MONOMIAL IDEALS, EDGE IDEALS OF HYPERGRAPHS, AND THEIR MINIMAL GRADED FREE RESOLUTIONS

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Abstract. We use the correspondence between hypergraphs and their associated edge ideals to study the minimal graded free resolution of squarefree monomial ideals. The theme of this paper is to understand how the combinatorial structure of a hypergraph $\mathcal{H}$ appears within the resolution of its edge ideal $\mathcal{I}(\mathcal{H})$. We discuss when recursive formulas to compute the graded Betti numbers of $\mathcal{I}(\mathcal{H})$ in terms of its sub-hypergraphs can be obtained; these results generalize our previous work [21] on the edge ideals of simple graphs. We introduce a class of hypergraphs, which we call properly-connected, that naturally generalizes simple graphs from the point of view that distances between intersecting edges are "well behaved". For such a hypergraph $\mathcal{H}$ (and thus, for any simple graph), we give upper and lower bounds for the regularity of $\mathcal{I}(\mathcal{H})$ via combinatorial information describing $\mathcal{H}$. We also introduce triangulated hypergraphs, a properly-connected hypergraph which is a generalization of chordal graphs. When $\mathcal{H}$ is a triangulated hypergraph, we explicitly compute the regularity of $\mathcal{I}(\mathcal{H})$ and show that the graded Betti numbers of $\mathcal{I}(\mathcal{H})$ are independent of the ground field. As a consequence, many known results about the graded Betti numbers of forests can now be extended to chordal graphs.

1. Introduction

Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ be a finite set, and let $\mathcal{E} = \{E_1, \ldots, E_s\}$ be a family of distinct subsets of $\mathcal{X}$. The pair $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ is called a **hypergraph** if $E_i \neq \emptyset$ for each i. The elements of $\mathcal{X}$ are called the **vertices**, while the elements of $\mathcal{E}$ are called the **edges** of $\mathcal{H}$. A hypergraph $\mathcal{H}$ is **simple** if: (1) $\mathcal{H}$ has no loops, i.e., $|E| \geq 2$ for all $E \in \mathcal{E}$, and (2) $\mathcal{H}$ has no multiple edges, i.e., whenever $E_i, E_j \in \mathcal{E}$ and $E_i \subseteq E_j$, then $i = j$. A hypergraph generalizes the classical notion of a graph; a graph is a hypergraph for which every $E \in \mathcal{E}$ has cardinality two.

Let k be a field. By identifying the vertex x_i with the variable x_i in the ring $R = k[x_1, \ldots, x_n]$, we can associate to every simple hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ a squarefree monomial ideal

$$\mathcal{I}(\mathcal{H}) = \left(\left\{ x^E = \prod_{x \in E} x \;\middle|\; E \in \mathcal{E} \right\} \right) \subseteq R = k[x_1, \ldots, x_n].$$

We call the ideal $\mathcal{I}(\mathcal{H})$ the **edge ideal** of $\mathcal{H}$.

In this paper we study the minimal graded free resolution of $\mathcal{I}(\mathcal{H})$. Since there is a natural bijection between the sets

$$\left\{ \begin{array}{c} \text{simple hypergraphs } \mathcal{H} = (\mathcal{X}, \mathcal{E}) \\ \text{with } \mathcal{X} = \{x_1, \ldots, x_n\} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{squarefree monomial} \\ \text{ideals } I \subseteq R = k[x_1, \ldots, x_n] \end{array} \right\}$$

we are in fact studying a fundamental problem in commutative algebra which asks for the minimal graded free resolution of a monomial ideal (for an introduction see [23]). The edge ideal approach allows us to study this problem from a new angle; the standard approach is to use the Stanley-Reisner dictionary to associate to a squarefree monomial ideal I a simplicial complex Δ where

2000 *Mathematics Subject Classification.* 13D40, 13D02, 05C90, 05E99.
Key words and phrases. hypergraphs, chordal graphs, monomial ideals, graded resolutions, regularity.
Version: June 21, 2006.

the generators of I correspond to the minimal nonfaces of Δ. Instead, we associate to I a new combinatorial object, namely, a hypergraph. The theme of this work is to understand how the algebraic invariants of $I = \mathcal{I}(\mathcal{H})$ encoded in its minimal free resolution relate to the combinatorial properties of $\mathcal{H}$.

The edge ideal of a hypergraph was first introduced by Villarreal [30] in the special case that $\mathcal{H} = G$ is a simple graph. Subsequently, many people, including [1, 11, 12, 14, 15, 16, 17, 26, 28, 27, 29, 31], have been working on a program to build a dictionary between the algebraic properties of $\mathcal{I}(G)$ and the combinatorial structure of G. Of particular relevance to this paper, the minimal graded resolution of $\mathcal{I}(G)$ was investigated in [6, 8, 18, 19, 20, 21, 25, 32] (see also [22] for a survey). In this paper we shall extend some of these results to the hypergraph case, most notably, the results of [21], thereby extending our understanding of quadratic squarefree monomial ideals to arbitrary squarefree monomial ideals. At the same time, we shall also derive new results which, even when restricted to graphs, give new and interesting corollaries.

The edge ideal $\mathcal{I}(\mathcal{H})$ of an arbitrary hypergraph was first studied by Faridi [10] but from a slightly different perspective. Recall that Δ is a **simplicial complex** on the vertex set $\mathcal{X}$ if $\{x_i\} \in \Delta$ for all i, and if $F \in \Delta$ then all subsets of F belong to Δ. The **facets** of Δ are the maximal elements of Δ under inclusion. The **facet ideal** of Δ is then defined to be the ideal $\mathcal{I}(\Delta) = \left(\left\{ x^F = \prod_{x \in F} x \;\middle|\; F \text{ is a facet of } \Delta \right\}\right) \subseteq R$. Note, however, that if $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$ denotes the set of facets of Δ, then $\mathcal{H}(\Delta) = (\mathcal{X}, \mathcal{F}(\Delta))$ is a hypergraph. In fact, what Caboara, Faridi and Selinger [3] call a **facet complex** is a hypergraph. It is immediate that $\mathcal{I}(\mathcal{H}(\Delta)) = \mathcal{I}(\Delta)$. Conversely, given any hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$, we can associate to $\mathcal{H}$ the simplicial complex $\Delta(\mathcal{H}) = \{F \subseteq \mathcal{X} \mid F \subseteq E_i \text{ for some } E_i \in \mathcal{E}\}$. It is again easy to verify that $\mathcal{I}(\mathcal{H}) = \mathcal{I}(\Delta(\mathcal{H}))$.

One may therefore take the viewpoint that the generators of a squarefree monomial ideal correspond to either the edges of a hypergraph or the facets of a simplicial complex. In this paper, we have chosen to take the first option for at least two reasons: first, the language of hypergraphs is more natural to describe our results; and second, we only require the edge structure of the hypergraph and never make use of the simplicial complex structure. (A hypergraph point of view is also taken in the recent paper [14].) Of course, all our results could be reinterpreted as statements about the facet ideal of some simplicial complex.

The starting point of this paper is to determine how the splitting technique used in [21] to study the resolution of edge ideals of graphs can be extended to hypergraphs. Recall that Eliahou and Kervaire [7] call a monomial ideal I **splittable** if $I = J + K$ for two monomial ideas J and K such that the minimal generators of J, K and $J \cap K$ satisfy a technical condition (see Definition 2.3 for the precise statement). When an ideal is splittable, the minimal resolutions (specifically the graded Betti numbers) of I, J, K and $J \cap K$ are then related. Given a hypergraph $\mathcal{H}$, we therefore want to split $\mathcal{I}(\mathcal{H})$ so that the ideals J, K, and $J \cap K$ correspond to edge ideals of sub-hypergraphs of $\mathcal{H}$. This allows us to derive recursive-type formulas to relate the graded Betti numbers of $\mathcal{I}(\mathcal{H})$ to those of sub-hypergraphs of $\mathcal{H}$. These formulas provide a systematic approach to investigating algebraic invariants and properties of $\mathcal{I}(\mathcal{H})$.

We now summarize the results of this paper. In Section 3 we extend the notion of a splitting edge of a graph as defined in [21] to the hypergraph setting. Precisely, let E be an edge of the hypergraph $\mathcal{H}$. If $\mathcal{H}\backslash E$ denotes the hypergraph with the edge E removed, then it is clear that $\mathcal{I}(\mathcal{H}) = (x^E) + \mathcal{I}(\mathcal{H}\backslash E)$. We call E a **splitting edge** precisely when $\mathcal{I}(\mathcal{H}) = (x^E) + \mathcal{I}(\mathcal{H}\backslash E)$ is a splitting of the ideal $\mathcal{I}(\mathcal{H})$. Our main result in Section 3 is the following classification of splitting edges, thus answering a question raised in [22].

Theorem 1.1 (Theorem 3.2)**.** *Let $\mathcal{H}$ be a hypergraph with two or more edges. Then an edge E is a splitting edge of $\mathcal{H}$ if and only if there exists a vertex $z \in E$ such that*

$$(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E) \subseteq (x^E) \cap \mathcal{I}(\mathcal{H}\backslash \{z\}).$$

Here, $\mathcal{H}\backslash\{z\}$ denotes the sub-hypergraph of $\mathcal{H}$ where every edge containing z is removed.

To make use of our classification of splitting edges, we need to be able to describe the resolution of $J \cap K = (x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)$. This resolution was described when $\mathcal{H} = G$ is a simple graph in [22]. However, this is a difficult problem for an arbitrary $\mathcal{H}$. We are therefore interested in families of hypergraphs, which includes all simple graphs, where one can say something about $J \cap K$.

In Section 4 we introduce one such family which we call **properly-connected** hypergraphs. A hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ is properly-connected if all its edges have the same cardinality, and furthermore, if $E, H \in \mathcal{E}$ with $E \cap H \neq \emptyset$, then the distance $\operatorname{dist}_{\mathcal{H}}(E, H)$ between E and H, that is, the length of the shortest path between E and H in $\mathcal{H}$, is determined by $|E \cap H|$. It is easy to see that all simple graphs are properly-connected. In fact, a re-examination of the results of [21] reveals that the properly-connected property of graphs is an essential ingredient implicitly used in the proofs. A properly-connected hypergraph is in some sense a natural generalization of a simple graph.

When $\mathcal{H}$ is properly-connected, we can describe the resolution of $J \cap K$ in terms of edge ideals of sub-hypergraphs of $\mathcal{H}$. Therefore, for any splitting edge $E \in \mathcal{H}$, we can derive the following recursive-type formula for $\beta_{i,j}(\mathcal{I}(\mathcal{H}))$.

Theorem 1.2 (Theorem 4.14)**.** *Let $\mathcal{H}$ be a properly-connected hypergraph and let E be a splitting edge of $\mathcal{H}$. Suppose $d = |E|$, $\mathcal{H}' = \{H \in \mathcal{H} \mid \operatorname{dist}_{\mathcal{H}}(E, H) \geq d+1\}$, and $t = |N(E)|$, where*

$$N(E) = \bigcup_{\{H \in \mathcal{H} \mid \operatorname{dist}_{\mathcal{H}}(E,H)=1\}} H\backslash E.$$

Then for all $i \geq 1$

$$\beta_{i,j}(\mathcal{I}(\mathcal{H})) = \beta_{i,j}(\mathcal{I}(\mathcal{H}\backslash E)) + \sum_{l=0}^{i} \binom{t}{l} \beta_{i-1-l,j-d-l}(\mathcal{I}(\mathcal{H}')).$$

Here, $\beta_{-1,j}(\mathcal{I}(\mathcal{H}')) = 1$ if $j = 0$ and 0 if $j \neq 0$.

