1} - \hat{x}_{k+1\,|\,k} \\ &= Ax_k + \nu_k Bu_k + \omega_k - A\hat{x}_{k\,|\,k} + \theta_k \nu_k Bu_k - (1-\theta_k)\bar{\nu} Bu_k \\ &= Ae_{k\,|\,k} + (\nu_k - \theta_k\nu_k - (1-\theta_k)\bar{\nu})Bu_k + \omega_k. \end{aligned} \tag{14}$$

We can then compute the associated error covariance one-step prediction:

$$\begin{aligned} P_{k+1\,|\,k} &= E[e_{k+1\,|\,k} e^T_{k+1\,|\,k} \,|\, E_k, \theta_k, \theta_k\nu_k] \\ &= E[Ae_{k\,|\,k} e^T_{k\,|\,k} A \,|\, E_k] + E[\omega_k \omega_k^T \,|\, E_k] \\ &\quad + E[(\nu_k - \theta_k\nu_k - (1-\theta_k)\bar{\nu})^2 \,|\, E_k, \theta_k, \theta_k\nu_k] Bu_k u_k^T B^T, \end{aligned}$$

obtaining

$$P_{k+1\,|\,k} = AP_{k\,|\,k}A^T + Q + (1-\theta_k)(1-\bar{\nu})\bar{\nu}[Bu_k u_k^T B^T]. \tag{15}$$

Equations (13)–(15) represent the predictions of the Kalman Filter for the systems (12). The correction steps, instead, are the classical ones reported in Schenato et al. (2007):

$$\hat{x}_{k+1\,|\,k+1} = \hat{x}_{k+1\,|\,k} + \gamma_{k+1}K_{k+1}(y_{k+1} - Cx_{k+1\,|\,k}) \tag{16}$$

$$P_{k+1\,|\,k+1} = P_{k+1\,|\,k} - \gamma_{k+1}K_{k+1}CP_{k+1\,|\,k} \tag{17}$$

$$K_{k+1} = P_{k+1\,|\,k}C^T(CP_{k+1\,|\,k}C^T + R)^{-1}. \tag{18}$$

**Remark 1:** Note that:

$$\begin{aligned} \theta_k = 1 &\Rightarrow P_{k+1\,|\,k} = AP_{k\,|\,k}A + Q \\ \theta_k = 0 &\Rightarrow P_{k+1\,|\,k} = AP_{k\,|\,k}A + Q + \bar{\nu}(1-\bar{\nu})[Bu_k u_k^T B^T]. \end{aligned}$$

This implies that, at each time $k$, the prediction switches between the 'TCP-like' predictions or the 'UDP-like' ones, depending on the instantaneous value of $\theta_k$.

### *3.2 Optimal control: general case*

In this section we will show that, in the presence of a stochastic acknowledgements, the optimal control law is not a linear function of the state and that the estimation and control design cannot be treated separately. In order to prove such a statement it is sufficient to consider the following simple counter example.

Consider a simple scalar discrete-time Linear Time-Invariant (LTI) system with a single sensor and a single actuator, i.e., $A = B = C = W_N = W_k = R = 1$, $U_k = Q = 0$. We can define

$$\begin{aligned} V(N) &= E[x_N^T W_N x_N \,|\, E_N] \\ &= E[x_N^2 \,|\, E_N]. \end{aligned}$$

For $k = N - 1$ we will have:

$$\begin{aligned} V_{N-1}(x_{N-1}) &= \min_{u_N} E[x_{N-1}^2 + V_N(x_N) \mid E_{N-1}] \\ &= \min_{u_N} E[x_{N-1}^2 + x_N^2 \mid E_{N-1}] \\ &= \min_{u_N} E[x_{N-1}^2 + (x_{N-1} + \nu_{N-1} u_{N-1})^2 \mid E_{N-1}], \end{aligned} \tag{19}$$

and then finally

$$V_{N-1}(x_{N-1}) = E[2x_{N-1 \mid N-1}^2 \mid E_{N-1}] + \min_{u_N} \bar{\nu}(u_{N-1}^2 + 2\hat{x}_{N-1 \mid N-1} u_{N-1}). \tag{20}$$

If we differentiate the latter, we obtain the following optimal input:

$$u_{N-1}^* = -\hat{x}_{N-1 \mid N-\hat{1}}. \tag{21}$$

If we substitute equation (21) in equation (19) the cost becomes:

$$\begin{aligned} V_{N-1}(x) &= E[2x_{N-1}^2 \mid E_{N-1}] - \bar{\nu}\, \hat{x}_{N-1 \mid N-1}^2 \\ &= (2 - \bar{\nu}) E[x_{N-1}^2 \mid E_{N-1}] - \bar{\nu} P_{N-1 \mid N-1}. \end{aligned} \tag{22}$$

Let us focus now on the covariance matrix:

$$\begin{aligned} P_{N-1 \mid N-1} &= P_{N-1 \mid N-2} - \gamma_{N-1} \frac{P_{N-1 \mid N-2}^2}{(P_{N-1 \mid N-2} + 1)} \\ &= P_{N-1 \mid N-2} - \gamma_{N-1} \left( P_{N-1 \mid N-2} - 1 + \frac{1}{(P_{N-1 \mid N-2} + 1)} \right) \end{aligned} \tag{23}$$

because of

$$P_{N-1 \mid N-2} = P_{N-2 \mid N-2} + (1 - \theta_{N-2})(1 - \bar{\nu})\bar{\nu} u_{N-2}^2 \tag{24}$$

then:

$$\begin{aligned} E[P_{N-1 \mid N-1} \mid E_{N-2}] = {} & P_{N-2 \mid N-2} + (1 - \bar{\theta})(1 - \bar{\nu})\bar{\nu} u_{N-2}^2 \\ & - \bar{\gamma} \Bigg( P_{N-2 \mid N-2} + (1 - \bar{\theta})(1 - \bar{\nu})\bar{\nu}\, u_{N-2}^2 - 1 + \bar{\theta} \frac{1}{P_{N-2 \mid N-2}} \\ & + (1 - \bar{\theta}) \frac{1}{P_{N-2 \mid N-2} + (1 - \bar{\nu})\bar{\nu} u_{N-2}^2} \Bigg). \end{aligned} \tag{25}$$

Finally we get

$$\begin{aligned} V_{N-2}(x) = {} & \min_{u_{N-2}} E[x_{N-2}^2 + V_{N-1}(x_{N-1}) \mid E_{N-2}] \\ = {} & (3 - \bar{\nu}) E[x_{N-1}^2 \mid E_{N-2}] + \min_{u_{N-2}} P_{N-2 \mid N-2} + (1 - \bar{\theta})(1 - \bar{\nu})\bar{\nu} u_{N-2}^2 \\ & - \bar{\gamma} \Bigg( P_{N-2 \mid N-2} + (1 - \bar{\theta})(1 - \bar{\nu})\bar{\nu} u_{N-2}^2 - 1 + \bar{\theta} \frac{1}{P_{N-2 \mid N-2}} \\ & + (1 - \bar{\theta}) \frac{1}{P_{N-2 \mid N-2} + (1 - \bar{\nu})\bar{\nu} u_{N-2}^2} \Bigg). \end{aligned} \tag{26}$$

The first terms within the last parenthesis in equation (26) are convex quadratic functions of the control input $u_{N-2}$, however the last term is not. Therefore, the optimal control law is, in general, a nonlinear function of the information set $E_k$.

There are only two cases where the optimal control is linear. The first case is if $\bar{\theta} = 1$ (TCP-Case). This corresponds to the TCP-like case studied in Schenato et al. (2007). The second case is when the measurement noise covariance is zero ($R = 0$) and any delivered packet contains all the state information, i.e., $\text{Rank}(C) = n$. In fact this would mean that, at each time instant $k$, $\gamma_k C$ is either zero or full column-rank. If it is zero the dependence is linear. If $\gamma_k C$ is full column-rank, then it is equivalent to having exact measurement of the actual state (see Schenato et al., 2007). However it is important to remark that in such a second case the separation principle still does not hold, since the control input affects the estimator error covariance. These results can be summarised in the following theorem.

