# Coordinating Production and Delivery Under a $(z, Z)$-type Vendor-Managed Inventory Contract


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April 12, 2001

**Abstract**

This paper models a type of vendor-managed inventory (VMI) agreement that occurs in practice called a $(z,\ Z)$ contract. We investigate the savings due to better coordination of production and delivery facilitated by such an agreement. The optimal behavior of both the supplier and the retailer are characterized. The optimal replenishment and production policies for a supplier are found to be up-to policies, which are shown to be easily computed by decoupling the periods when the supplier outsources from those when the supplier does not outsource. A simple application of the newsvendor relation is used to define the retailer's optimal policy. Numerical analysis is conducted to compare the performance of a single supplier and a single retailer operating under a $(z,\ Z)$ VMI contract with the performance of those operating under traditional retailer-managed inventory (RMI) with information sharing. Our results verify some observations made in industry about VMI and show that the $(z,\ Z)$ type of VMI agreement performs significantly better than RMI in many settings, but can perform worse in others.

# 1 Introduction

New information technologies and a continued increase in competition has led to the development of new inter-organizational relationships in the supply chain. One such relationship is vendor-managed inventory, or VMI (Andel (1996), Lucy (1995)). Under VMI, the supplier assumes management of the retailer's inventory, making such decisions as when and how much inventory to ship to the retailer. The supplier typically uses advanced information technology, such as Electronic Data Interchange (EDI) or the internet, to monitor the retailer's stock level and demand information. As these technologies continue to become less expensive and, hence, more common in industry, it is expected that programs such as VMI will also become more popular. Probably the best known VMI program is the partnership between Wal-Mart and Procter and Gamble, but other companies have implemented VMI as well including Campbell Soup (Clark and McKenney (1994)), Barilla SpA (Hammond (1994)), Shell Chemical, General Electric, Intel, and many others.

Reports from industry provide differing accounts on the success and benefits attributable to VMI (Saccomano (1997a), Schenck and McInerney (1998)). Due to a lack of analytical models describing VMI and the fact that VMI appears in many different forms within industry, it has been very difficult to determine exactly why VMI seems to work in some situations, but not in others. It is also well known that implementing VMI can be difficult since information systems and business processes must be made compatible between partners (Saccomano (1997b)). Therefore, it is extremely helpful for firms considering VMI to be able to estimate the expected benefits before actual implementation of a VMI program. Our research models one particular form of VMI agreement, which we call a $(z,\ Z)$ VMI contract. Using it, we examine whether or not VMI provides opportunities for savings over traditional retailer-managed inventory (RMI) for the overall supply chain, and if so, how much of a savings can be realized.

In this paper we model a specific VMI agreement between a single retailer and a single supplier to determine the benefits offered by coordinating production and delivery. We compare our VMI model to a model representing a traditional retailer-managed scenario.

Our analysis suggests that the $(z, Z)$-type of VMI agreement performs significantly better than RMI in many settings, but that it can perform worse in some scenarios.

## Literature Review

To the best of our knowledge, this paper is the only research that studies the specific type of VMI agreement, which we refer to as a $(z,\ Z)$ VMI contract. In addition to some very recent work that examines VMI from slightly different perspectives, the related literature includes research describing the benefits of information sharing in the supply chain.

There is considerable literature regarding information sharing in the supply chain. Here, we list some of the references most closely related to our work. Gavirneni et al. (1999) examine the value of information in the context of a single retailer and a single capacitated supplier. The authors examine the benefits of knowing, instead of deducing, the retailer's inventory position. They show that in 'most typical' situations, the benefits of information may be significant while they disappear in extreme conditions. Cachon and Fisher (2000) examine a single supplier and $N$ retailers to construct an upper bound on the value of shared information. They compare this upper bound on shared information with benefits offered from faster and cheaper order processing. Their analysis suggests that more efficient order processing leads to an acceleration of the physical flow of goods through a supply chain, which is significantly more beneficial than increasing the flow of information. Lovejoy and Whang (1995), under the assumption that information and processing are done in parallel, show that decreasing information flow is always more beneficial than physical flow. Chen (1998) considers a serial supply system and calculates the value of centralized demand information. Chen defines the benefit of centralized demand information to be the relative cost difference between an inventory policy based on echelon stock (which requires centralized information) to a policy based on local inventory. The benefit is derived from batching which causes some information to be lost when only local-inventory information is used. All of the above references assume, as we do, that the customer's demand process is stationary. Lee et al. (2000) examine a system with nonstationary demand. As in our model, Lee et al. and Gavirneni et al. assume that there is another source of material other than the supplier that can be used to replenish the retailer at an additional cost. Hariharan and Zipkin (1995), Bourland et al. (1996), Moinzadeh (1999), and Moinzadeh and Bassok (1998) also discuss the benefits of information sharing in the supply chain.

Many references examining information sharing deal with benefits that are commonly attributed to VMI by the business press but are actually accomplished through information sharing. For instance, reducing lead times or delaying allocation of scarce products among retailers and improving forecasts are often reasons given for implementing VMI. This is accomplished by the supplier being able to observe the demands at the retailer when they occur, instead of when the retailer places an order to the supplier.

Most of the research dealing specifically with VMI is quite recent and there are few published papers that attempt to develop analytical models for VMI. Narayanan and Raman (1997) provide the only other research we are aware of that examines VMI agreements between a single retailer and a single supplier. They model the effect of 'contract incompleteness' on both VMI and RMI. Narayanan and Raman compare RMI to a specific VMI agreement where the retailer basically rents space to the supplier. Clark and Hammond (1997) provide empirical evidence that suggests that VMI coupled with information sharing (in the form of EDI) provides greater benefits than information sharing alone. However, Clark and Hammond concede that "most of the benefits of [VMI] could be realized by information sharing without actually requiring . . . the VMI replenishment process." Cachon and Fisher (1997) use simulation to identify the savings offered by VMI to the Campbell Soup Company. They conclude that the savings could have been achieved without VMI through information sharing. Cachon (1999b) uses game theory to investigate how to obtain channel coordination from a two-echelon competitive supply chain using VMI. Cachon's analysis shows that VMI is not guaranteed to achieve the supply chain optimal solution unless both the retailers and the supplier are willing to make fixed transfer payments in order to participate in VMI, and both are willing to share the benefits. His numerical study shows that there is no improvement from VMI if fixed payments were forbidden. Bernstein and Federgruen (1999) analyze the constant-demand-rate case and consider a model of VMI where the replenishment decision is transferred to the supplier, but the retailer is able to make his own pricing decisions. The authors, however, assume that the supplier absorbs all holding costs, even for inventory at the retailer, which seems to more closely resemble a consignment type of system or shelf renting.

The remainder of the literature addressing VMI focuses on the flexibility that VMI offers for a supplier delivering stock to multiple retailers. These papers mostly concentrate on coordinating the delivery of products and possibly delaying allocation of products. Campbell et al. (1998) and Kleywegt et al. (2000a) derive benefits for VMI by allowing the supplier to construct better delivery routes for multiple retailers. Kleywegt et al. (2000b) focuses on the special case where only one customer is visited on each vehicle route. Cheung and Lee (1998) also address the flexibility of delivery to multiple retailers offered by VMI. Specifically, they model the benefits of shipment coordination and stock rebalancing. Çetinkaya and Lee (2000) investigate a VMI agreement where the supplier can delay small orders from multiple retailers to achieve economies of scale in delivery. Çetinkaya and Lee derive the optimal policy for such an agreement and conclude that shipment consolidation is beneficial in some, but not all, cases. Aviv and Federgruen (1998) analyze a VMI setting with one supplier and $N$ retailers under periodic inventory review. They conclude that VMI is always more beneficial than information sharing alone. However, this is based on the assumption that the supplier is minimizing a single system-wide performance measure. Our analysis allows the supplier to act competitively in his or her own interest and considers the effect of this decision-rights transfer.

## The Model

We assume a supply chain consisting of a single retailer (he) and a single supplier (she). The supplier can ship to the retailer in any period but produces only once every $T$ periods. This is appropriate for a setting in which a supplier may manufacture different goods on a set production schedule so that she can only produce material for a given retailer once every $T$ periods. We assume no bound on the amount produced. Alternatively, one could imagine a supplier which, because of limited availability of necessary raw materials, can produce for a given retailer only once every $T$ periods. There is a large body of literature that discusses the benefits of sequencing multiple jobs that must use common resources over a fixed time interval. These types of problems are collectively referred to as economic lot scheduling problems or lot-sizing problems. Elmaghraby (1978) examines both analytical and heuristic techniques for finding policies under no capacity restrictions. Cachon (1999a) provides additional reasons for using policies with fixed-time-review intervals as well as an excellent review of papers that use this assumption. Other references can be found in Graves (1981), which provides a review of production scheduling literature.

We further assume that the retailer (under RMI) and the supplier (under VMI) follow a periodic-review inventory policy so that they examine the retailer's inventory level at the beginning of each period, and that end-user demand occurs only at the retailer. Demand is independently and identically distributed (i.i.d.) according to the distribution function $\Phi(\cdot)$ and density function $\phi(\cdot)$. Excess demand at the retailer is backlogged. We allow the supplier to outsource in order to obtain material to ship to the retailer in a period in which the supplier cannot produce. In this context outsourcing could represent a form of expediting such as working overtime, producing using less efficient technology, transshipping from another location, or procuring from an outside source. Outsourcing results in an additional cost to the supplier of $b_0$ per unit. When salvage value corresponds to production costs, both salvage value and production costs can be ignored. Therefore, we do not include them in the model. Instead, we model only the premium for outsourced goods, $b_0$. Under both RMI and VMI, as soon as inventory arrives at the retailer, ownership of the inventory is transferred to the retailer. Thus, our situation does not represent a consignment system.

The relevant costs for the supplier are $h_S$, the supplier unit holding cost, and $b_0$, the unit cost of outsourcing. For the retailer we consider $h_R$, the retailer unit holding cost, and $p$, the retailer unit backorder penalty cost.

One shortcoming of many investigations of VMI is that there is little attention paid to why or how participating firms enter into such an agreement. If the supplier's control under VMI were unrestricted, then without some form of penalty costs at the supplier it would be optimal for the supplier to ship an extraordinarily large amount of inventory to the retailer. However, in actuality the supplier's control is not total (Cachon (1999b)). We have seen that in many instances VMI is initiated under some sort of agreement or contract between the involved parties (see also Aviv and Federgruen (1998) and Copacino (1993)). The contract or agreement may specify a minimum end-user customer service level that must be maintained by the supplier under VMI, and storage space (e.g., shelf space allocated by the retailer) may effectively play the role of an upper limit. Alternatively, the contract may set minimum and maximum retailer inventory levels that the supplier is expected to maintain. Despite many common features, the details of implementation of VMI contracts differ from company to company. Several of our specific experiences with VMI agreements used in practice are listed here.

