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Boltzmanngasse 9
A-1090 Wien, Austria

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Maria Hoffmann–Ostenhof
Thomas Hoffmann–Ostenhof
Nikolai Nadirashvili

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CRITICAL SETS OF SMOOTH SOLUTIONS TO ELLIPTIC EQUATIONS IN DIMENSION 3

M. Hoffmann-Ostenhof[1]
T. Hoffmann-Ostenhof[2,3]
N. Nadirashvili[3]

Institut für Mathematematik, Universität Wien[1]
Institut für Theoretische Chemie, Universität Wien[2]
International Erwin Schrödinger Institute for Mathematical Physics[3]

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Abstract. Let $u \not\equiv \text{const}$ satisfy an elliptic equation $L_0 u \equiv \Sigma a_{i,j} D_{ij} u + \Sigma b_j D_j u = 0$ with smooth coefficients in a domain in $\mathbb{R}^3$. It is shown that the critical set $|\nabla u|^{-1}\{0\}$ has locally finite 1-dimensional Hausdorff measure. This implies in particular that for a solution $u \not\equiv 0$ of $(L_0 + c)u = 0$, with $c \in C^\infty$, the critical zero set $u^{-1}\{0\} \cap |\nabla u|^{-1}\{0\}$ has locally finite 1-dimensional Hausdorff measure.

1. Introduction and Main Results

Let $u \not\equiv 0$ be a continuous real valued solution of an elliptic equation in a domain $\Omega \subset \mathbb{R}^n$

$$Lu := \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{j=1}^{n} b_j D_j u + cu = 0. \tag{1.1}$$

Under various assumptions on the coefficients of L the zero sets of such solutions have been investigated by various authors. In particular it has been shown that under suitable assumptions on the coefficients the zero set $u^{-1}\{0\}$ has Hausdorff dimension $n-1$ [CF], and various interesting estimates for the $n-1$-dimensional Hausdorff measure of this set have been achieved [CM, DF1, DF2, DF3, HS, L, N]. In other works concerning $u^{-1}\{0\}$ the behaviour of u near a zero is investigated showing that u can be approximated by certain polynomials [Al,B, HOHO1, HOHO2, HOHON, R].

For all such investigations the critical zero set

$$\Sigma_0 := u^{-1}\{0\} \cap |\nabla u|^{-1}\{0\}$$

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plays an important role. But in contrast to the 2-dimensional case [Al, D, DF3, N], where Σ_0 consists of finitely many isolated points the existing results on Σ_0 in dimensions ≥ 3 [CF, HS, H] are not as explicit [Y]. Essentially it has been shown, if the coefficients of L are sufficiently smooth, then Σ_0 is countable $n-2$-rectifiable [HS, H, RS].

But even for the smooth case it is not known that Σ_0 has (locally) finite $(n-2)$-dimensional Hausdorff measure. In this paper we investigate this problem for the case $n = 3$.

In the following we assume that $a_{ij}, b_j, c \in C^\infty(\Omega)$ and without loss let $a_{ij} = a_{ji}, \quad \forall i, j$. Further we assume that L is strictly elliptic in $\Omega, \Omega \subset \mathbb{R}^3$.

We shall not consider (1.1) in general directly. Instead we first investigate the critical set $|\nabla u|^{-1}\{0\}$ of a solution u to (1.1) with $c \equiv 0$. The result about the critical zero set of a solution of (1.1) with $c \not\equiv 0$ will then be an immediate consequence.

Theorem 1.1. *Let $u \not\equiv$ const satisfy in Ω*

$$L_0 u := \sum_{i,j=1}^{3} a_{ij} D_{ij} u + \sum_{j=1}^{3} b_j D_j u = 0 \tag{1.2}$$

and let

$$\Sigma(u, \Omega) = \{x \in \Omega | |\nabla u|(x) = 0\}. \tag{1.3}$$

Then for every subset Ω' of Ω with $\Omega' \subset\subset \Omega$ the 1-dimensional Hausdorff measure of the critical set of u in Ω'

$$H^1(\Sigma(u, \Omega')) < \infty. \tag{1.4}$$

Therefrom we obtain the result on critical zero sets of solutions of (1.1), namely

Corollary 1.1. *Let $u \not\equiv 0$ satisfy in Ω*

$$(L_0 + c)u = 0. \tag{1.5}$$

Then $\forall \Omega'$, $\Omega' \subset\subset \Omega$

$$H^1(\Sigma_0(u, \Omega')) < \infty \tag{1.6}$$

where $\Sigma_0(u, \Omega') = \{x \in \Omega' | u(x) = |\nabla u|(x) = 0\}$.

Proof of Corollary 1.1. Given $x_0 \in \Omega$ there is a neighbourhood $U(x_0)$ and a $u_0 \in C^\infty(U(x_0))$ with $u_0 > 0$ and $Lu_0 = L_0 u_0 + cu_0 = 0$ (see e.g. [BJS], p. 228). Let $\mu = \frac{u}{u_0}$, then an easy calculation shows that

$$\sum_{i,j=1}^{3} a_{ij} D_{ij} \mu + (b - 2A\frac{\nabla u_0}{u_0} - 2\frac{\nabla u_0}{u_0})\nabla \mu = 0,$$

where $b = (b_1, b_2, b_3)$ and $A = (a_{ij})$. Hence by Theorem 1.1, $H^1(\Sigma(\mu, U(x_0)) < \infty$, and since clearly $\Sigma_0(u, U(x_0)) \subseteq \Sigma(\mu, U(x_0))$, (1.6) follows $\square$

Given the assumptions of Theorem 1.1 and given $x_0 \in \Omega$, then there is a linear coordinate transform T_{x_0} such that with $x = T_{x_0} y$ and $v(y) = u(x)$, v satisfies the transformed equation (1.1)

$$\sum_{i,j} A_{ij} D_{ij} v + \sum_{j} B_j D_j v = 0 \tag{1.7}$$

where $A_{ij}(y_0) = \delta_{ij}$, $y_0 = T_{x_0}^{-1} x_0$, and D_j, D_{ij} denote the partial derivatives with respect to the coordinates y_j, $\quad 1 \leq j \leq 3$. Obviously $v - v(y_0)$ also satisfies (1.7). But this implies that there is a harmonic homogeneous polynomial $P_{M(y_0)} \not\equiv 0$, of degree $M(y_0) \geq 1$ such that for $y \to y_0$

$$v(y) = v(y_0) + P_{M(y_0)}(y - y_0) + \mathfrak{o}(|y - y_0|^{M(y_0)}). \tag{1.8}$$

For the smooth case which we consider (1.8) is a well known result (for a more general setting see e.g. [Bs]), and $P_{M(y_0)} \not\equiv 0$ due to the strong unique continuation property of (1.7).

These considerations will be of importance for the following. The investigations in [CF, HOHO1, HOHO2, HOHON, R] and also the present ones are certainly motivated by the desire to understand to which extent the zero set or critical set of a solution can be described locally qualitatively by zero sets respectively critical sets of harmonic polynomials. Noting that for a harmonic polynomial $p \not\equiv$ const of degree M in $\mathbb{R}^n$

$$H^{n-2}(\Sigma(P, B_1)) \leq c(n) M^2$$

(see [HS]) one might expect that a more explicit version of Theorem 1.1 holds. Namely, let $B_\rho(x_0)$ be a ball in Ω' with radius ρ and centre x_0, and let M be the maximal order of vanishing of u in $\overline{\Omega'}$, then $M^{-2} H^1(\Sigma(u, B_{\rho/2}(x_0)))$ is bounded (compare also conjecture 2 in [L]).

That the analogy between solutions of (1.2) and harmonic polynomials should not be stressed too much can be seen from the following example: Consider in $\mathbb{R}^3$ the function

$$u(x, y, z) = \begin{cases} u_1(x, y, z) \text{ for } z > 0 \\ u_2(x, y, z) \text{ for } z < 0, \end{cases} \quad \text{with}$$

$$u_1 = xy + \sum_{k=0}^{4} \frac{(-1)^k}{(2k)!} z^{2k} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^k (x^4 y^4)$$

and $u_2 = xy + x^4 y^4$.