**Theorem 1:** *Let us consider the stochastic system defined in equation* (12) *with horizon $N \geq 2$. Then:*

- *If $\bar{\theta} < 1$, the separation principle does not hold.*
- *The optimal control feedback $u_k = f_k^*(E_k)$ that minimises the cost functional defined in equation* (11) *is, in general, a nonlinear function of information set $E_k$.*
- *The optimal control feedback $u_k = f_k^*(E_k)$ is a linear function of the estimated state if and only if one of the following conditions holds true:*
  - $\bar{\theta} = 1$
  - *Rank*$(C) = n$ *and* $R = 0$.

In the next subsection we will focus on the case where $\text{Rank}(C) = n$, and $R = 0$. In particular we will compute the optimal control and we will show that, in the infinite horizon scenario, the optimal state-feedback gain is constant, i.e., $L_k^* = L^*$ and can be computed as the solution of a convex optimisation problem.

### *3.3 Optimal control: Rank$(C) = n$, $R = 0$ case*

Without loss of generality we can assume $C = I$. Because of the hypothesis of no measurement noise, i.e., $R = 0$, it is possible to simply measure the state $x_k$ when a packet is delivered. The estimator equations then simplify in the following way:

$$K_{k+1} = I \tag{27}$$

$$P_{k+1\,|\,k} = AP_{k\,|\,k}A + Q + (1-\theta_k)(1-\bar{\nu})\bar{\nu}[Bu_k u_k^T B^T] \tag{28}$$

$$P_{k+1\,|\,k+1} = (1-\gamma_{k+1})P_{k+1\,|\,k} \tag{29}$$

$$= (1-\gamma_{k+1})(AP_{k\,|\,k}A + Q + (1-\theta_k)(1-\bar{\nu})\bar{\nu}[Bu_k u_k^T B^T]) \tag{30}$$

$$E[P_{k+1\,|\,k+1}\,|\,E_k] = (1-\bar{\gamma})(AP_{k\,|\,k}A + Q + (1-\bar{\theta})(1-\bar{\nu})\bar{\nu}[Bu_k u_k^T B^T]). \tag{31}$$

In the last equation the independence of $E_k, \gamma_{k+1}, \theta_k$ is exploited. Following the classical dynamic programming approach to optimal control, we claim that the value function $V_k^*(x_k)$ can be written as follows:

$$\begin{aligned} V_k(x_k) &= \hat{x}_{k\,|\,k}^T S_k \hat{x}_{k\,|\,k} + \text{trace}(T_k P_{k\,|\,k}) + \text{trace}(D_k Q) \\ &= E[x_{k\,|\,k}^T S_k x_{k\,|\,k}] + \text{trace}(H_k P_{k\,|\,k}) + \text{trace}(D_k Q) \end{aligned} \tag{32}$$

for each $k = N, \ldots, 0$ where $H_k = T_k - S_k$. This is clearly true for $k = N$, in fact we have:

$$\begin{aligned} V_N(x_N) &= E[x_N^T W_N x_N \,|\, E_N] \\ &= \hat{x}_{N\,|\,N}^T W_N \hat{x}_{N\,|\,N} + \text{trace}(W_N P_{N\,|\,N}), \end{aligned}$$

and the statement is satisfied by $S_N = T_N = W_N, D_N = 0$. Let us suppose that equation (32) is true for $k+1$ and let us show by induction that it holds true for $k$:

$$\begin{aligned} V_k(x_k) &= \min_{u_k} E[x_k^T W_k x_k + \nu_k u_k^T U_k u_k + V_{k+1}(x_{k+1}) \,|\, E_k] \\ &= \min_{u_k} E[x_k^T W_k x_k \,|\, E_k] + \bar{\nu} u_k^T U_k u_k + E[x_{k+1}^T S_{k+1} x_{k+1} \,|\, E_k] \\ &\quad + \text{trace}(H_{k+1} P_{k+1\,|\,k+1}) + \text{trace}(D_{k+1} Q) \\ &= \min_{u_k} E[x_k^T W_k x_k \,|\, E_k] + \bar{\nu}_k u_k^T U_k u_k + \text{trace}(D_{k+1} Q) \\ &\quad + \text{trace}(H_{k+1}((1-\bar{\gamma})(A P_{k\,|\,k} A + Q + (1-\theta_k)\bar{\nu}(1-\bar{\nu})[B u_k u_k^T B^T]))) \\ &\quad + E[(A x_{k\,|\,k} + \theta_k \nu_k B u_k + (1-\theta_k)\bar{\nu} B u_k)^T S_{k+1} \\ &\quad\;\; (A x_{k\,|\,k} + \theta_k \nu_k B u_k + (1-\theta_k)\bar{\nu} B u_k) \,|\, E_k]. \end{aligned}$$

Further manipulation yields:

$$\begin{aligned} V_k(x_k) &= \min_{u_k} E[x_k^T W_k x_k + \bar{\nu}_k u_k^T U_k u_k + (x_{k\,|\,k}^T A^T S_{k+1} A x_{k\,|\,k}) \\ &\quad + (\theta_k \nu_k u_k^T B^T B u_k) + ((1-\theta_k)\bar{\nu} u_k^T B^T B u_k^T) + 2\theta_k \nu_k x_{k\,|\,k}^T A^T S_{k+1} B u_k \\ &\quad + 2(1-\theta_k)\bar{\nu} x_{k\,|\,k}^T A^T S_{k+1} B u_k \,|\, E_k] + \text{trace}(D_{k+1} Q) \\ &\quad + \text{trace}(H_{k+1}((1-\bar{\gamma})(A P_{k\,|\,k} A + Q + (1-\bar{\theta})\bar{\nu}(1-\bar{\nu})[B u_k u_k^T B^T]))) \\ &= E[x_{k\,|\,k}^T (W_k + A^T S_{k+1} A) x_{k\,|\,k}] + (1-\bar{\gamma})\text{trace}(H_{k+1}((A P_{k\,|\,k} A + Q))) \\ &\quad + \min_{u_k} \bar{\nu}(u_k^T (U_k + B^T (S_{k+1} + (1-\bar{\theta})(1-\bar{\nu})\bar{\nu} H_{k+1}) B) u_k^T) \\ &\quad + 2\bar{\nu}(x_{k\,|\,k}^T A^T S_{k+1} B u_k) + \text{trace}(D_{k+1} Q). \end{aligned}$$

Since $V_k(x_k)$ is a convex quadratic function w.r.t. $u_k$, the minimiser is the solution of $\partial V_k(x_k)/\partial u_k = 0$, given by:

$$u_k^* = -(U_k + B^T (S_{k+1} + \bar{\alpha} H_{k+1}) B)^{-1} (B^T S_{k+1} A x_{k\,|\,k}) = L_k x_{k\,|\,k}, \tag{33}$$

where $\bar{\alpha} = (1-\bar{\gamma})(1-\bar{\theta})(1-\bar{\nu})\bar{\nu}$. The optimal control is a linear function of the estimated state $x_{k\,|\,k}$. Substituting back equation (33) into the value function we get:

$$\begin{aligned} V_k(x_k) &= \text{trace}((1-\bar{\gamma})H_{k+1}((AP_{k\,|\,k}A))) + trace(((1-\bar{\gamma})T_{k+1} + D_{k+1})Q) \\ &\quad + E[x_{k\,|\,k}^T(W_k + A^T S_{k+1}A)x_{k\,|\,k}] - \bar{\nu}x_{k\,|\,k}^T(A^T S_{k+1}BL_k)x_{k\,|\,k}, \end{aligned}$$

which becomes

$$\begin{aligned} V_k(x_k) &= \text{trace}((1-\bar{\gamma})H_{k+1}((AP_{k\,|\,k}A))) \\ &\quad + E[x_{k\,|\,k}^T(W_k + A^T S_{k+1}A)x_{k\,|\,k} + (\bar{\nu}(x_{k\,|\,k}^T A^T S_{k+1}B)L_k x_{k\,|\,k})] \\ &\quad + \text{trace}((D_{k+1} + (1-\bar{\gamma})T_{k+1})Q) - \text{trace}((\bar{\nu}A^T S_{k+1}BL_k P_{k\,|\,k})). \end{aligned}$$