The sub-hypergraphs $\mathcal{H}\backslash E$ and $\mathcal{H}'$ in Theorem 1.2 may fail to have splitting edges, thus preventing us from recursively computing $\beta_{i,j}(\mathcal{I}(\mathcal{H}))$. However, in [21] (see also [18, 19] in the case of forests), it is proved that when $\mathcal{H}$ is a hyperforest (i.e., a simplicial forest in the sense of [10]) then $\beta_{i,j}(\mathcal{I}(\mathcal{H}))$ can be computed recursively. The goal of Section 5 is to introduce a subclass of properly-connected hypergraphs, which we call **triangulated** hypergraphs, for which Theorem 1.2 can used to completely resolve the graded Betti numbers of $\mathcal{I}(\mathcal{H})$ recursively. Triangulated hypergraphs generalize the notion of **chordal** graphs, which has attracted considerable attention lately (cf. [11, 12, 16, 17]). In fact, triangulated graphs are precisely chordal graphs. As a consequence of Theorem 1.2, we show also that the graded Betti numbers of a triangulated hypergraph are independent of the characteristic of the ground field (Corollary 5.9). Restricted to simple graphs, we obtain the following interesting corollary, which extends a result of [18, 19] (who proved the result for forests).

Corollary 1.3 (Corollary 5.10)**.** *Suppose that G is a chordal graph. Then the graded Betti numbers of $\mathcal{I}(G)$ are independent of the characteristic of the ground field and can be computed recursively.*

In Section 6 we study $\operatorname{reg}(\mathcal{I}(\mathcal{H}))$, the Castelnuovo-Mumford regularity of $\mathcal{I}(\mathcal{H})$, when $\mathcal{H}$ is properly-connected. Again, the key idea we need here is the notion of distance between edges. We say two edges $E, H \in \mathcal{H}$ are ***t*-disjoint** if $\operatorname{dist}_{\mathcal{H}}(E, H) \geq t$. When $\mathcal{H}$ is a properly-connected hypergraph and d is the common cardinality of the edges, then d-disjoint edges are disjoint edges in the usual sense. We then show the following:

Theorem 1.4. *Let $\mathcal{H}$ be a properly-connected hypergraph. Suppose d is the common cardinality of the edges in $\mathcal{H}$. Let n_1 (respectively, n_2) be the maximal number of pairwise $(d+1)$-disjoint edges (respectively, d-disjoint edges) of $\mathcal{H}$. Then*

(*i*) (Theorem 6.6) $(d-1)n_1 + 1 \leq \operatorname{reg}(\mathcal{I}(\mathcal{H})) \leq (d-1)n_2 + 1$.
(*ii*) (Theorem 6.11) *if $\mathcal{H}$ is also triangulated, then* $\operatorname{reg}(\mathcal{I}(\mathcal{H}))$ *equals the lower bound.*

By a **matching** of a hypergraph $\mathcal{H}$, we mean any subset $\mathcal{E}' \subseteq \mathcal{E}$ of edges in $\mathcal{H}$ which are pairwise disjoint. The **matching number** of $\mathcal{H}$, denoted by $\alpha'(\mathcal{H})$, is the largest size of a maximal matching of $\mathcal{H}$. Theorem 1.4 (i) gives a particularly nice corollary for simple graphs. This addresses a question J. Herzog had asked us.

Corollary 1.5 (Corollary 6.9)**.** *Let G be a finite simple graph. Then*

$$\operatorname{reg}(R/\mathcal{I}(G)) \leq \alpha'(G)$$

where $\alpha'(G)$ is the matching number of G.

Using Corollary 1.5, we can compare the regularity and projective dimension of $\mathcal{I}(G)$ to those of $\mathcal{I}(G)^\vee$, the **Alexander dual** of $\mathcal{I}(G)$.

Theorem 1.6 (Theorem 6.17)**.** *Let G be a simple graph.*

(1) *If G is unmixed (i.e., all the minimal vertex covers have the same cardinality), then*

$$\operatorname{reg}(\mathcal{I}(G)) \leq \operatorname{ht}\mathcal{I}(G) + 1 \leq \operatorname{reg}(\mathcal{I}(G)^\vee) + 1 \text{ and } \operatorname{pdim}(\mathcal{I}(G)^\vee) \leq \operatorname{ht}\mathcal{I}(G) \leq \operatorname{pdim}(\mathcal{I}(G)) + 1.$$

(2) *If G is not unmixed, then*

$$\operatorname{reg}(\mathcal{I}(G)) \leq \operatorname{ht}\mathcal{I}(G) + 1 \leq \operatorname{reg}(\mathcal{I}(G)^\vee) \text{ and } \operatorname{pdim}(\mathcal{I}(G)^\vee) \leq \operatorname{ht}\mathcal{I}(G) \leq \operatorname{pdim}(\mathcal{I}(G)).$$

When restricted to simple graphs, Theorem 1.4 (*ii*) also gives an interesting corollary, which was first proved by Zheng [32] in the special case that G was a forest.

Corollary 1.7 (Corollary 6.12)**.** *Let G be a chordal graph. Then*

$$\operatorname{reg}(\mathcal{I}(G)) = j + 1$$

where j is the maximal number of 3-disjoint edges in G.

Finally, in Section 7 we show that the first syzygy module of $\mathcal{I}(\mathcal{H})$ when $\mathcal{H}$ is properly-connected is generated by linear syzygies if and only if the diameter of the hypergraph $\mathcal{H}$ is small enough (Theorem 7.4). By **diameter** we mean the maximum distance between any two edges of $\mathcal{H}$. This result can be seen as the first step towards generalizing Fröberg's result [13] characterizing graphs whose edge ideals have a linear resolution. As an interesting corollary, if $\mathcal{H}$ is a triangulated hypergraph, and if $\mathcal{I}(\mathcal{H})$ only has linear first syzygies, then the resolution of $\mathcal{I}(\mathcal{H})$ must in fact be linear (Corollary 7.6).

2. Preliminaries

We recall the relevant results concerning hypergraphs, resolutions, and splittable ideals.

2.1. **Hypergraphs and edge ideals.** Our reference for the hypergraph material is Berge [2].

Throughout this paper we shall assume that our hypergraphs $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ are simple, i.e., $|E| \geq 2$ for all $E \in \mathcal{E}$, and there is no element of $\mathcal{E}$ which contains another. When there is no danger of confusion, we sometimes specify a hypergraph by describing only its set of edges.

If each $E \in \mathcal{E}$ has the same cardinality d, then we call $\mathcal{H}$ a d**-uniform** hypergraph. Note that a simple graph is a simple 2-uniform hypergraph. If $\mathcal{H}$ is d-uniform, then the associated simplicial complex $\Delta(\mathcal{H})$ is a **pure** simplicial complex, that is, all its facets have the same dimension.

If E is an edge of a hypergraph $\mathcal{H}$, then we let $\mathcal{H}\backslash E$ denote the hypergraph formed by removing the edge E from $\mathcal{H}$. Similarly, if x is a vertex of $\mathcal{H}$, we shall write $\mathcal{H}\backslash\{x\}$ to denote the hypergraph formed by removing x and all edges $E \in \mathcal{E}$ with the property that $x \in \mathcal{E}$. Note that x is an isolated vertex of $\mathcal{H}\backslash\{x\}$, or we can also consider the vertex set of $\mathcal{H}\backslash\{x\}$ to be $\mathcal{X}\backslash\{x\}$. If $\mathcal{Y} \subset \mathcal{X}$, then the **induced hypergraph on** $\mathcal{Y}$, denoted $\mathcal{H}_{\mathcal{Y}}$, is the sub-hypergraph of $\mathcal{H}$ whose edge set is $\{E \in \mathcal{E} \mid E \subseteq \mathcal{Y}\}$.

The notion of distance between edges in a hypergraph will play a fundamental role in later discussions. We introduce the relevant definitions here.

Definition 2.1. A **chain of length** n in $\mathcal{H}$ is a sequence $(E_0, x_1, E_1, \ldots, x_n, E_n)$ such that

(1) $x_1, \ldots, x_n$ are all distinct vertices of $\mathcal{H}$,
(2) $E_0, \ldots, E_n$ are all distinct edges of $\mathcal{H}$, and
(3) $x_1 \in E_0$, $x_n \in E_n$, and $x_k, x_{k+1} \in E_k$ for each $k = 1, \ldots, n-1$.

We sometimes denote the chain by $(E_0, \ldots, E_n)$ if the vertices in the chain are not being investigated. Note that (3) implies that $E_i \cap E_{i+1} \neq \emptyset$ for $i = 0, \ldots, n-1$. If E and E' are two edges, then E and E' are **connected** if there exists a chain $(E_0, \ldots, E_n)$ where $E = E_0$ and $E' = E_n$. If $|E| \geq |E'|$, then the chain connecting E to E' is a **proper chain** if $|E_i \cap E_{i+1}| = |E_{i+1}| - 1$ for all $i = 0, \ldots, n-1$. The (proper) chain is an **(proper) irredundant chain** of length n if no proper subsequence is a (proper) chain from E to E'.

Definition 2.2. If E and E' are two edges of a hypergraph $\mathcal{H}$ with $|E| \geq |E'|$, then we define the **distance** between E and E', denoted by $\operatorname{dist}_{\mathcal{H}}(E, E')$, to be

$$\operatorname{dist}_{\mathcal{H}}(E, E') = \min\{\ell \mid (E = E_0, \ldots, E_\ell = E') \text{ is a proper irredundant chain}\}.$$

If no proper irredundant chain between the two edges exists, we set $\operatorname{dist}_{\mathcal{H}}(E, E') = \infty$.

As in the introduction, the **edge ideal** of $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ is the squarefree monomial ideal

$$\mathcal{I}(\mathcal{H}) = \left(\left\{ x^E = \prod_{x \in E} x \;\middle|\; E \in \mathcal{E} \right\} \right) \subseteq R = k[x_1, \ldots, x_n].$$

We often abuse notation and write x^E for both the edge E and the corresponding monomial.

2.2. **Resolutions and splittable ideals.** Let M be a graded R-module where $R = k[x_1, \ldots, x_n]$. Associated to M is a **minimal graded free resolution** of the form

$$0 \to \bigoplus_j R(-j)^{\beta_{l,j}(M)} \to \bigoplus_j R(-j)^{\beta_{l-1,j}(M)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(M)} \to M \to 0$$

where $l \leq n$ and $R(-j)$ is the R-module obtained by shifting the degrees of R by j. The number $\beta_{i,j}(M)$, the ijth **graded Betti number** of M, equals the number of minimal generators of degree j in the ith syzygy module of M.

Of particular interest are the following invariants which measure the "size" of the minimal graded free resolution of I. The **regularity** of I, denoted $\operatorname{reg}(I)$, is defined by

$$\operatorname{reg}(I) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

The **projective dimension** of I, denoted $\operatorname{pdim}(I)$, is defined to be

$$\operatorname{pdim}(I) := \max\{i \mid \beta_{i,j}(I) \neq 0\}.$$

An ideal I generated by elements of degree d is said to have a **linear resolution** if $\beta_{i,j}(I) = 0$ for all $j \neq i + d$.

We now recall some results concerning splittable ideals. We use $\mathcal{G}(I)$ to denote the unique minimal set of generators of a monomial ideal I.

Definition 2.3 (see [7]). A monomial ideal I is **splittable** if I is the sum of two nonzero monomial ideals J and K, that is, $I = J + K$, such that

(1) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$.
(2) there is a **splitting function**

$$\begin{array}{rcl} \mathcal{G}(J \cap K) & \rightarrow & \mathcal{G}(J) \times \mathcal{G}(K) \\ w & \mapsto & (\phi(w), \psi(w)) \end{array}$$

satisfying
  (a) for all $w \in \mathcal{G}(J \cap K)$, $w = \mathrm{lcm}(\phi(w), \psi(w))$.
  (b) for every subset $S \subset \mathcal{G}(J \cap K)$, both $\mathrm{lcm}(\phi(S))$ and $\mathrm{lcm}(\psi(S))$ strictly divide $\mathrm{lcm}(S)$.