Consider first u_1 and u_2 in $\mathbb{R}^3$, then $u_1(x, y, 0) = u_2(x, y, 0)$ and $\frac{\partial u_1}{\partial z}(x, y, 0) = \frac{\partial u_2}{\partial z}(x, y, 0)$. Furthermore u_1 is harmonic and hence u satisfies $\Delta u = V u$ for, say $|x|, |y| < 1/2$ with

$$V(x, y, z) = \begin{cases} 0 \text{ for } z > 0 \\ 12 \dfrac{xy^3 + x^3 y}{1 + x^3 y^3} \text{ for } z \leq 0, \end{cases}$$

hence V is bounded for $|x|, |y| < 1/2$. But for the critical zero sets we obtain $\Sigma_0(u_2, \{z \leq 0\}) = \{(0,0,z)|z \leq 0\}$, i.e. a line of critical zeros, whereas $\Sigma_0(u_1, \{z > 0\})$ is empty. To see this just notice that u_1 has a term which depends only on z (in fact a term $\sim z^8$), so $|\nabla u_1|(x,y,z) \neq 0$ for small x, y and $z > 0$. So the example illustrates that a critical line of zeros can in fact stop, something that cannot happen for the analytic case.

The proof of Theorem 1.1 will be given in section 2. Thereby we need to show that we can apply some results from singularity theory [AGV]. This amounts to investigating properties of harmonic homogeneous polynomials or rather their complexification in $\mathbb{C}^3$. These results which might be of independent interest are presented in section 3.

2. Proof of Theorem 1.1

Let for $R > 0$, $B_R(\mathcal{O}) = \{x \in \mathbb{R}^3 | |x| < R\}$. We assume without loss that $\overline{B_R}(\mathcal{O}) \subset \Omega$, and that

$$a_{ij}(\mathcal{O}) = \delta_{ij}, \quad 1 \leq i, j \leq 3. \tag{2.1}$$

Since $u \in C^\infty(\Omega)$ and $\forall x_0 \in \Omega$, $u - u(x_0)$ is a solution of (1.2) which vanishes with some finite order $M(x_0)$ in the point x_0, i.e. for some homogeneous polynomial $p^{(x_0)}_{M(x_0)} \not\equiv 0$ with degree $M(x_0) \geq 1$

$$u(x) - u(x_0) = p^{(x_0)}_{M(x_0)}(x - x_0) + \mathfrak{o}(|x - x_0|^{M(x_0)}), \text{ for } x \to x_0 \tag{2.2}$$

and for every multiindex α, $|\alpha| \leq M(x_0)$

$$D^\alpha u(x) = D^\alpha p^{(x_0)}_{M(x_0)}(x - x_0) + \mathfrak{o}(|x - x_0|^{M(x_0) - |\alpha|}), \tag{2.3}$$

Therefrom it is easily seen that

$$\overline{M} := \sup\{M(x_0) | x_0 \in \Sigma(u, \overline{B_R}(\mathcal{O}))\} < \infty \tag{2.4}$$

Let $x_0 \in \Sigma(u, \overline{B_R}(\mathcal{O}))$, then by the coordinate transform T_{x_0} considered above, $v(y) = u(T_{x_0} y)$, v has a critical point in $y_0 = T_{x_0} x_0$ and due to (1.8) there is a harmonic homogeneous polynomial $P^{(y_0)}_{M(y_0)} \not\equiv 0$ corresponding to this critical point. It is well known (from Courant's nodal theorem, or e.g. [C]) that the number of critical points of a harmonic homogeneous polynomial P_M on the unit sphere is bounded by a constant $N(M)$. From the foregoing we obtain $P^{(y_0)}_{M(y_0)}(y) = p^{(x_0)}_{M(x_0)}(T_{x_0} y)$, where $M(x_0) = M(y_0)$, and $P^{(y_0)}_M$ and $p^{(x_0)}_M$ have the same number of critical points on the unit sphere. This together with (2.4) implies that with $N = N(\overline{M}) < \infty$

$$\sup_{x_0 \in \Sigma(u, \overline{B_r}(\mathcal{O}))} \#\Sigma(p^{x_0}_{M(x_0)}, S^2) \quad \leq \quad N \tag{2.5}$$

Lemma 2.1. *Let $S_N = \{\nu_j | \nu_j \in S^2, 1 \leq j \leq 2N+1\}$ be such that any three of the unit vectors ν_j are linear independent. Then the following holds: There is a $c = c(\overline{M}, R, S_N) > 0$ such that $\forall x_0 \in \Sigma(u, \overline{B_R}(\mathcal{O}))$ one can choose $\nu \in S_N$ with*

$$(a) \quad | < \omega | \nu > | > c \quad \forall \omega \in S^2 \text{ with } \nabla p^{(x_0)}_{M(x_0)}(\omega) = \mathcal{O} \tag{2.6}$$

$$(b) \quad \liminf_{\substack{x \to x_0 \\ x \in \Sigma(u, \overline{B_R}(\mathcal{O}))}} \quad | < \frac{x - x_0}{|x - x_0|} | \nu > | > c. \tag{2.7}$$

Proof of Lemma 2.1. We first show

Proposition 2.1. *For $K \in \mathbb{N}$ let $S_K = \{\nu_j | \nu_j \in S^2, 1 \leq j \leq 2K+1\}$ with the property that any three of the unit vectors ν_j are linear independent. Then there exists $c > 0$ such that for arbitrary $\omega_{1,\ldots,}\omega_K \in S^2$ one can find $\nu \in S_K$ with $| < \nu | \omega_j > | > c > 0$, $1 \leq j \leq K$.*

Proof of Proposition 2.1. Let $\omega_j \in S^2$ and denote $\omega_j^{\perp} = \{\omega \in S^2 | < \omega | \omega_j >= 0\}$. Then due to the definition of S_K at most two elements of S_K belong to $\omega_j^{\perp}$. Since this is true for $1 \leq j \leq K$, then, at most $2K$ elements of S_K are involved. Hence there is at least a $\nu \in S_K$ with $< \nu | \omega_l > \neq 0$ for $1 \leq l \leq K$.

To see that $c = c(K, S_K)$, but does not depend on the choice of $\omega_1, \ldots, \omega_K$, let $\omega^{(m)} = (\omega_1^{(m)}, \ldots, \omega_K^{(m)})$, $m \in \mathbb{N}$, $\omega_j^{(m)} \in S^2$, and let

$$c_m = \max_{\nu \in S_K} \quad \min_{1 \leq i \leq K} | < \omega_i^{(m)} | \nu > |.$$

Then due to the above $c_m > 0$. Suppose now for contradiction that $\liminf_{m \to \infty} c_m = 0$. Then clearly for some $\bar{\omega}_m$, $\bar{\omega} \in S^2$, $\bar{\omega}_m \to \bar{\omega}$ for $m \to \infty$ and $| < \bar{\omega}_m | \nu > | \to 0$ $\forall \nu \in S_K$. Hence $< \bar{\omega} | \nu >= 0$ $\forall \nu \in S_K$ which is a contradiction $\square$

Now take $K = N$, then Proposition 2.1 together with (2.5) immediately implies (a).