Then finally we obtain

$$\begin{aligned} V_k(x_k) &= \text{trace}((D_{k+1} + (1-\bar{\gamma})H_{k+1})Q) \\ &\quad + E[x_{k\,|\,k}^T(W_k + A^T S_{k+1}A)x_{k\,|\,k} + (\bar{\nu}(x_{k\,|\,k}^T A^T S_{k+1}B)L_k x_{k\,|\,k})] \\ &\quad + \text{trace}((((1-\bar{\gamma})A^T H_{k+1}A - \bar{\nu}A^T S_{k+1}BL_k)P_{k\,|\,k})). \end{aligned}$$

From the last equation we see that the value function can be written as in equation (32) if and only if the following equations are satisfied:

$$S_k = W_k + A^T S_{k+1}A + \bar{\nu}(A^T S_{k+1}B)L_k \tag{34}$$

$$T_k = (1-\bar{\gamma})A^T T_{k+1}A + W_k + \bar{\gamma}A^T S_{k+1}A \tag{35}$$

$$D_k = D_{k+1} + (1-\bar{\gamma})T_{k+1} + \bar{\gamma}S_{k+1}. \tag{36}$$

**Remark 2:** Notice that, if $\bar{\theta} \to 0$, control design system soon regresses to the UDP-like.

The optimal minimal cost for the finite horizon, $J_N^* = V_0(x_0)$ is then given by:

$$J_N^* = \bar{x}_0^T S_0 x_0 + \text{trace}(S_0 P_0) + \text{trace}(D_0 Q).$$

For the infinite horizon optimal controller, necessary and sufficient conditions for the average minimal cost $J_\infty^* = \lim_{N\to\infty} \frac{1}{N} J_N^*$ to be finite, are that the coupled iterative equations (35) and (34) should converge to a finite value $S_\infty$ and $T_\infty$ as $N \to \infty$.

**Theorem 2:** *Consider the system* (12) *and consider the problem of minimising the cost function* (11) *within the class of admissible policies* $u_k = f(E_k)$. *Assume also that* $R = 0$ *and* $\text{rank}\, C = n$. *Then:*

1 *The optimal estimator gain is constant and in particular $K_k = I$ if $C = I$.*

2 *The infinite horizon optimal control exists if and only if there exist positive definite matrices $S_\infty, T_\infty > 0$ such that $S_\infty = \Phi_S(S_\infty, T_\infty)$ and $T_\infty = \Phi_T(S_\infty, T_\infty)$, where $\Phi_S$ and $\Phi_T$ are:*

$$\Phi_S(S_k, W_k) = W_k + A^T S_k A - \bar{\nu}(A^T S_k B)$$
$$(U_k + B^T((1-\bar{\alpha})S_{k+1} + \bar{\alpha}T_{k+1})B)^{-1}(B^T S_{k+1} A) \tag{37}$$
$$\Phi_T(S_k, T_k) = (1-\bar{\gamma})A^T T_{k+1} A + W_k + \bar{\gamma} A^T S_{k+1} A. \tag{38}$$

3 *The infinite horizon optimal controller gain is constant:* $\lim_{k\to\infty} L_k = L_\infty$

$$L_\infty = -(U + B^T((1-\bar{\alpha})S_\infty + \bar{\alpha}T_\infty)B)^{-1}(B^T S_\infty A). \tag{39}$$

4 *A necessary condition for existence of $S_\infty, T_\infty > 0$ is*

$$\begin{aligned} &1 - |A|^2\left(1 - \frac{\bar{\nu}}{(1-\bar{\alpha}) + \bar{\alpha}\frac{\bar{\gamma}|A|2}{1-(1-\bar{\gamma})|A|^2}}\right) \geq 0 \\ &\bar{\gamma} > 1 - \frac{1}{|A|^2} \end{aligned} \tag{40}$$

*where $|A| = \max_i |\lambda_i(A)|$ is the largest eigenvalue of the matrix A. This condition is also sufficient if B is square and invertible.*

5 *The expected minimum cost for the infinite horizon scenario converges to:*

$$J_\infty^* = \lim_{k\to\infty} \frac{1}{N} J_N^* = \text{trace}(((1-\bar{\gamma})T_k + \bar{\gamma}S_k)Q). \tag{41}$$

*Proof*: (1) This fact follows from equation (27). Statements (2), (3) and (5) follow from Lemma 2 (see Appendix) and equations (34) and (35). Statement (4) corresponds to Lemmas 3 and 4 (see Appendix). □

## 4 Generalisation to the multichannel case

Following the same reasoning as in the single channel case, i.e., $m' = n' = 1$, it is possible to generalise the results above to the multichannel case. Due to space constraint, here we will just summarise the principal differences with the single channel case. Proofs are reported in Appendix.

### *4.1 Optimal observer*

The prediction step of the Kalman filter shown in equations (13)–(15) for the single input case, becomes

$$\hat{x}_{k+1\,|\,k} = A\hat{x}_{k\,|\,k} + B\Theta_k N_k u_k + B(1-\Theta_k)\bar{N}u_k \tag{42}$$
$$e_{k+1\,|\,k} = Ae_{k\,|\,k} + B(I-\Theta_k)(N_k - \overline{N})u_k + \omega_k. \tag{43}$$
$$P_{k+1\,|\,k} = AP_{k\,|\,k}A^T + Q + B(1-\Theta_k)(\Psi(u_k, \overline{N}))(1-\Theta_k)B^T \tag{44}$$

where the following quantities are introduced to improve clarity:

$$\overline{N} = E[N_k] = I_{m\times m} - \prod_{i=1}^{m'} (I_{m\times m} - \bar{\nu}'_i \mathrm{diag}\{\eta_i\}), \tag{45}$$

$$N_I = I_{m\times m} - \prod_{i\in I\cup\{0\}} (I_{m\times m} - \mathrm{diag}\{\eta_i\}), \tag{46}$$

$$\Psi(u_k, \overline{N}) = \sum_{I\in 2^{\Im}} \left( \left( \prod_{i\in I} \bar{\nu}'_i \prod_{i\notin I} (1-\bar{\nu}'_i) \right) [(N_I - \overline{N}) u_k u_k^T (N_I - \overline{N})] \right), \tag{47}$$

where $I \subseteq \Im \equiv \{1, \ldots, m'\}$ is a set of indices and $\eta_0 = 0_m$. The correction step remains the classical one as shown in Garone et al. (2007):

$$\hat{x}_{k+1\,|\,k+1} = \hat{x}_{k+1\,|\,k} + K_{k+1}\Gamma^m_{k+1}(y_{k+1} - Cx_{k+1\,|\,k}) \tag{48}$$

$$P_{k+1\,|\,k+1} = P_{k+1\,|\,k} - K_{k+1}\Gamma^m_{k+1} C P_{k+1\,|\,k} \tag{49}$$

$$K_{k+1} = P_{k+1\,|\,k} C^T \Gamma^{mT}_{k+1} (\Gamma^m_{k+1} C P_{k+1\,|\,k} C^T \Gamma^{mT}_{k+1} + \Gamma^m_{k+1} R \Gamma^{mT}_{k+1})^{-1} \tag{50}$$

where $\Gamma^m_k$ is the matrix of the nonzero rows of $\Gamma_k$.

## *4.2 Optimal control*

In this subsection we generalise the theorem statements for the multi-channel case. In particular, we can extend conditions that make the optimal control linear as follows:

**Theorem 3:** *Let us consider the stochastic system defined in equation* (12) *with horizon* $N \geq 2$. *Then:*

- *if* $\overline{\theta}_i < 1 \forall i$, *the separation principle does not hold*
- *the optimal control feedback* $u_k = f^*_k(E_k)$ *that minimises the cost functional defined in equation* (11) *is, in general, a nonlinear function of information set* $E_k$
- *the optimal control feedback* $u_k = f^*_k(E_k)$ *is a linear function of the estimated state if and only if one of the following conditions holds true:*
  - $\overline{\theta}_i = 1, \forall i$
  - $\mathrm{Rank}(\mathrm{diag}\{g_i\}C) = n, i = 1, \ldots, p'$ *and* $R = 0$.