If J and K satisfy the above properties, then we shall say $I = J + K$ is a **splitting** of I.

When $I = J + K$ is a splitting, then there is a relation between $\beta_{i,j}(I)$ and the graded Betti numbers of the "smaller" ideals. This relation was first observed for the total Betti numbers by Eliahou and Kervaire [7] and extended to the graded case by Fatabbi [9].

Theorem 2.4. *Suppose I is a splittable monomial ideal with splitting $I = J + K$. Then*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \text{ for all } i, j \geq 0$$

where $\beta_{i-1,j}(J \cap K) = 0$ if $i = 0$.

When I is a splittable ideal, Theorem 2.4 gives us the following corollary.

Corollary 2.5. *If I is a splittable monomial ideal with splitting $I = J + K$, then*

(*i*) $\mathrm{reg}(I) = \max\{\mathrm{reg}(J), \mathrm{reg}(K), \mathrm{reg}(J \cap K) - 1\}$.
(*ii*) $\mathrm{pdim}(I) = \max\{\mathrm{pdim}(J), \mathrm{pdim}(K), \mathrm{pdim}(J \cap K) + 1\}$.

Our goal is to study the numbers $\beta_{i,j}(\mathcal{I}(\mathcal{H}))$. It follows directly from the definition of $\mathcal{I}(\mathcal{H})$ that $\beta_{0,j}(\mathcal{I}(\mathcal{H}))$ is simply the number of edges $E \in \mathcal{H}$ with $|E| = j$. We can therefore restrict to investigating the numbers $\beta_{i,j}(\mathcal{I}(\mathcal{H}))$ with $i \geq 1$. When $\mathcal{H}$ is a d-uniform hypergraph, the following results implies that we only need to consider a finite range of values of j for each i.

Theorem 2.6. *Suppose that $\mathcal{H}$ is a d-uniform hypergraph. If $\beta_{i,j}(\mathcal{I}(\mathcal{H})) \neq 0$, then $i + d \leq j \leq \min\{n, d(i+1)\}$.*

Proof. Because $\mathcal{H}$ is a d-uniform hypergraph, $\mathcal{I}(\mathcal{H})$ is generated by monomials of degree d. So, $\beta_{i,j}(\mathcal{I}(\mathcal{H})) = 0$ for $j < i + d$, thus giving us the lower bound. For the upper bound, the Taylor resolution implies that $\beta_{i,j}(\mathcal{I}(\mathcal{H})) = 0$ if $j > d(i+1)$. On the other hand, Hochster's formula implies that $\beta_{i,j}(\mathcal{I}(\mathcal{H})) = 0$ if $j > n$. The conclusion now follows. □

3. Splitting edges

Let I be any squarefree monomial ideal, and suppose that $\mathcal{H}$ is the hypergraph associated to I, i.e., $I = \mathcal{I}(\mathcal{H})$. We would like to find splittings of I so that we can make use of Theorem 2.4. In this section we describe one possible splitting of $\mathcal{I}(\mathcal{H})$.

One of the simplest ways to partition $\mathcal{G}(I)$ is to pick any $m \in \mathcal{G}(I)$, and set $\mathcal{G}(J) = \{m\}$ and $\mathcal{G}(K) = \mathcal{G}(I) \backslash \{m\}$. Note that this is equivalent to picking any edge E of $\mathcal{H}$, and setting

$$J = (x^E) \text{ and } K = \mathcal{I}(\mathcal{H} \backslash E).$$

It is immediate that $I = \mathcal{I}(\mathcal{H}) = J + K$, and furthermore, J and K satisfy condition (1) of Definition 2.3. However, for an arbitrary edge E, J and K may fail to satisfy condition (2) of Definition 2.3. If E is chosen so that J and K satisfy this condition, then we give this edge the following name.

Definition 3.1. Let $\mathcal{H}$ be a hypergraph. An edge E is a **splitting edge** of $\mathcal{H}$ if

$$\mathcal{I}(\mathcal{H}) = (x^E) + \mathcal{I}(\mathcal{H}\backslash E)$$

is a splitting of $\mathcal{I}(\mathcal{H})$.

To make use of Theorem 2.4, one would therefore like a means to identify the splitting edges of a hypergraph. The main result of this section is the following theorem which provides a classification of the splitting edges of a hypergraph. This theorem answers Question 5.4.2 of [22] which asked the equivalent question of what facet could be a splitting facet of simplicial complex.

Theorem 3.2. *Let $\mathcal{H}$ be a hypergraph with two or more edges. Then an edge E is a splitting edge of $\mathcal{H}$ if and only if there exists a vertex $z \in E$ such that*

$$(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E) \subseteq (x^E) \cap \mathcal{I}(\mathcal{H}\backslash\{z\}).$$

Proof. Let E be an edge of $\mathcal{H}$, and set $J = (x^E)$ and $K = \mathcal{I}(\mathcal{H}\backslash E)$. To prove the "only if" direction, we prove the contrapositive. So, suppose that for every vertex $z \in E$, we have

$$(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E) \nsubseteq (x^E) \cap \mathcal{I}(\mathcal{H}\backslash\{z\}).$$

Thus, for each $z \in E$, there exists a minimal generator x^{L_z} of $J \cap K$ such that $x^{L_z} \notin (x^E) \cap \mathcal{I}(\mathcal{H}\backslash\{z\})$. Set $S = \{x^{L_z} \mid z \in E\} \subseteq \mathcal{G}(J \cap K)$.

We will now show that no splitting function can exist. Suppose there was a splitting function $s : \mathcal{G}(J \cap K) \to \mathcal{G}(J) \times \mathcal{G}(K)$ given by $s(w) = (\phi(w), \varphi(w))$. Then, since $J = (x^E)$, for each $x^{L_z} \in S$, we have $\phi(x^{L_z}) = x^E$. For each $z \in E$, let $x^{G_z} = \varphi(x^{L_z}) \in \mathcal{G}(K)$. So G_z is an edge of $\mathcal{H}$, and $\operatorname{lcm}(x^E, x^{G_z}) = x^{E \cup G_z} = x^{L_z}$.

We claim that for each $z \in E$, we have $z \in G_z$. Indeed, if $z' \notin G_{z'}$ for some $z' \in E$, then $G_{z'}$ is an edge of $\mathcal{H}\backslash\{z'\}$. But then $x^{L_{z'}} = \operatorname{lcm}(x^E, x^{G_{z'}}) = x^{E \cup G_{z'}}$ is an element of $(x^E) \cap \mathcal{I}(\mathcal{H}\backslash\{z'\})$, a contradiction to the choice of $x^{L_{z'}}$.

Now, since $z \in G_z$ for each $z \in E$, we have

$$\begin{aligned} \operatorname{lcm}(\varphi(S)) &= \operatorname{lcm}(\{x^{G_z} \mid z \in E\}) = x^{\cup_{z \in E} G_z} = x^{(\cup_{z \in E} G_z) \cup E} = x^{\cup_{z \in E}(G_z \cup E)} \\ &= x^{\cup_{z \in E} L_z} = \operatorname{lcm}(\{x^{L_z} \mid z \in E\}) = \operatorname{lcm}(S). \end{aligned}$$

But this contradicts the fact that we have a splitting function. This proves the "only if" direction.

Conversely, suppose that there exists a vertex z of E such that

$$(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E) \subseteq (x^E) \cap \mathcal{I}(\mathcal{H}\backslash\{z\}).$$

This implies that $\mathcal{G}(J \cap K) \subseteq \{x^{E \cup H} \mid H \in \mathcal{H}\backslash\{z\}\}$. We will construct a splitting function $s = (\phi, \varphi) : \mathcal{G}(J \cap K) \to \mathcal{G}(J) \times \mathcal{G}(K)$ which satisfies the conditions of Definition 2.3. For any $x^L \in \mathcal{G}(J \cap K)$ we define $\phi(x^L) = x^E \in \mathcal{G}(J)$. For each $x^L \in \mathcal{G}(J \cap K)$, $\varphi(x^L)$ is defined as follows: by our hypothesis, we have $L \in \{E \cup H \mid H \in \mathcal{H}\backslash\{z\}\}$. Thus, $\mathbb{A} = \{H \in \mathcal{H}\backslash\{z\} \mid L = E \cup H\}$ is not the empty set. We consider $\mathcal{X}$ as a set of alphabets (in some order of its elements) and identify each element of $\mathbb{A}$ with the word formed by its vertices (in increasing order). Let G_L be the unique maximal element of $\mathbb{A}$ with respect to the lexicographic word ordering (which is a total order). Observe that, by construction, $z \notin G_L$ and $E \cup G_L = L$. Define $\varphi(x^L) = x^{G_L}$.

It is easy to see that $s = (\phi, \varphi)$ is a well defined function on $\mathcal{G}(J \cap K)$ and that condition (a) of Definition 2.3 is satisfied. To show that condition (b) of Definition 2.3 is satisfied, we observe that for any $x^L \in \mathcal{G}(J \cap K)$, by construction, z does not divide $\varphi(x^L)$. Observe further that for any subset $S \subseteq \mathcal{G}(J \cap K)$, z divides x^E which strictly divides $\operatorname{lcm}(S)$. Thus, since $\operatorname{lcm}(\phi(S)) = x^E$ and since z does not divide $\operatorname{lcm}(\varphi(S))$, we must have that $\operatorname{lcm}(\phi(S))$ and $\operatorname{lcm}(\varphi(S))$ both strictly divide $\operatorname{lcm}(S)$. The "if" direction is proved. □

Remark 3.3. Theorem 3.2 could be reinterpreted as describing when a squarefree monomial ideal $I = (m_1, \ldots, m_s)$ in $R = k[x_1, \ldots, x_n]$ has a splitting $I = (m_i) + (m_1, \ldots, \hat{m}_i, \ldots, m_s)$ for some i. Precisely, $I = (m_i) + (m_1, \ldots, \hat{m}_i, \ldots, m_s)$ is a splitting if and only if there exists a variable x_j such that $x_j | m_i$ and $(m_i) \cap (m_1, \ldots, \hat{m}_i, \ldots, m_s) \subseteq (m_i) \cap I'R$ where by $I'R$ we mean the ideal $I' = I \cap k[x_1, \ldots, \hat{x}_j, \ldots, x_n]$, but viewed as ideal of R. The result follows from the fact that $\mathcal{I}(\mathcal{H} \backslash \{x_j\}) = I'R$. This reformulation nicely illustrates that in some cases the hypergraph point of view is conceptually easier (at least to us) to grasp.

Example 3.4. The following example illustrates that a hypergraph may not have a splitting edge. Let $\mathcal{H}$ be the hypergraph on vertex set $\mathcal{X} = \{a, b, c, d, e\}$ with edge set $\mathcal{E} = \{abe, ade, bce, cde\}$. The edge ideal is then $\mathcal{I}(\mathcal{H}) = (abe, ade, bce, cde)$. By symmetry it suffices to show that any one of the edges is not a splitting edge. So, consider the edge $E = abe$. Then

$$(x^E) \cap \mathcal{I}(\mathcal{H} \backslash E) = (abde, abce, abcde) = (abde, abce)$$

while

$$(x^E) \cap \mathcal{I}(\mathcal{H} \backslash \{a\}) = (abce), \quad (x^E) \cap \mathcal{I}(\mathcal{H} \backslash \{b\}) = (abde), \text{ and } (x^E) \cap \mathcal{I}(\mathcal{H} \backslash \{e\}) = (0).$$

Thus, there is no vertex $z \in E$ with the property that $(x^E) \cap \mathcal{I}(\mathcal{H} \backslash E) \subseteq (x^E) \cap \mathcal{I}(\mathcal{H} \backslash \{z\})$.

