Good formal structures for flat meromorphic connections, III: Towards functorial modifications

Kiran S. Kedlaya

unstable draft; version of February 14, 2010

Abstract

Given a formal flat meromorphic connection over an excellent scheme over a field of characteristic zero, we proved existence of good formal structures and a good Deligne-Malgrange lattice after suitably blowing up. For the corresponding situation over a complex analytic space, one immediately obtains the existence of suitable blowups locally, but it is not clear that these blowups can be glued together. We outline an approach to constructing a global blowup by making the result for excellent schemes functorial for regular morphisms. However, the approach remains conditional on resolution of a problem of birational geometry (canonical determination of nef Cartier b-divisors).

Introduction

The Hukuhara-Levelt-Turrittin decomposition theorem gives a classification of differential modules over the field $\mathbb{C}((z))$ of formal Laurent series resembling the decomposition of a finite-dimensional vector space equipped with a linear endomorphism into generalized eigenspaces. It implies that after adjoining a suitable root of z, one can express any differential module as a successive extension of one-dimensional modules. This classification serves as the basis for the asymptotic analysis of meromorphic connections around a (not necessarily regular) singular point. In particular, it leads to a coherent description of the *Stokes phenomenon*, i.e., the fact that the asymptotic growth of horizontal sections near a singularity must be described using different asymptotic series depending on the direction along which one approaches the singularity. (See [36] for a beautiful exposition of this material.)

The purpose of this paper, and its prequels [17, 18], is to give some higher-dimensional analogues of the Hukuhara-Levelt-Turrittin decomposition for *irregular* flat formal meromorphic connections on complex analytic or algebraic varieties. (The regular case is already well understood by work of Deligne [6].) We do not discuss asymptotic analysis or the Stokes phenomenon; these have been treated in the two-dimensional case by Sabbah [30] (building on work of Majima [25]), and one expects the higher-dimensional case to behave similarly.

In the remainder of this introduction, we recall what was established in [17] and [18], then explain what is added in this paper.

0.1 Resolution of turning points

In [17], we developed a numerical criterion for the existence of a *good decomposition* (in the sense of Malgrange [26]) of a formal flat meromorphic connection at a point where the polar divisor has normal crossings. This criterion is inspired by the treatment of the original decomposition theorem given by Robba [29] using spectral properties of differential operators on nonarchimedean rings; our treatment depends heavily on joint work with Xiao [16] concerning differential modules on some nonarchimedean analytic spaces.

We then applied this criterion to prove a conjecture of Sabbah [30, Conjecture 2.5.1] concerning formal flat meromorphic connections on a two-dimensional complex algebraic or analytic variety. We say that such a connection has a *good formal structure* at some point if it acquires a good decomposition after pullback along a finite cover ramified only over the polar divisor. In general, even if the polar divisor has normal crossings, one only has good formal structures away from some discrete set, the set of *turning points*. However, Sabbah conjectured that one can replace the given surface with a suitable blowup in such a way that the pullback connection admits good formal structures everywhere. Such a blowup might be called a *resolution of turning points*; we constructed it using the numerical criterion plus some analysis on a certain space of valuations (called the *valuative tree* by Favre and Jonsson [9]).

In [18], we constructed resolutions of turning points for formal flat meromorphic connections on excellent schemes of characteristic zero, which include algebraic varieties of all dimensions over any field of characteristic zero. This combined the numerical criterion of [17] with a more intricate valuation-theoretic argument, based on the properties of one-dimensional Berkovich nonarchimedean analytic spaces.

We also obtained a partial result for complex analytic varieties, using the fact that the local ring of a complex analytic variety at a point is an excellent ring. Namely, we obtained *local* resolution of turning points, i.e., we only construct a good modification in a neighborhood of a fixed starting point. For excellent schemes, one can always extend the resulting local modifications, by taking the Zariski closure of the graph of a certain rational map, then take a global modification dominating these. However, this approach is not available for analytic varieties.

0.2 Functorial modifications

To remedy this problem, we would like to refine our construction for excellent schemes, to obtain resolutions of turning points which are *functorial* for regular morphisms on the base space. This would include open immersions, so the local modifications will then be forced to patch together to give a global modification achieving good formal structures everywhere, even in the complex analytic setting. (Here the adjective *regular* does not carry its colloquial meaning of simply emphasizing that the morphism really is a true morphism of schemes, and not a rational morphism! Rather, it means that the morphism is flat with geometrically regular fibres; for instance, any smooth morphism is regular.)

The possibility of constructing functorial resolutions of turning points is suggested by the corresponding situation for the problem of resolution of singularities. It is possible to give a resolution of singularities for quasi-excellent schemes over a field of characteristic zero which is functorial for regular morphisms; this has been done recently by Temkin [33], using the resolution algorithm for complex algebraic varieties given by Bierstone and Milman [1], [2]. Similarly, Temkin has also given a functorial embedded resolution of singularities for quasi-excellent schemes over a field of characteristic zero [34].

Beware that functorial resolution of singularities is sometimes called *canonical* resolution of singularities, but this terminology is misleading because there is no uniqueness property. For instance, Temkin's proofs can in principle be adapted to other functorial resolution algorithms for complex analytic varieties (several of which are described in [12]); this should lead to different (but still functorial) nonembedded and embedded resolutions of singularities for quasi-excellent schemes over a field of characteristic zero.

0.3 Irregularity on Riemann-Zariski space

It is not straightforward to modify the construction of resolutions of turning points from [18] to obtain functoriality for regular morphisms, since the construction is highly local in nature. A better approach is to take the existence of nonfunctorial resolutions of turning points as a black box, then make further use of the numerical criterion for good formal structures to make a better construction.

Consider a meromorphic differential module $\mathcal{E}$ on an excellent scheme. After Malgrange, we construct from the differential module a canonical function, the *irregularity*, on the set of exceptional divisors on local modifications of the base space. One may view this function as a *Weil divisor on the Riemann-Zariski space* in the language of Boucksom-Favre-Jonsson [3], or as a *b-divisor* in the language of Shokurov [31].

In this language, the numerical criterion for good formal structures proved in [17] asserts that $\mathcal{E}$ admits good formal structures everywhere if and only if the irregularity functions of both $\mathcal{E}$ and $\mathrm{End}(\mathcal{E})$ are computed by a certain Cartier divisor on the base space. The existence of a resolution of turning points then implies that the irregularity b-divisor is computed by a Cartier divisor on some blowup of the original space, i.e., it is a *Cartier b-divisor*.

One can say somewhat more about the irregularity b-divisor. In order to construct functorial resolutions of turning points, we would like to know that the irregularity b-divisor is *relatively semiample*, i.e., that it is a nonnegative rational multiple of the Cartier b-divisor arising from some coherent fractional ideal sheaf on the base space. This turns out to be somewhat difficult, due to the slippery nature of the semiample condition. What we can show is the weaker statement that the irregularity b-divisor is *relatively nef*, that is, it is a limit (for an appropriate topology) of relatively semiample Cartier b-divisors. This follows easily from the spectral interpretation of irregularity.

0.4 The functorial determination problem

Given our inability to establish that irregularity b-divisors are semiample, our approach to functorial resolutions of turning points founders on a problem of birational geometry which has nothing to do with differential modules. Namely, given a Cartier b-divisor on an excellent scheme X over a field of characteristic zero, one must construct a modification of X on which this b-divisor is defined by a Cartier divisor, in a manner functorial for regular morphisms on X. We call this the *functorial determination problem* for Cartier b-divisors.

It may be helpful to restate this problem without reference to b-divisors. Consider triples (X, f, D) in which X is an excellent scheme over a field of characteristic zero, $f : Y \to X$ is a modification, and D is a Cartier divisor on Y. (We only consider integral Cartier divisors here, but the corresponding problems for rational or real Cartier divisors may also be considered.) The problem is to assign to each triple a pair (g, E) with $g : Z \to X$ another modification and E a Cartier divisor on Z, subject to the following conditions.

(a) If $(X, f, D) \mapsto (g, E)$, then the pullbacks of D and E to $Y \times_X Z$ coincide.

(b) If $h : X' \to X$ is a regular morphism and $(X, f, D) \mapsto (g, E)$, then $(X', h^*(f), h^*(D)) \mapsto (h^*(g), h^*(E))$.

(c) If $(X, f, D) \mapsto (g, E)$ and $f' : Y' \to Y$ is a modification, then $(X, f \circ f', (f')^*(D)) \mapsto (g, E)$ also.

It is condition (c) that prevents the trivial solution $(g, E) = (f, D)$.

If one restricts to the triples in which D is f-semiample, then there is a natural solution of the functorial determination problem. Namely, for m sufficiently divisible, the blowup of X along the coherent fractional ideal sheaf $f_*\mathcal{O}(mD)$ is independent of m, and is the unique minimal blowup of X on which there is a Cartier divisor defining the same b-divisor as D. Consequently, if we knew that irregularity b-divisors were always relatively semiample, we would be able to use Temkin's embedded resolution theorem to obtain functorial resolution of turning points.

Since we only know that irregularity b-divisors are relatively nef, we are stuck. It seems likely that given a nef Cartier b-divisor, one can construct functorially a finite-dimensional space of b-divisors containing the given one, which is spanned by its relatively semiample members. This would allow for a functorial determination of nef Cartier b-divisors, which would suffice for functorial resolution of turning points. We can construct several natural candidates for such a space, but we do not know how to establish finite dimensionality for any of them.

0.5 Further remarks

An alternate approach to resolution of turning points has been given Mochizuki [27, 28] in certain cases (notably algebraic connections on algebraic varieties), using algebraic and analytic properties of Deligne-Malgrange lattices (i.e., Malgrange's *canonical lattices*). However, it seems to be limited to true meromorphic connections, rather than formal meromorphic connections. We do not know whether Mochizuki's techniques can be made functorial for regular morphisms.

Another possible alternate approach to constructing resolutions of turning points in the analytic category is to give a purely analytic variant of the arguments in [18]. For this, one would replace the Riemann-Zariski space with its analytic analogue, the *voûte étoilée* of Hironaka [13]. We have made no attempt to carry out this approach.