It is worth to notice that the conditions $\mathrm{Rank}(\mathrm{diag}\{g_i\}C) = n$ and $R = 0$ are equivalent to the case where any sensor data packet contains the actual value of the whole state. System (1) is then equivalent to

$$\begin{aligned} x_{k+1} &= Ax_k + Bu^a_k + \omega_k, \\ u^a_k &= N_k u_k + [I_{m\times m} - N_k]\, u^l_k, \\ y(k) &= \gamma_k x_k, \end{aligned} \tag{51}$$

where

$$\gamma_k = 1 - \prod_{i=1}^{p'} (1 - \gamma'_{i,k}).$$

This means that the optimal control is linear only when the sensing apparatus is able to perfectly measure and deliver the full state. For such a case it is possible to extend the results previously derived in the following manner:

**Theorem 4:** *Consider the system* (1) *and consider the problem of minimising the cost function* (11) *within the class of admissible policies* $u_k = f(E_k)$. *Assume also that* $R = 0$ *and* $C$ *is square and invertible. Then:*

(*a*) *The optimal estimator gain is constant and in particular* $K_k = I$ *if* $C = I$.

(*b*) *The optimal control is linear and is*

$$u_k^* = -[\Omega_{\bar{N}\bar{\Theta}}(S_{k+1}, H_k)]^{-1}(\overline{N})B^T S_{k+1} A x_{k\,|\,k} = L_k x_{k\,|\,k} \tag{52}$$

*where*

$$\begin{aligned}\Omega_{\bar{N}\bar{\Theta}}(S_{k+1}, H_k) = \sum_{\substack{I \in 2^{\Im} \\ I_\theta \in 2^{\Im}}} \Bigg[ & \Bigg( \prod_{\substack{i \in I \\ j \in I_\Theta}} \bar{\nu}'_i \bar{\theta}'_j \prod_{\substack{i \notin I \\ j \notin I_\Theta}} (1 - \bar{\theta}'_j)(1 - \bar{\nu}'_i) \Bigg) \\ & \Big( N_I U_k N_I + N_I \Theta_{I_\Theta} B^T S_{k+1} B \Theta_{I_\Theta} N_I \\ & + (N_I - \overline{N})(I - \Theta_{I_\Theta}) B^T H_{k+1} B (I - \Theta_{I_\Theta})(N_I - \overline{N}) \\ & + 2 u_k^T \overline{N}(I - \Theta_{I_\Theta}) B^T S_{k+1} B \Theta_{I_\Theta} N_I u \\ & + \overline{N}(I - \Theta_{I_\Theta}) B^T S_{k+1} (I - \Theta_{I_\Theta}) \overline{N} B \Big) \Bigg]\end{aligned}$$

*and* $\Theta_I = N_I$. *Matrices* $T_k, S_k, D_k$ *remain the same defined in equations* (34)–(36) *and* $H_k = T_k - S_k$.

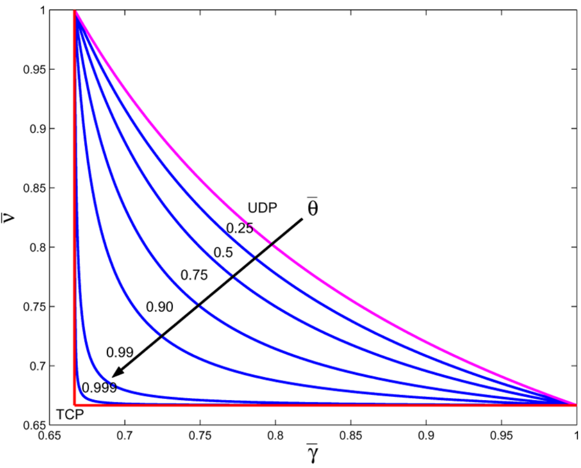
## 5 Example

This section is devoted to show how the probability to receive an acknowledgement from the actuators affects the stability regions of the LQG controller. In order to exploit necessary and sufficient conditions arising from equation (40), we will consider a very simple system with an invertible and square $B$:

$$\begin{aligned} x(t+1) &= 3x(t) + u(t) + w(t) \\ y(t) &= x(t) \end{aligned} \tag{53}$$

with $Q = 1$. Figure 3 shows the different stability regions with respect to $\bar{\nu}$ and $\bar{\gamma}$, parameterised by the acknowledgement probability $\bar{\theta}$. In particular it is possible to show that, as $\bar{\theta} \to 1$ the stability region converges to the one computed for the TCP-like case.

**Figure 3** Region of stability relative to measurement packet arrival probability $\bar{\gamma}$, and the control packet arrival probability $\bar{\nu}$, parameterised into the acknowledgement packet arrival probability $\bar{\theta}$ (see online version for colours)




## 6 Conclusions

In this paper we analysed a generalised version of the LQG control problem in the case where both observation and control packets may be lost during transmission over a communication channel. This situation arises frequently in distributed systems where sensors, controllers and actuators reside in different physical locations and have to rely on data networks to exchange information. In this context controller design heavily depends on the communication protocol used. In fact, in TCP-like protocols, acknowledgements of successful transmissions of control packets are provided to the controller, while in UDP-like protocols, no such feedback is provided. In the first case, the separation principle holds and the optimal control is a linear function of the state. As a consequence, controller and estimator design problems are decoupled. UDP-like protocols present a much more complex problem. We have shown that the even partial lack of acknowledgement of control packets results in the failure of the separation principle. Estimation and control are now intimately coupled. We have shown that the LQG optimal control is, in general, nonlinear in the estimated state. In the particular case, where we have access to full state information, the optimal controller is linear in the state. In this particular case we could show how the partial presence of acknowledgement increases the stability range of the overall system. The stability range for the Quasi-TCP-like case converges to the TCP-like one with deterministic acknowledgements as the arrival rate for the acknowledgement packets tends to one, as shown in the illustrative example.

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## Appendix: Proofs

### 6.1 Infinite horizon

**Lemma 1:** *Let $S, T \in \mathbb{M} = \{M \in \mathbb{R}^{n\times n} \mid M \geq 0\}$. Consider the operators $\Phi^S(S,T)$, and $\Phi^T(S,T)$ as defined in equations (34) and (35), and consider the sequences $S_{k+1} = \Phi^S(S_k, T_k)$ and $T_{k+1} = \Phi^T(S_k, T_k)$. Consider $L^*_{S,T} = -(U + B'((1-\bar{\alpha})S + \bar{\alpha}T)B)^{-1}B'SA$ and the operator:*

$$\Upsilon(S,T,L) = \left(1 - \frac{\bar{\nu}}{1-\bar{\alpha}}\right) A'SA + W + \frac{\bar{\nu}}{1-\bar{\alpha}}(A + (1-\bar{\alpha})BL)'S(A + (1-\bar{\alpha})BL) + \bar{\nu}L'UL + \bar{\nu}\bar{\alpha}L'B'TBL.$$

*Then the following facts are true:*

- $\Phi^S(S,T) = \min_L \Upsilon(S,T,L)$.
- $0 \leq \Upsilon(S,T,L^*_{S,T}) = \Phi^S(S,T) \leq \Upsilon(S,T,L)\ \forall L$.
- *If $S_{k+1} > S_k$ and $T_{k+1} > T_k$, then $S_{k+2} > S_{k+1}$ and $T_{k+2} > T_{k+1}$.*
- *If the pair $(A, W^{1/2})$ is observable and $S = \Phi^S(S,T)$ and $T = \Phi^T(S,T)$, then $S > 0$ and $T > 0$.*

*Proof*:

(a) If $U$ is invertible then it is easy to verify by direct substitution that

$$\Upsilon(S,T,L) = \Phi^S(S,T) + \bar{\nu}(L - L^*_{S,T})'(U + B'((1-\bar{\alpha})S + \bar{\alpha}T)B)(L - L^*_{S,T}) \geq \Phi^S(S,T).$$

(b) The nonnegativeness follows from the observation that $\Upsilon(S,T,L)$ is a sum of positive semi-definite matrices. In fact $\left(1 - \frac{\bar{\nu}}{1-\bar{\alpha}}\right) \geq 0$ and $0 \leq \bar{\alpha} \leq 1$. The equality $\Upsilon(S,T,L^*_{S,T}) = \Phi^S(S,T)$ can be verified by direct substitution. The last inequality follows directly from Fact (b).