There is a nice class of edges of a simple hypergraph that are easy to identify and also have the property that they are splitting edges. We now define this class.

Definition 3.5. Let $\mathcal{H}$ be a simple hypergraph. An edge E is a v-**leaf** if E contains a free vertex, that is, E contains a vertex $v \in \mathcal{X}$ such that v does not belong to any other edge of $\mathcal{H}$.

Remark 3.6. If $\mathcal{H} = G$ is a simple graph, then v-leaves are precisely the leaves in the usual sense.

Corollary 3.7. *Suppose E is a v-leaf of a hypergraph $\mathcal{H}$. Then E is a splitting edge of $\mathcal{H}$.*

Proof. Suppose v is the free vertex in E. Observe that for any $x^L \in \mathcal{G}(J \cap K)$, by definition we have $x^L = x^{E \cup H}$ for some edge H of $\mathcal{H} \backslash E$. Moreover, since v is a free vertex, we have $v \notin H$, which then implies that $H \in \mathcal{H} \backslash \{v\}$. Thus, $x^L \in (x^E) \cap \mathcal{I}(H \backslash \{v\})$. Since this is true for any $x^L \in \mathcal{G}(J \cap K)$, the conclusion now follows from Theorem 3.2. □

S. Faridi [10] introduced the notion of a leaf for a simplicial complex Δ. Precisely, a facet F of Δ is a **leaf** if F is the only facet of Δ, or there exists a facet $G \neq F$ in Δ such that $F \cap F' \subseteq F \cap G$ for all facets $F' \neq F$ in Δ. We can translate Faridi's definition into hypergraph language; we call the translated version of Faridi's leaf a f-leaf to distinguish it from a v-leaf.

Definition 3.8. An edge E of a hypergraph $\mathcal{H}$ is a f-**leaf** if E is the only edge of $\mathcal{H}$, or if there exists an edge H of $\mathcal{H}$ such that $E \cap E' \subseteq E \cap H$ for all edges $E' \neq E$ of $\mathcal{H}$.

We introduce two types of hypertrees and hyperforests based upon the two notions of leaves.

Definition 3.9. A hypergraph $\mathcal{H}$ is a v-**forest**, respectively, f-**forest**, if every induced subgraph of $\mathcal{H}$, including $\mathcal{H}$ itself, contains a v-leaf, respectively, a f-leaf. If $\mathcal{H}$ is connected, we call $\mathcal{H}$ a v-**tree**, respectively, f-**tree**. When $\mathcal{H}$ is a f-forest, the associated simplicial complex $\Delta(\mathcal{H})$ is called a **simplicial forest**.

Notice that when $\mathcal{H} = G$ is a simple graph, the notions of v-leaf and f-leaf coincide. So, with simple graphs, the notions of a v-forest and a f-forest coincide with the usual notion of a forest. These definitions, however, are not equivalent in a general hypergraph, as illustrated below.

Example 3.10. A f-leaf must always contain a free vertex (cf. [10, Remark 2.3]), thus every f-leaf is a v-leaf. However, a v-leaf need not be a f-leaf. For example, consider the hypergraph $\mathcal{H}$ on $\mathcal{X} = \{a,b,c,d,e,f\}$ with the edge set $\mathcal{E} = \{abf, bcd, def\} = \{E_1, E_2, E_3\}$. Each edge is a v-leaf since each edge has a vertex not in the other two edges. However, $\mathcal{H}$ has no f-leaf. By symmetry, it is enough to show that $E_1 = abf$ cannot be a f-leaf. Indeed, $E_1 \cap E_2 \not\subseteq E_1 \cap E_3$ and $E_1 \cap E_3 \not\subseteq E_1 \cap E_2$.

The hypergraph $\mathcal{H}$ is an example of a v-tree, but $\mathcal{H}$ is not a f-tree since $\mathcal{H}$ has no f-leaf, although all its induced subgraphs have a f-leaf.

Because a f-leaf is a v-leaf, Corollary 3.7 immediately gives:

Corollary 3.11. *If E is a f-leaf of a hypergraph $\mathcal{H}$, then E is a splitting edge of $\mathcal{H}$.*

4. Properly-connected hypergraphs

Given a hypergraph $\mathcal{H}$, we would like to express the numbers $\beta_{i,j}(\mathcal{I}(\mathcal{H}))$ in terms of the graded Betti numbers of edge ideals associated to subgraphs of $\mathcal{H}$; this would lead to recursive-type formulas. When E is a splitting edge of a hypergraph $\mathcal{H}$, Theorem 2.4 implies that $\beta_{i,j}(\mathcal{I}(\mathcal{H}))$ can be computed from the graded Betti numbers of the ideals (x^E), $\mathcal{I}(\mathcal{H}\backslash E)$, and $L = (x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)$. The Betti numbers of (x^E) are trivial to compute, while those of $\mathcal{I}(\mathcal{H}\backslash E)$ already correspond to the edge ideal of a sub-hypergraph of $\mathcal{H}$. Thus one only needs to relate the numbers $\beta_{i,j}(L)$ to the Betti numbers of an edge ideal of some other sub-hypergraph. For a general hypergraph, this appears to be a difficult problem.

The goal of this section is to introduce a family of d-uniform hypergraphs, which we call properly-connected, that among other things enables us to relate the graded Betti numbers of L to those of an edge ideal associated to a sub-hypergraph $\mathcal{H}$.

Definition 4.1. A d-uniform hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ is said to be **properly-connected** if for any two edges E and E' of $\mathcal{H}$ with the property that $E \cap E' \neq \emptyset$, then

$$\mathrm{dist}_{\mathcal{H}}(E, E') = d - |E \cap E'|.$$

Otherwise, we say $\mathcal{H}$ is **improperly-connected**.

Remark 4.2. Our definition of properly-connected is similar to (but not equivalent to) what Zheng [32, Definition 3.14] called the **intersection property** for a simplicial complex. If Δ is a pure simplicial forest, then Δ has the intersection property if for any two facets $F, F' \in \Delta$ the distance between F and F' (defined in terms of the length of chain of between the two facets) is determined by $|F \cap F'|$.

Example 4.3. Consider the 4-uniform hypergraph $\mathcal{H}$ with edge set

$$\mathcal{E} = \{x_1x_2x_3x_4, x_1x_2x_3x_7, x_1x_2x_6x_7, x_1x_5x_6x_7, x_1x_5x_6x_8\}.$$

There is a proper irredundant chain of length 4 from the edge $E = x_1x_2x_3x_4$ to $E' = x_1x_5x_6x_8$ (to form the chain, just take the edges as listed in $\mathcal{E}$). Furthermore, there is no shorter such chain. But E and E' have a nonempty intersection. So $\mathcal{H}$ is not properly-connected since $4 = \mathrm{dist}_{\mathcal{H}}(E, E') \neq 4 - |E \cap E'| = 3$. This hypergraph is improperly-connected.

Example 4.4. Every finite simple graph G is properly-connected. To see this, note that a graph is clearly a 2-uniform hypergraph. If E, E' are two edges of G such that $E \cap E' \neq \emptyset$, then either E and E' are the same edge, or E and E' share exactly one vertex. In the first case, $\mathrm{dist}_G(E, E') = 2 - |E \cap E'| = 2 - 2 = 0$, while in the second case $\mathrm{dist}_G(E, E') = 2 - |E \cap E'| = 1$. So, properly-connected hypergraphs is a generalization of simple graphs from the point of view that this class also has the property that intersecting edges force the edges to be a specific distance apart.

Properly-connected hypergraphs are appealing combinatorial objects to study because within this family, the notions of v-leaf and f-leaf become equivalent. As well, splitting edges of properly-connected hypergraphs can be described combinatorially. We prove both of these assertions.

Theorem 4.5. *Suppose $\mathcal{H}$ is properly-connected, and E is an edge of $\mathcal{H}$. Then E is a v-leaf if and only if E is a f-leaf.*

Proof. Suppose that the common cardinality of the edges in $\mathcal{H}$ is d. Since we know an f-leaf is a v-leaf, it suffices to prove the converse. If $\mathcal{H}$ has only one edge then we are done. So, suppose that $\mathcal{H}$ has at least two edges. Let E be a v-leaf with free vertex v. Let H be any edge of $\mathcal{H}\backslash E$ that is in the same connected component as E. (If there is no such H, then E is automatically a f-leaf.) Since $\mathcal{H}$ is properly-connected, there is a proper chain $E_0 = E, E_1, \ldots, E_k = H$ from E to H. Because $|E| = |E_1| = d$ and $|E \cap E_1| = d-1$, $E \cap E_1 = E\backslash\{v\}$. To see that E is a f-leaf, let G be any other edge of $\mathcal{H}$. Then $E \cap G \subset E\backslash\{v\} = E \cap E_1$. □

Let E be an edge of a properly-connected hypergraph $\mathcal{H}$. If H is any edge of $\mathcal{H}$ with $\operatorname{dist}_{\mathcal{H}}(E, H) = 1$, then $|H\backslash E| = 1$, or in other words, $H\backslash E = \{z\}$ for some vertex z. Before classifying splitting edges, we introduce the following definition.

Definition 4.6. If E is an edge of a properly-connected hypergraph $\mathcal{H}$, then the **vertex neighbor set of** E is the following subset of $\mathcal{X}$:

$$N(E) = \bigcup_{\{H \in \mathcal{H} \ | \ \operatorname{dist}_{\mathcal{H}}(E,H)=1\}} H\backslash E.$$

Example 4.7. When G is a finite simple graph, and x is a vertex, then $N(x)$ denotes all the neighbors of x. If $E = \{u, v\}$ is any edge of G, then $N(E) = (N(u) \cup N(v))\backslash\{u, v\}$.

Theorem 4.8. *Let E be an edge of a properly-connected hypergraph $\mathcal{H}$, and suppose $N(E) = \{z_1, \ldots, z_t\}$. Then E is a splitting edge if and only if there exists a vertex $z \in E$ such that $(E\backslash\{z\}) \cup \{z_i\} \in \mathcal{H}$ for each $z_i \in N(E)$.*

The proof of this theorem depends upon the following two lemmas.

Lemma 4.9. *Let $\mathcal{H}$ be a properly-connected hypergraph. Suppose that $E = E_0 = \{x_1, \ldots, x_d\}$ and E' are edges in $\mathcal{H}$ with $\operatorname{dist}_{\mathcal{H}}(E, E') = t \leq d$. Then, after relabelling, there exist edges $E_1, \ldots, E_t$ such that $E_i = \{y_1, \ldots, y_i, x_{i+1}, \ldots, x_d\}$, $E_t = E'$, and $y_i \notin E_j$ for all $j < i$.*

Proof. Since $\operatorname{dist}_{\mathcal{H}}(E, E') = t$, there must be a proper irredundant chain of edges $E_0 = E, \ldots, E_t = E'$. Since E_i differs from E_{i+1} by exactly one vertex, for each i, $|E \cap E_i| \geq d - i$ because at most one vertex changes at each stage. Since $(E_0, \ldots, E_t)$ is an irredundant chain and $\mathcal{H}$ is properly-connected, for $i < d$, we must have

$$i = \operatorname{dist}_{\mathcal{H}}(E_0, E_i) = d - |E_0 \cap E_i|.$$

Hence, $|E_0 \cap E_i| = d - i$ for any i less than d for which the expression makes sense. Moreover, if $i = t = d$, then $\operatorname{dist}_{\mathcal{H}}(E_0, E_i) = d$, and we have $E_0 \cap E_i = \emptyset$. That is, $|E_0 \cap E_i| = 0 = d - i$.

We will prove the result using induction on i. Let $E = E_0 = \{x_1, \ldots, x_d\}$, and assume the vertices are labeled so that $x_1 \notin E_1$. We know that $|E_0 \cap E_1| = d-1$ which implies that $E_1 = \{y_1, x_2, \ldots, x_d\}$ where $y_1 \notin E_0$, thus proving the base case.