We insert one remark concerning the analogy between irregularity of meromorphic connections and wild ramification of finite morphisms of schemes. Let X be a smooth scheme of characteristic $p > 0$. Suppose τ is a discrete linear representation of the étale fundamental group of X. Following the work of Kato [15] in the case of a one-dimensional representation, it would be desirable to define a *Swan conductor* of τ measuring the wild ramification of τ along different divisorial valuations on X. Our experience with good formal structures suggests that one might only be able to define the Swan conductor as a *virtual* sheaf on X.

Acknowledgments

Thanks to Mattias Jonsson, James McKernan, Rob Lazarsfeld, Mircea Mustaţă, and Michael Temkin for helpful discussions, and to the Institute for Advanced Study for hospitality during the 2009-2010 academic year. Financial support was provided by NSF CAREER grant DMS-0545904, DARPA grant HR0011-09-1-0048, MIT (NEC Fund, Cecil and Ida Green Career Development Professorship), and the Institute for Advanced Study (NSF grant DMS-0635607, James D. Wolfensohn Fund).

1 Tools from birational geometry

We start by gathering some tools from birational geometry. These include Temkin's functorial desingularization theorems, and divisors which are ample or nef relative to a morphism.

1.1 Basic notations

Notation 1.1.1. For X a noetherian integral separated scheme, let

$$\mathrm{Div}_{\mathbb{Z}} X, \mathrm{Div}_{\mathbb{Q}} X, \mathrm{Div}_{\mathbb{R}} X, \mathrm{CDiv}_{\mathbb{Z}} X, \mathrm{CDiv}_{\mathbb{Q}} X, \mathrm{CDiv}_{\mathbb{R}} X$$

denote respectively the groups of integral Weil divisors, rational Weil divisors, (real) Weil divisors, integral Cartier divisors, rational Cartier divisors, and (real) Cartier divisors on X. We topologize $\mathrm{Div}_{\mathbb{R}} X$ and $\mathrm{CDiv}_{\mathbb{R}} X$ as the *locally convex direct limits* of their finite-dimensional $\mathbb{R}$-subspaces, in the sense of [4, §II.6]. By Lemma 1.1.2 below, this means that a net converges if and only if it eventually lands in a finite-dimensional space and converges therein.

Although the spaces we are considering are not separable when their dimensions are not countable, one still has the following fact.

Lemma 1.1.2. *Let V be an $\mathbb{R}$-vector space topologized as the locally convex direct limit of its finite-dimensional subspaces. Let $\{x_i\}_{i\in I}$ be a net in V indexed by a directed set I, which converges to an element $x \in V$. Then there exist a finite-dimensional subspace T of V and an index $i \in I$ such that $x_j \in T$ for all $j \geq i$. In particular, any convergent net in V admits a convergent subsequence.*

Proof. We reduce at once to the case $x = 0$. Suppose no such T, i exist. We define a strictly increasing sequence $i_1 < i_2 < \cdots$ in I as follows: given $i_1, \dots, i_{n-1}$ for some $n \geq 1$, choose $i_n \geq i_{n-1}$ such that x_{i_n} is not in the $\mathbb{R}$-span of $x_{i_1}, \dots, x_{i_{n-1}}$. (If this were not possible, we would have $T = \mathbb{R}x_{i_1} + \cdots + \mathbb{R}x_{i_{n-1}}$, and $i = i_{n-1}$ would satisfy the conclusion of the lemma, contrary to hypothesis.)

Let W be the convex hull of the union of the sets $\{tx_{i_n} : t \in (-1,1)\}$ for $n = 1, 2, \dots$. By [4, §II.5, Exemple], W is an open neighborhood of 0 in V. However, it does not contain any of the x_{i_n}, contradicting the assumption that the net converges to 0. □

Notation 1.1.3. Recall that elements of $\mathrm{CDiv}_{\mathbb{Z}} X$ are by definition sections of the sheaf $\mathcal{K}_X^\times/\mathcal{O}_X^\times$, where $\mathcal{K}_X$ denotes the sheaf of total quotient rings on X. The definition of the line bundle $\mathcal{O}_X(D)$ associated to D follows the usual sign convention: if $D \in \mathrm{CDiv}_{\mathbb{Z}} X$ is represented on an open subset U of X by an element $f \in \Gamma(U, \mathcal{K}_X^\times)$, then we take $\mathcal{O}_X(D)|_U = f^{-1}\mathcal{O}_X$.

Notation 1.1.4. For X a noetherian integral separated scheme and $D \in \mathrm{Div}_{\mathbb{R}} X$, define the floor and ceiling $\lfloor D \rfloor, \lceil D \rceil \in \mathrm{Div}_{\mathbb{Z}} X$ by the formula

$$(\lfloor D \rfloor)(E) = \lfloor D(E) \rfloor, \qquad (\lceil D \rceil)(E) = \lceil D(E) \rceil,$$

in which $D(E)$ denotes the multiplicity of D along a geometric valuation E of X. Note that the floor and ceiling operations do not typically map $\mathrm{CDiv}_{\mathbb{R}} X$ to $\mathrm{CDiv}_{\mathbb{Z}} X$ unless X is locally factorial (e.g., if X is regular), in which case $\mathrm{Div}_{\mathbb{Z}} X = \mathrm{CDiv}_{\mathbb{Z}} X$.

1.2 Functorial resolution of singularities

The notion of quasi-excellent schemes was introduced into this series in [18, §1.1]. We have already made extensive use of the fact that over $\mathrm{Spec}(\mathbb{Q})$, such schemes admit nonembedded and embedded desingularization; this was originally proposed by Grothendieck, but only recently verified by Temkin [32].

In this paper, we need results of this form with the additional feature that the final modification is functorial for morphisms which are *regular*) (flat with geometrically regular fibres), such as smooth morphisms. Such results can be obtained by approximation arguments from a resolution algorithm for varieties over a field in which one repeatedly blows up so as to reduce some local invariant. One suitable algorithm is that of Bierstone and Milman [1] as refined by Bierstone, Milman, and Temkin [2]; using this algorithm, Temkin has established the following functorial desingularization theorems. (Temkin also obtains some control over the sequence of blowups used; we have not attempted to exert such control in the following statements.)

Theorem 1.2.1 (Temkin)**.** *Let* Sch *be the category of schemes. Let $\mathcal{C}$ be the subcategory of* Sch *whose objects are the reduced integral noetherian quasi-excellent schemes over* $\operatorname{Spec}(\mathbb{Q})$ *and whose morphisms are the regular morphisms of schemes. Let $\iota : \mathcal{C} \to \mathrm{Sch}$ denote the inclusion. There then exist a covariant functor $Y : \mathcal{C} \to \mathrm{Sch}$ and a natural transformation $F : Y \to \iota$ satisfying the following conditions.*

(a) For each $X \in \mathcal{C}$, the scheme $Y(X)$ is regular, and the morphism $F(X) : Y(X) \to X$ of schemes is a projective modification.

(b) For each regular $X \in \mathcal{C}$, $F(X)$ is an isomorphism.

(c) For each morphism $f : X' \to X$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} Y(X') & \xrightarrow{Y(f)} & Y(X) \\ \downarrow{\scriptstyle F(X')} & & \downarrow{\scriptstyle F(X)} \\ X' & \xrightarrow{f} & X \end{array}$$

is cartesian.

Proof. See [33, Theorem 1.2.1]. □

Definition 1.2.2. By a *schematic pair*, we will mean a pair (X, Z) in which X is a scheme and Z is a closed subscheme of X. We say such a pair is *regular* (and describe it for short as a *regular pair*) if X is regular and Z is a divisor of simple normal crossings on X.

By a *morphism* $f : (X', Z') \to (X, Z)$ of schematic pairs, we will mean a morphism $f : X' \to X$ of schemes for which $f^{-1}(Z) = Z'$. In other words, the inverse image ideal sheaf under f of the ideal sheaf $\mathcal{I}_Z$ defining Z should be the ideal sheaf $\mathcal{I}_{Z'}$ defining Z'.

Theorem 1.2.3 (Temkin)**.** *Let* Sch' *be the category of schematic pairs. Let $\mathcal{C}'$ be the subcategory of* Sch' *whose objects are the pairs for which the underlying schemes are regular integral noetherian quasi-excellent schemes over* $\operatorname{Spec}(\mathbb{Q})$*, and whose morphisms are those for which the underlying morphisms of schemes are regular. Let $\iota' : \mathcal{C}' \to \mathrm{Sch}'$ denote the inclusion. Then there exist a covariant functor $(Y, W) : \mathcal{C}' \to \mathrm{Sch}'$ and a natural transformation $F : (Y, W) \to \mathrm{id}_{\mathcal{C}'}$ satisfying the following conditions.*

(a) For each $(X, Z) \in \mathcal{C}'$, the pair $(Y, W)(X, Z)$ is regular, and the morphism $F(X, Z) : Y(X, Z) \to X$ is a projective modification.

(b) For each regular $(X, Z) \in \mathcal{C}'$, $F(X, Z)$ is an isomorphism.

(c) For each morphism $f : (X', Z') \to (X, Z)$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} Y(X', Z') & \xrightarrow{Y(f)} & Y(X, Z) \\ \downarrow{\scriptstyle F(X', Z')} & & \downarrow{\scriptstyle F(X, Z)} \\ X' & \xrightarrow{f} & X \end{array}$$

is cartesian.

Proof. See [34, Theorem 1.1.7]. □

1.3 Relatively ample, semiample, and nef divisors

We will use extensively the relative versions of the standard notions of ample, semiample, and nef divisors.

Hypothesis 1.3.1. Throughout § 1.3, let $f : X \to T$ be a proper morphism of noetherian schemes.

Definition 1.3.2. A line bundle $\mathcal{L}$ on X is *very ample relative to* f, or f*-very ample*, if the adjunction map $\rho : f^* f_* \mathcal{L} \to \mathcal{L}$ is surjective and defines an embedding $X \to \mathbb{P}(f_* \mathcal{L})$; this condition is local on T. The bundle $\mathcal{L}$ is *ample relative to* f, or f*-ample*, if locally on T, $\mathcal{L}^{\otimes m}$ is very ample relative to f for some positive integer m; this implies the same for each multiple of m. Since we are assuming T is noetherian, it follows that we can make a single good choice of m over all of T.