(c)
$$\begin{aligned} S_{k+2} &= \Phi^S(S_{k+1}, T_{k+1}) = \Upsilon(S_{k+1}, T_{k+1}, L^*_{S_{k+1},T_{k+1}}) \\ &\geq \Upsilon(S_k, T_k, L^*_{S_{k+1},T_{k+1}}) \geq \Upsilon(S_k, T_k, L^*_{S_k,T_k}) \\ &= \Phi^S(S_k, T_k) = S_{k+1} \\ T_{k+2} &= \Phi^T(S_{k+1}, T_{k+1}) \geq \Phi^T(S_k, T_k) = T_{k+1}. \end{aligned} \tag{54}$$

(d) First observe that $S = \Phi^S(S, T) \geq 0$ and $T = \Phi^T(S, T) \geq 0$. Thus, to prove that $S, T > 0$, we only need to establish that $S, T$ are nonsingular. Suppose they are singular, then there exist vectors $0 \neq v_s \in \mathcal{N}(S)$ and $0 \neq v_t \in \mathcal{N}(T)$, i.e., $Sv_s = 0$ and $Tv_t = 0$, where $\mathcal{N}(\cdot)$ indicates the null space. Then

$$\begin{aligned} 0 &= v_s' S v_s = v_s' \Phi^S(S, T) v_s = v_s' \Upsilon(S, T, L^*_{S,T}) v_s \\ &= \left(1 - \frac{\bar{\nu}}{1 - \bar{\alpha}}\right) v_s' A' S A v_s + v_s' W v_s + \star \end{aligned} \tag{55}$$

where $\star$ indicates other terms. Since all the terms are positive semi-definite matrices, this implies that they must be zero:

$$\begin{aligned} &v_s' A' S A v_s = 0 \Longrightarrow S A v_s = 0 \Longrightarrow A v_s \in \mathcal{N}(S) \\ &v_s' W v_s = 0 \Longrightarrow W^{1/2} v_s = 0. \end{aligned} \tag{56}$$

As a result, the null space $\mathcal{N}(S)$ is $A$-invariant. Therefore, $\mathcal{N}(S)$ contains an eigenvector of $A$, i.e., there exists $u \neq 0$ such that $Su = 0$ and $Au = \sigma u$. As before, we conclude that $Wu = 0$. This implies (using the PBH test) that the pair $(A, W^{1/2})$ is not observable, contradicting the hypothesis. Thus, $\mathcal{N}(S)$ is empty, proving that $S > 0$. The same argument can be used to prove that also $T > 0$. □

**Lemma 2:** *Consider the following operator:*

$$\Upsilon(S, T, L) = A'SA + W + 2\bar{\nu} A'SBL + \bar{\nu} L'(U + B'((1 - \bar{\alpha})S + \bar{\alpha} T)B)L. \tag{57}$$

*Assume that the pairs* $(A, W^{1/2})$ *and* $(A, B)$ *are observable and controllable, respectively. Then the following statements are equivalent:*

- *There exist a matrix* $\widetilde{L}$ *and positive definite matrices* $\widetilde{S}$ *and* $\widetilde{T}$ *such that:*

$$\widetilde{S} > 0, \quad \widetilde{T} > 0, \quad \widetilde{S} = \Upsilon(\widetilde{S}, \widetilde{T}, \widetilde{L}), \quad \widetilde{T} = \Phi^T(\widetilde{S}, \widetilde{T}).$$

- *Consider the sequences:*

$$S_{k+1} = \Phi^S(S_k, T_k), \quad T_{k+1} = \Phi^T(S_k, T_k)$$

*where the operators* $\Phi^S(\cdot), \Phi^T(\cdot)$ *are defined in equations* (34) *and* (35). *For any initial condition* $S_0, T_0 \geq 0$, *we have*

$$\lim_{k\to\infty} S_k = S_\infty, \quad \lim_{k\to\infty} T_k = T_\infty$$

*and $S_\infty, T_\infty > 0$ are the unique positive definite solution of the following equations*

$$S_\infty = \Phi^S(S_\infty, T_\infty), \quad T_\infty = \Phi^T(S_\infty, T_\infty)$$

*Proof*: See Schenato et al. (2007). □

**Lemma 3:** *Let us consider the fixed points of equations* (34) *and* (35)*, i.e., $S = \Phi^S(S,T)$, $T = \Phi^T(S,T)$ where $S,T \geq 0$. Let $A$ be unstable. A necessary condition for the existence of a solution is*

$$\begin{aligned} &1 - |A|^2 \left(1 - \frac{\bar\nu}{(1-\bar\alpha) + \bar\alpha \frac{\bar\gamma |A|^2}{1-(1-\bar\gamma)|A|^2}}\right) \geq 0 \\ &\bar\gamma > 1 - \frac{1}{|A|^2} \end{aligned} \tag{58}$$

*where $|A| \triangleq \max_i |\lambda_i(A)|$ is the largest eigenvalue of the matrix $A$.*

*Proof*: To prove the necessity condition it is sufficient to show that there exist some initial conditions $S_0, T_0 \geq 0$ for which the sequences $S_{k+1} = \Phi^S(S_k,T_k), T_{k+1} = \Phi^T(S_k,T_k)$ are unbounded, i.e., $\lim_{k\to\infty} S_k = \lim_{k\to\infty} T_k = \infty$. To do so, suppose that at some time-step $k$ we have $S_k \geq s_k vv'$ and $T_k \geq t_k vv'$, where $s_k, t_k > 0$, and $v$ is the eigenvector corresponding to the largest eigenvalue of $A'$, i.e., $A'v = \lambda_{\max} v$ and $|\lambda_{\max}| = |A'| = |A|$. Then, we have:

$$\begin{aligned} S_{k+1} &= \Phi^S(S_k,T_k) \geq \Phi^S(s_k vv', t_k vv') \\ &= \min_L \Upsilon(s_k vv', t_k vv', L) \\ &= \min_L (s_k A'vv'A + W + 2 s_k \bar\nu A'vv'BL \\ &\qquad + \bar\nu L'(U + B'((1-\bar\alpha)s_k vv' + \bar\alpha t_k vv')B)L) \\ &\geq \min_L (s_k |A|^2 vv' + 2 s_k \bar\nu \lambda_{\max} vv'BL + \bar\nu L'B'((1-\bar\alpha)s_k vv' + \bar\alpha t_k vv')BL) \\ &= \min_L \left( s_k |A|^2 vv' - \frac{|A|^2 \bar\nu s_k^2}{\xi_k} vv' \right. \\ &\qquad \left. + \bar\nu \xi_k \left(\lambda_{\max} s_k^2 I + \frac{1}{\xi_k} BL\right)' vv' \left(\lambda_{\max} s_k^2 I + \frac{1}{\xi_k} BL\right)\right) \\ &\geq s_k |A|^2 vv' - \frac{|A|^2 \bar\nu s_k^2}{(1-\bar\alpha)s_k + \bar\alpha t_k} vv' \\ &= |A|^2 s_k \left(1 - \frac{\bar\nu s_k}{(1-\bar\alpha)s_k + \bar\alpha t_k}\right) vv' \\ &= s_{k+1} vv', \end{aligned}$$

where $I$ is the identity matrix and $\xi_k = (1-\bar{\alpha})s_k + \bar{\alpha}t_k$. Similarly, we have:

$$\begin{aligned}
T_{k+1} &= \Phi^T(S_k, T_k) \geq \Phi^T(s_k vv', t_k vv') \\
&= (1-\bar{\gamma})t_k A'vv'A + \bar{\gamma}s_k A'vv'A + W \\
&\geq (1-\bar{\gamma})t_k |A^2| vv' + \bar{\gamma}s_k |A|^2 vv' \\
&= |A|^2((1-\bar{\gamma})t_k + \bar{\gamma}s_k)vv' \\
&= t_{k+1}vv'.
\end{aligned}$$

We can summarise the previous results as follows:

$$\begin{aligned}
&(S_k \geq s_k vv', T_k \geq t_k vv') \Rightarrow (S_{k+1} \geq s_{k+1}vv', T_{k+1} \geq t_{k+1}vv') \\
&s_{k+1} = \phi^s(s_k, t_k) = |A|^2 s_k \left(1 - \frac{\bar{\nu}s_k}{(1-\bar{\alpha})s_k + \bar{\alpha}t_k}\right), \\
&t_{k+1} = \phi^t(s_k, t_k) = |A|^2((1-\bar{\gamma})t_k + \bar{\gamma}s_k).
\end{aligned}$$