A large semiconductor manufacturer establishes VMI agreements with several of its largest spare parts suppliers. These parts include many different items that are used in a wafer-fabrication process. The end-user customers are the production engineers who demand a part whenever one is needed in the wafer fab because of machine breakdown, replacement, or maintenance. In this particular VMI agreement, each supplier is given access to the semiconductor's inventory programs so that they can view current inventory levels and demand histories for the items that they supply.[1] The suppliers are then expected to make all decisions on how much and when to ship inventory to the semiconductor manufacturer. The semiconductor firm also communicates suggested minimum inventory levels and reorder points for each of the parts involved in the VMI program as well as maximum inventory levels for some items (usually larger parts that had space constraints). For the most part, these levels are the same as the semiconductor firm uses for its own in-house inventory decision making process. The semiconductor manufacturer and the suppliers have scheduled meetings (monthly at the beginning of the program and less often as the VMI program progresses) to discuss the success of the VMI program and any problems that are occurring. Some of the main factors in evaluating the performance of a supplier in the VMI program is how many stockouts have occurred at the semiconductor manufacturer for its parts and whether the

[1] If a supplier cannot access the semiconductor firm's inventory computer system (due to incompatible software or such), then daily or weekly facsimile reports are sent to the supplier showing the supplier's parts current inventory levels and demand histories.

supplier maintained the minimum and the maximum allowable inventory levels.

A JIT assembler of electronic devices uses VMI-type agreements with suppliers of more than 85% of its parts – all of the important components except one supplied by a company considered as a monopolist. The minimum and maximum levels of inventory are expressed as seven and 14 days of production according to the forecast that the assembler publishes. The suppliers do not have any major benefits of scale in the delivery process and typically deliver several times a week (sometimes every day). The performance, in terms of maintaining inventory between seven and 14 days of production, is closely monitored and is used as a significant factor (one of five factors) in a "scorecard" evaluation of all suppliers. The actual parameters of seven and 14 days, when introduced and for a long period afterwards, were identical across all parts (and all suppliers). They were viewed as probably exceeding the actual needs, but maintained due to the weekly routine of updating forecasts (a newly published forecast might dramatically change the actual number of parts needed, and too small a buffer or too small a difference between the minimum and maximum could put the retailer's inventory outside of the desirable interval). A student team evaluated the appropriateness of the levels used. As a result of recommendations, the assembler has introduced a pilot program with parameters based on economic considerations (as we describe in this paper) instead of a fixed number of days, independent of product.

A producer of furniture that offers some partly customized configurations with two-day production lead time relies on minimum and maximum levels of inventory of all parts. The parts are stored in a nearby warehouse (three miles away) owned by the producer and replenished by the supplier. Lower and upper limits on each item are set by the producer.

Increasingly often, we observe evaluation techniques based on a "balanced scorecard," which make the inventory performance an important part of the evaluation. In most cases the supplier's ability to maintain the desired inventory level is collected over the year and becomes a significant factor in next year's market share and price decisions. The JIT assembler mentioned here performs evaluations on a quarterly basis and its suppliers feel that this puts more weight on being out of the defined inventory range than the assembler cares to admit. Less often, the explicit penalties are collected. Our conversations with the Advanced

Supply Chain Solutions Department of an auto manufacturer indicate that this company has considered explicitly setting penalties for falling below the minimum levels (in the spirit of currently collected penalties for any order that is not satisfied) but has not implemented such a system yet.

Our work with these firms has led us to define a type of initiating VMI contract, which we call a $(z,\ Z)$ VMI contract. Under a $(z,\ Z)$ VMI contract, the retailer sets a minimum inventory level, $z$, and a maximum inventory level, $Z$, that represent the lowest and highest inventory levels, respectively, that the retailer wishes his stock to experience when measured after customer demand. The values of $z$ and $Z$ represent either the actual minimum and maximum levels or the implicit values that are tied to inventory turns and customer service levels. The supplier agrees to pay a penalty amount of $b^-$ $(b^+)$ per unit to the retailer for every unit of the retailer's inventory that is less than $z$ (more than $Z$) after customer demand. In all of the VMI agreements mentioned previously, the penalties are not incurred immediately (i.e., on a daily basis), but are based on long-term (approximately yearly) performance, often as part of "balanced scorecard" evaluation. The companies that we worked with indicated that these measures and resulting penalties and awards, although often neither contractually enforced or even necessarily explicit, are nevertheless very strong incentives in to trying to remain within the agreed upon inventory levels. VMI agreements effectively serve to transfer part of the demand risk from the retailer to the supplier. While this model does not perfectly correspond to all of the intricacies of the examples given here, it is a useful abstraction of common VMI agreements, which we believe allows us to make a fair evaluation of VMI.

We wish to characterize the supplier's optimal policy under a $(z,\ Z)$ VMI contract. This means determining how much the supplier should produce once every $T$ periods, and how much the supplier should send to the retailer at the beginning of each period (associated with this decision is the decision of how much to outsource in each period). We assume that there is negligible lead time for shipments from supplier to retailer (an assumption that can be relaxed), but that the decision of how much to outsource must be made by the supplier before demand is experienced at the retailer. We use Markov Decision Processes (MDPs) to characterize the optimal policy for the supplier under VMI.

Once the optimal policy for the supplier under a $(z,\ Z)$ VMI contract is determined, we examine how a retailer would want to structure a $(z,\ Z)$-type of VMI contract. We do this by allowing the retailer to choose $Z$ and $z$ values so as to minimize its overall costs. A traditional RMI setting is defined and the optimal policies for the supplier are derived. In both cases we assume that all information is shared (whether it is useful or not), i.e., in addition to actual inventory position, the supplier has full knowledge of the end-user demand distribution as well as the inventory policy being followed by the retailer. We incorporate information sharing into the RMI case so that we can identify those benefits that are offered by VMI in addition to benefits offered by information sharing alone.

Once both the supplier's and the retailer's operating policies under VMI and RMI have been fully defined, we numerically compare the overall performance of the supply chain under VMI to the supply chain operating under a traditional RMI setting. Infinitesimal perturbation analysis (IPA) is used to determine the retailer's order up-to values under RMI and the supplier's optimal levels of replenishing the retailer under VMI as well as her optimal production quantities. The retailer's best $z$ and $Z$ values are found using a newsvendor-type relation. A computational study also allows us to describe when VMI contracts are most beneficial as well as to define general guidelines for setting the contractual parameters.

Section 2 of this paper defines the optimal policy that a supplier would follow under the $(z,\ Z)$ VMI contract that we have defined. Section 3 describes the retailer's optimal policy and explains how the VMI contractual parameters are set. Sections 4 and 5 describe our numerical solution technique and analysis. Section 6 translates our findings into implications and insights for practitioners of VMI. Section 7 lists possible problem extensions and Section 8 provides concluding remarks.

# 2 Supplier Behavior Under VMI and RMI

## Replenishment Policy Under VMI

Given that a certain amount of inventory has been produced at the beginning of the cycle of $T$ periods, the replenishment policy for the supplier consists of the decision of how much inventory to ship to the retailer at the beginning of each period under a $(z,\ Z)$ VMI contract. The sequence of events relating to the replenishment decision under VMI is as follows: 1) the supplier examines the retailer's inventory level and decides how much to outsource and how much to ship to the retailer; 2) the material is shipped to the retailer with negligible lead time; 3) demand occurs at the retailer; 4) VMI contractual costs, as well as penalty and holding costs, are assessed.

Let $x_S$ and $x_R$ denote the inventory level at the beginning of a period at the supplier and retailer, respectively. The following Lemma 1 allows us to characterize the optimal policy for the supplier by first considering two special cases for replenishment.

**Lemma 1** *As long as $x_S > 0$, it is cost effective for the supplier to replenish the retailer from on-hand inventory rather than outsourcing.*

**Proof** The proof is based on a simple interchange argument.

#### Case 1: Supplier Has No Stock and Considers Outsourcing

Once the supplier has allocated all of her on-hand inventory ($x_S = 0$), outsourcing is the only means for the supplier to obtain more stock with which she can replenish the retailer. Let $V_n^{\mathrm{VMI}}(x_R)$ be the supplier's expected cost-to-go of the optimal policy for replenishing the retailer, from $n$ periods after the last production period until the end of the production cycle, when the retailer has $x_R$ units of inventory and the supplier has none. $V_n^{\mathrm{VMI}}(x_R)$ can be decoupled into the term, $-b_0 x_R$ and a term that depends only on $y$, the retailer's inventory after being replenished by the supplier. Thus, $V_n^{\mathrm{VMI}}(x_R) = -b_0 x_R + \min_{y \geq x_R} J_n^{\mathrm{VMI}}(y)$, where $J_n^{\mathrm{VMI}}(y)$ is defined below. We can determine the optimal amount for the supplier to send to the retailer by finding the value of $y$ that minimizes $J_n^{\mathrm{VMI}}(y)$.

Consider a period $n$ where $x_S = 0$. If $x_S = 0$ in period $n$, then $x_S = 0$ in period $n+1$, therefore

$$
\begin{aligned}
J_n^{\text{VMI}}(y) &= b^- E\left[(z+D-y)^+\right] + b^+ E\left[(y-Z-D)^+\right] + b_0 y \\
&\quad + E\left[V_{n+1}^{\text{VMI}}(y-D)\right] \\
&= L(y) + b_0 y + E\left[V_{n+1}^{\text{VMI}}(y-D)\right], \qquad (1)
\end{aligned}
$$

where $L(y) = b^- E\left[(z+D-y)^+\right] + b^+ E\left[(y-Z-D)^+\right]$. Assume that $J_T^{\text{VMI}}(\cdot) = V_T^{\text{VMI}}(\cdot) = 0$. Theorem 1 shows that a replenish up-to policy is optimal for the supplier when $x_S = 0$; by which we mean that the supplier replenishes the retailer up to a value of $y_n^*$ by outsourcing $y_n^* - x_R$ units. Although $L$, $J_{n+1}^{\text{VMI}}$, and $V_{n+1}^{\text{VMI}}$ may not be differentiable, due to the convexity of $L$, $J_{n+1}^{\text{VMI}}$, and $V_{n+1}^{\text{VMI}}$, right-hand-side first and second derivatives exist. We use $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $'$ and $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$, $''$ to denote them, respectively. Although Theorems 1-4 below apply to any production cycle, we prove them here for the last cycle (of $T$ periods). Later (see Theorem 5) we argue that the results apply generally.

**Theorem 1** *If $x_S = 0$ in period $n$, then it is optimal for the supplier to follow a replenish up-to policy using outsourcing. The replenish up-to values may depend on $n$.*

**Proof** We show that a replenish up-to policy is optimal by using induction to show that (1) is convex in $y$, and that $V_n^{\text{VMI}}(x)$ is convex in $x$.