Now assume that $E_0, \ldots, E_i$ satisfy the claim, i.e., that $E_i = \{y_1, \ldots, y_i, x_{i+1}, \ldots, x_d\}$ with $y_i \notin E_j$ for all $j < i$. We know that $|E_i \cap E_{i+1}| = d - 1$, so that E_{i+1} is constructed from E_i by removing some vertex and adding a vertex that we will call y_{i+1} which is not in E_i. First, we claim that the vertex that we remove from E_i cannot be a y_j. If we were to replace some y_j with a vertex y_{i+1}, then $|E_0 \cap E_i| = d - i \leq |E_0 \cap E_{i+1}|$ which contradicts our earlier assumption that

$|E_0 \cap E_{i+1}| = d-i-1$. So, we may assume that y_{i+1} replaces x_{i+1}. If, $y_{i+1} = x_j$ for some $j \leq i$ then $|E_0 \cap E_{i+1}| = |E_0 \cap E_i|$ which is a contradiction as before. Therefore, $y_{i+1} \notin E_j$ for any $j \leq i$. □

Lemma 4.10. *Let E be any edge of a properly-connected hypergraph $\mathcal{H}$, and suppose $|E| = d$. Then*

$$\begin{aligned}(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E) &= (\{\mathrm{lcm}(x^E, x^H) \mid H \in \mathcal{H} \text{ and } \mathrm{dist}_{\mathcal{H}}(E,H) = 1\}) + \\ &\quad (\{\mathrm{lcm}(x^E, x^H) \mid H \in \mathcal{H} \text{ and } \mathrm{dist}_{\mathcal{H}}(E,H) \geq d+1\}).\end{aligned}$$

Proof. Set

$$\begin{aligned}A &= (\{\mathrm{lcm}(x^E, x^H) \mid H \in \mathcal{H}\backslash E \text{ and } \mathrm{dist}_{\mathcal{H}}(E,H) \leq d\}) \text{ and} \\ B &= (\{\mathrm{lcm}(x^E, x^H) \mid H \in \mathcal{H} \text{ and } \mathrm{dist}_{\mathcal{H}}(E,H) \geq d+1\}).\end{aligned}$$

By definition $(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E) = A + B$. Thus, if we set

$$C = (\{\mathrm{lcm}(x^E, x^H) \mid H \in \mathcal{H} \text{ and } \mathrm{dist}_{\mathcal{H}}(E,H) = 1\}),$$

then it suffices to show that $A = C$. Since $C \subseteq A$ is clear, we now show the reverse containment.

Let $x^{E\cup H} = \mathrm{lcm}(x^E, x^H)$ be a generator of A, i.e., suppose $H \in \mathcal{H}\backslash E$ and $t = \mathrm{dist}_{\mathcal{H}}(E,H) \leq d$. Note that we can assume that $2 \leq t \leq d$ because if $t = \mathrm{dist}_{\mathcal{H}}(E,H) = 1$, then $x^{E\cup H} \in C$. So there exists a proper irredundant chain $E = H_0, H_1, H_2, \ldots, H_t = H$ whose length is minimal among all proper irredundant chains from E to H.

Since the chain $(H_0, \ldots, H_t)$ is proper, $|H_i \cap H_{i+1}| = d-1$ for $i = 0, \ldots, t-1$. So H_i differs from H_{i+1} by exactly one vertex. In other words, H_{i+1} can be constructed by removing exactly one vertex of H_i and replacing it with a vertex not in H_i. This implies that $|E \cap H_i| \geq d-i$ since we remove at most i vertices of E to form H_i.

Now if $E = \{x_1, \ldots, x_d\}$, then $H_1 = \{x_1, \ldots, \hat{x}_i, \ldots, x_d, z\}$ where by $\hat{x}_i$ we mean the vertex x_i is removed, and z is not one of $x_1, \ldots, x_d$. From this observation, we have

$$\mathrm{lcm}(x^E, x^{H_1}) = x^{E\cup\{z\}} = x^E z.$$

Now $x^E z$ is a generator of C. To finish the proof, Lemma 4.9 implies that $z \in H_i$ for $i = 2, \ldots, t$. Therefore, $\mathrm{lcm}(x^E, x^{H_i}) = x^{E\cup H_i}$ is divisible by $x^E z$, and thus is in C. In particular $x^{E\cup H} \in C$. □

Proof of Theorem 4.8. Suppose that E is a splitting edge. By Theorem 3.2 there is a vertex $z \in E$ such that $(x^E) \cap \mathcal{I}(H\backslash E) \subseteq (x^E) \cap \mathcal{I}(H\backslash\{z\})$. Let $z_i \in N(E)$. We will show that $(E\backslash\{z\}) \cup \{z_i\}$ is an edge of $\mathcal{H}\backslash\{z\} \subseteq \mathcal{H}$. Since $z_i \in N(E)$, there exists an edge H with $\mathrm{dist}_{\mathcal{H}}(E,H) = 1$ such that $H\backslash E = \{z_i\}$. Thus, $x^{E\cup H}$ is a generator of $(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)$. We thus must have $x^{E\cup H} \in (x^E) \cap \mathcal{I}(\mathcal{H}\backslash\{z\})$. Hence there is an edge $H' \in \mathcal{H}\backslash\{z\}$ such that $E \cup H = E \cup H'$. Because $|E \cap H| = d-1$, we must have that $|E \cap H'| = d-1$. Since $z \notin H'$ and $z_i \notin E$, we must have $H' = (E\backslash\{z\}) \cup \{z_i\}$. So, $(E\backslash\{z\}) \cup \{z_i\} \in \mathcal{H}\backslash\{z\}$ as desired.

Conversely, suppose there exists a vertex $z \in E$ such that $(E\backslash\{z\}) \cup \{z_i\} \in \mathcal{H}$ for each $z_i \in N(E)$. Let x^L be any minimal generator of $(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)$. By Lemma 4.10, we have $L = E \cup H$ with $\mathrm{dist}_{\mathcal{H}}(E,H) = 1$ or $L = E \cup H$ with $\mathrm{dist}_{\mathcal{H}}(E,H) \geq d+1$. If $\mathrm{dist}_{\mathcal{H}}(E,H) \geq d+1$, then $z \notin H$ since $E \cap H = \emptyset$. So $H \in \mathcal{H}\backslash\{z\}$, and hence $x^L \in (x^E) \cap \mathcal{I}(\mathcal{H}\backslash\{z\})$. So, suppose $L = E \cup H$ with $\mathrm{dist}_{\mathcal{H}}(E,H) = 1$. Then, the exists $z_i \in N(E)$ such that $E \cup H = E \cup \{z_i\}$. By hypothesis, the edge $E' = (E\backslash\{z\}) \cup \{z_i\} \in \mathcal{H}$. But then $E' \in \mathcal{H}\backslash\{z\}$. Furthermore, $L = E \cup H = E \cup E'$. So $x^L \in (x^E) \cap \mathcal{I}(\mathcal{H}\backslash\{z\})$. We have now shown that $(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E) \subseteq (x^E) \cap \mathcal{I}(\mathcal{H}\backslash\{z\})$, so by Theorem 3.2 the edge E must be a splitting edge. □

Notation 4.11. Suppose E is an edge of a properly-connected hypergraph $\mathcal{H}$ with $|E| = d$, the common cardinality of edges in $\mathcal{H}$. For simplicity of notation, throughout the rest of the paper, when not specified, $\mathcal{H}'$ refers to the sub-hypergraph

$$\mathcal{H}' = \{H \in \mathcal{H} \mid \operatorname{dist}_{\mathcal{H}}(E,H) \geq d+1\}.$$

As a byproduct of Lemma 4.10, we can rewrite $(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)$ in terms of the edge ideal of $\mathcal{H}'$.

Corollary 4.12. *Let E be any edge of a properly-connected hypergraph $\mathcal{H}$, and suppose $N(E) = \{z_1,\ldots,z_t\}$. Then*

$$(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E) = x^E((z_1,\ldots,z_t) + \mathcal{I}(\mathcal{H}')).$$

Proof. It is straight forward to verify that

$$x^E(z_1,\ldots,z_t) = (\{\operatorname{lcm}(x^E,x^H) \mid H \in \mathcal{H} \text{ and } \operatorname{dist}_{\mathcal{H}}(E,H) = 1\}).$$

If $H \in \mathcal{H}\backslash E$ with $\operatorname{dist}_{\mathcal{H}}(E,H) \geq d+1$, then because $\mathcal{H}$ is properly-connected, $|E \cap H| = \emptyset$. So

$$x^E\mathcal{I}(\mathcal{H}') = (\{\operatorname{lcm}(x^E,x^H) \mid H \in \mathcal{H} \text{ and } \operatorname{dist}_{\mathcal{H}}(E,H) \geq d+1\}).$$

The result now follows from Lemma 4.10. □

When E is an edge of a properly-connected hypergraph, we can also describe the graded Betti numbers of $(x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)$ in terms of those of $\mathcal{I}(\mathcal{H}')$.

Lemma 4.13. *Let E be any edge of a properly-connected hypergraph $\mathcal{H}$. Let $d = |E|$ and $t = |N(E)|$. Then*

$$\beta_{i-1,j}((x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)) = \sum_{l=0}^{i} \binom{t}{l} \beta_{i-1-l,j-d-l}(\mathcal{I}(\mathcal{H}'))$$

where $\beta_{-1,j}(\mathcal{I}(\mathcal{H}')) = 1$ if $j = 0$ and 0 otherwise.

Proof. If $N(E) = \{z_1,\ldots,z_t\}$, then by the previous corollary,

$$\begin{aligned}\beta_{i-1,j}((x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)) &= \beta_{i-1,j}(x^E((z_1,\ldots,z_t) + \mathcal{I}(\mathcal{H}'))) \\ &= \beta_{i-1,j-d}((z_1,\ldots,z_t) + \mathcal{I}(\mathcal{H}')) \\ &= \beta_{i,j-d}(R/((z_1,\ldots,z_t) + \mathcal{I}(\mathcal{H}'))).\end{aligned}$$

None of the generators of $\mathcal{I}(\mathcal{H}')$ are divisible by z_i for $i = 1,\ldots,t$. To see this, suppose that $x^H \in \mathcal{I}(\mathcal{H}')$ is divisible by some z_i, i.e., z_i is a vertex of the edge H. Now there is a edge H_i with $z_i \in H_i$ and $\operatorname{dist}_{\mathcal{H}}(E,H_i) = 1$. Since $H \cap H_i \neq \emptyset$ and because $\mathcal{H}$ is properly-connected, $p = \operatorname{dist}_{\mathcal{H}}(H,H_i) = d - |H \cap H_i| < d$. So there is a proper irredundant chain $H_i = H'_0,\ldots,H'_p = H$. But then $E, H_i = H'_0,\ldots,H'_p = H$ forms a proper irredundant chain of length $p+1 \leq d$, and thus $\operatorname{dist}_{\mathcal{H}}(E,H) \leq d$, contradicting the fact that $\operatorname{dist}_{\mathcal{H}}(E,H) \geq d+1$.