To verify (b) let $x^{(l)} \in \Sigma(u, \overline{B_R}(\mathcal{O}))$, $l \in \mathbb{N}$ with $x^{(l)} \to x_0$ for $l \to \infty$, and using polar coordinates $x^{(l)} - x_0 = r_l \omega_l$ with $\omega_l = \omega(r_l)$. Without loss we assume that $\omega_l \to \bar{\omega}$ for $l \to \infty$, for some $\bar{\omega} \in S^2$. From the following it is straight forward to see that $\nabla p^{x_0}_{M(x_0)}(\bar{\omega}) = \mathcal{O}$: From (2.3) we obtain with $x = x^{(l)}$ for $1 \leq j \leq 3$

$$0 = D_j u(x^{(l)}) = D_j p^{(x_0)}_{M(x_0)}(r_l \omega_l) + \mathfrak{o}(r_l^{M(x_0)-1}).$$

Since $D_j p^{(x_0)}_{M(x_0)}$ is homogeneous of degree $M(x_0) - 1$ this implies $D_j p^{(x_0)}_{M(x_0)}(\bar{\omega}) = 0$ for $l \to \infty$.

Now we conclude from (a) with $\omega = \bar{\omega}$ that $\lim_{l \to \infty} | < \omega_l | \nu > | = | < \bar{\omega} | \nu > | > c$ verifying (b). This finishes the proof of Lemma 2.1 $\square$

Lemma 2.1 provides some information on the location of the critical points of u in the neighbourhood of a critical point x_0 of u, which will be relevant later on.

We now assume without loss that u has a critical zero in the origin. So according to (1.8) there is a harmonic homogeneous polynomial $P_M \not\equiv 0$ of degree $M \geq 2$ such that

$$u(x) = P_M(x) + \mathfrak{o}(|x|^M) \text{ for } |x| \to 0 \tag{2.8}$$

Thereby $M \leq \overline{M}$, $\overline{M}$ given accordingly to (2.4).

In the next step of the proof of Theorem 1.1 we need some properties of homogeneous polynomials of complex variables.

In the following a polynomial $p : \mathbb{C}^n \to \mathbb{C}$ is said to be harmonic if

$$\Delta p \equiv \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2} p = 0.$$

Lemma 2.2. *Let $P_M(z), z \in \mathbb{C}^3$ be a harmonic homogeneous polynomial of degree M, $2 \leq M \leq \overline{M}$, with real coefficients and let $N = N(\overline{M})$ be given according to (2.5). Then there exists $S_N = \{\nu_j | \nu_j \in S^2, 1 \leq j \leq 2N+1\}$, where any three of the unit vectors are linear independent, with the following property:*

(2.9)

$$\begin{aligned}
&\forall j, 1 \leq j \leq 2N+1, \quad P_M(z) \textit{ restricted to the complex plane} \\
&\varepsilon_j = \{z \in \mathbb{C}^3 | < \nu_j | z >= 0\}, \textit{ has an isolated critical point in the origin.}
\end{aligned}$$

Now suppose that Lemma 2.2 is proven (The proof is given in section 3). Then Lemma 2.2 together with Lemma 2.1 implies that

(2.10)

$$\begin{aligned}
&\Sigma(u, \overline{B_R}(\mathcal{O})) = \bigcup_{j=1}^{2N+1} E_j, \text{ where} \\
&\qquad E_j := \{x_0 \in \Sigma(u, \overline{B_R}(\mathcal{O})) | (a) \text{ and } (b) \text{ hold with } \nu = \nu_j\}.
\end{aligned}$$

Clearly the sets E_j are not necessarily pairwise disjoint.

Since every critical point of u is also a critical point of u restricted to a plane through this point, it will suffice to obtain information on the critical points of u restricted to the planes

$$\begin{aligned}
&\varepsilon_{j,t} = \{x \in \mathbb{R}^3 | < \nu_j | x - \nu_j t >= 0\}, \quad 1 \leq j \leq 2N+1 \\
&\text{with } 0 \leq |t| \leq t_0, \quad t_0 \text{ small enough .}
\end{aligned} \tag{2.11}$$

In the following we consider $u|_{\varepsilon_{j,t}}$ as a perturbation of $P_M|_{\varepsilon j}$, and because of (2.9) this will allow us via arguments from [AGV] to show

Lemma 2.3. *There exists $\tilde{r} > 0$ small enough, and a constant $d = d(M)$, such that $\forall t$, $|t| \leq t_0, t_0$ sufficiently small*

$$\#\Sigma_0(u, \varepsilon_{j,t} \cap Z_{j,\tilde{r}}) \leq d \tag{2.12}$$

Thereby $Z_{j,\tilde{r}}$ is the open cylinder with radius $\tilde{r}$ and axis $\{\nu_j t | t \in \mathbb{R}\}$.

Proof of Lemma 2.4. The proof of this Lemma is based on

Proposition 2.2. *Let $p(z_1, z_2)$ be a homogeneous polynomial in $\mathbb{C}^2$ of degree k with real coefficients, and assume that p has an isolated critical point in the origin in $\mathbb{C}^2$. Let further $\varphi \in C^\infty(D_r(\mathcal{O}))$, $D_r(\mathcal{O}) = \{x \in \mathbb{R}^2 | |x| < r\}$, $r > 0$, with*

$$\varphi(y) = p(y) + \mathfrak{o}(|y|^k) \quad \text{for } |y| \to 0$$

and let $\varphi_t(y) \in C^\infty(D(\mathcal{O}) \times I)$ for $t \in I, I = [-t_0, t_0]$ for some $t_0 > 0$, with $\varphi_0 = \varphi$. Then there exists $\tilde{r}$, $0 < \tilde{r} < r$ such that for $|t| \leq t_0$, t_0 small enough, the number of critical points of $\varphi_t(.)$ in $D_{\tilde{r}}(\mathcal{O})$ is uniformly bounded by a constant $d(k)$.

Proof of Proposition 2.2. For the proof we shall use some results of [AGV]. For convenience we repeat some definitions given there: Let $f : (\mathbb{C}^n, z_0) \to (\mathbb{C}^n, \mathcal{O})$ be a holomorphic map germ at a point z_0. Let $\mathbb{C}\{z\}_{z_0}$ denote the algebra of all holomorphic function-germs at z_0, and let I_{f,z_0} denote the ideal in this algebra, which is generated by the germ of the components of f.

Definition: The multiplicity of f at z_0 is the dimension of the local algebra Q_{f,z_0}

$$\mu_{z_0}[f] := \dim_{\mathbb{C}} Q_{f,z_0},$$

where Q_{f,z_0} denotes the quotient algebra $\mathbb{C}\{z\}_{z_0}/I_{f,z_0}$.

By Theorem 2 ([AGV] p. 86) a holomorphic map germ fails to be of finite multiplicity at a point z_0, if and only if z_0 is a non-isolated inverse image of zero of the germ.

Due to our assumption $p(z)$ is a homogeneous polynomial with an isolated critical point in the origin. By the above Theorem the multiplicity of ∇p in $\mathcal{O}$ is finite, $\mu_{\mathcal{O}}[\nabla p] =: d < \infty$.

Definition: A critical point $\mathcal{O}$ of a smooth function $f : (\mathbb{R}^n, \mathcal{O}) \to (\mathbb{R}, 0)$ is said to be of finite multiplicity $\mu(f)$, if the gradient map ∇f is of finite multiplicity, i.e.

$$\mu(f) := \dim_{\mathbb{R}} \mathbb{R}[[x]] / \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) < \infty$$

Thereby $\mathbb{R}[[x]]$ denotes the algebra of formal power series and $\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$ denotes the ideal generated by the components of ∇f.

By the subadditivity of the multiplicity (see [AGV], Proposition 1, p. 94) we conclude from the above that $\mu(\varphi) \leq d$, and again by the subadditivity of the multiplicity there exists a $D_{\tilde{r}}(\mathcal{O})$, $\tilde{r} > 0$, such that for $|t| \leq t_0$, t_0 small enough the number of critical points of φ_t in $D_{\tilde{r}}(\mathcal{O})$ counted with their multiplicities is bounded from above by d. $\square$

Remark 2.1. We note that since p is semiquasi-homogeneous (compare [AGV], p. 193) it follows from Bezout's formula (see [AGV], Corollary 1, p. 200) that $d = (k-1)^2$.