Let us define the following sequences:

$$\begin{aligned}
&S_{k+1} = \Phi^S(S_k, T_k), \quad T_{k+1} = \Phi^T(S_k, T_k), \quad S_0 = T_0 = vv' \\
&s_{k+1} = \phi^s(s_k, t_k), \quad t_{k+1} = \phi^t(s_k, t_k), \quad s_0 = t_0 = 1 \\
&\widetilde{S}_k = s_k vv', \quad \widetilde{T}_k = t_k vv'.
\end{aligned}$$

From the previous derivations we have that $S_k \geq \tilde{S}_k, T_k \geq \tilde{T}_k$ for all time instants $k$. Therefore, in order to establish necessary conditions, we need to study the divergence of the scalar sequences $s_k, t_k$. It should be evident that also the operators $\phi^s(s,t), \phi^t(s,t)$ are monotonic in their arguments and that the only fixed points of $s = \phi^s(s,t), t = \phi^t(s,t)$ are $s = t = 0$. Therefore we need to establish when the origin is an unstable equilibrium point, since in this case $\lim_{k\to\infty} s_k, t_k = \infty$. Notice that $t = \phi^t(s,t)$ can be written as:

$$\begin{aligned}
t &= \Phi^t(s,t) = (1-\bar{\gamma})|A|^2 t + \bar{\gamma}|A|^2 s \\
&= \psi(s) = \frac{\bar{\gamma}|A|^2 s}{1-(1-\bar{\gamma})|A|^2}
\end{aligned}$$

with the additional constraint $1-(1-\bar{\gamma})A^2 > 0$. A necessary condition for the stability of the origin is that the origin of restricted map $z_{k+1} = \phi(z_k, \psi(z_k))$ is also stable. The restricted map is given by:

$$\begin{aligned}
z_{k+1} &= |A|^2 z_k \left(1 - \bar{\nu}\frac{z_k}{(1-\bar{\alpha})z_k + \bar{\alpha}\frac{\bar{\gamma}|A|^2}{1-(1-\bar{\gamma})A^2}z_k}\right) \\
&= |A|^2 \left(1 - \frac{\bar{\nu}}{(1-\bar{\alpha}) + \bar{\alpha}\frac{\bar{\gamma}|A|^2}{1-(1-\bar{\gamma})A^2}}\right) z_k.
\end{aligned}$$

This is a linear map and it is stable if and only if the term inside the parenthesis is smaller than unity, i.e.,

$$1 - |A|^2 \left( 1 - \frac{\bar{\nu}}{(1-\bar{\alpha}) + \bar{\alpha} \frac{\bar{\gamma}|A|^2}{1-(1-\bar{\gamma})|A|^2}} \right) > 0 \tag{59}$$

which concludes the lemma. □

**Lemma 4:** *Let us consider the fixed points of equations* (34) *and* (35), *i.e.,* $S = \Phi^S(S,T)$, $T = \Phi^T(S,T)$ *where* $S,T \geq 0$. *Let* $A$ *be unstable,* $(A, W^{1/2})$ *observable and* $B$ *square and invertible. Then a sufficient condition for the existence of a solution is*

$$\begin{aligned} &1 - |A|^2 \left( 1 - \frac{\bar{\nu}}{(1-\bar{\alpha}) + \bar{\alpha} \frac{\bar{\gamma}|A|^2}{1-(1-\bar{\gamma})|A|^2}} \right) > 0 \\ &\bar{\gamma} > 1 - \frac{1}{|A|^2} \end{aligned} \tag{60}$$

*where* $|A| \triangleq \max_i |\lambda_i(A)|$ *is the largest eigenvalue of the matrix* $A$.

*Proof*: The proof is constructive. In fact we find a control feedback gain $\widetilde{L}$ that satisfies the conditions stated in Theorem 2(a). Let $\widetilde{L} = -\eta B^{-1} A$ where $\eta > 0$ is a positive scalar that is to be determined. Also consider $S = sI$, $T = tI$, where $I$ is the identity matrix and $s, t > 0$ are positive scalars. Then, we have

$$\begin{aligned} \Upsilon(sI, tI, \widetilde{L}) &= A'sA + W - 2\bar{\nu}\eta A'sA + \bar{\nu} A' B^{-'} U B^{-1} A + \bar{\nu}\eta^2 A'((1-\bar{\alpha})s + \bar{\alpha} t)A \\ &\leq |A|^2 (s - 2\bar{\nu} s \eta + \bar{\nu}((1-\bar{\alpha})s + \bar{\alpha} t)\eta^2) I + wI \\ &= \varphi^s(s,t,\eta) I \end{aligned} \tag{61}$$

$$\begin{aligned} \Phi^T(sI, tI) &= \bar{\gamma} A'sA + (1-\bar{\gamma}) A'tA + W \\ &\leq (\bar{\gamma}|A|^2 s + (1-\bar{\gamma})|A|^2 t) I + wI \\ &\leq \varphi^t(s,t) I \end{aligned} \tag{62}$$

where $w = |W + \bar{\nu} A' B^{-'} U B^{-1} A| > 0$ and $I$ is the identity matrix. Let us consider the following scalar operators and sequences:

$$\begin{aligned} \varphi^s(s,t,\eta) &= |A|^2 (1 - 2\bar{\nu}\eta + \bar{\nu}(1-\bar{\alpha})\eta^2) s + \bar{\nu}\bar{\alpha}\eta^2 t + w \\ \varphi^t(s,t) &= \bar{\gamma}|A|^2 s + (1-\bar{\gamma})|A|^2 t + w \\ s_{k+1} &= \varphi^s(s_k, t_k, \eta), \quad t_{k+1} = \varphi^t(s_k, t_k), \quad s_0 = t_0 = 0. \end{aligned}$$

The operators are clearly monotonically increasing in $s, t$, and since $s_1 = \varphi^s(s_0, t_0, \eta) = w \geq s_0$ and $t_1 = \varphi^t(s_0, t_0) = w \geq t_0$, it follows that the sequences $s_k, t_k$ are monotonically increasing. If these sequences are bounded, then they must converge to $\tilde{s}, \tilde{t}$. Therefore $s_k, t_k$ are bounded if and only if there exist $\tilde{s}, \tilde{t} > 0$ such that $\tilde{s} = \varphi^s(\tilde{s}, \tilde{t}, \eta)$ and $\tilde{t} = \varphi^t(\tilde{s}, \tilde{t})$. Let us find the fixed points:

$$\tilde{t} = \varphi^t(\tilde{s}, \tilde{t}) \Rightarrow$$
$$\tilde{t} = \frac{\bar{\gamma}|A|^2}{1-(1-\bar{\gamma})|A|^2}\tilde{s} + w_t$$

where $w_t \triangleq \frac{w}{1-(1-\bar{\gamma})|A|^2} > 0$, and we must have $1-(1-\bar{\gamma})|A|^2 > 0$ to guarantee that $\tilde{t} > 0$. Substituting back into the operator $\varphi^s$, we have:

$$\begin{aligned}\tilde{s} &= |A|^2(1-2\bar{\nu}\eta + \bar{\nu}(1-\bar{\alpha})\eta^2)\tilde{s} + \bar{\nu}\bar{\alpha}\eta^2\frac{\bar{\gamma}|A|^2}{1-(1-\bar{\gamma})|A|^2}\tilde{s} + \bar{\nu}\bar{\alpha}\eta^2 w_t + w \\ &= |A|^2\left(1-2\bar{\nu}\eta + \bar{\nu}\left((1-\bar{\alpha}) + \frac{\bar{\gamma}\bar{\alpha}|A|^2}{1-(1-\bar{\gamma})|A|^2}\right)\eta^2\right)\tilde{s} + w(\eta) \\ &= a(\eta)\tilde{s} + w(\eta),\end{aligned}$$

where $w(\eta) \triangleq \bar{\nu}\bar{\alpha}\eta^2 w_t + w > 0$. For a positive solution $\tilde{s}$ to exist, we must have $a(\eta) < 1$. Since $a(\eta)$ is a quadratic function of the free parameter $\eta$, we can try to increase the range of existence of solutions by choosing $\eta^* = \operatorname{argmin}_\eta a(\eta)$, which can be found by solving $\frac{\mathrm{d}a}{\mathrm{d}\eta}(\eta^*) = 0$ and is given by:

$$\eta^* = \frac{1}{(1-\alpha) + \frac{\gamma\alpha|A^2|}{1-(1-\gamma)A^2}}. \tag{63}$$

Therefore a sufficient condition for the existence of solutions is given by:

$$a(\eta^*) < 1$$
$$|A|^2\left(1 - \frac{\nu}{\left((1-\alpha) + \frac{\gamma\alpha|A|^2}{1-(1-\gamma)A^2}\right)}\right) < 1$$

which is the same bound encountered in the computation of the necessary condition of convergence in Lemma 3.