Because $J_T^{\text{VMI}}(\cdot) = 0$ and $V_T^{\text{VMI}}(\cdot) = 0$, $J_T^{\text{VMI}}(\cdot)$ and $V_T^{\text{VMI}}(\cdot)$ are obviously convex in $y$ and $x$, respectively. We assume that $V_{n+1}^{\text{VMI}}(x)$ is convex in $x$ and $J_{n+1}^{\text{VMI}}(y)$ is convex in $y$. We then show that $J_n^{\text{VMI}}(y)$ and $V_n^{\text{VMI}}(x)$ are convex in $y$ and $x$, respectively. Using (1),

$$
\begin{aligned}
\frac{\partial}{\partial y} J_n^{\text{VMI}}(y) &= L'(y) + b_0 + \frac{\partial}{\partial y} E\left[V_{n+1}^{\text{VMI}}(y-D)\right] \\
&= L'(y) + b_0 + E\left[V_{n+1}'^{\text{VMI}}(y-D)\right] \\
&= -b^-(1-\Phi(y-z)) + b^+\Phi(y-Z) + b_0 + \int_0^\infty V_{n+1}'^{\text{VMI}}(y-\xi)\, d\Phi(\xi),
\end{aligned}
$$

where the second equality follows since $V_{n+1}^{\text{VMI}}(\cdot)$ is continuous and convex by the inductional assumption (see Wets (1989), page 585, Proposition 2.10).

$$
\frac{\partial^2}{\partial y^2} J_n^{\text{VMI}}(y) = b^-\phi(y-z) + b^+\phi(y-Z) + \int_0^\infty V_{n+1}''^{\text{VMI}}(y-\xi)\, d\Phi(\xi)
$$

which is $\geq 0$ since $\phi(\cdot) \geq 0$ and $V''^{VMI}_{n+1}(y-\xi) \geq 0$ by our inductional assumption on the convexity of $V^{\text{VMI}}_{n+1}(x)$. Thus, $J^{\text{VMI}}_n(y)$ is convex in $y$ (as a sum of convex functions), and hence, $V^{\text{VMI}}_n(x)$ is convex (see Heyman and Sobel (1984), page 525, Proposition B-4). Therefore, since $J^{\text{VMI}}_n(y)$ is convex in $y$, it is optimal to replenish the retailer up to $y^*_n$, where $y^*_n$ is the minimizing value of $y$ in $J^{\text{VMI}}_n(y)$. □

The following Theorem 2 shows that as the number of periods remaining in the production cycle increases, the replenish up-to values, $y^*_n$, increase. The proof is similar to that in Arrow et al. (1958) and is included in the Appendix.

**Theorem 2** *The values of $y^*_n$ are ordered such that $y^*_0 \geq y^*_1 \geq \cdots \geq y^*_{T-1}$.*

**Proof** See Appendix.

**Case 2: Supplier Replenishes Retailer from On-Hand Inventory**

Since we have derived the optimal policy for the supplier once she has allocated all of her on-hand inventory to the retailer, we should optimize the remaining periods in which the supplier has inventory to replenish the retailer. For periods in which the supplier has on-hand inventory, we hypothesize that this will again be a replenish up-to policy where the supplier replenishes the retailer up to $\bar{y}^*_n$ in period $n$. Let $x_E$ be the combined units of inventory at the supplier and retailer ($x_E = x_S + x_R$). We adapt the analysis of Clark and Scarf (1960) dealing with multi-echelon inventory systems to express $K^{\text{VMI}}_n(y, x_E)$, the expected cost-to-go for the supplier from period $n$ until the end of the production cycle when the retailer has $y$ units of inventory, as

$$\begin{aligned} K^{\text{VMI}}_n(y, x_E) &= b^- E\left[(z + D - y)^+\right] + b^+ E\left[(y - Z - D)^+\right] \\ &\quad - h_S y + h_S x_E + E\left[U^{\text{VMI}}_{n+1}(y - D, x_E - D\right] \\ &= \bar{L}(y) + h_S x_E + E\left[U^{\text{VMI}}_{n+1}(y - D, x_E - D)\right], \end{aligned} \tag{2}$$

where $\bar{L}(y) = b^- E\left[(z + D - y)^+\right] + b^+ E\left[(y - Z - D)^+\right] - h_S y$ and

$$U^{\text{VMI}}_n(x_1, x_2) = \min_{x_1 \leq y \leq x_2} K^{\text{VMI}}_n(y, x_2). \tag{3}$$

Theorem 3 shows that there exists an optimal amount, independent of $x_E$, up to which the supplier should replenish the retailer without outsourcing. Corollary 1 shows that these up-to values are independent of the number of periods remaining. We introduce here notation which will be used in Theorem 3:

$$\bar{J}_n^{\mathrm{VMI}}(y) = \bar{L}(y) + \int_0^\infty \bar{V}_{n+1}^{\mathrm{VMI}}(y-\xi)d\Phi(\xi) \tag{4}$$

where

$$\bar{V}_n^{\mathrm{VMI}}(x_1) = \min_{y \geq x_1} \bar{J}_n^{\mathrm{VMI}}(y) \tag{5}$$

and $\bar{V}_T^{\mathrm{VMI}}(\cdot) = \bar{J}_T^{\mathrm{VMI}}(\cdot) = 0$.

**Theorem 3** *There exists a sequence of functions, $g_n(x_2)$, such that $U_n^{VMI}(x_1, x_2) = \bar{V}_n^{VMI}(x_1) + g_n(x_2)$ Furthermore, it is optimal for the supplier to replenish the retailer up to $\bar{y}_n^*$ from her on-hand inventory in period $n$.*

**Proof** See Appendix.

**Corollary 1** *The values $\bar{y}_n^*$ that minimize $\bar{J}_n^{VMI}$ are all equal, i.e., $\bar{y}_0^* = \bar{y}_1^* = \cdots = \bar{y}_{T-1}^*$.*

**Proof** See Appendix.

Notice that we derive the structure of the optimal policy in two steps. In the first step (Theorem 3) we show that the value function decouples into two parts, first, a function of inventory at the retailer, and second, a function of total supply chain inventory. Because both of these functions are convex, the optimal policy of the supplier under VMI is to replenish the retailer up to a level independent of total inventory in the supply chain. By explicitly showing that the first function is $\bar{V}_n^{\mathrm{VMI}}$, we make it easy to show in Corollary 1 that the optimal up-to levels are the myopic levels which minimize $\bar{L}$. Since $\bar{y}_0^* = \bar{y}_1^* = \cdots = \bar{y}_{T-1}^*$, we denote the optimal replenish up-to value without outsourcing as $\bar{y}^*$.

**Corollary 2** $\bar{y}^* \geq y_n^*\ \forall\ n = 0, \ldots, T-1$.

**Proof** See Appendix.

Based on Lemma 1 and the fact that the derivative of future costs is identical for the non-outsourcing case and for the general case, we can now state the supplier's optimal replenishment policy as follows:

- If $x_S + x_R \geq \bar{y}^*$, replenish the retailer up to $\bar{y}^*$.
- If $\bar{y}^* > x_S + x_R > y_n^*$, replenish the retailer up to $x_S + x_R$.
- If $x_S + x_R \leq y_n^*$ in period $n$, replenish the retailer up to $y_n^*$.

## Production Policy Under VMI

The production policy for the supplier consists of the decision of how much inventory to produce at the beginning of each production cycle, which will be used to replenish the retailer. Let $q$ be the amount of inventory at the supplier and retailer at the beginning of the final production cycle. We can recursively define the supplier's cost over the final production cycle when the supplier and retailer begin the cycle with a combined total of $q$ units of on-hand inventory as

$$\begin{aligned} F^{\mathrm{VMI}}(q) &= h_S \left[\sum_{n=0}^{T-1} I_n\right] + b_0 \left[\sum_{n=1}^{T-1} O_n\right] + b^+ \left[\sum_{n=0}^{T-1} (y_n - D_n - Z)^+\right] \\ &\quad + b^- \left[\sum_{n=0}^{T-1} (z - y_n + D_n)^+\right] \end{aligned} \tag{6}$$

where

$I_n$ = Inventory at the supplier, after delivery to the retailer, in period $n$,

$D_n$ = Demand experienced at retailer in period $n$,

$O_n$ = Amount outsourced in period $n$.

Notice that in the first period of a production cycle (period 0) the supplier will never outsource, since it is always cheaper for her to produce enough inventory to replenish the retailer up-to $\bar{y}^*$ (as production has no associated cost and supplier holding costs are assessed after replenishment of the retailer). Therefore, $I_n$ and $O_n$ are defined as

$$
\begin{aligned}
I_0 &= \bar{x}_S + (q - \bar{x}_S - \bar{x}_R)^+ - (\bar{y}^* - \bar{x}_R)^+ \text{ where} \\
&\quad \bar{x}_S = \text{inventory position at supplier just before production,} \\
&\quad \bar{x}_R = \text{inventory position at retailer just before production,} \\
I_n &= \left[I_{n-1} - (\bar{y}^* - (y_{n-1} - D_{n-1}))^+\right]^+ \text{ for } n = 1, \ldots, T-1, \\
O_n &= [y_n^* - I_{n-1} - (y_{n-1} - D_{n-1})]^+ \text{ for } n = 1, \ldots, T-1 \text{ where} \\
&\quad y_n = \text{retailer's inventory position just after being replenished by the supplier} \\
&\quad \text{in period } n, \\
y_0 &= \bar{x}_R + (\bar{y}^* - \bar{x}_R)^+, \text{ and for } n = 1, \ldots, T-1, \\
y_n &= \begin{cases} y_{n-1} - D_{n-1} & : \; y_{n-1} - D_{n-1} \geq \bar{y}^* \\ \bar{y}^* & : \; y_{n-1} - D_{n-1} < \bar{y}^* \text{ and } I_{n-1} \geq \bar{y}^* - (y_{n-1} - D_{n-1}) \\ I_{n-1} + O_n + y_{n-1} - D_{n-1} & : \; y_{n-1} - D_{n-1} < \bar{y}^* \text{ and } I_{n-1} < \bar{y}^* - (y_{n-1} - D_{n-1}). \end{cases}
\end{aligned}
$$

We can demonstrate that $F^{\text{VMI}}(q)$ is convex in $q$ by using a modified sample path argument which then allows us to derive the optimal production policy (Theorem 4).

**Theorem 4** *The cost for the supplier of having (producing up to) $q$ units in the production period is convex in $q$.*

**Proof** See Appendix.

**Theorem 5** *All properties expressed in Theorems 1-4 hold for multiple production cycles.*

**Idea of Proof** The production decision and replenish up-to decisions from periods 0 to $T-1$ can be interpreted as a single action set. Then following the same logic as in Veinott (1965) it is possible to prove that the end of period inventory levels at the supplier and the retailer will be no larger than the optimal up-to values solved for under a myopic policy. It is then feasible for the supplier to bring her inventory level up to the myopic policy (since we do not consider a production capacity). As the retailer's inventory levels are no larger than the myopic replenish up-to levels, it is also feasible for the supplier to bring the retailer's inventory level back to these values. Thus, Veinott's Theorem 3.1 can be used to show that this feasible policy is indeed optimal. □

This result is fairly intuitive: the next production period becomes the end of the planning horizon, which serves as a renewal point. Consequently, we do not need to consider actions taken at and after the next production period. We denote the minimizing value of $E\left[F^{\text{VMI}}(q)\right]$ as $q^*_{\text{VMI}}$, and thus it is optimal for the supplier to produce up to $q^*_{\text{VMI}}$ (i.e., to produce $(q^*_{\text{VMI}} - \bar{x}_S - \bar{x}_R)^+$ at the beginning of each production cycle).

## Replenishment Policy Under RMI

To make fair comparisons between VMI and RMI, we consider a retailer-managed scenario where the retailer can expect to have his orders fully replenished by the supplier. Otherwise, the retailer would have to either procure the remaining items from another source (most likely at a higher cost) or the retailer might order a higher amount than would actually be optimal, expecting that the supplier will only fulfill a portion of the quantity ordered. Both of these situations add additional complexities that are not captured in the VMI model and, therefore, would not lead to fair comparisons. Also, we believe that this assumption is very realistic. Based on anecdotal evidence, many suppliers in RMI environments are not given an option of not satisfying retailer's orders, and thus they attempt to fully ship the ordered amount of goods to the retailer, even if it implies higher costs.