Now we apply Proposition 2.2 to our case: Representing the planes $\varepsilon_{j,t}$ as $\varepsilon_{j,t} = \{\mathfrak{n}_1 y_1 + \mathfrak{n}_2 y_2 + \nu_j t | (y_1, y_2) \in \mathbb{R}^2\}$ with some $\mathfrak{n}_1, \mathfrak{n}_2 \in S^2$, $< \mathfrak{n}_1 | \mathfrak{n}_2 >= 0$, $< \mathfrak{n}_l | \nu_j >= 0$, $l = 1, 2$, we then identify $p(y)$ with $P_M(\mathfrak{n}_1 y_1 + \mathfrak{n}_2 y_2)$, and $\varphi_t(y)$ with $u(\mathfrak{n}_1 y_1 + \mathfrak{n}_2 y_2 + t\nu_j)$. Due to (2.8) we have

$$\varphi_0(y) = \varphi(y) = u(\mathfrak{n}_1 y_1 + \mathfrak{n}_2 y_2) = P_M(\mathfrak{n}_1 y_1 + \mathfrak{n}_2 y_2) + \mathfrak{o}(|y|^M).$$

Then by Proposition 2.2 there is a $\tilde{r} > 0$, such that for $|t| \leq t_0$, t_0 small enough the number of critical points of $u|_{\varepsilon_{j,t}\cap Z_{j,\tilde{r}}}$ is uniformly bounded by $d(M)$ (actually $d = (M-1)^2$). This finishes the proof of Lemma 2.3. $\square$

In the following let $\tilde{t} > 0$ such that

$$\tilde{t} < \min(\tilde{r}, t_0, 1) \equiv \tilde{t}_0. \tag{2.13}$$

In this last step of the proof of Theorem 1.1 we shall use Lemma 2.3 and Lemma 2.1 to show that the 1-dimensional Hausdorff measure of the critical set of u is finite.

Lemma 2.4. *For some $C = C(\overline{M}, c) < \infty$, c given according to Lemma 2.1,*

$$H^1(\Sigma(u, \overline{B_{\tilde{t}}}(\mathcal{O}))) \leq C.$$

Given Lemma 2.4, then by translation of the coordinate system it follows that $\forall x_0 \in \Sigma(u, \Omega)$ the H^1-measure of the critical set of u is finite in a neighbourhood of x_0. Let $\Omega' \subset\subset \Omega$, then $\Sigma(u, \overline{\Omega'})$ is a compact set and it follows via Heine-Borel that $H^1(\Sigma(u, \overline{\Omega'}))$ is finite finishing the proof of Theorem 1.1.

Proof of Lemma 2.4. We first show

Proposition 2.3. *Let $\Sigma_j = E_j \cap \overline{B_{\tilde{t}}}(\mathcal{O})$, $1 \leq j \leq 2N+1$. Then for some $C = C(\overline{M}, c) < \infty$, $H^1(A) \leq C$ for every closed subset A of $\Sigma_j, \forall j$.*

Proof of Proposition 2.3. Let I_j denote the line segment $[-\tilde{t}_0\nu_j, \tilde{t}_0\nu_j]$ in $\mathbb{R}^3$ and denote by Z_j the compact truncated cylinder with radius $\tilde{t}_0$ and axis $\{\nu_j t \,|\, |t| \leq \tilde{t}_0\}$. Let A be a closed subset of Σ_j and define $\forall x \in A$

$$\pi_j(x) = < x|\nu_j > \nu_j$$

i.e. the orthogonal projection to the line segment I_j. For $y \in I_j$ let $\varepsilon_{j,y}$ denote the plane orthogonal to ν_j with $y \in \varepsilon_{j,y}$.

Then we conclude from Lemma 2.3 that $\varepsilon_{j,y} \cap \Sigma(u, Z_j)$ consists of at most $k = k(y, j)$ points, where $k \leq d(M)$, hence

$$\forall y \in I_j, \quad \#\pi_j^{-1}(y) \leq d. \tag{2.14}$$

So let $y \in I_j$ and $\pi_j^{-1}(y) = \{x^{(1)}, \ldots, x^{(k)}\}$. Now we use Lemma 2.1.: According to the Lemma there is a $c > 0$ and $\rho_l := \rho(x^{(l)}) > 0$, $1 \leq l \leq k$ such that for some $c_1 > c$

$$|< \frac{x - x^{(l)}}{|x - x^{(l)}|}|\nu_j > | \geq c_1 \quad \forall x \in \Sigma(u, \overline{B_{\rho_l}}(x^{(l)})). \tag{2.15}$$

Let $\rho_0(y) = \min_{1\leq l\leq k} \rho(x^{(l)})$ and $\delta_0(y) := c\rho_0(y)$. Further, for $\delta > 0$, we denote the line segment $[y - \delta\nu_j, y + \delta\nu_j]$, by I_δ (suppressing the y-dependence).

We claim: There is a $\delta(y) > 0$, $\delta(y) \leq \min(\delta_0(y), \tilde{t}_0 - \tilde{t}) =: \tilde{\delta}$, such that

$$\forall \delta \text{ with } 0 < \delta \leq \delta(y) \tag{2.16}$$

$$\pi_j^{-1}(I_\delta) \subset \bigcup_{l=1}^{k} \overline{B_\rho}(x^{(l)}), \text{ where } \rho = \frac{\delta}{c}.$$

Note first that because of (2.15) we know that

$$\forall \delta \text{ with } \delta \leq \tilde{\delta}, \quad x \in \pi_j^{-1}(I_\delta) \setminus \bigcup_{l=1}^{k} \overline{B_\rho}(x^{(l)})$$

implies

$$x \in \pi_j^{-1}(I_\delta) \setminus \bigcup_{l=1}^{k} \overline{B_{\rho_0}}(x^{(l)}).$$

Now suppose for contradiction that (2.16) is not true. Then because of the above there is a sequence

$$\delta_m \downarrow 0 \text{ and } \bar{x}^{(m)} \in \pi_j^{-1}(I_{\delta_m}) \setminus \bigcup_{l=1}^{k} \overline{B_{\rho_0}}(x^{(l)}), \quad \forall m.$$

Since $\bar{x}^{(m)} \in A$ and A is compact, there is a convergent subsequence with limit $\bar{x} \in A$ and without loss let $\bar{x}^{(m)} \to \bar{x}$ for $m \to \infty$. Obviously $\bar{x} \in \pi_j^{-1}(y) \setminus \bigcup_{l=1}^{k} B_{\rho_0}(x^{(l)})$, which is a contradiction since $\bar{x}$ must be equal to some $x^{(l)}$, $1 \leq l \leq k$. This verifies (2.16).

Now let $J = \{I_{\delta(y)/5} | y \in I_j\}$. It is a standard covering result (see e.g. Lemma 1.9., p. 10 [F]) that there exists a (finite or) countable disjoint subcollection J' of J such that the following holds: with $J' = \{I_{\delta(\bar{y}_i)/5} | i \in \mathbb{N}\}$, $I_j \subset \bigcup_{I \in J} I \subset \bigcup_{i \in \mathbb{N}} I_{\delta(\bar{y}_i)}$. Therefrom and from (2.16) we obtain

$$\begin{aligned} A = \pi_j^{-1}(I_j) &\subset \pi_j^{-1}\Big(\bigcup_{i \in \mathbb{N}} I_{\delta(\bar{y}_i)}\Big) \\ &= \bigcup_{i \in \mathbb{N}} \pi_j^{-1}(I_{\delta(\bar{y}_i)}) \subset \bigcup_{i \in \mathbb{N}} \bigcup_{l=1}^{k(\bar{y}_i)} \overline{B_{\delta(\bar{y}_i)/c}}(x^{(l)}). \end{aligned} \tag{2.17}$$