If this condition is satisfied then $\lim_{k\to\infty} s_k = \tilde{s}$ and $\lim_{k\to\infty} t_k = \tilde{t}$. Let us consider now the sequences $\bar{S}_k = s_k I$, $\bar{T}_k = t_k I$, $S_{k+1} = \Upsilon(S_k, T_k, \tilde{L})$ and $T_{k+1} = \Phi^T(S_k, T_k)$, where $\tilde{L} = -\eta^* B^{-1} A$, $S_0 = T_0 = 0$, and $s_k, t_k$ defined above.

These sequences are all monotonically increasing. From equations (61) and (62) it follows that $(S_k \leq s_k I, T_k \leq t_k I) \Rightarrow (S_{k+1} =\leq s_{k+1} I, T_{k+1} \leq t_k I)$. Since this is verified for $k = 0$ we can claim that $S_k < \tilde{s}I$ and $T_k < \tilde{t}I$ for all $k$. Since $S_k, T_k$ are monotonically increasing and bounded, then they must converge to positive semidefinite matrices $\widetilde{S}, \widetilde{T} \geq 0$, solutions of the equations $\widetilde{S} = \Upsilon(\widetilde{S}, \widetilde{T}, \widetilde{L})$ and $\widetilde{T} = \Phi^T(\widetilde{S}, \widetilde{T})$. Since by hypothesis the pair $(A, W^{1/2})$ is observable, using similar arguments as in Lemma 1(d), it is possible to show that $\widetilde{S}, \widetilde{T} > 0$. Therefore $\widetilde{S}, \widetilde{T}, \widetilde{L}$ satisfy the conditions of Theorem 2(a), from which statement (b) follows. This implies that the sufficient conditions derived here guarantee the claim of the lemma. □

### 6.2 Multichannel optimal estimator

**Lemma 5:** *Given a system in form* (1), *the associated error covariance one step prediction is equation* (44).

*Proof*: By using equation (43), we can compute the associated error covariance one-step prediction in the following way:

$$\begin{aligned} P_{k+1\,|\,k} &= E[e_{k+1\,|\,k}e^T_{k+1\,|\,k}\,|\,E_k, \Theta_k, \Theta_k N_k] \\ &= E[(Ae_{k\,|\,k} + B(1-\Theta_k)(N_k - \overline{N})u_k + \omega_k) \\ &\quad (Ae_{k\,|\,k} + B(1-\Theta_k)(N_k - \overline{N})u_k + \omega_k)^T] \\ &= AP_{k\,|\,k}A^T + Q + E[B(1-\Theta_k)(N_k - \overline{N})u_k u_k^T(N_k - \overline{N})(1-\Theta_k)B^T]. \end{aligned}$$

Then:

$$\begin{aligned} P_{k+1\,|\,k} &= AP_{k\,|\,k}A^T + Q + B(1-\Theta_k)(E[(N_k - \overline{N})u_k u_k^T(N_k - \overline{N})])(1-\Theta_k)B^T \\ &= AP_{k\,|\,k}A^T + Q + B(1-\Theta_k)(\Psi(u_k, \overline{N}))(1-\Theta_k)B^T. \end{aligned}$$

□

### 6.3 Theorem 4 Proof

If $\operatorname{rank}(g_i C) = n, i = 1, \ldots, m'$ and $R = 0$, then at each time instant $k$ two situations are possible

- $\Gamma_k C = 0$ and no correction is possible, i.e., $P_{k+1|k+1} = P_{k+1|k}$.
- $\operatorname{Rank}(\Gamma_k^m C) = n$. Since $R = 0$, in this case we can select $n$ measurement corresponding to $n$ independent lines of $\Gamma_k C$. Simple algebraic manipulations shows that this is equivalent to saying that, any time a packet arrives, the state $x_k$ is known and $\Gamma_k C = I$.

The above condition is equivalent to having the following system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k^a + \omega_k, \\ u_k^a &= N_k u_k + [I_{m\times m} - N_k]\, u_k^l, \\ y(k) &= \gamma_k(x_k), \end{aligned} \tag{64}$$

where

$$\gamma_k = 1 - \prod_{i=1}^{p'} (1 - \gamma_i').$$

For such a system, the estimator equations simplify as follows:

$$\begin{aligned} K_{k+1} &= I \\ P_{k+1\,|\,k} &= AP_{k\,|\,k}A^T + Q + B(1-\Theta_k)(\Psi(u_k, \overline{N}))(1-\Theta_k)B^T \end{aligned}$$

$$
\begin{aligned}
P_{k+1\,|\,k+1} &= (1-\gamma_{k+1})P_{k+1\,|\,k} \\
&= (1-\gamma)(AP_{k\,|\,k}A^T + Q + B(1-\Theta_k)(\Psi(u_k,\overline{N}))(1-\Theta_k)B^T) \\
E[P_{k+1\,|\,k+1}\,|\,E_k] &= (1-\bar{\gamma})(AP_{k\,|\,k}A^T + Q) \\
&\quad + (1-\bar{\gamma})E[B(1-\Theta_k)(\Psi(u_k,\overline{N}))(1-\Theta_k)B^T],
\end{aligned} \tag{65}
$$

where in the last equation the independence of $E_k, \Gamma_{k+1}, N_k, Q_k$ is exploited. Let us define:

$$
\begin{aligned}
\Phi_{\bar{N}\bar{\Theta}}(u_k u_k^T) &\triangleq E[(1-\Theta_k)(\Psi(u_k,\overline{N}))(1-\Theta_k)] \\
&= \sum_{I\in 2^{\Im}} \sum_{I_\theta\in 2^{\Im}} \Bigg[\left(\prod_{j\in I_\Theta}\theta'_j \prod_{j\notin I_\Theta}(1-\theta'_j)\prod_{i\in I}\nu_i\prod_{i\notin I}(1-\nu_i)\right) \\
&\qquad ((1-\Theta_{I_\Theta})(N_I-\overline{N})u_k u_k^T(N_I-\overline{N})(1-\Theta_{I_\Theta}))\Bigg],
\end{aligned} \tag{66}
$$

where $I_\Theta \subseteq \Im$ and $\Theta_I \equiv N_i$. Equation (65) becomes

$$
E[P_{k+1\,|\,k+1}\,|\,E_k] = (1-\bar{\gamma})(AP_{k\,|\,k}A^T + Q + B\Phi_{\bar{N}\bar{\Theta}}B^T). \tag{67}
$$

Following the classical dynamic programming approach to optimal control, we claim that the value function $V_k^*(x_k)$ can be written as follows:

$$
\begin{aligned}
V_k(x_k) &= \hat{x}_{k\,|\,k}^T S_k \hat{x}_{k\,|\,k} + \mathrm{trace}(T_k P_{k\,|\,k}) + \mathrm{trace}(D_k Q) \\
&= E[x_{k\,|\,k}^T S_k x_{k\,|\,k}] + \mathrm{trace}(H_k P_{k\,|\,k}) + \mathrm{trace}(D_k Q)
\end{aligned} \tag{68}
$$

for each $k = N, \ldots, 0$, with $H_k \triangleq T_k - S_k$.