If the retailer can expect to receive his ordered amount in full each period, then the retailer will simply order up to the traditional, well-known newsvendor amount. Let $y^*_{\text{RMI}}$ denote this amount, so that $y^*_{\text{RMI}} = \Phi^{-1}\left(\frac{p}{p+h_R}\right)$. To ensure that the retailer can expect to receive his ordered amount in full each period, we assume that the supplier must completely satisfy the retailer's order. In other words, production takes place only at the beginning of period 0, and in each period the sequence of events under RMI is that 1) the supplier decides how much to outsource; 2) the amount ordered in the previous period by the retailer is shipped to the retailer with negligible lead time; 3) demand occurs at the retailer; 4) penalty and holding costs are incurred; 5) the retailer places the order with the supplier to be filled in the next period. This is basically the same sequence of events as under VMI (discussed previously): both outsourcing and shipping decisions are made at the very beginning of the period based on the retailer's inventory after the previous period's demand. The RMI replenishment policy can be characterized as an order up-to policy where the retailer orders up to $y^*_{\text{RMI}}$ in each period.

### Production Policy Under RMI

Under RMI, we assume that all information is available to the supplier. However, the only relevant information is the distribution of customer demand and current orders. The decision of how much the supplier should produce is very similar to that of the VMI case. The cost resulting from having $q$ units in inventory at the supplier and retailer at the beginning of the production cycle is

$$F^{\text{RMI}}(q) = h_S \left[\sum_{n=0}^{T-1} I_n\right] + b_0 \left[\sum_{n=1}^{T-1} O_n\right]$$

where

$$\begin{aligned}
I_0 &= \bar{x}_S + (q - \bar{x}_S - \bar{x}_R)^+ - (y^*_{\text{RMI}} - \bar{x}_R)^+ \\
I_n &= \left[I_{n-1} - (y^*_{\text{RMI}} - (y_{n-1} - D_{n-1}))^+\right]^+ \text{ for } n = 1, \ldots, T-1 \\
O_n &= [y^*_{\text{RMI}} - I_{n-1} - (y_{n-1} - D_{n-1})]^+ \text{ for } n = 1, \ldots, T-1 \\
y_0 &= \bar{x}_R + (y^*_{\text{RMI}} - \bar{x}_R)^+, \text{ and for } n = 1, \ldots, T-1 \\
y_n &= \begin{cases} y_{n-1} - D_{n-1} & : \; y_{n-1} - D_{n-1} \geq y^*_{\text{RMI}} \\ y^*_{\text{RMI}} & : \; y_{n-1} - D_{n-1} < y^*_{\text{RMI}}. \end{cases}
\end{aligned}$$

As in the VMI case, $F^{\text{RMI}}(q)$ can be shown to be a convex function of $q$ through a sample path argument, and thus it is optimal for the supplier to produce $(q^*_{\text{RMI}} - \bar{x}_S - \bar{x}_R)^+$ at the beginning of the production cycle under an RMI scenario.

## 3 Retailer Behavior Under VMI

Although we have now defined the optimal replenishment and production policy for the supplier under a $(z, \ Z)$ VMI contract, we have made no mention of how a retailer would operate under VMI or how the model parameters such as $b^-, b^+, z$, and $Z$ are chosen. There are many different approaches one could take for deciding the value of these parameters. One possibility is to assume that $b^-, b^+, z$, and $Z$ are all set by a central decision maker who wishes to optimize the performance of the entire supply chain. While this is guaranteed to result in the supply chain optimal policy (as is shown in Theorem 6)[2], we wish to examine the supplier and retailer as individual decision makers. Therefore, we consider a somewhat different approach that we believe is closer to what happens in actual VMI agreements in practice.

**Theorem 6** *If a central controller is allowed to choose the values of $b^-, b^+, z$, and $Z$, then the supply chain optimal solution can be obtained.*

**Proof** Let the expected cost to the supply chain from period $n$ until the end of the production cycle be $J_n^{SC}(y)$.

$$\begin{aligned} J_n^{SC}(y) \;&=\; h_R E\left[(y-D)^+\right] + pE\left[(D-y)^+\right] + b_0(y - x_S - x_R)^+ \\ &\quad + h_S(x_S - y + x_R)^+ + E\left[V_{n+1}^{SC}(y-D)\right], \end{aligned} \tag{7}$$

where $V_n^{SC}(x) = \min_{y \geq x} J_n^{SC}(y)$. By referring to Equations (1) and (4), we can see that one solution to obtaining the supply chain optimal replenishment values under VMI is to set $z = Z = 0$, $b^- = p$, and $b^+ = h_R$. By doing so, we have forced the supplier to minimize her expected costs under VMI as if she was minimizing the total supply chain costs. Denote the minimizing value of (7) as $y_n^{SC}$. Solving (6) with $y_n^* = y_n^{SC}$ provides the supply chain optimal production amount. □

We assume that the values of $z$ and $Z$ are set by the retailer. The purpose of these values is to provide a measure of control over the supplier. Without these values, one could imagine a situation in which the supplier ships all of her inventory to the retailer at the beginning of a production period to save on holding costs. Or, once the supplier has run out

[2] Unfortunately, this particular supply chain optimal solution transfers all costs to the supplier and makes her responsible for making the optimal decision for the entire supply chain.

of on-hand inventory, she could decide not to replenish the retailer at all rather than incur outsourcing costs. When the retailer sets the values of $z$ and $Z$, he knows the supplier's optimal replenishment and production policies.

We assume that the value of $Z - z$ is set through a mutual agreement between the supplier and the retailer before any other concerns are discussed. From the retailer's point of view $Z - z = 0$ is preferred, thereby maximizing the contractual penalties paid to him by the supplier. The supplier, however, would like to set $Z - z = \infty$, and thus not have to pay any penalty for over-stocking or under-stocking the retailer. Given the contrary nature of these two viewpoints, it seems likely that a mutual agreement will be reached between the supplier and retailer. In order to continue our analysis, we define $Q = Z - z$ and assume it has been reached by negotiation (later we examine the effect that $Q$ has on supply chain costs).

The retailer is then left with choosing the value of $Z$ for the VMI contract. For a value of $Z$ specified by the retailer, let $y_n^*(Z)$ be the amount that the supplier replenishes the retailer up-to in period $n$ when outsourcing is required. Also let $\bar{y}^*(Z)$ be the similar value when the supplier does not utilize outsourcing. Consider the supplier's cost of replenishing the retailer with and without outsourcing ($J_n^{\text{VMI}}$ and $\bar{J}_n^{\text{VMI}}$) as functions of both $y$ and $Z$. Lemma 2 shows that $y_n^*(Z)$ and $\bar{y}^*(Z)$ are linear functions of $Z$, and Lemma 3 shows a similar result for the production up-to quantity $q_{\text{VMI}}^*(Z)$. The exact procedure for finding the minimizing value of $Z$ is outlined in the next section.

**Lemma 2** *(a)* $y_n^*(Z) = y_n^*(0) + Z$*; (b)* $\bar{y}^*(Z) = \bar{y}^*(0) + Z$*.*

**Proof** See Appendix.

**Lemma 3** *Assume* $x_S = x_R = 0$*. Then,* $q_{VMI}^*(Z) = q_{VMI}^*(0) + Z$*.*

**Proof** See Appendix.

# 4 Solution Procedure

The optimal replenish up-to values under a $(z,\ Z)$ VMI contract and the optimal production values under both VMI and RMI are obtained using infinitesimal perturbation analysis, or

IPA (see Glasserman and Tayur (1995) for a discussion of IPA). The value of $Z$ that minimizes the retailer costs is found using an analogous version of a newsvendor-type problem based on the following Theorem 7 and supporting arguments.

**Lemma 4** *For fixed values of $Q, b^-$, and $b^+$, the total expected payment from the supplier to the retailer is independent of the retailer's choice of $Z$.*

**Proof** See Appendix.

**Corollary 3** *The choice of $Z$ by the retailer influences only his holding and penalty costs.*

Lemma 4 and Corollary 3 show that the supplier is indifferent to the choice of $Z$; therefore, the assignment of the right to choose $Z$ to the retailer is fully justified. For the rest of the analysis, we intend to use an analog of shortfall calculations, therefore, we set $Z$ to be our reference point.

**Definition 1** *For a given $Z$ and corresponding $\bar{y}^*, y_n^*$ and $q_{VMI}^*$, which are linear in $Z$, define $\Theta_n(x) = Pr(u_n \geq Z - x)$ and $\Psi(x) = \sum_{n=0}^{T-1} \Theta_n(x)/T$ where $u_n$ is the ending inventory at the retailer in period $n$.*

Notice that based on Lemmas 2 and 3 and Proposition 4, $\Theta_n(\cdot)$ and $\Psi(\cdot)$ do not depend on $Z$ and the definition given above is correct. This allows us to state the following Theorem 7 which shows that the retailer's optimal choice of $Z$ satisfies a newsvendor type relation.

**Theorem 7** *The retailer's expected cost, $C^{VMI}(Z)$, is convex in $Z$ and the optimal choice of $Z$ satisfies $\Psi(Z) = \frac{p}{p+h_R}$.*

**Proof** Let $C_n^{\text{VMI}}(Z)$ be the expected cost paid by the retailer under VMI when he chooses a value of $Z$ in period $n$ in the production horizon (we exclude here the payment from the supplier which is independent of $Z$). Further, let $C^{\text{VMI}}(Z)$ be the expected cost paid by the retailer under VMI during the entire production cycle. Then,

$$C^{\text{VMI}}(Z) = \sum_{n=0}^{T-1} C_n^{\text{VMI}}(Z)$$

$$
\begin{aligned}
&= \sum_{n=0}^{T-1} \left( h_R E\left[u_n^+\right] + pE\left[u_n^-\right] \right) \\
&= \sum_{n=0}^{T-1} \left[ h_R \int_{-\infty}^{Z} (Z-\xi) d\Theta_n(\xi) + p \int_{Z}^{\infty} (\xi - Z) d\Theta_n(\xi) \right]
\end{aligned}
$$

and

$$
\begin{aligned}
C'^{\text{VMI}}(Z) &= \sum_{n=0}^{T-1} \left[ h_R \int_{-\infty}^{Z} d\Theta_n(\xi) - p \int_{Z}^{\infty} d\Theta_n(\xi) \right] \\
&= \sum_{n=0}^{T-1} \left[ h_R \Theta_n(Z) - p(1-\Theta_n(Z)) \right] \\
&= \left( h_R \Psi(Z) - p + p\Psi(Z) \right) T.
\end{aligned}
$$

$C''^{\text{VMI}}(Z)$ is obviously non-negative, thus, $C^{\text{VMI}}$ is convex. Setting $C'^{\text{VMI}}(Z) = 0$, we see that the optimal $Z$ satisfies $\Psi(Z) = \frac{p}{p+h_R}$. □

Thus, our solution procedure is as follows: 1) we set $Z$ to some arbitrary value, say $\hat{Z}$; 2) we solve for $\bar{y}^*(\hat{Z}), y_n^*(\hat{Z})$ and $q_{\text{VMI}}^*(\hat{Z})$ using IPA; 3) we collect statistics on the retailer's inventory shortfall at the end of each period (defined as $s = \bar{y}^* - u_n$); 4) the optimal shortfall amount, say $s^*$, is defined to be the value of $s$ where $\Psi(s) = \frac{p}{p+h_R}$; 5) the optimal value of $Z$, say $Z^*$, is then computed as $Z^* = \hat{Z} - (\bar{y}^* - s^*)$; and 6) we find the supplier's optimal replenishment values ($\bar{y}^*(Z^*)$ and $y_n^*(Z^*)\ \forall\ n$) and production value ($q_{\text{VMI}}^*$) using the relationships given in Lemmas 2 and 3, and the associated costs are found through simulation.