Now we use that J' is a disjoint subcollection of J, so that $\sum_{i \in \mathbb{N}} \delta(\bar{y}_i)/5 \leq |I_j|$. Since $k(\bar{y}_i) \leq d(\overline{M})$ the above leads to

$$\sum_{i \in \mathbb{N}} \sum_{l=1}^{k(\bar{y}_i)} \delta(\bar{y}_i)/c \leq \frac{5|I_j| d(\overline{M})}{c} \leq \frac{10(\overline{M}-1)^2}{c}. \tag{2.18}$$

Because of (2.16), $\delta(y)$ can be taken arbitrarily small. Hence by (2.17) and (2.18), $A \subset \bigcup_i \bigcup_l \overline{B_{i,l}}$, where the diameters d_{il} of the balls $B_{i,l}$ are bounded from above by some $\delta > 0$. Taking δ arbitrarily small and applying (2.18) we obtain

$$\sum_{i,l} d_{il}(\delta) \leq \frac{10(M-1)^2}{c} =: C(\overline{M}, c) \tag{2.19}$$

Let $\{U_i\}$ denote a δ-cover of A, $d(U_i)$ denoting the diameter of U_i, so that $d(U_i) \leq \delta$. Let

$$H^1_\delta(A) := \inf_{\{U_i\}} \sum_{i=1}^{\infty} d(U_i).$$

Then the Hausdorff 1-dimensional outer measure of A is defined by $H^1(A) := \lim_{\delta\to 0} H^1_\delta$ and $H^1(A) = \sup_{\delta>0} H^1_\delta(A)$. But because of (2.17) and (2.19) we obtain for $\delta > 0$ arbitrarily small, $H^1(A) \le C(\overline{M}, c)$ and hence $H^1(A) \le C(\overline{M}, c)$, verifying Proposition 2.3. $\square$

We proceed in the proof of Lemma 2.4 by showing that $\Sigma(u, \overline{B_{\tilde{t}}}(\mathcal{O}))$ can be represented in a particular way as countable union of closed subsets of the Σ_j's. For this we introduce the following: For $k \ge 2$ let

$$\Gamma_k = \{\bar{x} \in \Sigma(u, \overline{B_{\tilde{t}}}(\mathcal{O})) | u \text{ vanishes of order } k \text{ in } \bar{x}\}.$$

This means for each $\bar{x}$ there is a homogeneous polynomial $p_k^{(\bar{x})}$ of degree k, $p_k^{(\bar{x})} \not\equiv 0$, such that

$$u(x) - u(\bar{x}) = p_k^{(\bar{x})}(x - \bar{x}) + \mathfrak{o}(|x - \bar{x}|^k) \quad \text{ for } x \to \bar{x}.$$

Let

$$\Gamma_{k,n} = \{x \in \Gamma_k | \text{ dist } (x, \bigcup_{j \ge k+1} \Gamma_j) \ge \frac{1}{n}\}, \quad n \in \mathbb{N}$$

Note that since $k \le \overline{M}$ (compare (2.4)), $\Gamma_j = \emptyset, \forall_j > \overline{M}$. $\Gamma_{k,n}$ is a closed subset of $\Sigma(u, \overline{B_{\tilde{t}}}(\mathcal{O}))$, which follows easily from the smoothness of u and from (2.3). Furthermore

$$\Gamma_k = \bigcup_{n \in \mathbb{N}} \Gamma_{k,n}. \tag{2.20}$$

This can be seen as follows: suppose for contradiction that for some $\bar{x} \in \Gamma_k$, $\bar{x} \notin \Gamma_{k,n}$ $\forall n$. Then there is a sequence $\{x^{(m)}\}, x^{(m)} \in \Gamma_{k_m}$, $k_m \ge k+1, \forall m$ with $x^{(m)} \to \bar{x}$ for $m \to \infty$. Because of (2.2) and (2.3) we obtain for every multiindex α with $|\alpha| = k$, that for $m \to \infty$

$$D^\alpha u(x^{(m)}) = D^\alpha p_k^{(\bar{x})}(x^{(m)} - \bar{x}) + \mathfrak{o}(1).$$

Since for suitable α, $D^\alpha p_k^{(\bar{x})}(\mathcal{O}) \neq 0$ and $D^\alpha u$ is continuous we obtain

$$D^\alpha u(\bar{x}) = \lim_{m\to\infty} D^\alpha u(x^{(m)}) = D^\alpha p_k^{(\bar{x})}(\mathcal{O}) \neq 0.$$

On the other hand since $k_m \ge k+1$ $\forall m$, $D^\alpha u(x^{(m)}) = 0$ $\forall m$, which leads to a contradiction. ¿From (2.20) we obtain that

$$\Sigma(u, \overline{B_{\tilde{t}}}(\mathcal{O})) = \bigcup_{k=2}^{\overline{M}} \bigcup_{n \in \mathbb{N}} \Gamma_{k,n} \tag{2.21}$$

To show that $H^1(\Gamma_{k,n})$ can be bounded uniformly in n we need:

Proposition 2.4. *Let $\bar{x} \in \Gamma_{k,n}$ and assume that $\bar{x} \in \Sigma_j$, then there exists $\rho(\bar{x}) > 0$ such that $\Gamma_{k,n} \cap \overline{B_{\rho(\bar{x})}}(\bar{x}) \subset \Sigma_j$.*

Proof of Proposition 2.4. Suppose for contradiction that there is a sequence $\{x^{(m)}\}$, $x^{(m)} \in \Gamma_{k,n}$ $\forall m$, $x^{(m)} \to \bar{x}$ for $m \to \infty$, with $x^{(m)} \notin \Sigma_j$, $\forall m$. Due to Lemma 2.1(a) and the definition of Σ_j this implies that $\forall m$ there is a $\omega^{(m)} \in S^2$ with

$$\nabla p_k^{(x^{(m)})}(\omega^{(m)}) = \mathcal{O}, \quad | < \omega^{(m)} | \nu_j > | \leq c.$$

Let $\bar{\omega}$ be an accumulation point of $\{\omega^{(m)}\}$, and without loss assume $\omega^{(m)} \to \bar{\omega}$ for $m \to \infty$. Then clearly we obtain

$$| < \bar{\omega} | \nu_j > | \leq c. \tag{2.22}$$

Suppose we have shown that $\bar{\omega}$ is a critical point of $p_k^{(\bar{x})}$, then since $\bar{x} \in \Sigma_j$, Lemma 2.1(a) is in contradiction to (2.22) and the proof of the Proposition is finished.

Therefore we show that the critical points $\omega^{(m)}$ of $p_k^{(x^{(m)})}$ tend for $m \to \infty$ to the critical point $\bar{\omega}$ of $p_k^{(\bar{x})}$: According to (2.3) we have for every multiindex $\beta, |\beta| = k$

$$D^\beta u(x) = D^\beta p_k^{(x^{(m)})}(x - x^{(m)}) + \mathfrak{o}(1) \text{ for } x \to x^{(m)}$$

where $p_k^{(x^{(m)})}(x) := \sum_{|\alpha|=k} a_\alpha^{(m)} x^\alpha$, $\forall m$ and analogously we have for $\bar{x}$, $p_k^{(\bar{x})}(x) := \Sigma_{|\alpha|=k} \bar{a}_\alpha x^\alpha$. Therefore $D^\beta u(x^{(m)}) = \beta! a_\beta^{(m)}$, $\forall m$, $\forall \beta$ with $|\beta| = k$. By the smoothness of u

$$\lim_{m \to \infty} D^\beta u(x^{(m)}) = D^\beta u(\bar{x})$$

and hence $\lim_{m\to\infty} a_\beta^{(m)} = \bar{a}_\beta$, $\forall \beta$, $|\beta| = k$. This implies in particular that

$$|\nabla p_k^{(x^{(m)})} - \nabla p_k^{(\bar{x})}| \to 0 \text{ for } m \to \infty \tag{2.23}$$

pointwise uniformly. By the triangle inequality we have

$$|\nabla p_k^{(\bar{x})}(\bar{\omega})| \leq |\nabla p_k^{(\bar{x})}(\bar{\omega}) - \nabla p_k^{(x^{(m)})}(\bar{\omega})| + |\nabla p_k^{(x^{(m)})}(\bar{\omega}) - \nabla p_k^{(x^{(m)})}(\omega^{(m)})|. \tag{2.24}$$