This is clearly true for $k = N$, in fact we have:

$$
\begin{aligned}
V_N(x_N) &= E[x_N^T W_N x_N\,|\,E_N] \\
&= \hat{x}_{N\,|\,N}^T W_N \hat{x}_{N\,|\,N} + \mathrm{trace}(W_N P_{N\,|\,N}).
\end{aligned}
$$

The statement is satisfied by setting $S_N = T_N = W_N, D_N = 0$. Let us suppose that equation (68) is true for $k+1$ and let us show by induction it holds true for $k$:

$$
\begin{aligned}
V_k(x_k) &= \min_{u_k}\ E[x_k^T W_k x_k + u_k^T N_k U_k N_k u_k + V_{k+1}(x_{k+1})\,|\,E_k] \\
&= \min_{u_k} E[x_k^T W_k x_k + u_k^T N_k U_k N_k u_k\,|\,E_k] \\
&\quad + E[x_{k+1}^T S_{k+1} x_{k+1} + \mathrm{trace}(H_{k+1}P_{k+1\,|\,k+1})\,|\,E_k] + \mathrm{trace}(D_{k+1}Q) \\
&= \min_{u_k} E[x_k^T W_k x_k\,|\,E_k] + E[u_k^T N_k U_k N_k u_k] + \mathrm{trace}(D_{k+1}Q) \\
&\quad + \mathrm{trace}(H_{k+1}((1-\bar{\gamma})(AP_{k\,|\,k}A^T + Q + B\Phi_{\bar{N}\bar{\Theta}}(u_k u_k^T)B^T))) \\
&\quad + E[(Ax_k + B\Theta_k N_k u_k + (I-\Theta_k)\overline{N}Bu_k)^T S_{k+1} \\
&\qquad (Ax_k + B\Theta_k N_k u_k + (I-\Theta_k)\overline{N}Bu_k)|E_k].
\end{aligned}
$$

Let us focus on the term:

$$\begin{aligned}
&\mathrm{trace}(H_{k+1}((B\Phi_{\bar{N}\bar{\Theta}}(u_k u_k^T)B^T))) \\
&\quad = \mathrm{trace}\left(\sum_{\substack{I\in 2^{\Im}\\ I_\theta \in 2^{\Im}}}\left[\left(\prod_{\substack{i\in I\\ j\in I_\Theta}}\bar{\nu}_i'\bar{\theta}_j' \prod_{\substack{i\notin I\\ j\notin I_\Theta}}(1-\bar{\theta}_j')(1-\bar{\nu}_i')\right)(H_{k+1}B(1-\Theta_{I_\Theta})\right.\right. \\
&\qquad\qquad \left.\left.(N_I-\overline{N})u_k u_k^T(N_I-\overline{N})(1-\Theta_{I_\Theta})B^T)\right]\right) = u_k^T(\Xi_{\bar{N}\bar{\Theta}}(H_{k+1}))u_k.
\end{aligned}$$

Then

$$\begin{aligned}
V_k(x_k) = \min_{u_k} & E[x_k^T(W_k)x_k \mid E_k] + E[u_k^T(N_kU_kN_k + (1-\bar{\gamma})\Xi_{\bar{N}\bar{\Theta}}(H_{k+1}))u_k] \\
&+ \mathrm{trace}(H_{k+1}((1-\bar{\gamma})(AP_{k\,|\,k}A^T+Q))) + \mathrm{trace}(D_{k+1}Q), \\
&+ E[(Ax_k + B\Theta_kN_ku_k + (I-\Theta_k)\overline{N}Bu_k)^T S_{k+1} \\
&\quad (Ax_k + B\Theta_kN_ku_k + (I-\Theta_k)\overline{N}Bu_k)E_k]
\end{aligned}$$

which becomes

$$\begin{aligned}
V_k(x_k) = \min_{u_k} & E[x_k^TW_kx_k + x_k^TA^TS_{k+1}Ax_k \mid E_k] + \mathrm{trace}(D_{k+1}Q) \\
&+ \mathrm{trace}(H_{k+1}(1-\overline{\Gamma})(AP_{k\,|\,k}A^T+Q)) + E[u_k^T(N_kU_kN_k \\
&+ N_k\Theta_kB^TS_{k+1}B\Theta_kN_k + 2u_k^T\overline{N}(I-\Theta_k)B^TS_{k+1}B\Theta_kN_ku) \\
&+ (\Xi_{\bar{N}\bar{\Theta}}(H_{k+1}) + \overline{N}(I-\Theta_k)B^TS_{k+1}(I-\Theta_k)\overline{N}B)u_k] \\
&+ 2u_k^T(N_k\Theta_k + \overline{N}(I-\Theta_k))B^TS_{k+1}Ax_{k\,|\,k}.
\end{aligned}$$

Let us introduce another operator

$$\begin{aligned}
\Omega_{\bar{N}\bar{\Theta}}(S_{k+1},H_k) = \sum_{\substack{I\in 2^{\Im}\\ I_\theta \in 2^{\Im}}}&\left[\left(\prod_{\substack{i\in I\\ j\in I_\Theta}}\bar{\nu}_i'\bar{\theta}_j' \prod_{\substack{i\notin I\\ j\notin I_\Theta}}(1-\bar{\theta}_j')(1-\bar{\nu}_i')\right)\right. \\
&(N_IU_kN_I + N_I\Theta_{I_\Theta}B^TS_{k+1}B\Theta_{I_\Theta}N_I + \Xi_{\bar{N}\bar{\Theta}}(H_{k+1}) \\
&+ 2u_k^T\overline{N}(I-\Theta_{I_\Theta})B^TS_{k+1}B\Theta_{I_\Theta}N_Iu \\
&\left.+\overline{N}(I-\Theta_{I_\Theta})B^TS_{k+1}(I-\Theta_{I_\Theta})\overline{N}B)\right].
\end{aligned}$$

We can write

$$V_k(x_k) = \min_{u_k} E[x_k^T W_k x_k + x_k^T A^T S_{k+1} A x_k^{|} E_k]$$
$$+ 2u_k^T (\overline{N}\,\overline{\Theta} + \overline{N}(I - \overline{\Theta})) B^T S_{k+1} A x_{k\,|\,k} + \text{trace}(D_{k+1}Q) + \text{trace}(S_{k+1}Q)$$
$$+ \text{trace}(H_{k+1}(1-\bar{\gamma})(AP_{k\,|\,k}A^T + Q)) + u_k^T \Omega_{\bar{N}\bar{\Theta}}(S_{k+1}, H_k) u_k.$$

Since $V_k(x_k)$ is a convex quadratic function w.r.t. $u_k$, the minimiser is the solution of $\partial V_k(x_k)/\partial u_k = 0$ which is given by:

$$u_k^* = -[\Omega_{\bar{N}\bar{\Theta}}(S_{k+1}, H_k)]^{-1}(\overline{N}) B^T S_{k+1} A x_{k\,|\,k} = L_k x_{k\,|\,k} \tag{69}$$

which is linear function of the estimated state $x_{k\,|\,k}$. Substituting back into the value function we get:

$$V_k(x_k) = \min_{u_k} E[x_k^T W_k x_k + x_k^T A^T S_{k+1} A x_k \,|\, E_k] + x_{k\,|\,k}^T (A^T S_{k+1} B \overline{N} L_k) x_{k\,|\,k}$$
$$+ \text{trace}(H_{k+1}(1-\bar{\gamma})(AP_{k\,|\,k}A^T + Q)) + \text{trace}(D_{k+1}Q),$$

i.e.,

$$V_k(x_k) = \min_{u_k} x_k^T (W_k + A^T S_{k+1} A + A^T S_{k+1} B \overline{N} L_k) x_{k\,|\,k}$$
$$+ \text{trace}(H_{k+1}(1-\bar{\gamma})(AP_{k\,|\,k}A^T + Q)) + \text{trace}(D_{k+1}Q)$$
$$+ \text{trace}((W_k + A^T S_{k+1} A) P_{k\,|\,k}). \tag{70}$$

From the last equation and from that fact that $H_k = T_k - S_k$ we can write the value function as in equation (68) if and only if the following equations are satisfied:

$$S_k = W_k + A^T S_{k+1} A + (A^T S_{k+1} B) \overline{N} L_k \tag{71}$$
$$T_k = (1-\bar{\gamma}) A^T T_{k+1} A + W_k + \bar{\gamma} A^T S_{k+1} A \tag{72}$$
$$D_k = D_{k+1} + (1-\bar{\gamma}) T_{k+1} + \bar{\gamma} S_{k+1}. \tag{73}$$

Note that, if $\bar{\theta} \to 0$, the result reverts to the UDP-like special case presented in Sinopoli et al. (2006).