# 5 Numerical Analysis

Through numerical analysis, we examine the question of why VMI appears to outperform RMI in some situations, but performs worse than RMI in other scenarios. We then consider the question of how retailers and suppliers can structure a VMI contract (i.e., choose the contractual parameters $b^-, b^+$, and $Q$) such that VMI is given the best opportunity to outperform RMI. We present broad guidelines for choosing the contractual parameters. Given contractual parameters which allow VMI to perform well (not necessarily optimally), we ex-

Figure 1: Example of VMI Performing Poorly

amine the effect of different outsourcing costs and demand variabilities on the performance of VMI. Finally, we examine the effect of VMI on the individual supply chain members.

## Replenish up-to Values

Our numerical results show that a $(z,\ Z)$-type VMI contract can perform better or worse than RMI, depending on the scenario and the chosen contract parameters ($b^{-}, b^{+}$, $z$, and $Z$). Figure **??** shows the optimal VMI, RMI, and supply chain replenish up-to values for a situation where VMI performs better than RMI in terms of overall supply chain cost. Figure 1 shows the same information for a situation where VMI performs worse than RMI. In cases where VMI performs better than RMI, the VMI replenish up-to values closely match those of the supply chain optimal values (see Figure **??**). In Figure 1 due to the poor selection of $b^{-}$ and $b^{+}$ values, under VMI the supplier is sending much more inventory to the retailer than is optimal for the supply chain. However, by simply changing the $b^{-}$ value to 1100 and $b^{+}$ to 75, we can significantly improve the performance of VMI so that it now performs better than RMI, as shown in Figure 2. The VMI replenish up-to values are now very close to the supply chain optimal values for this scenario.

Figure 2: Example of Improving VMI Performance

## Choosing Contractual Parameters

Figure 3: VMI for Low Demand Variance

While it is possible to search over all $b^-, b^+$, and $Q$ values to attain the minimum supply chain cost achievable under VMI, finding these values requires an extremely time-intensive search procedure. However, we have examined a variety of scenarios which we consider to be realistic, and we have identified general guidelines which may aid in the choosing of

VMI contractual parameters. These guidelines will not, in general, achieve the supply chain optimal cost; however, they will provide a robust VMI contract which performs (in most realistic scenarios) significantly better than RMI agreements.

Figure 4: VMI for High Demand Variance

The recommended settings for the VMI contractual parameters are actually quite intuitive. Consider Figures 3 and 4 which show the total supply chain costs in cases of low and high demand variance, respectively. Each line in the figures corresponds to a different value of $b^+$. The $b^-$ values are along the $x$-axis. As demand variance increases, it is optimal (from the supply chain perspective) to allow more flexibility to the supplier in determining how much inventory to ship to the retailer. This is done by increasing the value of $Q$. Based on our sampling of different scenarios, our recommendation for the value of $Q$ does not appear to be sensitive to the length of the horizon or other parameters. Thus, the costs in Figure 3 are calculated with $Q = 40$, while the costs in Figure 4 are calculated using $Q = 200$).

When $b^+ < h_S$ it is cost effective for the supplier to pay the penalty costs rather than to hold inventory, which is obviously not in the best interest of the supply chain. Therefore, we exclude consideration of $b^+ < h_S$. We examined values of $b^+$ between 10 and 200 in increments of 10, as well as larger increments up to a maximum value of 1000 for this example (although for clarity not all of these lines are shown in the figures).

Figure 5: VMI for Low Demand Variance with Incorrect Choice of $Q$

The cost under RMI for the scenario of low demand variance (standard deviation=20) is approximately \$3463, and most combinations of $b^-$ and $b^+$ produce supply chain costs lower than under an RMI agreement. For the high demand variance case (standard deviation=100), the RMI cost is \$10965, and all of the VMI contracts shown in Figure 4 result in supply chain savings. In both cases values of $b^+$ near $h_R$ serve to transfer the cost of holding inventory from the retailer to the supplier to promote inventory levels that are more in-line with those of a coordinated channel. Therefore, we suggest that $b^+$ be set near the retailer's holding cost (so $b^+ = 10$, in this example).

Notice that in both the low demand variance case (Figure 3) and the high demand variance case (Figure 4), given that $b^+ = h_R$, the lowest supply chain costs are attained when $b^-$ is in the range of 250 to 300. At $Z = z = 0$, we would expect a $b^-$ value around the retailer's penalty cost $p$ to produce the lowest supply chain costs (see Theorem 6). Intuitively, as $Q$ (since $Q = Z - z$) increases, the supplier enjoys more flexibility under VMI, and to balance this a higher value of $b^-$ is necessary in order to create inventory levels which are best for the supply chain (as is the case in the scenario depicted in Figure 4). For $Q > 0$, therefore, we expect that the best values of $b^-$ to be $(1+\epsilon)p$ where $\epsilon$ depends on the value of $Q$, and is increasing in $Q$. For the examples shown here, $b^-$ values of about 1.2 to 1.4 times the value of $p$ give the lowest total supply chain costs.

Figure 5 shows the importance of choosing the appropriate value of $Q$ for a given scenario. It recreates the low demand variance scenario shown in Figure 3, except that here $Q$ is set to a value of 100, and our suggested guidelines for setting $b^-$ and $b^+$ would result in total supply chain costs greater than those under RMI. Figure 5 shows that when $Q$ is very large, the total supply chain costs can be more sensitive to changing $b^+$ values. Furthermore, Figure 5 suggests that when $Q$ is very large, large values of $b^+$ and $b^-$ actually result in lower total supply chain costs than our recommended settings.

## Effect of Outsourcing Cost

Table 1 compares the performance of VMI versus RMI at different costs of outsourcing ($b_0$). For these examples $h_S = 5, h_R = 10, p = 200$, $T = 3$, and demand is normally distributed with mean 100 and standard deviation of 50. Following our guidelines we set $Q$ to 100, $b^-$ to 300, and $b^+$ to 10. In these examples VMI results in savings over RMI of just over 8% at an outsourcing cost of 50 to over 23% at an outsourcing cost of 1000. Clearly, as outsourcing becomes more expensive, the supplier is able to send less inventory to the retailer under VMI. Thus, the savings from VMI increases as the cost of outsourcing increases. The increase in savings is driven by the widening gap between RMI costs and supply chain costs. For all cases in Table 1, more than 99% of the potential savings[3] are captured by VMI.

[3]Measured as $\frac{1-\text{VMI Cost - SC Opt. Cost}}{\text{RMI Cost - SC Opt. Cost}} \times 100\%$

| $b_0$ | SC Opt. Cost | VMI Cost | RMI Cost | % Savings Over RMI | % of Potential Savings Captured by VMI | Opt. VMI Production Amount | Opt. RMI Production Amount |
|---|---|---|---|---|---|---|---|
| 50 | 5332.33 | 5333.72 | 5813.26 | 8.25% | 99.71% | 393.75 | 425.27 |
| 100 | 5476.90 | 5481.42 | 6195.69 | 11.53% | 99.37% | 408.46 | 458.12 |
| 150 | 5512.90 | 5516.70 | 6394.52 | 13.73% | 99.57% | 413.38 | 474.77 |
| 200 | 5523.43 | 5527.80 | 6526.69 | 15.30% | 99.56% | 415.34 | 484.96 |
| 250 | 5523.43 | 5528.19 | 6628.35 | 16.60% | 99.57% | 416.31 | 491.70 |
| 500 | 5523.43 | 5528.22 | 6923.36 | 20.15% | 99.66% | 416.46 | 514.86 |
| 1000 | 5523.43 | 5528.22 | 7184.05 | 23.05% | 99.71% | 416.46 | 535.99 |

Table 1: Effect of Cost of Outsourcing

Notice that the optimal production amount under VMI levels off around 416 units once the cost of outsourcing ($b_0$) surpasses 200. Once outsourcing becomes prohibitively expensive, the supplier will choose to not ever outsource. Thus, further increases in $b_0$ have no effect under VMI, and the greater flexibility of VMI (specifically due to the supplier being in a better position to decide whether or not to outsource) translates into larger supply-chain savings.

## Effect of Variance

| Standard Deviation | $Q$ Value | SC Opt. Cost | VMI Cost | RMI Cost | % Savings Over RMI | % of Potential Savings Captured by VMI |
|---|---|---|---|---|---|---|
| 20 | 0 | 3113.24 | 3115.78 | 3463.78 | 10.04% | 99.27% |
| 50 | 100 | 5513.90 | 5522.22 | 6394.52 | 13.64% | 99.06% |
| 80 | 150 | 7788.11 | 7802.40 | 9186.52 | 15.07% | 98.98% |
| 100 | 200 | 9235.65 | 9255.93 | 10965.46 | 15.59% | 98.83% |

Table 2: Effect of Variance

Table 2 displays the effect of increasing demand variance on the performance of VMI for the supply chain. As variance increases, VMI provides greater savings for the supply chain over RMI. With higher variance we would expect a greater chance of the supplier outsourcing. As stated previously, VMI allows the supplier to make better decisions about when to outsource, and this translates into savings for the supply chain. Additionally, by following our general rules for setting the contractual parameters, VMI comes very close to the supply chain optimal. For the cases shown in Table 2, VMI captures at least 98.8% of the potential benefits for each value of demand variance.

## Cost to Individual Agents

We can also look at the cost to the individual agents (supplier and retailer) under VMI. To do this, we use a scenario which we consider "realistic", and we follow the guidelines we suggest for choosing VMI contractual parameters by setting $b^-$ to a value somewhat greater than $p$ ($b^- = 1.5p$ in these examples) and $b^+ = h_R$. We assume that $h_S = 5, h_R = 10, p = 200, b_0 = 150$, mean demand = 100, and $T = 3$. We will examine three scenarios: 1) low demand variance (standard deviation = 20); 2) medium demand variance (standard deviation = 50); and 3) high demand variance (standard deviation = 100).

For each instance of demand variance (see Figure 6) there is actually a range of $Q$ values for which a move from RMI to VMI results in significant total supply chain savings. Within this range, however, the value of $Q$ significantly affects the distribution of costs between individual agents. At very small values of $Q$, the costs under VMI are very unevenly distributed; the supplier is incurring most of the costs, while the retailer can actually make money from the contractual penalty payments paid to him by the supplier. As $Q$ increases the costs become much more equally divided between the retailer and supplier. When $Q$ becomes very large, the retailer incurs most of the costs. This illustrates the ability to adjust $Q$ in order to achieve different divisions of the total supply chain costs. This option, however, is attractive for values of $Q$ which are not 'too large' – very large values of $Q$ do not necessarily result in savings to the supply chain. Figure 6 suggests that when demand variance is high, not only is the potential savings from VMI contracts higher, but also the range in which $Q$ can be used as an instrument of dividing savings is much broader.