But because of (2.23) and because

$$|D_i p_k^{(x^{(m)})}(\bar{\omega}) - D_i p_k^{(x^{(m)})}(\omega^{(m)})| \leq \sum_{|\alpha|=k} |a_\alpha^{(m)}| \cdot |(D_i x^\alpha)|_{x=\bar{\omega}} - (D_i x^\alpha)|_{x=\omega^{(m)}}| \to 0$$

for $m \to \infty$, the right hand side of (2.24) tends to zero for $m \to \infty$. Thus $\bar{\omega}$ is a critical point of $p^{(\bar{x})}$. $\square$

Clearly we have $\Gamma_{k,n} \subset \bigcup_{\bar{x} \in \Gamma_{k,n}} B_{\rho(\bar{x})}(\bar{x})$, with $\rho(\bar{x})$ given according to Proposition 2.4. Since $\Gamma_{k,n}$ is a compact set, there is due to Heine Borel a finite cover of balls centered in $\bar{x}_i$, $1 \leq i \leq L_{k,n}$ for some $L_{k,n} < \infty$ such that

$$\Gamma_{k,n} \subseteq \bigcup_{i=1}^{L_{k,n}} A_i, \text{ where } A_i = \overline{B_{\rho(\bar{x}_i)}}(\bar{x}_i) \cap \Gamma_{k,n}. \tag{2.25}$$

¿From Proposition 2.4 we concluded that $\forall i$, A_i is a closed subset of some $\Sigma_{j(i)}$. Since this is a finite union of closed sets we rewrite (2.25) as

$$\Gamma_{k,n} \subseteq \bigcup_{j=1}^{2N+1} \mathcal{A}_j^{(k,n)}, \tag{2.26}$$

where $\mathcal{A}_j^{(k,n)}$ are closed subsets of Σ_j.

¿From Proposition 2.3 we have

$$H^1(\mathcal{A}_j^{(k,n)}) \leq C, \quad 1 \leq j \leq 2N+1$$

which together with (2.26) leads to

$$H^1(\Gamma_{k,n}) \leq \sum_{j=1}^{2N+1} H^1(\mathcal{A}_j^{(k,n)}) \leq (2N+1)C. \tag{2.27}$$

Noting that obviously $\Gamma_{k,n} \subset \Gamma_{k,n+1}$ $\forall n$, $\{\Gamma_{k,n}\}_{n\in\mathbb{N}}$ is an increasing sequence of sets and therefore (see e.g. Lemma 1.3 [F])

$$H^1(\bigcup_{n\in\mathbb{N}} \Gamma_{k,n}) = \lim_{n\to\infty} H^1(\Gamma_{k,n}).$$

This together with (2.27) yields

$$H^1(\bigcup_{n\in\mathbb{N}} \Gamma_{k,n}) \leq (2N+1)C, \quad 1 \leq k \leq \overline{M}. \tag{2.28}$$

Finally using (2.21) we obtain

$$H^1(\Sigma(u, \overline{B_{\tilde{t}}}(\mathcal{O}))) \leq \overline{M}(2N+1)C$$

finishing the proof of Lemma 2.4 and of Theorem 1.1. $\square$

3. Properties of harmonic homogeneous polynomials and the proof of Lemma 2.2

To prove Lemma 2.2 we first collect some properties of harmonic homogeneous polynomials:

Theorem 3.1. *Let $P : \mathbb{C}^3 \to \mathbb{C}$, $P \not\equiv 0$ be a harmonic homogeneous polynomial with real coefficients. Then the set of critical points of P is the union of at most finitely many straight lines in $\mathbb{C}^3$ through the origin.*

The proof of Theorem 3.1 will be given later.

Remark 3.1. (i) The assumption that the coefficients of P are real is essential as can be seen from the example $P(z_1, z_2, z_3) = (z_1 - iz_2)^2$: P is harmonic and homogeneous, but all points (z_1, z_2, z_3) with $z_1 = iz_2$ are critical points of P. (ii) That P is harmonic is also necessary as can be easily seen from the example $P(z_1, z_2, z_3) = z_1^2$ where the critical set of P is the plane $z_1 = 0$.

Via Theorem 3.1 we obtain

Proposition 3.1. *Let $P : \mathbb{C}^3 \to \mathbb{C}$ be a harmonic homogeneous polynomial with real coefficients and let P_ν denote the restriction of P to the complex plane $\varepsilon_\nu = \{z \in \mathbb{C}^3 | < \nu | z >= 0\}$, $\nu \in S^2$. Then the set*

$$\mathcal{F} = \{\nu \in S^2 | \mathcal{O} \text{ is not an isolated critical point of } P_\nu\}$$

is the union of at most finitely many analytic curves and finitely many isolated points.

Proof of Proposition 3.1. From Theorem 3.1 it follows that the critical set of P is the union of straight lines in $\mathbb{C}^3, g_j, 1 \leq j \leq N$ (N depending on the degree of P). Let $\mathcal{F}_j = \{\nu \in \mathcal{F} | g_j \subset \varepsilon_\nu\}$ and $g_j = \{\lambda z^{(j)} | \lambda \in \mathbb{C}\}$, then $x \in \mathbb{R}^3$ satisfies $< x | z^{(j)} >= 0$ if and only if $\nu = \frac{x}{|x|} \in \mathcal{F}_j$.

Since $z^{(j)} \neq \mathcal{O}$, the general solution of $< x | z^{(j)} >= 0$ is at most a 2-dimensional subspace of $\mathbb{R}^3$, and therefore $\mathcal{F}_j$ is either a great circle on S^2, isolated points or empty.

Now it remains to consider $\mathcal{F} \setminus \bigcup_{j=1}^N \mathcal{F}_j$: So let $\nu \in \mathcal{F} \setminus \bigcup_{j=1}^N \mathcal{F}_j$ which implies that the complex plane ε_ν has the following property: $\mathcal{O}$ is not an isolated critical point of P_ν and $g_j \cap \varepsilon_\nu = \{\mathcal{O}\}$ $\forall 1 \leq j \leq N$. Since P_ν is again a homogeneous polynomial the critical set of P_ν consists of finitely many complex straight lines $\gamma_i, 1 \leq i \leq n$, through the origin, and every critical point of P_ν is a zero of P_ν. Hence $\forall z \in \gamma_i$, $P(z) = 0$ and clearly ε_ν is tangent to the surface $P = 0$ in these points, so $\nabla P \neq \mathcal{O}$.

Due to the homogeneity of P we can locally represent the zero set of P (away from the critical set) by a holomorphic function f, so that for a domain $U \subset \mathbb{C}$, $P(\Gamma(\omega)) = 0$ $\forall \omega \in U$, with $\Gamma(\omega) = (\omega, f(\omega), 1)$ and $\nabla P(\Gamma(\omega)) \neq \mathcal{O}$.

Let $\varepsilon(\omega)$ denote the tangent plane to $P = 0$ in the point $\Gamma(\omega)$ and let $\mathfrak{n}(\omega) = \nabla P(\Gamma(\omega)) \; / |\nabla P(\Gamma(\omega))|$. Then the components of $\mathfrak{n}(\omega)$, $n_j(\omega)$, $1 \leq j \leq 3$ are holomorphic functions. Clearly $\varepsilon(\omega) \cap \mathbb{R}^3$ is a 2-dimensional manifold in $\mathbb{R}^3$ if and only if for some $\lambda \in \mathbb{C}$, $\lambda \mathfrak{n}(\omega) \in \mathbb{R}^3$.