For $D \in \mathrm{CDiv}_{\mathbb{Z}} X$, we say D is (very) ample relative to f if the associated line bundle $\mathcal{O}_X(D)$ is. For $D \in \mathrm{CDiv}_{\mathbb{Q}} X$, we say D is ample relative to f if for some (and hence any) positive integer m such that $mD \in \mathrm{CDiv}_{\mathbb{Z}} X$, $\mathcal{O}_X(mD)$ is f-ample. Note that any positive rational linear combination of f-ample divisors is f-ample.

The condition of relative ampleness may be checked fibrewise in the following sense.

Theorem 1.3.3. *Let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is f-ample if and only if for each $t \in T$, if we identify t with* $\mathrm{Spec}(\kappa_t)$ *for κ_t the residue field of t, then $\mathcal{L}_t$ is ample relative to $f_t : X_t \to t$.*

Proof. (Compare [22, Theorem 1.7.8].) Amplitude is stable under base change, so the condition for $\mathcal{L}$ implies the same for each $\mathcal{L}_t$. Conversely, Grothendieck's criterion for amplitude [10, Théoréme 4.7.1] asserts that if $\mathcal{L}_t$ is f_t-ample, then $\mathcal{L}|_{f^{-1}(U)}$ is f-ample for some neighborhood U of t. Since amplitude is local on the base, it follows that $\mathcal{L}_t$ is f-ample. $\square$

Definition 1.3.4. A line bundle $\mathcal{L}$ on X is *free relative to* f, or f*-free*, if the adjunction map $f^* f_* \mathcal{L} \to \mathcal{L}$ is surjective. The bundle $\mathcal{L}$ is *semiample relative to* f, or f*-semiample*, if locally on T, $\mathcal{L}^{\otimes m}$ is f-free relative to f for some positive integer m; this implies the same for each multiple of m. Again, since T is noetherian, it follows that we can make a single good choice of m over all of T.

For $D \in \mathrm{CDiv}_{\mathbb{Z}} X$, we say D is free (resp. semiample) relative to f if the associated line bundle $\mathcal{O}_X(D)$ is. For $D \in \mathrm{CDiv}_{\mathbb{Q}} X$, we say D is semiample relative to f if for some (and hence any) positive integer m such that $mD \in \mathrm{CDiv}_{\mathbb{Z}} X$, $\mathcal{O}_X(mD)$ is f-semiample. Note that any positive rational linear combination of f-semiample divisors is f-semiample. Note also that if D is f-ample, then it is also f-semiample, but not conversely.

Definition 1.3.5. A line bundle $\mathcal{L}$ on X is *nef relative to* f, or f*-nef*, if for each $t \in T$ and each complete curve C in X_t, the restriction of $\mathcal{L}$ to C has nonnegative degree. For D a rational Cartier divisor on X, we say D is nef relative to f if some (and hence any) positive integral multiple of D which is an integral Cartier divisor corresponds to a line bundle which is nef relative to f. Note that if D is f-semiample, then it is also f-nef, but not conversely.

Definition 1.3.6. A (real) Cartier divisor D on X is *f-ample* (resp. *f-semiample*, *f-nef*) if it can be written as a positive (resp. nonnegative, nonnegative) real linear combination of rational Cartier divisors which are f-ample (resp. f-semiample, f-nef). By this definition, $D \in \mathrm{CDiv}_{\mathbb{Q}} X$ is f-ample (resp. f-semiample, f-nef) if and only its image in $\mathrm{CDiv}_{\mathbb{R}} X$ is.

Let $\mathrm{Amp}_f(X)$ and $\mathrm{Nef}_f(X)$ be the sets of f-ample and f-nef Cartier divisors on X, respectively. These are also called the *f-ample cone* and *f-nef cone*, respectively. The word "cone" is used because $\mathrm{Nef}_f(X)$ is closed under addition and nonnegative scalar multiplication, whereas $\mathrm{Amp}_f(X)$ is closed under addition and positive scalar multiplication. (One may also define an *f-semiample* cone closed under addition and nonnegative scalar multiplication.)

Over a point, the following is a consequence of Kleiman's criterion for ampleness [22, Corollary 1.4.10]; the general assertion follows from this observation plus Theorem 1.3.3.

Theorem 1.3.7. *Let D be an f-nef Cartier divisor on X, and let H be an f-ample Cartier divisor on X. Then $D + \epsilon H$ is f-ample for any $\epsilon > 0$.*

Corollary 1.3.8. *Suppose that f is projective. Then $\mathrm{Nef}_f(X)$ is the closure of $\mathrm{Amp}_f(X)$, and $\mathrm{Amp}_f(X)$ is the interior of $\mathrm{Nef}_f(X)$.*

Proof. (Compare [22, Theorem 1.4.23].) If $D \in \mathrm{Nef}_f(X)$, then for any $H \in \mathrm{Amp}_f(X)$ (which exists because f is projective and X is noetherian), $D + \epsilon H \in \mathrm{Amp}_f(X)$ for all $\epsilon > 0$ by Theorem 1.3.7. Hence $\mathrm{Amp}_f(X)$ is dense in $\mathrm{Nef}_f(X)$; since it is easy to see that $\mathrm{Nef}_f(X)$ is closed, it follows that $\mathrm{Nef}_f(X)$ is the closure of $\mathrm{Amp}_f(X)$.

On the other hand, if D is in the interior of $\mathrm{Nef}_f(X)$, then for any $H \in \mathrm{Amp}_f(X)$, $D - \epsilon H$ is still f-nef for $\epsilon > 0$ sufficiently small, and so $D = (D - \epsilon H) + \epsilon H$ is f-ample by Theorem 1.3.7 again. Hence $\mathrm{Amp}_f(X)$ contains the interior of $\mathrm{Nef}_f(X)$; since $\mathrm{Nef}_f(X)$ contains $\mathrm{Amp}_f(X)$, and it is again easy to see that $\mathrm{Amp}_f(X)$ is open (as in [22, Example 1.3.14]), it follows that $\mathrm{Amp}_f(X)$ is the interior of $\mathrm{Nef}_f(X)$. □

2 b-divisors and functorial determinations

In this section, we introduce the notions of Weil and Cartier divisors on a Riemann-Zariski space, more commonly known as b-divisors and Cartier b-divisors. We then introduce the notion of a *functorial determination* for a given class of Cartier b-divisors on a given class of schemes.

Hypothesis 2.0.1. Throughout § 2, let X be a noetherian integral separated excellent $\mathbb{Q}$-scheme.

2.1 Riemann-Zariski spaces

Definition 2.1.1. Recall that a *centered valuation* on X is a (Krull) valuation v on the function field $K(X)$ of X such that for some $x \in X$, the local ring $\mathcal{O}_{X,x}$ is contained in the valuation ring $\mathfrak{o}_v$. The set of equivalence classes of centered valuations on X is denoted $\mathrm{RZ}(X)$ and called the *Riemann-Zariski space*; we may also interpret $\mathrm{RZ}(X)$ as the inverse limit over modifications of X. Since X is an excellent $\mathbb{Q}$-scheme, this inverse limit may also be taken over projective modifications $Y \to X$ with Y regular, e.g., by Theorem 1.2.1.

Definition 2.1.2. A *geometric valuation* on X is a real valuation measuring order of vanishing along some prime divisor on some modification of X; in particular, it must be normalized in order to surject onto $\mathbb{Z}$. Let $\mathrm{RZ}^{\mathrm{geom}}(X)$ be the subset of $\mathrm{RZ}(X)$ consisting of equivalence classes of geometric valuations; each such class contains a single geometric valuation, so we typically identify it with that valuation.

Remark 2.1.3. For X of dimension n, a valuation is equivalent to a geometric valuation if and only if it is *divisorial*, i.e., discretely valued with residue field of transcendence degree $n-1$ over the residue field of the generic center of the valuation. See [35, Proposition 10.1].

2.2 b-divisors and Cartier b-divisors

Definition 2.2.1. The group of *integral b-divisors on X*, denoted $\mathbf{Div}_{\mathbb{Z}} X$, is the inverse limit of $\mathrm{Div}_{\mathbb{Z}} Y$ over all modifications $f : Y \to X$, in which transition maps are pushforwards. Define similarly the groups $\mathbf{Div}_{\mathbb{Q}} X, \mathbf{Div}_{\mathbb{R}} X$ of *rational b-divisors* and *b-divisors* on X. Note that for any modification $f : Y \to X$, the restriction maps $\mathbf{Div}_* X \to \mathbf{Div}_* Y$ are isomorphisms. For $D \in \mathbf{Div}_{\mathbb{R}} X$, the component $D(Y)$ of D on a modification Y of X is called the *trace* of D on Y.

We may identify b-divisors on X with certain real-valued functions on $\mathrm{RZ}^{\mathrm{geom}}(X)$. The functions that occur are those having the following finiteness property: for any modification $Y \to X$, the support of the function must only include finitely many geometric valuations corresponding to prime divisors on Y. Similarly, integral and rational b-divisors on X correspond to functions on $\mathrm{RZ}^{\mathrm{geom}}(X)$ with values in $\mathbb{Z}$ and $\mathbb{Q}$, respectively, satisfying the same finiteness condition.

Note that each transition map $\mathrm{Div}_{\mathbb{R}} Y \to \mathrm{Div}_{\mathbb{R}} X$ has finite-dimensional kernel and cokernel. Since both source and target are topologized as the locally convex direct limit of their finite-dimensional subspaces, the map is not only continuous, but *strict*; that is, the quotient and subspace topologies on its image coincide. It follows that if we equip $\mathbf{Div}_{\mathbb{R}} X$ with the inverse limit topology (as we will do hereafter), then a sequence (or net) converges in the inverse limit topology if its projection onto each $\mathrm{Div}_{\mathbb{R}} Y$ converges.

Remark 2.2.2. The term *b-divisor* was introduced by Shokurov [31] in his construction of 3-fold and 4-fold flips, but has since become standard in birational geometry. See [5] for further discussion. A very similar notion appears in the work of Boucksom-Favre-Jonsson [3], under the guise of Weil divisors on Riemann-Zariski spaces, albeit using only point blowups rather than arbitrary modifications.