# 6 Managerial Insights

First, we see that VMI can provide significant savings for the supply chain over RMI in almost all scenarios capturing nearly all benefits of the centralized model. Second, we see that VMI can be particularly beneficial for products with high demand variance or high outsourcing cost. However, care must be taken in determining the contract parameters such as $b^-, b^+$ and $Q$ values. As we have shown, an incorrect choice of $b^-$, $b^+$, and $Q$ values can cause the supply chain to incur larger costs under VMI than RMI, however, such situations can be avoided.

We have also provided some very general guidelines that may aid in determining appropriate contractual parameters for many realistic scenarios. Our analysis has shown that if the value of $Q$ is adjusted based on the amount of variance present in the demand distribution, then guidelines for choosing $b^-$ and $b^+$ can be given to ensure that total supply chain costs under VMI are lower than those under RMI. We have also shown how changing the value of $Q$ can (within some limits) be used to obtain different divisions of costs between the retailer and its supplier. The potential for this tool is greater when demand is more variable.

Our computational analysis can also show that in many situations where VMI performs poorly, it is the supplier who bears the brunt of the additional costs. This appears to match some criticisms of VMI implementation in industry that VMI can be quite detrimental to suppliers. The performance of individual members of the supply chain also seems to depend to a large extent on the amount of flexibility given to the supplier in the form of greater values of $Q$. For example, our numerical analysis shows that given a large $Q$ and appropriate $b^-$ and $b^+$ values, the supplier benefits from a move to VMI. This validates claims made on a Harvard Business School videotape (HBS #695504) that the supplier may receive great benefit from VMI.

Figure 6: Cost to Individual Supply Chain Agents

Finally, although we provide guidelines for choosing parameters that benefit the supply chain as a whole, our computational analysis suggests that finding parameters where both the supplier and the retailer benefit from VMI is difficult in most cases. Thus, it may be appropriate to consider a fixed payment or other pricing scheme which more fairly distributes the savings from VMI to make VMI attractive to all parties involved.

# 7 Model Extensions

One can easily imagine many variations and extensions of the model presented here. Some possible extensions of our model include accounting for situations where the length of the production cycle may vary, incorporating multiple retailers under a single supplier, and including a lead time for delivery of product from supplier to retailer. We now discuss these changes to our model and their impact.

If we allow the length of the production cycle ($T$) to change, our structural results remain unchanged if we assume that the value of $T$ is known at the beginning of that production cycle (i.e., the value of $T$ is known at the time the amount of production for that cycle is decided). However, the actual production decision is no longer independent of the other production cycles as Theorem 5 will no longer hold since the inventory at the end of a production cycle cannot be guaranteed to be less than the desired up-to value for the next production cycle. Without transshipment, positive lead time can be easily incorporated into our model. As in Hariharan and Zipkin (1995), the actual uncertainty of demand corresponds to the sum of information delay and physical flow delay. Our analysis remains unchanged for this case, but the uncertainty is increased accordingly. If we now allow for multiple retailers, as long as the supplier is able to transship inventories among retailers and negotiated penalty coefficients are identical, the demand from the retailers can be aggregated and then viewed exactly as in our model. Notice also that positive lead time can be incorporated if we allow for transshipment (which implies that no portion of the inventory is assigned to a specific retailer).

# 8 Conclusions

In this paper we have modeled a specific type of VMI agreement that occurs in practice called a $(z,\ Z)$ VMI contract in which the supplier makes all decisions regarding the amount and timing of deliveries to the retailer, but is penalized if the retailer's inventory level falls below $z$ or remains above $Z$ after customer demand. Our goal has been to investigate the benefit of better coordination between production and delivery allowed by VMI. Our model applies to a single supplier and single retailer.

We characterize the optimal behavior of a supplier operating under such an agreement where the supplier is constrained to producing only once every $T$ periods, but can deliver and outsource in any period. The supplier's optimal policy is characterized as a replenish up-to policy, which is shown to be easily computed since the periods when the supplier outsources and when the supplier does not outsource can be decoupled. If the supplier employs outsourcing in a period, the replenishment amount is dependent on the number of periods remaining in the production cycle; however, the optimal up-to values when the supplier does not outsource are stationary. The optimal production policy for the supplier is also defined. We describe the policy of a retailer operating under a $(z,\ Z)$ VMI contract where the retailer can exert some measure of control over the supplier by setting the values of $z$ and $Z$ in the contract, and we relate his decision to the well-known newsvendor problem. It is also shown that the costs incurred by the supplier are driven by the contractual parameter $Q$, while the retailer's costs are driven by his choice of $Z$ and his transfer benefits by $Q$.

We use computer simulation to compare the performance of this type of VMI agreement with a traditional RMI setting with some information sharing. Our numerical results suggest that the savings from this type of VMI agreement can be significant (particularly when outsourcing is very expensive or variance is high), but that this type of VMI contract can also perform worse than RMI in some situations. These results correspond with industry reports, which provide differing accounts on the effectiveness of VMI. Our analysis suggests that the choice of VMI contract parameters during the negotiation process can make the difference between effective VMI agreements and agreements that are detrimental to one or all members of the supply chain, and thus we present some general guidelines that may be of use in determining VMI contractual parameters.

## Appendix

**Proof of Theorem 2** This will be proved using induction on the following statements:

1. $y^*_{n-1} \geq y^*_n$,

2. $V'^{\text{VMI}}_{n-1}(x) \leq V'^{\text{VMI}}_{n}(x) \; \forall \; x < y^*_{n-1}$.

Assume (1) and (2) hold for $n = i+1$. Recall that

$$\begin{aligned}
V^{\text{VMI}}_{i-1}(x) &= -b_0 x + \min_{y \geq x} J^{\text{VMI}}_{i-1}(y), \\
J^{\text{VMI}}_{i-1}(y) &= b_0 y + L(y) + \int_0^\infty V^{\text{VMI}}_{i}(y-\xi)d\Phi(\xi), \text{ and} \\
\frac{d}{dy} J^{\text{VMI}}_{i-1}(y) &= b_0 + L'(y) + \int_0^\infty V'^{\text{VMI}}_{i}(y-\xi)d\Phi(\xi).
\end{aligned}$$

Define

$$H_i(\omega) = b_0 + L'(\omega) + \int_0^\infty V'^{\text{VMI}}_{i}(\omega-\xi)d\Phi(\xi).$$

Since $J^{\text{VMI}}_{i}(y)$ is convex, we see that $H_i(\omega)$ is non-decreasing. We also note that $\lim_{\omega\to\infty} H_i(\omega) > 0$. Therefore, either $H_i(\omega)$ has at least one zero, or is positive for all $\omega$'s. We define $y^*_i$ as the smallest of its zeros or $-\infty$ if none exists.

We now show that (1) and (2) hold for $n = i$. From the definition of $H_i(\omega)$ and the second induction assumption, $H_i(\omega) \leq H_{i+1}(\omega)$. Thus, $y^*_{i-1} \geq y^*_i$. To show that $V'^{\text{VMI}}_{i-1}(x) \leq V'^{\text{VMI}}_{i}(x) \; \forall x < y^*_{i-1}$, we note that under an optimal policy,

$$V^{\text{VMI}}_{i-1}(x) = \begin{cases} b_0(y^*_{i-1} - x) + L(y^*_{i-1}) + \int_0^\infty V^{\text{VMI}}_{i}(y^*_{i-1}-\xi)d\Phi(\xi) & : \quad x < y^*_{i-1} \\ L(x) + \int_0^\infty V^{\text{VMI}}_{i}(x-\xi)d\Phi(\xi), & : \quad x \geq y^*_{i-1}, \end{cases}$$

and

$$V'^{\text{VMI}}_{i-1}(x) = \begin{cases} -b_0 & : \quad x < y^*_{i-1} \\ L'(x) + \int_0^\infty V'^{\text{VMI}}_{i}(x-\xi)d\Phi(\xi), & : \quad x \geq y^*_{i-1}. \end{cases}$$

Thus for $x < y^*_i \leq y^*_{i-1}$, $V'^{\text{VMI}}_{i-1}(x) = V'^{\text{VMI}}_{i}(x) = -b_0$.

For $y^*_i \leq x < y^*_{i-1}$, $V'^{\text{VMI}}_{i-1}(x) = -b_0$. But $V'^{\text{VMI}}_{i}(x) = 0$ at $y^*_i$ and since $V^{\text{VMI}}_{i}(x)$ is convex, $V'^{\text{VMI}}_{i}(x)$ is non-decreasing in $x$ for $x > y^*_i$. Therefore, $V'^{\text{VMI}}_{i}(x) \geq 0$ for $x \geq y^*_i$, and

$V_i'^{\mathrm{VMI}}(x) \geq V_{i+1}'^{\mathrm{VMI}}(x)$ for $y_i^* \leq x < y_{i+1}^*$. Combining, we see that $V_{i-1}'^{\mathrm{VMI}}(x) \leq V_i'^{\mathrm{VMI}}(x)$. In order to conclude our proof, we now must show that $y_{T-2}^* \geq y_{T-1}^*$ and that $V_{T-2}'^{\mathrm{VMI}}(x) \leq V_{T-1}'^{\mathrm{VMI}}(x)$. However, the above argument remains true with $n$ replaced by $T-1$ and recalling that $V_T^{\mathrm{VMI}}(\cdot) = 0$. □

**Proof of Theorem 3** The cost equations for the non-outsourcing case are given by (2) and (3). Consider only the costs that are dependent on the retailer's inventory, Equations (4) and (5). A slight adaptation of Theorem 1 can be used to show that (4) and (5) are convex in $y$ and $x_1$, respectively. We denote the $y$ that minimizes (4) as $\bar{y}_n^*$.