If all components of $\mathfrak{n}$ are constants, then $P = 0$ is a plane and since P is homogeneous and harmonic, P must be linear. Hence P has no critical points at all.

Therefore it suffices to consider the case that two of the components, say j, l, are not constant. Since $n_l(\omega)$ is holomorphic, n_l has only isolated zeros and away from them $n_{jl}(\omega) = \frac{n_j(\omega)}{n_l(\omega)}$ is holomorphic. To find out for which ω, $n_{jl}(\omega) \in \mathbb{R}$, note that $\mathrm{Im} n_{jl}$ is a real valued harmonic function in two real variables. Since $\mathrm{Im} n_{jl} \not\equiv 0$ this implies that the zero set of $\mathrm{Im} n_{jl}$ considered as a subset of $\mathbb{R}^2$ is locally the union of finitely many analytic curves.

This finishes the proof of Proposition 3.2 $\square$

Lemma 2.2 is now an immediate consequence of the foregoing Proposition.

Finally we give the

Proof of Theorem 3.1.

The proof is based on some well known results from algebraic geometry (see e.g. the books [S, M]), which are collected in the following:

Proposition 3.2. *(i) Let V be the complex algebraic set defined by the single polynomial equation $f(z) = 0, z \in \mathbb{C}^3$, with f irreducible. Then every polynomial which vanishes on V is divisible by f.*

(ii) Let $f(z), z \in \mathbb{C}^3$ be an irreducible polynomial, then the zero set of f has complex dimension 2 whereas the critical set of f is at most 1-dimensional.

Remark 3.2. A k-dimensional algebraic set in $\mathbb{C}^n$ is away from its critical points a smooth manifold of complex dimension k (see e.g. [S]).

Note first that since the polynomial P is homogeneous the critical set of P is given by

$$\Sigma(P) = \{z \in \mathbb{C}^3 | P(z) = |\nabla P(z)| = 0\}.$$

We shall exclude that $\Sigma(P)$ is 2-dimensional. Then since the critical points of P lie on straight lines through the origin (because of the homogeneity of P) it follows by the analyticity of P that there are only finitely many such lines.

We suppose now for contradiction that $\Sigma(P)$ is 2-dimensional. Then Proposition 3.2 (ii) implies that P is reducible. Furthermore P can be represented as

$$P = p^2 \cdot q, \text{ where } p, q \text{ are homogeneous polynomials and } p \text{ is irreducible .} \tag{3.1}$$

This can be seen as follows: suppose for contradiction that $P = \prod_{j=1}^{k} q_j$, $(k \geq 2)$, q_j irreducible, and let N_j denote the zero set of $q_j \cdot$. If for $i \neq j$, $N_i \cap N_j$ has dimension < 2, then it is easily seen that $\Sigma(P)$ cannot have dimension 2. So assume without loss that $N_1 \cap N_2$ is 2-dimensional. Then due to Proposition 3.2 (i) and the irreducibility of q_1 and q_2, $q_1 =$ const q_2 follows verifying (3.1).

Consider now the zero set of p, $N(p) = \{z \in \mathbb{C}^3 | p(z) = 0\}$. Let $z_0 \in N(p)$ with $|\nabla p(z_0)| \neq 0$.

Case (i): There is a neighbourhood Ω_0 of z_0, such that $N(p) \cap \Omega_0$ is not characteristic with respect to the Laplacian in $\mathbb{C}^3$. Then the Cauchy problem $\Delta u = 0$, with Cauchy data $u = 0$, $\partial_{\mathfrak{n}} u = 0$ on $N(p) \cap \Omega_0$ has due to the Theorem of Cauchy-Kowalewsky for the complex case (see e.g. [Hö]) only the trivial solution $u \equiv 0$. But $\Delta P = 0$ in $\mathbb{C}^3$, $P = 0$ in $N(p)$, and since

$$\nabla P = 2pq\nabla p + p^2 \nabla q,$$

also $|\nabla P| = 0$ in $N(p)$. This implies $P \equiv 0$ which is a contradiction to our assumption.

Case (ii): $N(p)$ is characteristic with respect to the Laplacian, i.e.

$$\sum_{j=1}^{3} \left(\frac{\partial p}{\partial z_j}\right)^2 (z) = 0$$

Lemma 3.1. *Let p be a homogeneous irreducible polynomial in $\mathbb{C}^3$, where $N(p)$ is characteristic with respect to the Laplacian. Then p is either linear or equals (up to a constant multiplicative factor) $z_1^2 + z_2^2 + z_3^2$.*

Suppose we have proven Lemma 3.1, then the case $P = (z_1^2 + z_2^2 + z_3^2)^2 q$ can be excluded as follows: Let $M, M \geq 4$, denote the degree of P. It is well known (see e.g. [SW]) that a homogeneous polynomial q of degree $M-4$ (with real coefficients) can be written in polar coordiantes $\omega = \frac{x}{r}$, $r = |x|$ for $x \in \mathbb{R}^3$ as

$$q(r\omega) = r^{M-4} \sum_{j=0}^{2m} Y_{M-4-j}(\omega)$$

with

$$m = \begin{cases} (M-4)/2 & \text{for } M \text{ even} \\ (M-5)/2 & \text{for } M \text{ odd}, \end{cases}$$

where $Y_l(\omega)$ are suitable spherical harmonics, i.e. the restriction of a harmonic homogeneous polynomial of degree l to the unitsphere S^2. But since $P(r\omega) = r^M Y_M(\omega) = r^4 q$ we obtain from the above for $r = 1$

$$Y_M = \sum_{j=0}^{2m} Y_{M-4-j} \text{ on } S^2$$

which is a contradiction, since

$$\int_{S^2} Y_M^2 d\omega \neq 0 \text{ and } \int_{S^2} Y_M Y_k d\omega = 0 \quad \text{for} \quad M \neq k.$$

Therefore we obtain from (3.1), Lemma 3.1 and the above that

$$P(z) = L^2(z) q(z) \text{ with } L(z) = < a|z >, \quad a \in \mathbb{C}^3 \backslash \{\mathcal{O}\} \tag{3.2}$$

where the plane $\varepsilon = \{z \in \mathbb{C}^3 | < a|z >= 0\}$

is characteristic with respect to the Laplacian. Hence

$$\sum_{j=1}^{3} a_j^2 = 0, \quad a = (a_1, a_2, a_3).$$

Clearly $\varepsilon \cap \mathbb{R}^3$ is a 1-dimensional subspace of $\mathbb{R}^3$. Without loss we choose our coordinate system such that $\varepsilon = \{z \in \mathbb{R}^3 / z_1 - i z_z = 0\}$. Then $\varepsilon \cap \mathbb{R}^3 = \{(0,0,\lambda) | \lambda \in \mathbb{R}\}$ and denoting the real parts of z_1, z_2, z_3 by x, y, z we have

$$P(x,y,z) = (x - iy)^2 q(x,y,z) \text{ for } (x,y,z) \in \mathbb{R}^3. \tag{3.3}$$

Now we use that P has real coefficients and that P is harmonic: Since P restricted to S^2 is a spherical harmonic Y it follows (see e.g. [C]) that the order of vanishing of P in $(0,0,1)$, say M, is equal to the number of nodal lines of Y passing through $(0,0,1)$. Since $(x-iy)^2|_{S^2}$ has the only zeros $(0,0,1)$ and $(0,0,-1)$, these nodal lines are nodal lines of $q|_{S^2}$. But then the order of vanishing of q in $(0,0,1)$ is at least M. On the other hand the order of vanishing of P in a point is equal to the sum of the orders of vanishing of the factors q and $(x-iy)^2$, which is in contradiction to the foregoing.