Definition 2.2.3. The group of *integral Cartier b-divisors on X*, denoted $\mathbf{CDiv}_{\mathbb{Z}} X$, is the *direct* limit of $\mathrm{CDiv}_{\mathbb{Z}} Y$ over all modifications $f : Y \to X$, in which transition maps are pullbacks. Define similarly the groups $\mathbf{CDiv}_{\mathbb{Q}} X, \mathbf{CDiv}_{\mathbb{R}} X$ of *rational Cartier b-divisors* and *Cartier b-divisors* on X. Note that for any modification $f : Y \to X$, the transition maps $\mathbf{CDiv}_* X \to \mathbf{CDiv}_* Y$ are isomorphisms.

For any X, we have a map $\operatorname{CDiv}_* X \to \operatorname{Div}_* X$ which is injective if X is normal, and bijective if X is locally factorial. Thanks to the compatibility between pullback and pushforward for Cartier divisors, we also get a map $\mathbf{CDiv}_* X \to \mathbf{Div}_* X$; this map is injective because every modification of X is dominated by a normal modification (since X is excellent). We may thus view Cartier b-divisors as a subclass of all b-divisors.

For $D \in \mathbf{CDiv}_{\mathbb{R}} X$, a *determination* of D is a modification $f : Y \to X$ such that D belongs to the image of $\operatorname{CDiv}_{\mathbb{R}} Y$ in $\mathbf{CDiv}_{\mathbb{R}} X$.

Remark 2.2.4. Note that the floor and ceiling functions induce well-defined maps $\lfloor\cdot\rfloor, \lceil\cdot\rceil : \mathbf{Div}_{\mathbb{R}} X \to \mathbf{Div}_{\mathbb{Z}} X$, but again do not take Cartier divisors to Cartier divisors.

Definition 2.2.5. Let $\mathcal{I}$ be a nonzero ideal sheaf on X. Let $f : Y \to X$ be the blowup of X along $\mathcal{I}$; then the inverse image ideal sheaf $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ equals $\mathcal{O}_Y(Z(\mathcal{I}))$ for a certain Cartier divisor $Z(\mathcal{I}) \in \operatorname{CDiv}_{\mathbb{Z}} Y$, which we identify with its image in $\mathbf{CDiv}_{\mathbb{Z}} X$. We may similarly define $Z(\mathcal{I})$ for $\mathcal{I}$ a nonzero coherent fractional ideal sheaf on X, by taking the blowup of X along $\mathcal{I}\mathcal{J}$ for any nonzero locally principal ideal sheaf $\mathcal{J}$ on X for which $\mathcal{I}\mathcal{J} \subseteq \mathcal{O}_X$.

This definition incorporates a sign convention which has mixed virtues. On the positive side, for $\mathcal{I}, \mathcal{J}$ two nonzero coherent fractional ideal sheaves on X, we have $\mathcal{I} \subseteq \mathcal{J}$ if and only if $Z(\mathcal{I}) \leq Z(\mathcal{J})$. On the negative side, for $g \in K(X)^\times$, if we write $Z(g)$ for the b-divisor corresponding to the principal fractional ideal sheaf generated by g, then $Z(g)(v) = -v(g)$ for all $v \in \operatorname{RZ}^{\text{geom}}(X)$.

One can partly invert the passage from a nonzero coherent fractional ideal sheaf to its associated b-divisor by taking sections.

Definition 2.2.6. Given $D \in \mathbf{Div}_{\mathbb{R}} X$, define $\mathcal{L}^\infty(D)$ to be the (possibly zero) coherent fractional ideal sheaf on X such that for $U \subseteq X$ a nonempty open affine subset,

$$\mathcal{L}^\infty(D)(U) = \{g \in K(U) : Z(g) \leq D|_U\}.$$

We record some easy but useful properties.

- For $D \in \mathbf{Div}_{\mathbb{R}} X$ with $\mathcal{L}^\infty(D) \neq 0$, $Z(\mathcal{L}^\infty(D))$ is the supremum of $Z(\mathcal{I})$ over all nonzero coherent fractional ideal sheaves $\mathcal{I}$ on X with $Z(\mathcal{I}) \leq D$.

- For $D_1, D_2 \in \mathbf{Div}_{\mathbb{R}} X$ with $\mathcal{L}^\infty(D_1), \mathcal{L}^\infty(D_2) \neq 0$, we have $\mathcal{L}^\infty(D_1)\mathcal{L}^\infty(D_2) \subseteq \mathcal{L}^\infty(D_1 + D_2)$. Moreover, if $D_2 \in \operatorname{CDiv}_{\mathbb{Z}} X$, we have $\mathcal{L}^\infty(D_1)\mathcal{L}^\infty(D_2) = \mathcal{L}^\infty(D_1)\mathcal{O}_X(D_2)$.

In particular, if $\mathcal{I}$ is a nonzero coherent fractional ideal sheaf on X, then $\mathcal{I} \subseteq \mathcal{L}^\infty(Z(\mathcal{I}))$. Equality may fail to hold; for instance, if $\mathcal{I}$ is an ideal sheaf, then $\mathcal{L}^\infty(Z(\mathcal{I}))$ is the integral closure of $\mathcal{I}$ [14, Theorem 6.8.3], which may differ from $\mathcal{I}$ even in simple examples [14, Example 1.1.2].

2.3 Functorial determination

Recall that in a direct system of abelian groups, each element of each group gives rise to an element of the direct limit, but an element of the direct does not remember exactly which group it came from. In particular, a Cartier b-divisor on X does not remember any particular determination of it. The functorial determination problem is to provide a distinguished such choice.

Definition 2.3.1. Define a *schematic (integral, rational) Cartier b-pair* to be a pair (X, D) in which X is a noetherian integral separated excellent $\mathbb{Q}$-scheme and D is an (integral, rational) Cartier b-divisor on X. We view these as forming a category whose morphisms $f : (X', D') \to (X, D)$ are those dominant regular morphisms $f : X' \to X$ of schemes for which the induced pullback map $f^* \, \mathbf{CDiv}_{\mathbb{R}} X \to \mathbf{CDiv}_{\mathbb{R}} X'$ carries D to D'. (The map f^* may be described as follows: for any modification $g : Y \to X$, if $h : Y' \to Y$ denotes the proper transform of f under g, then f^* carries $\mathrm{CDiv}_{\mathbb{R}} Y$ to $\mathrm{CDiv}_{\mathbb{R}} Y'$ via the map h^*.)

Definition 2.3.2. Let $\mathcal{C}$ be a subcategory of the category of schematic Cartier b-pairs. Let $\pi : \mathcal{C} \to \mathrm{Sch}$ be the functor carrying the pair (X, D) to X. A *functorial determination* of $\mathcal{C}$ consists of a covariant functor $Y : \mathcal{C} \to \mathrm{Sch}$ and a natural transformation $F : Y \to \pi$ satisfying the following conditions.

(a) For all $(X, D) \in \mathcal{C}$, the map $F(X) : Y(X) \to X$ of schemes is a modification, and is a determination of D.

(b) For each morphism $f : (X', D') \to (X, D)$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} Y(X', D') & \xrightarrow{Y(f)} & Y(X, D) \\ \downarrow{\scriptstyle F(X',D')} & & \downarrow{\scriptstyle F(X,D)} \\ X' & \xrightarrow{f} & X \end{array}$$

is cartesian.

Example 2.3.3. Let $\mathcal{C}$ be the category of schematic rational Cartier b-pairs (X, D) for which D is relatively semiample. Then for each $(X, D) \in \mathcal{C}$, there exists a unique minimal determination $f : Y \to X$ of D, namely the blowup of X along $\mathcal{L}^\infty(mD)$ for any positive integer m such that $Z(\mathcal{L}^\infty(mD)) = mD$. (This is immediate from the universal property of blowing up; e.g., see [11, Proposition 7.14].) These evidently form a functorial determination of $\mathcal{C}$.

One can generalize the previous example as follows.

Definition 2.3.4. Let V be a finite-dimensional subspace of $\mathbf{CDiv}_{\mathbb{Q}} X$. We say V is *relatively semiample* if V is spanned by those $D \in V$ which are relatively semiample.

Lemma 2.3.5. *Let V be a finite-dimensional subspace of $\mathbf{CDiv}_{\mathbb{Q}} X$ which is relatively semiample. Then the set of modifications $f : Y \to X$ which are determinations for each semiample $D \in V$ has a unique minimal element.*

Proof. The minimality forces uniqueness, so we need only check existence. Let $D_1, \dots, D_k \in V$ be semiample elements which span X. We construct a sequence of modifications $f_i : X_i \to X_{i-1}$ for $i = 1, \dots, k$, with $X_0 = X$, as follows. Given $f_1, \dots, f_{i-1}$, let E_i be the pullback of D_i along $f_1 \circ \cdots \circ f_{i-1}$. Then let $f_i : X_i \to X_{i-1}$ be the unique minimal determination of E_i (as in Example 2.3.3). The composite $f = f_1 \circ \cdots \circ f_k$ is then a minimal modification which serves as a determination of each of $D_1, \dots, D_k$. □

This suggests the following definition. For some potential constructions realizing this definition, see Conjecture 2.4.7 and Conjecture 2.5.10.

Definition 2.3.6. Let $\mathcal{C}$ be a subcategory of the category of schematic rational Cartier b-pairs and dominant regular morphisms. Let $\mathrm{Vect}_{\mathbb{Q}}$ be the category of vector spaces over $\mathbb{Q}$. Let $\psi : \mathcal{C} \to \mathrm{Vect}_{\mathbb{Q}}$ be the contravariant functor carrying (X, D) to $\mathbf{CDiv}_{\mathbb{Q}} X$. A *functorial spread* of $\mathcal{C}$ consists of a contravariant functor $V : \mathcal{C} \to \mathrm{Vect}_{\mathbb{Q}}$ and a natural transformation $F : V \to \psi$, satisfying the following conditions.