We assume that $U_{n+1}^{\mathrm{VMI}}(x_1, x_2) = \bar{V}_{n+1}^{\mathrm{VMI}}(x_1) + g_{n+1}(x_2)$ and we show that $U_n^{\mathrm{VMI}}(x_1, x_2) = \bar{V}_n^{\mathrm{VMI}}(x_1) + g_n(x_2)$. The $y$ that minimizes $K_n^{\mathrm{VMI}}(y, x_E)$ is $y = \bar{y}_n^*$. Hence, $x_1 = \bar{y}_n^*$ also minimizes $U_n^{\mathrm{VMI}}(x_1, x_2)$ in $x_1$. Thus, if $x_2 \geq \bar{y}_n^*$ (and therefore outsourcing is not required to reach the replenish up-to point),

$$U_n^{\mathrm{VMI}}(x_1, x_2) = \bar{V}_n^{\mathrm{VMI}}(x_1) + h_S x_2 + \int_0^\infty g_{n+1}(x_2 - \xi) d\Phi(\xi).$$

If $x_2 < \bar{y}_n^*$, it is optimal to set $y$ as close to $\bar{y}_n^*$ as possible due to convexity. Therefore, if $x_2 < \bar{y}_n^*$,

$$\begin{aligned} U_n^{\mathrm{VMI}}(x_1, x_2) &= \bar{L}(x_2) + h_S x_2 + \int_0^\infty \bar{V}_{n+1}^{\mathrm{VMI}}(x_2 - \xi) d\Phi(\xi) \\ &\quad + \int_0^\infty g_{n+1}(x_2 - \xi) d\Phi(\xi). \end{aligned}$$

Therefore, if we can show that $U_n^{\mathrm{VMI}}(x_1, x_2) - \bar{V}_n^{\mathrm{VMI}}(x_1)$ is a function of $x_2$ alone then we are done.

$$U_n^{\mathrm{VMI}}(x_1, x_2) - \bar{V}_n^{\mathrm{VMI}}(x_1) = \Lambda_n(x_1, x_2) + \int_0^\infty g_{n+1}(x_2 - \xi) d\Phi(\xi)$$

where

$$\Lambda_n(x_1, x_2) = \begin{cases} \bar{L}(x_2) + h_S x_2 + \int_0^\infty \bar{V}_{n+1}^{\mathrm{VMI}}(x_2 - \xi) d\Phi(\xi) - \bar{V}_n^{\mathrm{VMI}}(x_1) & : \quad x_2 < \bar{y}_n^* \\ 0 & : \quad x_2 \geq \bar{y}_n^*. \end{cases}$$

However, when $x_2 < \bar{y}_n^*$,

$$\bar{V}_n^{\mathrm{VMI}}(x_1) = \bar{L}(\bar{y}_n^*) + \int_0^\infty \bar{V}_{n+1}^{\mathrm{VMI}}(\bar{y}_n^* - \xi) d\Phi(\xi).$$

Therefore,

$$\Lambda_n(x_1,x_2) = \bar{L}(x_2) + h_S x_2 - \bar{L}(\bar{y}_n^*) + \int_0^\infty \left[\bar{V}_{n+1}^{\text{VMI}}(x_2-\xi) - \bar{V}_{n+1}^{\text{VMI}}(\bar{y}_n^*-\xi)\right] d\Phi(\xi)$$

which is independent of $x_1$. Therefore, $U_n^{\text{VMI}}(x_1,x_2) = \bar{V}_n^{\text{VMI}}(x_1) + g_n(x_2)$. Letting $n = T-1$, the same arguments can be used to show that $U_{T-1}^{\text{VMI}}(x_1,x_2) = \bar{V}_{T-1}^{\text{VMI}}(x_1) + g_{T-1}(x_2)$ in order to conclude the inductive argument. $\square$

**Proof of Corollary 1** Notice that $\bar{J}_{T-1}'^{\text{VMI}}(\bar{y}_{T-1}^*) = \bar{L}'(\bar{y}_{T-1}^*) = 0$. Assume $\bar{J}_{n+1}'^{\text{VMI}}(\bar{y}_{T-1}^*) = 0$ (and thus, $\bar{V}_{n+1}'^{\text{VMI}}(y) = 0\ \forall\ y < \bar{y}_{T-1}^*$). This implies $\int_0^\infty \bar{V}_{n+1}'^{\text{VMI}}(\bar{y}_{T-1}^* - \xi)d\Phi(\xi) = 0$. Therefore, $\bar{J}_n'^{\text{VMI}}(\bar{y}_{T-1}^*) = \bar{L}'(\bar{y}_{T-1}^*) + \int_0^\infty \bar{V}_{n+1}'^{\text{VMI}}(\bar{y}_{T-1}^* - \xi)d\Phi(\xi) = 0$. $\square$

**Proof of Corollary 2** Since $\bar{y}^*$ is independent of $n$, it is sufficient to show that $J_n'^{\text{VMI}}(y) \geq \bar{J}_{T-1}'^{\text{VMI}}(y)\ \forall\ n = 0,\ldots,T-1$.

The proof is by induction. For $n = T-1$

$$\begin{aligned} J_{T-1}'^{\text{VMI}}(y) &= L'(y) + b_0 \text{ and} \\ \bar{J}_{T-1}'^{\text{VMI}}(y) &= L'(y) - h_S. \end{aligned}$$

So, $\bar{J}_{T-1}'^{\text{VMI}}(y) \leq J_{T-1}'^{\text{VMI}}(y)$ for all $b_0 \geq 0$ and $h_S \geq 0$. Thus, $\bar{y}^* \geq y_{T-1}^*$. Assume that $\bar{J}_{T-1}'^{\text{VMI}}(y) \leq J_{n+1}'^{\text{VMI}}(y)$. We will show that $\bar{J}_{T-1}'^{\text{VMI}}(y) \leq J_n'^{\text{VMI}}(y)$.

$$\begin{aligned} J_n'^{\text{VMI}}(y) &= L'(y) + b_0 + E[V_{n+1}'^{\text{VMI}}(y-D)] \text{ and} \\ \bar{J}_{T-1}'^{\text{VMI}}(y) &= L'(y) - h_S. \end{aligned}$$

Since $V_{n+1}'^{\text{VMI}}(x) = -b_0$ if $x < y_{n+1}^*$ and $V_{n+1}^{\text{VMI}}$ is convex, $V_{n+1}'^{\text{VMI}}(x) \geq -b_0$ for all $x$. Therefore, $\bar{J}_{T-1}'^{\text{VMI}}(y) \leq J_n'^{\text{VMI}}(y)$ which implies that $\bar{y}^* \geq y_n^*$. $\square$

**Proof of Theorem 4** Let $\bar{L}(x)$ be the one period cost to the supplier as defined in Equation 2. Define $\hat{L}(x)$ as

$$\hat{L}(x) = \begin{cases} \bar{L}(x) + h_S x & : \quad x < \bar{y}^* \\ \bar{L}(\bar{y}^*) + h_S x & : \quad x \geq \bar{y}^*. \end{cases}$$

From Theorem 3, $\bar{L}$ is convex and from Corollary 1, $\bar{L}'(\bar{y}^*) = 0$, thus, $\hat{L}$ is convex.

The proof of Theorem 4 is provided by a modified sample path argument. Assume that the supplier initially has (produces up to) $q$ units. Let $x_i(q)$ be the amount of inventory at the retailer and the supplier (combined) in period $i$. For a given realization of demands, let $M(q)$ be the first period in which $x_{M(q)}(q) \le y^*_{M(q)}$. For $\delta \ge 0$, we have $x_i(q+\delta) = x_i(q) + \delta$, for $i \in \{0, \ldots, M(q) - 1\}$ since in the initial $M(q) - 1$ periods we do not outsource and, along a fixed sample path, increasing the production value by $\delta$ corresponds to increasing the system inventory by $\delta$. If $F_i^{\text{VMI}}(q)$ denotes the supplier's costs in period $i$ ($i \le M(q)$), it is easy to verify that

$$F_i^{\text{VMI}}(q+\delta) - F_i^{\text{VMI}}(q) = \hat{L}(x_i(q) + \delta) - \hat{L}(x_i(q)).$$

Consider the following cases:

1. There exists a period such that $x_{M(q)}(q) \le y^*_{M(q)}$ ($M(q) \in \{0, \ldots, T-1\}$).

   We group all the paths that have the same realizations of demand in periods 0 to $M(q)$. This is a correctly defined partition of all paths (based on our definition of $M(q)$), which allows us to use the expected cost-to-go values starting from period $M(q)$, $V^{\text{VMI}}_{M(q)}(\cdot)$.

   Consider two subcases:

   (a) $x_{M(q)}(q) < y^*_{M(q)}$.
   Choose a $\delta$ such that $x_{M(q)}(q+\delta) = x_{M(q)}(q) + \delta < y^*_{M(q)}$ and $x_{M(q)-1}(q-\delta) = x_{M(q)-1}(q) - \delta > y^*_{M(q)-1}$. In period $M(q)$, the retailer's inventory will be brought up to $y^*_{M(q)}$ when we start with $q-\delta$, $q$, or $q+\delta$. Therefore,

   $$F^{\text{VMI}}(q+\delta) - F^{\text{VMI}}(q) = \sum_{i=0}^{M(q)} \left[\hat{L}(x_i(q) + \delta) - \hat{L}(x_i(q))\right],$$

   and we have a similar result for $F^{\text{VMI}}(q) - F^{\text{VMI}}(q-\delta)$. Thus, by convexity of $\hat{L}$,

   $$F^{\text{VMI}}(q+\delta) - F^{\text{VMI}}(q) \ge F^{\text{VMI}}(q) - F^{\text{VMI}}(q-\delta).$$

   (b) $x_{M(q)}(q) = y^*_{M(q)}$.
   For $b_0 > 0$, $y^*_n < \bar{y}^*$ for all $n \in \{0, \ldots, T-1\}$ (see Proof of Corollary 2). Thus, we choose a $\delta$ such that $\delta < \bar{y}^* - y^*_{M(q)}$ and $\delta < x_{M(q)-1}(q) - y^*_{M(q)-1}$. Notice

that $F^{\text{VMI}}(q) - F^{\text{VMI}}(q-\delta) = -b_0\delta$ since $x_{M(q)}(q-\delta) = y^*_{M(q)}$. Also, since no inventory is left at the supplier after the retailer is replenished in period $M(q)$, $V^{\text{VMI}}_{M(q)}(\cdot)$ describes the cost-to-go.

$$\begin{aligned} F^{\text{VMI}}(q+\delta) - F^{\text{VMI}}(q) &= V^{\text{VMI}}_{M(q)}(x_{M(q)}(q)+\delta) - V^{\text{VMI}}_{M(q)}(x_{M(q)}(q)) \\ &\geq V^{\text{VMI}}_{M(q)}(x_{M(q)}(q)) - V^{\text{VMI}}_{M(q)}(x_{M(q)}(q)-\delta) \\ &= -b_0\delta \end{aligned}$$

from convexity of $V^{\text{VMI}}_{M(q)}$ and the definition of $y^*_{M(q)}$. Thus,

$$F^{\text{VMI}}(q+\delta) - F^{\text{VMI}}(q) \geq F^{\text{VMI}}(q) - F^{\text{VMI}}(q-\delta).$$

2. $x_n(q) > y^*_n$ in each period $n \in \{0, \ldots, T-1\}$.

   If $x_n > y^*_n$ for each period $n$, then for $\delta > 0$, $x_n(q+\delta) > y^*_n$ for each period $n$. Choose a $\delta$ such that $x_n(q-\delta) > y^*_n$ for each period $n$.

$$F^{\text{VMI}}(q) - F^{\text{VMI}}(q-\delta) = \sum_{i=0}^{T-1} \left[\hat{L}(x_i(q)) - \hat{L}(x_i(q)-\delta)\right].$$

   Thus, by convexity of $\hat{L}$,

$$F^{\text{VMI}}(q+\delta) - F^{\text{VMI}}(q) \geq F^{\text{VMI}}(q) - F^{\text{VMI}}(q-\delta).$$

□

**Proof of Lemma 2** (a) The minimizing value of $y$ in period $n$ can be found by solving $\frac{\partial}{\partial y} J^{\text{VMI}}_n(y, Z) = 0$ [4], or

$$\begin{aligned} & L'(y,Z) + \int_0^\infty V'^{\text{VMI}}_{n+1}(y-\xi, Z)\,d\Phi(\xi) = -b_0 \\ \Longleftrightarrow\ & b^-\Phi(y-Z+Q) + b^+\Phi(y-Z) + \int_0^\infty V'^{\text{VMI}}_{n+1}(y-\xi, Z)\,d\Phi(\xi) = b^- - b_0 \qquad \text{(A1)} \end{aligned}$$

Let $y^*_n(\hat{Z})$ be the minimal solution to (A1) when $Z = \hat{Z}$. We will use induction on the following statements:

1. $y^*_n(\hat{Z}) = y^*_n(0) + \hat{Z}$

[4] $J^{\text{VMI}}_n(y, Z)$ and $V^{\text{VMI}}_n(y, Z)$ are equivalent to the equations defined for $J^{\text{VMI}}_n(y)$ and $V^{\text{VMI}}_n(y)$ previously but now we need these equations to depend explicitly upon $Z$.