By abusing notation, we write $R = k[z_1,\ldots,z_t,x_1,\ldots,x_s]$ where $\{x_1,\ldots,x_s\} = \mathcal{X}\backslash N(E)$. Then

$$R/((z_1,\ldots,z_t) + \mathcal{I}(\mathcal{H}')) \cong R_1/(z_1,\ldots,z_t) \otimes R_2/\mathcal{I}(\mathcal{H}')$$

where $R_1 = k[z_1,\ldots,z_t]$ and $R_2 = k[x_1,\ldots,x_s]$, and where we view $\mathcal{I}(\mathcal{H}')$ as an ideal of R and as the ideal of R_2 generated by the same elements. By tensoring the resolutions of $R_1/(z_1,\ldots,z_t)$ and $R_2/\mathcal{I}(\mathcal{H}')$ together we get (see, for example, Lemma 2.1 and Corollary 2.2 of [19])

$$\beta_{i,j-d}(R/L) = \sum_{l_1=0}^{i}\sum_{l_2=0}^{j-d} \beta_{l_1,l_2}(R_1/(z_1,\ldots,z_t))\beta_{i-l_1,j-d-l_2}(R_2/(\mathcal{I}(\mathcal{H}'))$$

where $L = (z_1,\ldots,z_t) + \mathcal{I}(\mathcal{H}')$. Since $z_1,\ldots,z_t$ is a regular sequence on R_1

$$\beta_{l_1,l_2}(R_1/(z_1,\ldots,z_t)) = \begin{cases} 0 & \text{if } l_2 \neq l_1 \\ \binom{t}{l} & \text{if } l = l_2 = l_1. \end{cases}$$

As a consequence, the previous expression reduces to

$$\beta_{i,j-d}(R/L) = \sum_{l=0}^{i} \binom{t}{l} \beta_{i-l,j-d-l}(R_2/\mathcal{I}(\mathcal{H}')).$$

We are now done since

$$\beta_{i-l,j-d-l}(R_2/\mathcal{I}(\mathcal{H}')) = \beta_{i-l,j-d-l}(R/\mathcal{I}(\mathcal{H}')) = \beta_{i-l-1,j-d-l}(\mathcal{I}(\mathcal{H}'))$$

for all l (where we adopt the convention that $\beta_{-1,j}(\mathcal{I}(\mathcal{H}')) = 1$ if $j = 0$ and 0 if $j \neq 0$). □

When $\mathcal{H}$ is a properly-connected hypergraph, we obtain the following recursive like formula for $\beta_{i,j}(\mathcal{I}(\mathcal{H}))$. This result generalizes a similar result for simple graphs found in [21].

Theorem 4.14. *Let $\mathcal{H}$ be a properly-connected hypergraph and let E be a splitting edge of $\mathcal{H}$. Suppose $d = |E|$, $\mathcal{H}' = \{H \in \mathcal{H} \mid \operatorname{dist}_{\mathcal{H}}(E, H) \geq d+1\}$, and $t = |N(E)|$. Then for all $i \geq 1$*

$$\beta_{i,j}(\mathcal{I}(\mathcal{H})) = \beta_{i,j}(\mathcal{I}(\mathcal{H}\backslash E)) + \sum_{l=0}^{i} \binom{t}{l} \beta_{i-1-l,j-d-l}(\mathcal{I}(\mathcal{H}')).$$

Here, $\beta_{-1,j}(\mathcal{I}(\mathcal{H}')) = 1$ if $j = 0$ and 0 if $j \neq 0$.

Proof. Since E is a splitting edge, by Theorem 2.4 we have

$$\beta_{i,j}(\mathcal{I}(\mathcal{H})) = \beta_{i,j}((x^E)) + \beta_{i,j}(\mathcal{I}(\mathcal{H}\backslash E)) + \beta_{i-1,j}((x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)).$$

When $i \geq 1$, $\beta_{i,j}((x^E)) = 0$. Now substitute the formula of Lemma 4.13 into the last expression. □

5. Triangulated properly-connected hypergraphs

If $\mathcal{H}$ is a properly-connected hypergraph with splitting edge E, the sub-hypergraphs $\mathcal{H}\backslash E$ and $\mathcal{H}'$ in Theorem 4.14 may or may not have a splitting edge. This fact prevents us from using Theorem 4.14 to recursively compute $\beta_{i,j}(\mathcal{I}(\mathcal{H}))$ for any hypergraph. One is lead to ask if there is any subfamily of properly-connected hypergraphs for which the formula is recursive. In this section, we introduce one such family which generalizes the notion of a chordal graph. In [21] it was shown that hyperforests (i.e., a simplicial forest in the sense of [10]) is a family of hypergraphs for which the graded Betti numbers can be computed recursively. Since a hyperforest need not be properly-connected, the results of this section give a partial generalization of [21].

We begin by recalling the definition of a chordal graph.

Definition 5.1. A graph G is called **chordal** if every cycle of length 4 or larger has a chord, that is, an edge joining two nonadjacent vertices in the the cycle.

An alternative characterization for chordal graphs can be found in [24] (due to Dirac [5]). This characterization will prove more suitable when generalizing to properly-connected hypergraphs.

Theorem 5.2. *A graph G is chordal if and only if every induced subgraph of G contains a vertex v whose neighborhood $N(v)$ is a complete graph.*

To extend this definition, we first introduce an analog of complete graphs.

Definition 5.3. The **d-complete hypergraph of order** n, denoted by $\mathcal{K}_n^d$, is the hypergraph consisting of all the d-subsets of the vertex set $\mathcal{X}$, where $|\mathcal{X}| = n$. When $d = 2$, then $\mathcal{K}_n^2$ is the usual complete graph $\mathcal{K}_n$. When $n < d$, we consider $\mathcal{K}_n^d$ as the hypergraph with n isolated vertices.

Definition 5.4. Two vertices $x, y \in \mathcal{X}$ are **neighbors** if there is an edge $E \in \mathcal{H}$ such that $x, y \in E$. For any vertex $x \in \mathcal{X}$, the **neighborhood of** x, denoted $N(x)$, is the set

$$N(x) = \{y \in \mathcal{X} \mid y \text{ is a neighbor of } x\}.$$

Observe that if E is any edge of $\mathcal{H}$ and $x \in E$, then $E \subseteq N(x) \cup \{x\}$.

Definition 5.5. A d-uniform properly-connected hypergraph $\mathcal{H}$ is said to be **triangulated** if for every nonempty subset $\mathcal{Y} \subseteq \mathcal{X}$, the induced subgraph $\mathcal{H}_{\mathcal{Y}}$ contains a vertex $x \in \mathcal{Y} \subseteq \mathcal{X}$ such that the induced hypergraph of $\mathcal{H}_{\mathcal{Y}}$ on $N(x)$ is a d-complete hypergraph of order $|N(x)|$.

By virtue of Theorem 5.2, the simple graphs that are triangulated are precisely the chordal graphs. We shall show that properly-connected hyperforests are triangulated hypergraphs.

Theorem 5.6. *Suppose that $\mathcal{H}$ is a properly-connected hypergraph that is a v-forest (or equivalently, f-forest). Then $\mathcal{H}$ is a triangulated hypergraph.*

Proof. For any $\mathcal{Y} \subseteq \mathcal{X}$, the induced subgraph $\mathcal{H}_{\mathcal{Y}}$ must contain a v-leaf, say E. Since E is a v-leaf, E contains a free vertex, say x. Suppose $d = |E|$ and $E = \{x, x_2, \ldots, x_d\}$. Then $N(x) = \{x_2, \ldots, x_d\}$. But the induced graph on $N(x)$ is the set of $d-1$ isolated vertices of $N(x)$, which is the d-uniform complete hypergraph $\mathcal{K}^d_{d-1}$. So $\mathcal{H}$ is a triangulated hypergraph. □

The following lemma is the key result needed to prove that Theorem 4.14 is recursive for triangulated hypergraphs.

Lemma 5.7. *Let $\mathcal{H}$ be a triangulated hypergraph. Then there exists an edge $E \in \mathcal{H}$ such that*

(a) *E is a splitting edge, and*
(b) *the subgraphs $\mathcal{H}\backslash E$ and $\mathcal{H}'$ are triangulated hypergraphs.*

Proof. Since $\mathcal{H}$ is a triangulated hypergraph, there exists a vertex $x \in \mathcal{X}$ such that the induced hypergraph on $N(x)$ is a d-complete hypergraph. Let E be any edge of $\mathcal{H}$ that contains x. We will show that E is an edge that satisfies (a) and (b).

(a) Suppose that $N(E) = \{z_1, \ldots, z_t\}$. For each $z_i \in N(E)$, there must be an edge $E_i \in \mathcal{H}$ such that $\text{dist}_{\mathcal{H}}(E, E_i) = 1$ and $E \cup E_i = E \cup \{z_i\}$. For each i, either $x \in E_i$ or $x \notin E_i$. If $x \notin E_i$, then $(E\backslash\{x\}) \cup \{z_i\} = E_i \in \mathcal{H}$. Now, suppose $x \in E_i$. Since $z_i \in E_i$, we have $z_i \in N(x)$. If $E = \{x, x_2, \ldots, x_d\}$, then $\{x_2, \ldots, x_d, z_i\} \subseteq N(x)$ is a subset of size d in $N(x)$. But since the induced hypergraph on $N(x)$ is a d-complete hypergraph, that means that $\{x_2, \ldots, x_d, z_i\}$ is an edge of $\mathcal{H}$. This edge is simply $(E\backslash\{x\}) \cup \{z_i\}$. So, E is a splitting edge by Theorem 4.8.

(b) We first show that $\mathcal{H}\backslash E$ is a triangulated hypergraph. If the vertex $x \in E$ only appears in E, then E is a v-leaf. Then $\mathcal{H}\backslash E = \mathcal{H}\backslash\{x\} = \mathcal{H}_{\mathcal{X}\backslash\{x\}}$, and it is clear that $\mathcal{H}_{\mathcal{X}\backslash\{x\}}$ is a triangulated hypergraph. So, suppose that there are two or more edges that contain x. If $\mathcal{Y} \subseteq \mathcal{X}$ with $x \notin \mathcal{Y}$, then the induced hypergraph of $\mathcal{H}\backslash E$ on $\mathcal{Y}$ is the same as the induced hypergraph of $\mathcal{H}$ on $\mathcal{Y}$, so there exists a vertex $z \in \mathcal{Y}$ such that the induced hypergraph on $N(z)$ is a d-complete hypergraph. It remains to consider the case when $x \in \mathcal{Y}$. Let $N_{\mathcal{Y}}(x)$ denote the neighbors of x in $(\mathcal{H}\backslash E)_{\mathcal{Y}}$. Note that $N_{\mathcal{Y}}(x) \subseteq N(x)$. Since the induced hypergraph on $N(x)$ is a d-complete hypergraph, any induced subgraph on a subset of $N(x)$ is also a d-complete hypergraph. So the induced hypergraph $(\mathcal{H}\backslash E)_{N_{\mathcal{Y}}(x)}$ is a d-complete hypergraph. Thus $\mathcal{H}\backslash E$ is triangulated.

Finally, the reason that $\mathcal{H}'$ is triangulated follows from the fact that

$$\mathcal{H}' = \mathcal{H}\backslash\{x, x_2, \ldots, x_d, z_1, \ldots, z_t\} = \mathcal{H}_{\mathcal{X}\backslash\{x, x_2, \ldots, x_d, z_1, \ldots, z_t\}}$$

where $E = \{x, x_2, \ldots, x_d\}$ and $N(E) = \{z_1, \ldots, z_t\}$. □

We come to the main result of this section.

Theorem 5.8. *Suppose that $\mathcal{H}$ is a triangulated hypergraph. Then the graded Betti numbers of $\mathcal{I}(\mathcal{H})$ can be computed recursively using the formula*

$$\beta_{i,j}(\mathcal{I}(\mathcal{H})) = \beta_{i,j}(\mathcal{I}(\mathcal{H}\backslash E)) + \sum_{l=0}^{i} \binom{t}{l} \beta_{i-1-l,j-d-l}(\mathcal{I}(\mathcal{H}'))$$

where E is a splitting edge, $d = |E|$, $t = |N(E)|$, and $\mathcal{H}'$ and $\mathcal{H}\backslash E$ are also triangulated hypergraphs. Here, $\beta_{-1,j}(\mathcal{I}(\mathcal{H}')) = 1$ if $j = 0$ and 0 if $j \neq 0$.