(a) For each $(X, D) \in \mathcal{C}$, the image of $F(X, D)$ (as a subspace of $\mathbf{CDiv}_{\mathbb{Q}} X$) is relatively semiample and contains D.

(b) For each morphism $f : (X', D') \to (X, D)$ in $\mathcal{C}$, we have $f^*(V(X, D)) = V(X', D')$.

Using Lemma 2.3.5, given a functorial spread, we obtain a functorial determination of (X, D) by taking the minimal modification which serves as a determination for each relatively semiample element of $V(X, D)$. Conversely, given a functorial determination, we obtain a functorial spread by taking the space of rational Cartier divisors on $Y(X, D)$ whose support is contained in the support of $D(Y)$.

2.4 Relatively semiample and nef b-divisors

There is no good notion of ampleness for a Cartier b-divisor: if $f : Y \to X$ is a modification and D is an f-ample divisor on Y, then the pullback of D along a modification $g : Z \to Y$ is almost never $(f \circ g)$-ample. Thus there is no "ample cone" of Cartier b-divisors. However, the relative semiample and nef conditions do make sense for Cartier b-divisors, and we have a Kleiman-style result to the effect that the relative nef cone is the closure of the relative semiample cone. This will allows us to show that irregularity b-divisors are always nef, so that we need only solve the functorial determination problem in the nef case.

Definition 2.4.1. For $D \in \mathbf{CDiv}_{\mathbb{R}} X$, we say D is *relatively semiample* if for some determination $f : Y \to X$ of D, $Y(D)$ is f-semiample. The same then holds for any determination.

Definition 2.4.2. Note that the convex cone generated by the $Z(\mathcal{I})$ for all nonzero coherent fractional ideal sheaves $\mathcal{I}$ on X is also the convex cone generated by the relatively semiample Cartier b-divisors. We say $D \in \mathbf{Div}_{\mathbb{R}} X$ is *relatively nef* if it belongs to the closure of this cone.

Note that for $\mathcal{I}, \mathcal{J}$ two nonzero coherent fractional ideal sheaves on X,

$$Z(\mathcal{I}\mathcal{J}) = Z(\mathcal{I}) + Z(\mathcal{J})$$
$$Z(\mathcal{I} + \mathcal{J}) = \max\{Z(\mathcal{I}), Z(\mathcal{J})\}.$$

The first observation implies that the b-divisors of the form $m^{-1}Z(\mathcal{I})$, for m a positive integer and $\mathcal{I}$ a nonzero coherent fractional ideal sheaf, are dense among the relatively nef b-divisors. The second observation implies that the set of relatively nef b-divisors is closed under taking the pointwise supremum of an arbitrary collection of elements $\{D_i\}_{i \in I}$, provided that for each modification $f : Y \to X$, the traces $\{D_i(Y)\}_{i \in I}$ belong to a finite-dimensional subspace of $\operatorname{Div}_{\mathbb{R}} Y$ and are bounded above therein. (This in particular implies the existence of the supremum in $\mathbf{Div}_{\mathbb{R}} X$ itself.)

If $D \in \mathbf{Div}_{\mathbb{R}} X$ is relatively nef and $f : Y \to X$ is a modification, it does not follow that $D(Y)$ is f-nef even if D is Cartier. However, this does hold if D is Cartier and f is a determination; this amounts to a Kleiman-style criterion for relative nefness of Cartier b-divisors.

Proposition 2.4.3. *For any modification $f : Y \to X$ and any $D \in \operatorname{CDiv}_{\mathbb{R}} Y$, D is relatively nef as an element of $\mathbf{Div}_{\mathbb{R}} X$ if and only if D is f-nef.*

Proof. (Compare [3, Proposition 2.2].) Suppose first that D is f-nef. By replacing f by a further modification, we may assume f is projective. Choose a finite-dimensional subspace V of $\operatorname{CDiv}_{\mathbb{Q}} Y$ containing an f-ample divisor, such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contains D. By Corollary 1.3.8, the f-ample divisors in $V \otimes_{\mathbb{Q}} \mathbb{R}$ comprise the interior of the (closed convex) cone of f-nef divisors in $V \otimes_{\mathbb{Q}} \mathbb{R}$, and by construction this interior is nonempty. Hence D can be written as the limit in $V \otimes_{\mathbb{Q}} \mathbb{R}$ of a sequence $E_1, E_2, \dots$ of f-ample rational Cartier divisors. For $i = 1, 2, \dots$, choose a positive integer m_i for which $m_i E_i$ is f-very ample; then $m_i E_i = Z(\mathcal{I}_i)$ for $\mathcal{I}_i$ the image of $f_* \mathcal{O}_Y(mE_i)$ in $\mathcal{K}_X$. It follows that D is the limit of the $m_i^{-1} Z(\mathcal{I}_i)$, and thus is relatively nef.

Conversely, suppose D is relatively nef but not f-nef. Then there exists a complete curve C within some fibre of f such that the degree of $D(Y)|_C$ is negative. In particular, the Néron-Severi (i.e., numerical, or equivalently algebraic) class of $D(Y)|_C$ is not pseudoeffective, that is, it is not a limit of classes of effective rational Weil divisors on C. By Theorem 1.2.1, we can construct a modification $g : W \to Y$ dominating the blowup of C in Y, with W regular. Let E be an exceptional prime divisor of g dominating C. Then $D(W)|_E$ coincides with the pullback of $D(Y)|_C$, so its Néron-Severi class also fails to be pseudoeffective.

Since the set of pseudoeffective classes of E is a closed convex cone, we will derive a contradiction by checking that for any nonzero coherent fractional ideal sheaf $\mathcal{I}$ on X, for $W = Z(\mathcal{I})(Y)$, the Néron-Severi class of $W|_E$ is pseudoeffective. To wit, W is the divisorial part of $f^{-1}\mathcal{I}$, so the base locus of the global sections of $\mathcal{O}_Y(W)$ has codimension at least 2 on Y. It hence cannot contain E, so there must be a global section of $\mathcal{O}_Y(W)$ which does not vanish identically on E. In particular, the Néron-Severi class of $W|_E$ is pseudoeffective, yielding the desired contradiction. □

One key property of relatively nef b-divisors is that they can be successively approximated from above by their traces.

Lemma 2.4.4. *Let $D \in \mathbf{Div}_{\mathbb{R}} X$ be relatively nef. Let $f : Y \to X$ be a modification with Y locally factorial, so that $D(Y) \in \mathrm{Div}_{\mathbb{R}} Y$ may be identified with an element of $\mathbf{CDiv}_{\mathbb{R}} X$. Then $D \leq D(Y)$.*

Proof. (Compare [3, Proposition 2.4].) Since D is also relatively nef as an element of $\mathbf{Div}_{\mathbb{R}} Y$, it suffices to check that $D \leq D(X)$ when X is locally factorial. It further suffices to check the case $D = Z(\mathcal{I})$ for $\mathcal{I}$ a nonzero coherent fractional ideal sheaf on X. Moreover, we may subtract off the divisorial part of $\mathcal{I}$, so that $D(X) = 0$ and $\mathrm{Supp}(\mathcal{I})$ has codimension at least 2.

Since X is locally factorial and hence normal, a rational function on X has poles in codimension 1. Hence $\Gamma(X, \mathcal{O}_X) = \Gamma(X \setminus \mathrm{Supp}(\mathcal{I}), \mathcal{O}_X)$, so we must have $\mathcal{I} \subseteq \mathcal{O}_X$ and $D \leq 0 = D(X)$, as desired. □

Definition 2.4.5. Suppose $D \in \mathbf{Div}_{\mathbb{R}} X$ is such that $\mathcal{L}^\infty(D) \neq 0$. Then the set S of relatively nef b-divisors E on X satisfying $E \leq D$ is nonempty. Moreover, if we pick any $E_0 \in S$, then for each $E \in S$ we have $E_0 \leq \max\{E_0, E\} \leq D$. Hence on each modification $f : Y \to X$, the traces of the $\max\{E_0, E\}$ belong to a finite-dimensional $\mathbb{R}$-subspace of $\mathrm{Div}_{\mathbb{R}} Y$. We conclude that

$$\widehat{D} = \sup\{\max\{E_0, E\} : E \in S\}$$

exists in $\mathbf{Div}_{\mathbb{R}} X$ and is relatively nef. It is easy to see that the definition of $\widehat{D}$ does not depend on the choice of E_0.

We call $\widehat{D}$ the *nef envelope* of D. As suggested in [21], the construction of the nef envelope constitutes a higher-dimensional analogue of the classical Zariski decomposition of an effective divisor on a surface. (Compare [3, Definition 3.17].)

In certain cases, one can give an explicit convergent sequence of lower approximations to a relatively nef Cartier b-divisor as follows. (This can be generalized somewhat; see [3, Lemma 3.22].)

Proposition 2.4.6. *For $D \in \mathbf{CDiv}_{\mathbb{R}} X$, we have*

$$\widehat{D} = \sup_{k \in \mathbb{Z}, k>0} \{k^{-1} Z(\mathcal{L}^\infty(kD))\} = \sup_{k \in \mathbb{Z}, k>0} \{k^{-1} Z(\mathcal{L}^\infty(k\widehat{D}))\}.$$

Proof. For k a positive integer, put $D_k = k^{-1} Z(\mathcal{L}^\infty(kD))$. For any nonzero coherent fractional ideal sheaf $\mathcal{I}$ on X, $k^{-1} Z(\mathcal{I})$ is relatively nef, and so $k^{-1} Z(\mathcal{I}) \leq D$ if and only if $k^{-1} Z(\mathcal{I}) \leq \widehat{D}$. It follows that $D_k = k^{-1} Z(\mathcal{L}^\infty(k\widehat{D}))$. Since D_k is relatively nef and $D_k \leq k^{-1}(kD) = D$, we have $D_k \leq \widehat{D}$ for all k. It thus remains to prove that $\sup_k\{D_k\} \geq \widehat{D}$.