2. $V_n'^{\text{VMI}}(x+\hat{Z},\hat{Z}) = V_n'^{\text{VMI}}(x,0)$.

By definition of $y^*_{T-1}(0)$,

$$\begin{aligned} b^-\Phi(y^*_{T-1}(0)+Q)+b^+\Phi(y^*_{T-1}(0)) &= b^- - b_0 \\ \Rightarrow b^-\Phi(y^*_{T-1}(0)+\hat{Z}-\hat{Z}+Q)+b^+\Phi(y^*_{T-1}(0)+\hat{Z}-\hat{Z}) &= b^- - b_0. \end{aligned}$$

Therefore, $y^*_{T-1}(\hat{Z}) = y^*_{T-1}(0)+\hat{Z}$.

$$V_{T-1}'^{\text{VMI}}(x+\hat{Z},\hat{Z}) = \begin{cases} -b_0 & : \quad x+\hat{Z} < y^*_{T-1}(\hat{Z}) \\ L'(x+\hat{Z},\hat{Z}) & : \quad x+\hat{Z} \geq y^*_{T-1}(\hat{Z}) \end{cases}$$

and

$$V_{T-1}'^{\text{VMI}}(x,0) = \begin{cases} -b_0 & : \quad x < y^*_{T-1}(0) \\ L'(x,0) & : \quad x \geq y^*_{T-1}(0). \end{cases}$$

But,

$$\begin{aligned} L'(x+\hat{Z},\hat{Z}) &= b^-\Phi(x+\hat{Z}-\hat{Z}+Q)+b^+\Phi(x+\hat{Z}-\hat{Z}) \\ &= b^-\Phi(x+Q)+b^+\Phi(x) \\ &= L'(x,0) \end{aligned}$$

and $y^*_{T-1}(\hat{Z}) = y^*_{T-1}(0)+\hat{Z}$. Therefore, $V_{T-1}'^{\text{VMI}}(x+\hat{Z},\hat{Z}) = V_{T-1}'^{\text{VMI}}(x,0)$.

Now we assume that $y^*_{n+1}(\hat{Z}) = y^*_{n+1}(0)+\hat{Z}$ and $V_{n+1}'^{\text{VMI}}(x+\hat{Z},\hat{Z}) = V_{n+1}'^{\text{VMI}}(x,0)$ and we will show that the same is true with $n$ periods remaining. Recall that $\frac{\partial}{\partial y}J_n^{\text{VMI}}(y,Z)=0$ is equivalent to (A1). Thus, if $Z=0$

$$b^-\Phi(y^*_n(0)+Q)+b^+\Phi(y^*_n(0))+\int_0^\infty V_{n+1}'^{\text{VMI}}(y^*_n(0)-\xi,0)d\Phi(\xi) \quad = \quad b^- - b_0$$

by definition of $y^*_n(0)$. By the induction assumption,

$$\begin{aligned} & b^-\Phi(y^*_n(0)+Q)+b^+\Phi(y^*_n(0)) \\ & \qquad + \int_0^\infty V_{n+1}'^{\text{VMI}}(y^*_n(0)-\xi,0)d\Phi(\xi) = b^- - b_0 \\ \Rightarrow\; & b^-\Phi(y^*_n(0)+\hat{Z}-\hat{Z}+Q)+b^+\Phi(y^*_n(0)+\hat{Z}-\hat{Z}) \\ & \qquad + \int_0^\infty V_{n+1}'^{\text{VMI}}(y^*_n(0)+\hat{Z}-\xi,\hat{Z})d\Phi(\xi) = b^- - b_0. \end{aligned}$$

Therefore,

$$b^- \Phi(y_n^*(\hat{Z}) - \hat{Z} + Q) + b^+ \Phi(y_n^*(\hat{Z}) - \hat{Z}) + \int_0^\infty V_{n+1}^{\prime\mathrm{VMI}}(y_n^*(\hat{Z}) - \xi, \hat{Z}) d\Phi(\xi) = b^- - b_0$$

and $y_n^*(\hat{Z}) = y_n^*(0) + \hat{Z}$.

$$V_n^{\prime\mathrm{VMI}}(x + \hat{Z}, \hat{Z}) = \begin{cases} -b_0 & : \quad x + \hat{Z} < y_n^*(\hat{Z}) \\ L'(x + \hat{Z}, \hat{Z}) + \int_0^\infty V_{n+1}^{\prime\mathrm{VMI}}(x + \hat{Z} - \xi, \hat{Z}) d\Phi(\xi) & : \quad x + \hat{Z} \geq y_n^*(\hat{Z}) \end{cases}$$

and

$$V_n^{\prime\mathrm{VMI}}(x, 0) = \begin{cases} -b_0 & : \quad x < y_n^*(0) \\ L'(x, 0) + \int_0^\infty V_{n+1}^{\prime\mathrm{VMI}}(x - \xi, 0) d\Phi(\xi) & : \quad x \geq y_n^*(0). \end{cases}$$

By our previous argument, $L'(x, 0) = L'(x + \hat{Z}, \hat{Z})$ and from the induction assumption, $V_{n+1}^{\prime\mathrm{VMI}}(x + \hat{Z}, \hat{Z}) = V_{n+1}^{\prime\mathrm{VMI}}(x, 0)$. Thus, $\int_0^\infty V_{n+1}^{\prime\mathrm{VMI}}(x + \hat{Z} - \xi, \hat{Z}) d\Phi(\xi) = \int_0^\infty V_{n+1}^{\prime\mathrm{VMI}}(x - \xi, 0) d\Phi(\xi)$. Combined with the fact that $y_n^*(\hat{Z}) = y_n^*(0) + \hat{Z}$, we see that $V_n^{\prime\mathrm{VMI}}(x + \hat{Z}, \hat{Z}) = V_n^{\prime\mathrm{VMI}}(x, 0)$.

(b) The proof for the non-outsourcing case is similar. □

**Proof of Lemma 3** For a given $Z$, the supplier's optimal policy, $\pi(Z)$, can be fully described by the quantity produced ($q_{\mathrm{VMI}}^*$), the up-to level when the supplier has sufficient on-hand inventory to replenish the retailer ($\bar{y}^*$), and the outsourcing up-to levels ($y_n^*$); i.e., $\pi(Z) := \{q_{\mathrm{VMI}}^*, \bar{y}^*, y_n^*, \text{ for } n = 0, \ldots, T-1\}$. Based on Lemma 2, if we increase $Z$ by $\delta$ (to $Z + \delta$, $\delta > 0$), all up-to levels ($\bar{y}^*$ and $y_n^*$, for $n = 0, \ldots, T-1$) increase by $\delta$.

Notice that if $q_{\mathrm{VMI}}^*$ is also increased by $\delta$, then policy $\pi(Z + \delta) := \{q_{\mathrm{VMI}}^* + \delta, \bar{y}^* + \delta, y_n^* + \delta, \text{ for } n = 1, \ldots, T\}$ has the same costs for the supplier as $\pi(Z)$. Consider case (a) where the retailer specifies the level $Z$ and the supplier chooses policy $\pi(Z)$ and case (b) where the retailer specifies $Z + \delta$ and the supplier chooses $\pi(Z + \delta)$. These two cases result in the same amount of holding costs for the supplier, since the inventory levels at the supplier are identical for the two cases for a given set of demand realizations. The inventory levels at the retailer, however, are larger by a value of $\delta$ for case (b) than for case (a) for a given set of demand realizations. Let $V(Z)$ and $V(Z + \delta)$ be the minimum costs to the supplier over the entire horizon from contractual penalties under VMI when the retailer has specified $Z$ and $Z + \delta$, respectively. If there exists $\tilde{V}(Z + \delta) < V(Z)$, then we should be able to reduce the $\bar{y}^*$ and $y_n^*$ values chosen in $V(Z + \delta)$ by $\delta$ such that $\tilde{V}(Z) < V(Z)$ (where $\tilde{V}(Z)$

is the cost associated with the new $\bar{y}^*$ and $y_n^*$ values). But, this is a contradiction as $V(Z)$ is defined as the minimum cost if the retailer specifies $Z$. Therefore, $V(Z) = V(Z+\delta)$, and the supplier's costs (because of contractual penalties) are the same under case (a) and case (b). $\square$

**Proof of Lemma 4** The proof is by a sample path argument. Consider the effect of changing $Z$ to $Z+\delta$ ($\delta > 0$). From Lemma 3 the production quantity for $Z+\delta$, $q^*_{\text{VMI}}(Z+\delta)$, is equivalent to $q^*_{\text{VMI}}(Z)+\delta$, and similarly from Lemma 2, the replenish up-to levels satisfy $\bar{y}^*(Z+\delta) = \bar{y}^*(Z)+\delta$ and $y_n^*(Z+\delta) = y_n^*(Z)+\delta$. Thus, by a simple induction it is possible to show that in each period the actual payment made by the supplier to the retailer is the same, regardless of the value of $Z$. First, notice that in the first period, before demand is realized, that the retailer's inventory is $\bar{y}^*(Z)$ if the retailer chooses $Z$, and $\bar{y}^*(Z)+\delta$ if the retailer chooses $Z+\delta$, and that the resulting inventory at the supplier is the same for $Z$ and $Z+\delta$ (since $q^*_{\text{VMI}}(Z) - \bar{y}^*(Z) = q^*_{\text{VMI}}(Z)+\delta - (\bar{y}^*(Z)+\delta)$). Clearly, the retailer's inventory at the end of the period when he chooses $Z$ is $\bar{y}^*(Z) - D_n$, and is $\bar{y}^*(Z) - D_n + \delta$ when the retailer chooses $Z+\delta$ (where $D_n$ is the realization of demand in period $n$). In each of the following periods, if the retailer chooses $Z$, he can only reach the inventory level of $\bar{y}^*(Z)$ if a retailer choosing $Z+\delta$ could reach an inventory level of $\bar{y}^*(Z)+\delta$. If that level cannot be reached, then the retailer's inventory level under $Z+\delta$ is clearly $\delta$ more than the retailer choosing $Z$ since the same amount of inventory is available at the supplier. This implies that the supplier where the retailer chooses $Z$ and the supplier where the retailer chooses $Z+\delta$ will run out of inventory in the same period. A similar argument applies in analyzing the outsourcing replenish up-to values ($y_n^*$). Therefore, once the retailer begins outsourcing, his inventory levels after demand satisfy $y_n^*(Z) - D_n$ under $Z$, and $y_n^*(Z) - D_n + \delta$ under $Z+\delta$. Thus, the payments under the two scenarios are equal. $\square$

**Acknowledgments**

The authors would like to thank the Editor-in-Chief, the Senior Editor, and the referees for their detailed and very helpful comments. This research was partially funded through a grant from the University of Michigan Tauber Manufacturing Institute, NSF Career Grant DMI 9875202, and by a grant from the Intel Foundation. The authours would also like to thank Sara Hammerschmidt for her assistance with their numerical calculations.

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