Proof. By Lemma 5.7, the triangulated hypergraph $\mathcal{H}$ has a splitting edge E. Furthermore, since both hypergraphs $\mathcal{H}\backslash E$ and $\mathcal{H}'$ are triangulated hypergraphs, they also have a splitting edges. Thus, by repeatedly using the formula of Theorem 4.14 we get the recursive formula. □

It is well known that for an arbitrary monomial ideal, its graded Betti numbers may depend upon $\mathrm{char}(k)$. However, as a consequence of the above formula we obtain the following corollary.

Corollary 5.9. *Suppose that $\mathcal{H}$ is a triangulated hypergraph. Then the graded Betti numbers of $\mathcal{I}(\mathcal{H})$ are independent of the characteristic of the ground field and can be computed recursively.*

When restricted to simple graphs, we get a particularly nice corollary.

Corollary 5.10. *Suppose that G is a chordal graph. Then the graded Betti numbers of $\mathcal{I}(G)$ are independent of* $\mathrm{char}(k)$ *and can be computed recursively.*

Jacques [18] and Jacques and Katzman [19] first proved Corollary 5.10 in the special case that G is a forest, a subclass of chordal graphs.

6. Properly-connected hypergraphs and regularity

In this section we investigate the Castelnuovo-Mumford regularity of the edge ideal $\mathcal{I}(\mathcal{H})$ associated to a properly-connected hypergraph $\mathcal{H}$. For such a hypergraph, we bound $\mathrm{reg}(\mathcal{I}(\mathcal{H}))$ above and below by combinatorial invariants of the hypergraph. In the case that $\mathcal{H}$ is also triangulated, we explicitly compute $\mathrm{reg}(\mathcal{I}(\mathcal{H}))$. Our exact formula for $\mathrm{reg}(\mathcal{I}(\mathcal{H}))$ generalizes Zheng's formula [32] for the regularity of $\mathcal{I}(\mathcal{H})$ when $\mathcal{H} = G$ is a forest.

We begin by relating the regularity of $\mathcal{I}(\mathcal{H})$ to the regularity of edge ideals associated to sub-hypergraphs of $\mathcal{H}$. We produce similar results for the projective dimension of $\mathcal{I}(\mathcal{H})$. We first make the convention that $\mathrm{reg}(0) = 1$ and $\mathrm{pdim}(0) = 0$.

Lemma 6.1. *Let E be any edge of a properly-connected hypergraph $\mathcal{H}$ such that $\mathcal{H}\backslash E$ is nonempty. Let $d = |E|$, $t = |N(E)|$, and $\mathcal{H}' = \{H \in \mathcal{H} \mid \mathrm{dist}_{\mathcal{H}}(H, E) \geq d + 1\}$. If $L = (x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)$, then*

(*a*) $\mathrm{reg}(L) = \mathrm{reg}(\mathcal{I}(\mathcal{H}')) + d$, *and*
(*b*) $\mathrm{pdim}(L) = \mathrm{pdim}(\mathcal{I}(\mathcal{H}')) + t$.

Proof. We shall prove both results using Lemma 4.13. For (a) suppose $s = \mathrm{reg}(L)$. So, there exists a such that $\beta_{a,a+s}(L) \neq 0$. By Lemma 4.13

$$\beta_{a+1-1,a+s}(L) = \sum_{l=0}^{a+1} \binom{t}{l} \beta_{a+1-1-l,a+s-d-l}(\mathcal{I}(\mathcal{H}')).$$

Since every number in the summation on the right hand side is nonnegative, there exists some l such that $\beta_{a-l,a+s-d-l}(\mathcal{I}(\mathcal{H}')) \neq 0$. Hence, $\mathrm{reg}(\mathcal{I}(\mathcal{H}')) \geq s - d$, or equivalently, $\mathrm{reg}(\mathcal{I}(\mathcal{H}')) + d \geq \mathrm{reg}(L)$.

Conversely, if $r = \operatorname{reg}(\mathcal{I}(\mathcal{H}'))$, then there exists b such that $\beta_{b,b+r}(\mathcal{I}(\mathcal{H}')) \neq 0$. But then since $b+r = (b+r+d)-d$, by Lemma 4.13

$$0 \neq \beta_{b,(b+r+d)-d}(\mathcal{I}(\mathcal{H}')) \leq \sum_{l=0}^{b+1} \binom{t}{l} \beta_{b+1-1-l,b+r+d-d-l}(\mathcal{I}(\mathcal{H}')) = \beta_{b,b+r+d}(L).$$

So $\operatorname{reg}(\mathcal{I}(\mathcal{H}') + d \geq \operatorname{reg}(L) \geq \operatorname{reg}(\mathcal{I}(\mathcal{H}')) + d$, as desired.

To prove (b), suppose $N(E) = \{z_1, \ldots, z_t\}$. In the proof of Lemma 4.13 it was shown that

$$R/L \cong R_1/(z_1, \ldots, z_t) \otimes R_2/\mathcal{I}(\mathcal{H}').$$

where $R_1 = k[z_1, \ldots, z_t]$ and $R_2 = k[x_1, \ldots, x_s]$, with $\{x_1, \ldots, x_s\} = \mathcal{X}\backslash N(E)$. By tensoring the resolutions of $R_1/(z_1, \ldots, z_t)$ and $R_2/\mathcal{I}(\mathcal{H}')$, we get

$$\begin{aligned} \operatorname{pdim}(L) + 1 = \operatorname{pdim}(R/L) &= \operatorname{pdim}(R_1/(z_1, \ldots, z_t)) + \operatorname{pdim}(R_2/\mathcal{I}(\mathcal{H}')) \\ &= t + \operatorname{pdim}(R/\mathcal{I}(\mathcal{H}')) = t + \operatorname{pdim}(\mathcal{I}(\mathcal{H}')) + 1. \end{aligned}$$

The desired identity is obtained by comparing the first and last values of the above equality. □

Theorem 6.2. *Let E be any edge of a properly-connected hypergraph $\mathcal{H}$ such that $\mathcal{H}\backslash E$ is nonempty. Let $d = |E|$ and $t = |N(E)|$. Then*

- (a) $\operatorname{reg}(\mathcal{I}(\mathcal{H})) \leq \max\{\operatorname{reg}(\mathcal{I}(\mathcal{H}\backslash E)), \operatorname{reg}(\mathcal{I}(\mathcal{H}')) + d - 1\}$.
- (b) $\operatorname{pdim}(\mathcal{I}(\mathcal{H})) \leq \max\{\operatorname{pdim}(\mathcal{I}(\mathcal{H}\backslash E)), \operatorname{pdim}(\mathcal{I}(\mathcal{H}')) + t + 1\}$.

Furthermore, if E is a splitting edge, then we have equality in both (a) and (b).

Proof. Set $L = (x^E) \cap \mathcal{I}(\mathcal{H}\backslash E)$. The two inequalities then follow by using the short exact sequence

$$0 \rightarrow L \rightarrow (x^E) \oplus \mathcal{I}(\mathcal{H}\backslash E) \rightarrow \mathcal{I}(\mathcal{H}) \rightarrow 0$$

and Lemma 6.1 to bound $\operatorname{reg}(I(\mathcal{H}))$ and $\operatorname{pdim}(\mathcal{I}(\mathcal{H}))$, noting that since $\mathcal{H}\backslash E$ is nonempty, $\operatorname{reg}(\mathcal{H}\backslash E) \geq d$. When E is a splitting edge, the equalities are a result of the formulas of Corollary 2.5. □

We now focus our attention on using combinatorial information from $\mathcal{H}$ to bound $\operatorname{reg}(\mathcal{I}(\mathcal{H}))$. More precisely, the regularity will be expressed using the following terminology.

Definition 6.3. Let $\mathcal{H}$ be a properly-connected hypergraph. Two edges E, H of $\mathcal{H}$ are **t-disjoint** if $\operatorname{dist}_{\mathcal{H}}(E, H) \geq t$. A set of edges $\mathcal{E}' \subseteq \mathcal{E}$ is **pairwise t-disjoint** if every pair of edges of $\mathcal{E}'$ is t-disjoint. (We thank Jeremy Martin for suggesting this name.)

Remark 6.4. When $\mathcal{H}$ is a d-uniform hypergraph, then two edges E and H are d-disjoint if and only if $E \cap H = \emptyset$; that is, E and H are disjoint in the usual sense. When $\mathcal{H} = G$ is a simple graph, Zheng's definition [32, Definition 2.15] for two edges to be **disconnected** is equivalent to our definition that the two edges be 3-disjoint in G.

We shall require one more lemma.

Lemma 6.5. *Let $\mathcal{H}$ be a d-uniform properly-connected hypergraph. Then $\beta_{i-1,di}(\mathcal{I}(\mathcal{H}))$ equals the number of sets of i pairwise $(d+1)$-disjoint edges of $\mathcal{H}$.*

Proof. A set of i edges is pairwise $(d+1)$-disjoint if and only if the induced graph on the vertices of the i edges consists of the i disconnected edges. Katzman [20, Lemma 2.2] proved this lemma in the special case that $d = 2$ by making use of the Taylor resolution. His proof generalizes naturally to the case that $d > 2$; the details are left to the reader. □

We come to one of the main results of this section:

Theorem 6.6. *Let $\mathcal{H}$ be a properly-connected hypergraph. Let n_1 (respectively, n_2) be the maximal number of pairwise $(d+1)$-disjoint edges, (respectively, d-disjoint edges) of $\mathcal{H}$. Then*

$$(d-1)n_1 + 1 \leq \operatorname{reg}(\mathcal{I}(\mathcal{H})) \leq (d-1)n_2 + 1$$

where d is the common cardinality of edges in $\mathcal{H}$.

Proof. We first prove the lower bound. If $\mathcal{H}$ has n_1 pairwise $(d+1)$-disjoint edges, then by Lemma 6.5 we must have $\beta_{n_1-1,dn_1}(\mathcal{I}(\mathcal{H})) \neq 0$. So $dn_1 - n_1 + 1 = (d-1)n_1 + 1 \leq \operatorname{reg}(\mathcal{I}(\mathcal{H}))$.

To prove the upper bound we proceed by induction on the number of edges of $\mathcal{H}$. If $\mathcal{H}$ only has one edge, say E, then $\mathcal{I}(\mathcal{H}) = (x^E)$. It is immediate that $\mathcal{H}$ has one pairwise d-disjoint edge, and $\operatorname{reg}(\mathcal{I}(\mathcal{H})) \leq (d-1)\cdot 1 + 1$ (and in fact, we have equality in this case).

So, suppose $\mathcal{H}$ has two or more edges, and suppose n_2 is the maximal number of pairwise d-disjoint edges of $\mathcal{H}$. Let $E \in \mathcal{H}$ be any edge (then $\mathcal{H}\backslash E$ is nonempty). By Corollary 6.2

$$\operatorname{reg}(\mathcal{I}(\mathcal{H})) \leq \max\{\operatorname{reg}(\mathcal{I}(\mathcal{H}\backslash E)), \operatorname{reg}(\mathcal{I}(\mathcal{H}')) + d - 1\}.$$

Let j_1 be the maximal number of pairwise d-disjoint edges of $\mathcal{H}\backslash E$, and j_2 the maximal number of pairwise d-disjoint edges of $\mathcal{H}'$. By induction, $\operatorname{reg}(\mathcal{I}(\mathcal{H}\backslash E)) \leq (d-1)j_1+1$ and $\operatorname{reg}(\mathcal{I}(\mathcal{H}'))+d-1 \leq (d-1)j_2 + 1 + d - 1 = (d-1)(j_2+1)+1$. It therefore suffices to show that $n_2 \geq \max\{j_1, j_2+1\}$.