Let $f : Y \to X$ be a determination of D with Y regular. By Lemma 1.1.2, we may choose a sequence σ in $\mathbf{Div}_{\mathbb{R}} X$ converging to $\widehat{D}$, each of whose elements has the form $m^{-1} Z(\mathcal{I})$ for some positive integer m and some nonzero coherent fractional ideal sheaf $\mathcal{I}$ on

X, and all of whose elements have traces on Y lying in a finite-dimensional $\mathbb{R}$-subspace V of $\mathrm{Div}_{\mathbb{R}} Y = \mathrm{CDiv}_{\mathbb{R}} Y$. Let S be the finite set of prime divisors on Y occurring among the supports of elements of V; we may choose $J \in \mathrm{CDiv}_{\mathbb{Z}} X$ with $J \leq 0$ such that the multiplicity of J along each element of S is negative.

Let $\epsilon > 0$ be any positive rational. Beyond some point, each element E of σ satisfies $E(Y) + \epsilon J(Y) \leq \widehat{D}(Y) \leq D(Y)$, and hence $E + \epsilon J \leq D$ by Lemma 2.4.4 and the fact that D is Cartier. Let $E = m^{-1} Z(\mathcal{I})$ be one such element of σ. For $k > 0$ such that $k, km^{-1}, k\epsilon$ are all positive integers, we have $Z(\mathcal{I}^{k/m}\mathcal{O}_X(J^{k\epsilon})) = k(E + \epsilon J) \leq kD$, so $Z(\mathcal{I}^{k/m}\mathcal{O}_X(J^{k\epsilon})) \leq \mathcal{L}^\infty(kD)$. This yields $D_k \geq E + \epsilon J$ and hence $\sup_k\{D_k\} \geq E + \epsilon J$.

Since σ converges to $\widehat{D}$, we deduce that $\sup_k\{D\} \geq \widehat{D} + \epsilon J$. Since ϵ can be taken arbitrarily close to 0, we deduce $\sup_k\{D_k\} \geq \widehat{D}$ as desired. □

This suggests one approach to constructing a fundamental determination of nef b-divisors.

Conjecture 2.4.7. *Let $D \in \mathbf{CDiv}_{\mathbb{Q}} X$ be relatively nef. Then the divisors $Z(\mathcal{L}^\infty(nD))$ for $n = 1, 2, \dots$ span a finite-dimensional subspace of $\mathbf{CDiv}_{\mathbb{Q}} X$.*

Remark 2.4.8. If Conjecture 2.4.7 holds, then the space $V(X, D)$ spanned by the $Z(\mathcal{L}^\infty(nD))$ contains D by Proposition 2.4.6. It thus constitutes a functorial spread on the category of schematic rational Cartier b-pairs (X, D) in which D is relatively nef. This would follow if the $\mathcal{O}_X$-algebra $\oplus_{n=0}^\infty \mathcal{L}^\infty(nD)$ were always finitely generated; however, we expect the latter to fail to hold, just as as the section ring of a divisor on a smooth projective variety may fail to be finitely generated.

2.5 Multiplier ideals

We propose one further potential construction of fundamental determinations, now restricting to the case of regular schemes. This approach uses the construction of multiplier ideals; see [23, §9] for an introduction.

Hypothesis 2.5.1. Throughout § 2.5, in addition to Hypothesis 2.0.1, assume that X is regular. The correct definition in the singular case is subtler; see [23, §9.3.G].

Definition 2.5.2. Let K_X be the *canonical b-divisor*, whose trace on a modification $f : Y \to X$ equals the relative canonical divisor $K_{Y/X}$ (see [23, §9.1.B]). For $D \in \mathbf{Div}_{\mathbb{R}} X$, we define the *multiplier ideal sheaf* of D to be the coherent fractional ideal sheaf

$$\mathcal{L}^2(D) = \mathcal{L}^\infty(K_X + \lceil D \rceil).$$

For $\mathcal{I}$ a nonzero coherent fractional ideal sheaf on X, we write $\mathcal{L}^2(\mathcal{I})$ as shorthand for $\mathcal{L}^2(Z(\mathcal{I}))$.

Remark 2.5.3. We collect some easy consequences of Definition 2.5.2.

(a) For $\mathcal{I}$ a nonzero coherent fractional ideal sheaf on X, we always have $\mathcal{I} \subseteq \mathcal{L}^2(\mathcal{I})$ because $K_X \geq 0$ (as in [23, Proposition 9.2.32]).

(b) The formation of $\mathcal{L}^2$ commutes with taking the supremum of a sequence (or net), because $\lceil \cdot \rceil$ does so.

(c) For $D \in \mathbf{Div}_{\mathbb{R}} X$ and $E \in \mathrm{CDiv}_{\mathbb{Z}} X$, $\mathcal{L}^2(D + E) = \mathcal{L}^2(D)\mathcal{O}(E)$. (Compare [23, Proposition 9.2.31].)

In order to use existing results about multiplier ideals, we will need to reconcile this construction with the usual definition for Cartier b-divisors.

Definition 2.5.4. For $D \in \mathbf{CDiv}_{\mathbb{R}} X$, a *log-resolution* of X is a determination $f : Y \to X$ of D such that (Y, W) is a regular pair for $W = \mathrm{Supp}(D(Y)) \cup \mathrm{Supp}(\mathrm{except}(f))$. Such a determination always exists by Theorem 1.2.3. (The definition in [23, Definition 9.1.10] puts $W = \mathrm{Supp}(D(Y) + \mathrm{except}(f))$; this is merely a typo.)

Lemma 2.5.5. *For $D \in \mathbf{CDiv}_{\mathbb{R}} X$, for any log-resolution $f : Y \to X$ of D,*

$$\mathcal{L}^2(D) = f_* \mathcal{O}_Y(K_{Y/X} + \lceil D(Y) \rceil).$$

Proof. (Compare [23, Theorem 9.2.18].) Let $E_1, \dots, E_r$ be the irreducible components of $W = \mathrm{Supp}(D(Y)) \cup \mathrm{Supp}(\mathrm{except}(f))$ containing the center of v on Y. Let $g : Z \to Y$ be the blowup of Y along $E_1 \cap \dots \cap E_r$, and let E be the irreducible component of $\mathrm{except}(g)$ containing the center of v on Z. Then the multiplicity of $K_{Z/Y} = K_{Z/X} - f^* K_{Y/X}$ along E is $r - 1$. By contrast, if the multiplicity of $D(Y)$ along E_i equals e_i for $i = 1, \dots, r$, then the multiplicity of $\lceil D(Y) \rceil$ along E equals $\lceil e_1 \rceil + \dots + \lceil e_r \rceil$, while the multiplicity of $\lceil D(Z) \rceil$ along E equals

$$\lceil e_1 + \dots + e_r \rceil \geq \lceil e_1 \rceil + \dots + \lceil e_r \rceil - r + 1.$$

We deduce that $K_{Y/X} + \lceil D(Y) \rceil \leq K_{Z/X} + \lceil D(Z) \rceil$ as elements of $\mathbf{Div}_{\mathbb{R}} X$. By repeating this argument, we deduce that

$$K_{Y/X} + \lceil D(Y) \rceil \leq K_X + \lceil D \rceil,$$

that is, $K_X + \lceil D \rceil$ is bounded below by its trace on Y. This yields the claim. □

The following subadditivity property of multiplier ideals is originally due to Demailly, Ein, and Lazarsfeld [7] in the case of ideal sheaves, but it extends readily to relatively nef Cartier b-divisors. As noted in [3], the approximation argument we use here is essentially due to Ein, Lazarsfeld, and Smith [8]; it can also be found in [23, Chapter 11] in the language of graded systems of ideals.

Theorem 2.5.6. *For $D_1, D_2 \in \mathbf{CDiv}_{\mathbb{R}} X$ relatively nef,*

$$\mathcal{L}^2(D_1 + D_2) \subseteq \mathcal{L}^2(D_1)\mathcal{L}^2(D_2).$$

Proof. (Compare [3, Theorem 3.10].) We first consider the special case in which $D_i = c_i Z(\mathcal{I}_i)$ for some $c_i \geq 0$ and some nonzero coherent fractional ideal sheaf $\mathcal{I}_i$ on X. By twisting by suitable integral Cartier divisors on X and using Remark 2.5.3(c), we may reduce to the case where $\mathcal{I}_i \subseteq \mathcal{O}_X$. In this case, this is [23, Theorem 9.5.20]. (Note that we are using the regularity of X here.)

We next consider the special case in which D_i is the nef envelope of a Cartier b-divisor E_i. Let D_3 be the nef envelope of $E_1 + E_2$; since $D_1 + D_2$ is nef and $D_1 + D_2 \leq E_1 + E_2$, we have $D_1 + D_2 \leq D_3$ and $\mathcal{L}^2(D_1 + D_2) \subseteq \mathcal{L}^2(E_3)$. For $i \in \{1, 2, 3\}$ and k a positive integer, put $E_{i,k} = k^{-1} Z(\mathcal{L}^\infty(kE_i))$. We apply the previous paragraph to obtain

$$\sup_k \{Z(\mathcal{L}^2(E_{3,k}))\} \leq \sup_k \{Z(\mathcal{L}^2(E_{1,k}))\} + \sup_k \{Z(\mathcal{L}^2(E_{2,k}))\}.$$

By Remark 2.5.3(b) and Proposition 2.4.6,

$$\sup_k \{Z(\mathcal{L}^2(E_{i,k}))\} = Z(\mathcal{L}^2(\sup_k \{E_{i,k}\})) = Z(\mathcal{L}^2(D_i)),$$

so the desired result follows.