Finally we present two versions of the proof of Lemma 3.1, namely an analytical and a geometrical one:

Analytical proof of Lemma 3.1. Let $p(z), z \in \mathbb{C}^3$ be a homogeneous and irreducible polynomial, such that

$$\sum_{j=1}^{3} \left(\frac{\partial p}{\partial z_j} \right)^2 (z) = 0 \text{ for } z \in N(p).$$

(Note that due to Proposition 3.1 (ii) $\Sigma(p)$ is at most 1-dimensional). For the following it will suffice to consider $N(p) \backslash \Sigma(p) \cap \{z_3 = 1\} \equiv N_1(p)$. Since p is irreducible, $N_1(p)$ can be represented as $\Gamma(w) = (w, f(w), 1)$, $w \in \mathbb{C}$, with some holomorphic function f, so

$$p(\Gamma(w)) = 0 \quad \forall w \tag{3.4}$$

which implies

$$0 = \frac{d}{dw} p(\Gamma(w)) =< \overline{\nabla p(\Gamma(w))} | \Gamma'(w) > \tag{3.5}$$

Noting that $\Gamma'(w) = (1, f'(w), 0)$, (3.5) becomes

$$\frac{\partial p}{\partial z_1}(\Gamma(w)) + \frac{\partial p}{\partial z_2}(\Gamma(w)) f'(w) = 0 \tag{3.6}$$

Let ε denote the tangent plane to $N(p)$ in the point $\tilde{z} = \Gamma(\tilde{w})$, $\tilde{z} \in N_1(p)$. Then ε is given by

$$< \overline{\nabla p(\tilde{z})} | z - \tilde{z} >= 0, \quad z \in \mathbb{C}^3. \tag{3.7}$$

Since $p(\lambda \tilde{z}) = 0 \; \forall \lambda \in \mathbb{C}$ we have

$$0 = \frac{d}{d\lambda} p(\lambda \tilde{z}) =< \overline{\nabla p(\tilde{z})} | \tilde{z} >,$$

which for $\lambda = 1$ implies $\forall z \in N_1(p)$

$$\begin{aligned} 0 &=< \overline{\nabla p(z)} | z > \\ &= \frac{\partial p}{\partial z_1}(\Gamma(w)) w + \frac{\partial p}{\partial z_2}(\Gamma(w)) f(w) + \frac{\partial p}{\partial z_3}(\Gamma(w)). \end{aligned} \tag{3.8}$$

Combining (3.6) with (3.8) we obtain

$$\frac{\partial p}{\partial z_3}(\Gamma(w)) = \frac{\partial p}{\partial z_2}(\Gamma(w))(f'(w) w - f(w)). \tag{3.9}$$

Furthermore from (3.6) and (3.9) we have

$$\nabla p(\Gamma(w)) = \frac{\partial p}{\partial z_2}(\Gamma(w))(-f'(w), 1, f'(w) w - f(w)). \tag{3.10}$$

Now we use that $N(p)\backslash\Sigma(p)$ is characteristic, which together with (3.6) leads to

$$\left(\frac{\partial p}{\partial z_3}(\Gamma(w))\right)^2 = -\left(\frac{\partial p}{\partial z_2}(\Gamma(w))\right)^2 (1+f^{'2}(w)). \tag{3.11}$$

For $\frac{\partial p}{\partial z_2}(\Gamma(w)) \neq 0$ we conclude via (3.9) and (3.11) that

$$1+f^{'2}(w)+(f'(w)w-f(w))^2 = 0. \tag{3.12}$$

Let $\tilde{w}$ be as before and let $a=(a_1,a_2,a_3)$ with

$$a_1 = -f'(\tilde{w}), \quad a_2 = 1, \quad a_3 = f'(\tilde{w})\tilde{w} - f(\tilde{w}).$$

We define $L(z) =< \bar{a}|z >$. Clearly for $z_3 = 1$ the zero set of L can be represented by

$$L(w,g(w),1) = 0, \quad w \in \mathbb{C} \tag{3.13}$$

for some holomorphic function g. For $\tilde{z} = \Gamma(\tilde{w})$, $\nabla p(\tilde{z}) = \frac{\partial p}{\partial z_2}(\tilde{z})a$, so that due to (3.8) $L(\tilde{z}) = 0$. Therefore $f(\tilde{w}) = g(\tilde{w})$. Furthermore f and g satisfy the differential equation

$$1+F^{'2}+(wF^{'2}-F)^2 = 0. \tag{3.14}$$

For f this is true due to (3.12) and for g this is straight forward to verify:

Because of (3.12), $a_1^2+1+a_3^2 = 0$, and because of (3.13), $a_1w+g(w)+a_3 = 0$, $\forall w$. Therefore $g(w) = -a_1w-a_3$ and $g'(w) = -a_1 = -f'(\tilde{w})$. Hence $a_3 = g'(w)w-g(w)$ implying (3.14) with $F = g$.

Let without loss $1+w^2 \neq 0$, then we obtain from (3.14)

$$F' = G_{\pm}(w,F), \text{ where} \tag{3.15}$$

$$G_{\pm}(w,F) = (1+w^2)^{-1}(wF \pm i\sqrt{1+w^2+F^2}).$$

If $1+\tilde{w}^2+f^2(\tilde{w}) \neq 0$, there are neighbourhoods $U(\tilde{w}) \subset \mathbb{C}$ and $V(f(\tilde{w})) \subset \mathbb{C}$ where $G_{\pm}(w,F)$ is holomorphic and $|G_{\pm}(w,F)|$ bounded. Hence the initial value problems

$$F' = G_{\pm}(w,F), F(\tilde{w}) = f(\tilde{w})$$

have unique solutions in a neighbourhood of $\tilde{w}$. Now the foregoing implies that $f = g$ there, and by the analyticity $f \equiv g$. Since p is irreducible it follows from Proposition 3.2(i) that p equals L up to a multiplicative constant.

If $1+w^2+f^2(w) = 0$, $\forall w$, then p equals up to a multiplicative constant $q(z) = z_1^2+z_2^2+z_3^2$: Since q is homogeneous and irreducible the zero set of q for $z_3 = 1$ can be represented by $w^2+g(w)^2+1 = 0$, with g holomorphic. We obtain $f^2 = g^2$ which leads again by Proposition 3.2(i) to the desired result.

Geometrical proof of Lemma 3.1.. Denote by α the projective quadric $\Sigma_{j=1}^3 z_j^2$ in $\mathbb{C}^3$ and let α^* be tangent to the quadric α in $\mathbb{C}^{3*}$. If ξ_j, $1 \leq j \leq 3$, is a dual basis in $\mathbb{C}^{3*}$, then α^* has the form $\Sigma_{j=1}^3 \xi_j^2$. Let $\gamma^* \subset P(\mathbb{C}^{3*})$ be the set of tangent planes to the characteristic surface $N(p)\backslash\Sigma(p)$. Evidently, the characteristic property of the surface implies that $\gamma^* \subset \text{im}(\alpha^*)$ (see [B], ch.14.5). Since $\text{im}(\alpha^*)$ is an algebraic curve in $P(\mathbb{C}^{3*})$ we have: either γ^* is a point and thus p is linear, or γ^* coincides with $\text{im}(\alpha^*)$. In the last case it follows from Proposition 3.2(i) that γ^* is the image of the quadric $\Sigma_{j=1}^3 \xi_j^2$ and hence $p = \text{ const } \Sigma_{j=1}^3 z_j^2$.

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M. Hoffmann-Ostenhof: Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria
E-mail address: `mho@nelly.mat.univie.ac.at`

T. Hoffmann-Ostenhof: Institut für Theoretische Chemie, Universität Wien, Währinger Strasse 17, A-1090 Wien, Austria and International Erwin Schrödinger Institute for Mathematical Physics, Pasteurgasse 6/7, A-1090 Wien, Austria
E-mail address: `thoffman@esi.ac.at`

N. Nadirashvili: International Erwin Schrödinger Institute for Mathematical Physics, Pasteurgasse 6/7, A-1090 Wien, Austria and Institute of Earth Physics, Moscow, Russia