To deduce the general case, put $D_3 = D_1 + D_2$, then apply Lemma 2.4.4 to write D_i as the infimum of $D_i(Y)$ as $f : Y \to X$ runs over modifications of X. Let $E_{i,Y} \in \mathbf{Div}_\mathbb{R} X$ denote the nef envelope of $D_i(Y)$; then $D_i \leq E_{i,Y} \leq D_i(Y)$ since D_i is relatively nef, so D_i is also the infimum of the $E_{i,Y}$. We claim that

$$\mathcal{L}^2(D_i) = \bigcap_Y \{\mathcal{L}^2(E_{i,Y})\},$$

from which we may deduce the desired result using the previous paragraph. Namely, it is clear that the left side is contained in the right side. To check the reverse containment, let $Z(\mathcal{I})$ be a nonzero coherent fractional ideal sheaf such that $\mathcal{I} \subseteq \mathcal{L}^2(E_{i,Y}) = \mathcal{L}^\infty(K_X + \lceil E_{i,Y} \rceil)$ for each modification $f : Y \to X$. Then $Z(\mathcal{I})(Y) \leq (K_X + \lceil E_{i,Y} \rceil)(Y) \leq K_{Y/X} + \lceil D_i(Y) \rceil$ for each f, and so $Z(\mathcal{I}) \leq K_X + \lceil D_i \rceil$ and $\mathcal{I} \subseteq \mathcal{L}^2(D_i)$. (Beware that this argument does not imply that the formation of $\mathcal{L}^2$ commutes with *arbitrary* infima.) $\square$

An important consequence of subadditivity is the following result, which states that the multiplier ideal of a relatively nef b-divisor is a reasonably close approximation to the original divisor.

Definition 2.5.7. The *thin b-divisor* $A_X \in \mathbf{Div}_\mathbb{Z} X$ has trace on a modification $f : Y \to X$ equal to the relative canonical divisor $K_{Y/X}$ plus the exceptional divisor $\mathrm{except}(f)$.

Theorem 2.5.8. *For $D \in \mathbf{Div}_\mathbb{R} X$ relatively nef,*

$$D \leq Z(\mathcal{L}^2(D)) \leq D + A_X.$$

Proof. As in the proof of Theorem 2.5.6, it suffices to consider the case where D is the nef envelope of a Cartier b-divisor. The inequality $Z(\mathcal{L}^2(D)) \leq D + A_X$ is evident from the definition of $\mathcal{L}^2(D)$, because $K_{Y/X} + \lceil D(Y) \rceil \leq (D + A_X)(Y)$. To prove the other inequality, for k a positive integer, put $D_k = k^{-1} Z(\mathcal{L}^\infty(kD))$. Then

$$\begin{aligned} \mathcal{L}^\infty(kD) &\subseteq \mathcal{L}^2(\mathcal{L}^\infty(kD)) = \mathcal{L}^2(kD_k) \qquad \text{(by Remark 2.5.3(a))} \\ &\subseteq \mathcal{L}^2(D_k)^{\otimes k} \qquad \text{(by Theorem 2.5.6)} \\ &\subseteq \mathcal{L}^2(D)^{\otimes k} \qquad \text{(since } D_k \leq D\text{)}, \end{aligned}$$

or in other words $kD_k \leq kZ(\mathcal{L}^2(D))$. By Proposition 2.4.6, this implies $D \leq Z(\mathcal{L}^2(D))$ as desired. □

One can continue in this fashion, generalizing many other properties of multiplier ideals from ideal sheaves to relatively nef b-divisors. We leave this as an exercise for the interested reader.

Remark 2.5.9. The procedure used to prove Theorems 2.5.6 and 2.5.8 is rather closely modeled on the proofs of [3, Theorems 3.9, 3.10]. The main difference is that [3] uses a definition of the multiplier ideal of a b-divisor in the style of Lipman [24], which only works properly when considering blowups over a single point.

We conclude the discussion of multiplier ideals with the following variant of Conjecture 2.4.7.

Conjecture 2.5.10. *For $D \in \mathbf{CDiv}_{\mathbb{Q}} X$ relatively nef, the sequence $\{Z(\mathcal{L}^2(nD))\}_{n=1}^\infty$ spans a finite-dimensional subspace of $\mathbf{CDiv}_{\mathbb{R}} X$.*

Remark 2.5.11. If Conjecture 2.5.10 holds, then the space $V(X, D)$ spanned by the $Z(\mathcal{L}^2(nD))$ contains D by Theorem 2.5.8. It thus constitutes a functorial spread on the category of schematic rational Cartier b-pairs (X, D) in which X is regular and D is relatively nef.

Remark 2.5.12. Conjecture 2.5.10 would follow from the existence of positive integers m, N such that for any $n \geq N$,

$$Z(\mathcal{L}^2((m+n)D)) = Z(\mathcal{L}^2(nD)) + mD. \tag{2.5.12.1}$$

For $D = Z(\mathcal{I})$ for $\mathcal{I}$ a nonzero coherent fractional ideal sheaf, Skoda's theorem [23, Theorem 9.6.21] gives this equality for $m = 1, N = \dim(X) - 1$.

In general, by Theorem 2.5.8, for any log-resolution $f : Y \to X$ of D, the difference between the two sides of (2.5.12.1) is bounded above by $A_X(Y)$ and bounded below by $-A_X(Y)$. Unfortunately, this does not restrict the difference to even a countable set, let alone a finite set (which would imply Conjecture 2.5.10).

3 Irregularity and functorial good formal structures

In this section, we construct the irregularity b-divisor associated to a formal meromorphic connection on a nondegenerate differential scheme. We then (conditionally on the existence of a fundamental determination for rational relatively nef Cartier b-divisors) construct functorial resolutions of turning points in both the algebraic and analytic categories.

3.1 Irregularity b-divisors

Hypothesis 3.1.1. Throughout § 3.1, let X be a nondegenerate differential integral scheme. Let Z be a nowhere dense closed subscheme of X. Let $\mathcal{E}$ be a ∇-module over $\mathcal{O}_{\widehat{X|Z}}(*Z)$.

Definition 3.1.2. The *irregularity b-divisor* of $\mathcal{E}$, denoted $\mathrm{Irr}(\mathcal{E})$, is the integral b-divisor on X whose component on $v \in \mathrm{RZ}^{\mathrm{geom}}(X)$ computes the irregularity along each geometric valuation on X. Note that it is supported on geometric valuations supported within Z.

The numerical criterion for good formal structures [17, Theorem 4.4.2] may be reformulated as follows [18, Proposition 6.2.2].

Theorem 3.1.3. *Let $f : Y \to X$ be a modification such that $(Y, f^{-1}(Z))$ is a regular pair. Then $f^*\mathcal{E}$ admits good formal structures everywhere if and only if* $\mathrm{Irr}(\mathcal{E})$ *and* $\mathrm{Irr}(\mathrm{End}(\mathcal{E}))$ *are Cartier b-divisors for which Y is a determination.*

Using this criterion, the main result of [18], and a spectral interpretation of irregularity, we deduce the following key result.

Theorem 3.1.4. *The b-divisor* $\mathrm{Irr}(\mathcal{E})$ *is Cartier. If (X, Z) is a regular pair, then* $\mathrm{Irr}(\mathcal{E})$ *is also relatively nef.*

We do not know whether $\mathrm{Irr}(\mathcal{E})$ is always relatively nef even without the hypothesis that (X, Z) is a regular pair.

Proof. The Cartier condition follows from Theorem 3.1.3 plus the existence of a modification $f : Y \to X$ such that $f^*\mathcal{E}$ admits good formal structures everywhere [18, Theorem 8.2.2]. To check the relatively nef condition in case (X, Z) is a regular pair, it is enough to work locally in a neighborhood of a point $z \in X$. By shrinking X, we may assume that $\mathcal{E}$ admits a free lattice $\mathcal{E}_0$.

For s a positive integer, let $\mathcal{E}_s$ be the span of the images of $\mathcal{E}_0$ under all differential operators on X which act on $\mathcal{O}_X(-Z)$. The construction of $\mathcal{E}_s$ obviously commutes with replacing X with an open subscheme; it also commutes with blowing up in an intersection of components of Z. We also claim that it commutes with replacing Z by a larger closed subscheme Z' such that (X, Z') again forms a regular pair. It suffices to check this in case Z' is obtained from Z by adding an additional component; in this case, we need only compare the constructions at the generic point η of Z. If $\tilde{\mathcal{E}}_s$ represents the analogue of $\mathcal{E}_s$ made with Z replaced by Z', we have $\tilde{\mathcal{E}}_{s,\eta} \subseteq \mathcal{E}_{s,\eta}$ because $\tilde{\mathcal{E}}_s$ is the image of $\mathcal{E}_0$ under a smaller set of derivations. On the other hand, we have $\mathcal{E}_{0,\eta} \subseteq \tilde{\mathcal{E}}_{s,\eta}$ evidently, and $\mathcal{E}_{s,\eta} = \mathcal{E}_{0,\eta}$ because $\mathcal{E}$ has no singularities at η. We end up with a circle of containments forcing $\tilde{\mathcal{E}}_{s,\eta} = \mathcal{E}_{s,\eta}$.

It follows that the formation of $\mathcal{E}_s$ commutes with arbitrary blowups with regular centers. We now check that

$$\operatorname{Irr}(\mathcal{E}) = \lim_{s \to \infty} \frac{1}{s} Z((\wedge^{\operatorname{rank}(\mathcal{E})}\mathcal{E}_s) \otimes (\wedge^{\operatorname{rank}(\mathcal{E})}\mathcal{E}_0)^\vee). \tag{3.1.4.1}$$

To compare the components of (3.1.4.1) at any $v \in \mathrm{RZ}^{\mathrm{geom}}(X)$, use resolution of singularities to make a sequence of blowups with regular centers, ending in a variety in which v appears as the order of vanishing along a component of the inverse image of Z. The preceding analysis shows that the construction of $\mathcal{E}_s$ commutes with the blowups; in the final position, the comparison with the formula in [17, Proposition 2.5.6] becomes evident in local coordinates. This shows that (3.1.4.1) holds for the topology of pointwise convergence on $\mathbf{Div}_{\mathbb{R}} X$. However, on any modification $f : Y \to X$, the divisors $Z((\wedge^{\operatorname{rank}(\mathcal{E})}\mathcal{E}_s) \otimes (\wedge^{\operatorname{rank}(\mathcal{E})}\mathcal{E}_0)^\vee)(Y)$ are all supported within $f^{-1}(Z)$, and hence all belong to a finite-dimensional subspace of $\mathrm{CDiv}_{\mathbb{R}} Y$. We conclude that (3.1.4.1) also holds for the inverse limit topology on $\mathbf{Div}_{\mathbb{R}} X$.

Since $\operatorname{Irr}(\mathcal{E})$ is a limit of relatively nef Cartier b-divisors by (3.1.4.1), it is relatively nef. This completes the proof. □

Remark 3.1.5. The proof technique of Theorem 3.1.4 can also be used to give another proof of [17, Proposition 5.4.1]. Namely, we may use (3.1.4.1) to verify condition (b$''$) of [17, Lemma 5.3.5], as the latter is stable under pointwise limits.

Remark 3.1.6. It would be interesting to know whether the purely combinatorial argument used to prove [17, Proposition 5.4.1] has a higher-dimensional analogue. In other words, do the conditions that $\operatorname{Irr}(\mathcal{E}) \in \mathbf{Div}_{\mathbb{Z}} X$, $\operatorname{Irr}(\mathcal{E}) \geq 0$, and $\operatorname{Irr}(\mathcal{E})$ is relatively nef force $\operatorname{Irr}(\mathcal{E}) \in \mathbf{CDiv}_{\mathbb{Z}} X$?

On one hand, the possibility of an affirmative answer is suggested by corresponding results for convex functions. For instance, a continuous convex function $f : [0,1]^n \to \mathbb{R}$ with the property that

$$f(x_1, \dots, x_n) \in \mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n \qquad (x_1, \dots, x_n \in [0,1] \cap \mathbb{Q})$$

is necessarily polyhedral. See [20] for discussion of numerous results in this spirit.

On the other hand, we suspect that even a purely combinatorial proof that $\operatorname{Irr}(\mathcal{E})$ is Cartier will require the sort of valuation-theoretic local analysis that was used in [18], as the latter seems to be the correct generalization of the analysis made in [17].

3.2 Functorial modifications

We can now turn the numerical criterion around, to give a conditional construction of functorial resolutions of turning points in both the algebraic and analytic categories.

Theorem 3.2.1. *Suppose that there exists a fundamental determination of the category of schematic rational Cartier b-pairs (X, D) in which X is regular and D is relatively nef. Let*

X be either a nondegenerate differential scheme (the algebraic case*), or a complex analytic variety (the* analytic case*). Let Z be a closed subspace of X containing no irreducible component of X. Let $\mathcal{E}$ be a ∇-module over $\mathcal{O}_{\widehat{X|Z}}(*Z)$. Then there exists a projective modification $f : Y \to X$ such that $(Y, f^{-1}(Z))$ is a regular pair and $f^*\mathcal{E}$ admits good formal structures everywhere. Moreover, the construction of f is functorial for regular morphisms on X; in particular, on the locus where (X, Z) is a regular pair, f is an isomorphism away from the turning locus of $\mathcal{E}$.*

Proof. We first deal with the algebraic case, for which we reduce immediately to the case of X integral. Using Theorem 1.2.3, we may also force (X, Z) to be a regular pair. By Theorem 3.1.4, the integral b-divisors $\mathrm{Irr}(\mathcal{E})$ and $\mathrm{Irr}(\mathrm{End}(\mathcal{E}))$ on X are both Cartier and relatively nef. By assumption, there exists a functorial determination $f : Y \to X$ of both $\mathrm{Irr}(\mathcal{E})$ and $\mathrm{Irr}(\mathrm{End}(\mathcal{E}))$. By applying Theorem 1.2.1 and then Theorem 1.2.3, we may force $(Y, f^{-1}(Z))$ to be a smooth pair. By Theorem 3.1.3, $f^*\mathcal{E}$ admits good formal structures everywhere.

To treat the analytic case, choose pairs (X_i, K_i) with X_i Stein and K_i compact, such that the K_i cover X. By [18, Corollary 3.1.7], the localization of $\Gamma(X_i, \mathcal{O}_{X_i})$ at K_i is a nondegenerate differential ring. We may thus apply the algebraic case to construct a functorially determined modification f_i in a neighborhood of K_i. By functoriality, the f_i glue to give the desired global modification f. □

Remark 3.2.2. We mention in passing that arguments of Mochizuki [28] can be used to construct global modifications achieving good formal structures for flat meromorphic connections on analytic spaces. The theorem in this case is stated in [28] under the condition that the connection is actually algebraic, but most of the arguments take place in the analytic category. The only use of the algebraicity hypothesis is to invoke the corresponding result for surfaces, which Mochizuki had proved in [27] using algebraic rather than analytic methods. However, our prior result for surfaces [17, Theorem 6.4.1] applies to the analytic category, so one may start from this result and then follow Mochizuki to obtain global modifications achieving good formal structures in the analytic category. Beware, though, that we do not know whether the resulting construction is functorial for smooth morphisms, in contrast with Theorem 3.2.1. Moreover, this construction does not apply to *formal* flat meromorphic connections.

References

[1] E. Bierstone and P. D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, *Invent. Math.* **128** (1997), 207–302.

[2] E. Bierstone, P. Milman, and M. Temkin, $\mathbb{Q}$-universal desingularization, arXiv preprint `0905.3580v1` (2009).

[3] S. Boucksom, C. Favre, and M. Jonsson, Valuations and plurisubharmonic singularities, *Publ. RIMS, Kyoto Univ.* **44** (2008), 449–494.

[4] N. Bourbaki, *Espaces Vectoriels Topologiques*, Springer-Verlag, Berlin, 2007.

[5] A. Corti (ed.), *Flips for 3-folds and 4-folds*, Oxford Lecture Series in Math. 35, Oxford Univ. Press, Oxford, 2007.

[6] P. Deligne, *Équations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math. 163, Springer-Verlag, 1970.

[7] J.-P. Demailly, L. Ein, and R. Lazarsfeld, A subadditivity property of multiplier ideals, *Michigan Math. J.* **48** (2000), 137–156.

[8] L. Ein, R. Lazarsfeld, and K. Smith, Uniform approximation of Abhyankar valuation ideals in smooth function fields, *Amer. J. Math* **125** (2003), 409–440.

[9] C. Favre and M. Jonsson, *The Valuative Tree*, Lecture Notes in Math. 1853, Springer-Verlag, Berlin, 2004.

[10] A. Grothendieck, Éléments de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents, I, *Publ. Math. IHÉS* **11** (1961), 5–167.

[11] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. 52, Springer-Verlag, New York, 1977.

[12] H. Hauser, The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand), *Bull. Amer. Math. Soc. (N.S.)* **40** (2003), 323–403.

[13] H. Hironaka, La voûte étoilée, Singularités à Cargése, *Astérisque* **7-8** (1973), 415–440.

[14] C. Huneke and I. Swanson, *Integral Closure of Ideals, Rings, and Modules*, London Math. Society Lecture Note Series 336, Cambridge Univ. Press, Cambridge, 2006.

[15] K. Kato, Class field theory, $\mathcal{D}$-modules, and ramification on higher dimensional schemes, *Amer. J. Math.* **116** (1994), 757–784.

[16] K.S. Kedlaya and L. Xiao, Differential modules on p-adic polyannuli, arXiv preprint `0804.1495v4` (2008); to appear in *J. Institut Math. Jussieu.*

[17] K.S. Kedlaya, Good formal structures for flat meromorphic connections, I: Surfaces, *Duke Math. J.*, to appear; arXiv preprint `0811.0190v4` (2009).

[18] K.S. Kedlaya, Good formal structures for flat meromorphic connections, II: Excellent schemes, arXiv preprint `1001.0544v1` (2010).

[19] K.S. Kedlaya, *p-adic Differential Equations*, Cambridge Univ. Press, to appear (2010); book draft (of 30 Sep 09) available at `http://math.mit.edu/~kedlaya/papers/`.

[20] K.S. Kedlaya and P. Tynan, Detecting integral polyhedral functions, *Confluentes Math.* **1** (2009), 87–109.

[21] A. Küronya and C. Maclean, Zariski decomposition of b-divisors, arXiv preprint `0807.2809v2` (2008).

[22] R. Lazarsfeld, *Positivity in Algebraic Geometry I*, Ergebnisse der Math. 48, Springer, Berlin, 2004.

[23] R. Lazarsfeld, *Positivity in Algebraic Geometry II*, Ergebnisse der Math. 49, Springer, Berlin, 2004.

[24] J. Lipman, Adjoints of ideals in regular local rings, *Math. Res. Lett.* **1** (1994), 739–755.

[25] H. Majima, *Asymptotic Analysis for Integrable Connecitons with Irregular Singular Points*, Lecture Notes in Math. 1075, Springer-Verlag, Berlin, 1984.

[26] B. Malgrange, Connexions méromorphes, in J.P. Brasselet (ed.), *Singularities Lille 1991*, London Math. Soc. Lecture Notes 201, Cambridge Univ. Press, 1994.

[27] T. Mochizuki, Good formal structure for meromorphic flat connections on smooth projective surfaces, in T. Miwa et al. (eds.), *Algebraic Analysis and Around*, Advanced Studies in Pure Math. 54, Math. Soc. Japan, 2009, 223–253.

[28] T. Mochizuki, Wild harmonic bundles and wild pure twistor D-modules, arXiv preprint `0803.1344v1` (2008).

[29] P. Robba, Lemme de Hensel pour des opérateurs différentiels, *Ens. Math.* **26** (1980), 279–311.

[30] C. Sabbah, Équations différentielles à points singuliers irréguliers et phenomene de Stokes en dimension 2, *Astérisque* **263** (2000).

[31] V.V. Shokurov, Prelimiting flips (translated from the Russian), *Proc. Steklov Inst. Math.* **240** (2003), 75–213.

[32] M. Temkin, Desingularization of quasi-excellent schemes in characteristic zero, *Advances in Math.* **219** (2008), 488–522.

[33] M. Temkin, Functorial desingularization of quasi-excellent schemes in characteristic zero: the non-embedded case, arXiv preprint `0904.1592v1` (2009).

[34] M. Temkin, Functorial desingularization over $\mathbb{Q}$: boundaries and the embedded case, arXiv preprint `0912.2570v1` (2009).

[35] M. Vaquié, Valuations, in *Resolution of singularities (Obergurgl, 1997)*, Progr. Math. 181, Birkhäuser, Basel, 2000, 539–590.

[36] V.S. Varadarajan, Linear meromorphic differential equations: a modern point of view, *Bull. Amer. Math. Soc.* **33** (1996), 1–42.