Note that H and H' are two d-disjoint edges if and only if $H \cap H' = \emptyset$. If $j_1 > n_2$, then since $\mathcal{H}\backslash E$ contains a subset of j_1 pairwise d-disjoint edges, $\mathcal{H}$ will also contain this subset, contradicting the maximality of n_2. So, $j_1 \leq n_2$. Similarly, if $j_2 + 1 > n_2$, then $j_2 \geq n_2$. Thus, $\mathcal{H}'$ contains a subset S of n_2 pairwise d-disjoint edges. But for every edge $H \in S$ in this subset, we must have $H \cap E = \emptyset$ (since E is disjoint from every edge in $\mathcal{H}'$). Therefore, $S \cup \{E\}$ is a set of n_2+1 pairwise d-disjoint edges in $\mathcal{H}$, again contradicting our definition of n_2. This completes the proof. □

Example 6.7. It is possible for the number of d-disjoint edges of a hypergraph $\mathcal{H}$ to be arbitrarily larger than the number of $(d+1)$-disjoint edges of $\mathcal{H}$. For example, let $\mathcal{K}_{a,b}$ denote the complete bipartite graph of order a, b with $a \leq b$. Then clearly $\mathcal{K}_{a,b}$ has a pairwise 2-disjoint edges. However, $\mathcal{K}_{a,b}$ has only one 3-disjoint edge. Indeed, if $E = a_1b_1, E' = a_2b_2$ are any two edges of $\mathcal{K}_{a,b}$, then either $E \cap E' \neq \emptyset$, (and thus not 3-disjoint) or $E \cap E' = \emptyset$, in which case, there exists an edge $E'' = a_1b_2$, thus showing that $\operatorname{dist}_{\mathcal{K}_{a,b}}(E, E') = 2$.

Definition 6.8. Let $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ be a hypergraph. A **matching** of $\mathcal{H}$ is defined to be a subset $\mathcal{E}' \subseteq \mathcal{E}$ consisting of pairwise disjoint edges. The **matching number** of $\mathcal{H}$, denoted $\alpha'(\mathcal{H})$, is the largest size of a maximal matching in $\mathcal{H}$.

Any set of pairwise disjoint edges in a graph forms a matching. Theorem 6.6 has an especially appealing consequence when $\mathcal{H} = G$ is a simple graph.

Corollary 6.9. *Let G be a finite simple graph. Then*

$$\operatorname{reg}(R/\mathcal{I}(G)) \leq \alpha'(G)$$

where $\alpha'(G)$ is the matching number of G.

Remark 6.10. Corollary 6.9 can also be proved directly via Taylor's resolution (the proof does not extend to the hypergraph case though). It can be seen from the Taylor resolution that

$$\operatorname{reg}(\mathcal{I}(G)) \leq \max\{\deg \operatorname{lcm}(x^{E_1}, \ldots, x^{E_i}) - i \mid \{E_1, \ldots, E_i\} \subseteq \mathcal{E}\} + 1.$$

Since any edge of G has 2 vertices, it can be seen that $i + 1 \leq \deg \operatorname{lcm}(x^{E_1}, \ldots, x^{E_i}) \leq 2i$. Let $\deg \operatorname{lcm}(x^{E_1}, \ldots, x^{E_i}) = i + k$ for some $1 \leq k \leq i$. It suffices to show that we can always find a matching of size k among $\{E_1, \ldots, E_i\}$. To this end, we shall use induction on $i + k$.

If $i + k = 2$, i.e., $i = k = 1$, then the statement is clear. Suppose now that $i + k > 2$. If $k = 1$ or $k = i$ then the statement is also clear. Assume that $1 < k < i$. If E_i is disjoint from E_j for all

$j < i$, then $\deg \operatorname{lcm}(x^{E_1}, \ldots, x^{E_{i-1}}) = i + k - 2 = (i-1) + (k-1)$. By induction, there exists a matching $S \subset \{E_1, \ldots, E_{i-1}\}$ of size $(k-1)$. It is easy to see that $S \cup \{E_i\}$ is now a matching of size k. It remains to consider the case that at least a vertex of E_i is also a vertex of E_j for some $j < i$. In this case, we have $\deg \operatorname{lcm}(x^{E_1}, \ldots, x^{E_{i-1}}) \geq i + k - 1 = (i-1) + k$. By induction, there is a matching $S \subset \{E_1, \ldots, E_{i-1}\}$ of size k, and the statement is proved.

Corollary 6.9 seems to give an interesting bound for the regularity of edge ideals with a simple proof which may have been overlooked.

When $\mathcal{H}$ is a triangulated hypergraph, the lower bound of Theorem 6.6 turns out to be the exact formula for the regularity of $\mathcal{I}(\mathcal{H})$.

Theorem 6.11. *Suppose that $\mathcal{H}$ is a properly-connected triangulated hypergraph, and let d be the common cardinality of edges in $\mathcal{H}$. If j is the maximum number of pairwise $(d+1)$-disjoint edges of $\mathcal{H}$, then*

$$\operatorname{reg}(\mathcal{I}(\mathcal{H})) = (d-1)j + 1.$$

Proof. The proof is similar to the one given by [21] in the case for forests. We proceed by doing induction on the number of edges of $\mathcal{H}$. If $\mathcal{H}$ only has one edge E, then $\mathcal{I}(\mathcal{H}) = (x^E)$. Because $\mathcal{I}(\mathcal{H})$ is principal, it is clear that $\operatorname{reg}(\mathcal{I}(\mathcal{H})) = d$. But then it is clear that the formula holds since 1 is the maximal number of pairwise $(d+1)$-disjoint edges.

So, suppose $\mathcal{H}$ has at least two edges. Since $\mathcal{H}$ is triangulated, by Lemma 5.7 there is a splitting edge $E \in \mathcal{H}$ ($\mathcal{H} \backslash E$ is nonempty in this case). Since E is a splitting edge, by Corollary 6.2 we have

$$\operatorname{reg}(\mathcal{I}(\mathcal{H})) = \max\{\operatorname{reg}(\mathcal{I}(\mathcal{H} \backslash E)), \operatorname{reg}(\mathcal{I}(\mathcal{H}')) + d - 1\}.$$

By induction $\operatorname{reg}(\mathcal{I}(\mathcal{H} \backslash E)) = (d-1)j_1 + 1$ where j_1 is the maximal number of pairwise $(d+1)$-disjoint edges of $\mathcal{H} \backslash E$, and $\operatorname{reg}(\mathcal{I}(\mathcal{H}')) = (d-1)j_2 + 1$ where j_2 is the maximal number of pairwise $(d+1)$-disjoint edges of $\mathcal{H}'$. Since $\mathcal{H} \backslash E$ has at least one edge, $(d-1)j_1 + 1 \geq d$. So

$$\operatorname{reg}(\mathcal{I}(\mathcal{H})) = \max\{(d-1)j_1 + 1, (d-1)j_2 + d\}.$$

If we let j denote the maximal number of pairwise $(d+1)$-disjoint edges of $\mathcal{H}$, then since $(d-1)j_2 + d = (d-1)(j_2+1) + 1$ to complete the proof it suffices for us to show that $j = \max\{j_1, j_2 + 1\}$.

Let $\mathcal{E}_1$ be the set of the j_1 pairwise $(d+1)$-disjoint edges of $\mathcal{H} \backslash E$. The edges of $\mathcal{E}_1$ are also a set of pairwise $d+1$-disjoint edges of $\mathcal{H}$. Thus $|\mathcal{E}_1| = j_1 \leq j$. If $\mathcal{E}_2$ is a set of j_2 pairwise $(d+1)$-disjoint edges of $\mathcal{H}'$, we claim that $\mathcal{E}_2 \cup \{E\}$ is a set of pairwise $(d+1)$-disjoint edges of $\mathcal{H}$. Indeed, for any edge $H \in \mathcal{H}'$, $\operatorname{dist}_{\mathcal{H}}(E, H) > d$, and so in particular, E and H is $(d+1)$-disjoint for every edge $H \in \mathcal{E}_2$. Thus $|\mathcal{E}_2 \cup \{E\}| = j_2 + 1 \leq j$. Thus $j \geq \max\{j_1, j_2 + 1\}$.

Suppose that $j > \max\{j_1, j_2 + 1\}$. Let $\mathcal{E}$ be a set of j pairwise $(d+1)$-disjoint edges of $\mathcal{H}$. If $E \notin \mathcal{E}$, then $\mathcal{E}$ is also a set of pairwise $(d+1)$-disjoint edges of $\mathcal{H} \backslash E$, and so $j = |\mathcal{E}| \leq j_1$, a contradiction. If $E \in \mathcal{E}$, then $\mathcal{E} \backslash \{E\}$ is a set of pairwise $(d+1)$-disjoint edges of $\mathcal{H}'$ since any other edge $H \in \mathcal{E}$ must have $\operatorname{dist}_{\mathcal{H}}(E, H) > d$. But this would imply that $j - 1 \leq j_2$, again a contradiction. Hence $j = \max\{j_1, j_2 + 1\}$. □

Theorem 6.11 gives the following interesting corollary for simple graphs, which was first proved by Zheng [32] in the special case that G was a forest.

Corollary 6.12. *Suppose that G is chordal graph. If j is the maximum number of pairwise 3-disjoint edges of G, then*

$$\operatorname{reg}(\mathcal{I}(G)) = j + 1.$$

Example 6.13. The bounds for the regularity in Theorem 6.6 are sharp. If $\mathcal{H}$ is any triangulated hypergraph, then the lower bound is achieved by Theorem 6.11. To show that the upperbound is achieved, consider the the edge ideal of C_5, the five-cycle. So $\mathcal{I}(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$.

Then $\alpha'(G) = 2$ (for example, take edges $E_1 = x_1x_2$ and $E_2 = x_3x_4$). So $\operatorname{reg}(\mathcal{I}(G)) \leq 3$. In fact we have equality since the resolution of $\mathcal{I}(G)$ is

$$0 \to R(-5) \to R^5(-3) \to R^5(-2) \to \mathcal{I}(G) \to 0.$$

In the study of squarefree monomial ideals, the theory of Alexander duality has proved to be significant in many ways. We round out this section by relating some algebraic invariants of edge ideals and their Alexander duals.

Definition 6.14. Let $I = (x_{11} \cdots x_{1i_1}, \dots, x_{r1} \cdots x_{ri_r}) \subseteq k[x_1, \dots, x_n]$ be a squarefree monomial ideal. Then the **Alexander dual** of I is defined to be

$$I^\vee = (x_{11}, \dots, x_{1i_1}) \cap \cdots \cap (x_{r1}, \dots, x_{ri_r}).$$

Definition 6.15. Let G be a graph. A subset V of the vertices of G is called a **vertex cover** if every edge in G is incident to at least a vertex in V; a **minimal vertex cover** is a vertex cover V with the property that no proper subset of V is. The smallest size of a minimal vertex cover of G is denoted by $\nu(G)$. The graph G is **unmixed** if all its minimal vertex covers have the same cardinality $\nu(G)$.

Remark 6.16. The operation of taking the Alexander dual of a squarefree monomial ideal brings generators to primary components. The minimal generators of $\mathcal{I}(G)^\vee$ correspond to minimal vertex covers of G.

Theorem 6.17. *Let G be a simple graph.*

(1) *If G is unmixed, then*

$$\operatorname{reg}(\mathcal{I}(G)) \leq \operatorname{ht}\mathcal{I}(G) + 1 \leq \operatorname{reg}(\mathcal{I}(G)^\vee) + 1 \text{ and } \operatorname{pdim}(\mathcal{I}(G)^\vee) \leq \operatorname{ht}\mathcal{I}(G) \leq \operatorname{pdim}(\mathcal{I}(G)) + 1.$$

(2) *If G is not unmixed, then*

$$\operatorname{reg}(\mathcal{I}(G)) \leq \operatorname{ht}\mathcal{I}(G) + 1 \leq \operatorname{reg}(\mathcal{I}(G)^\vee) \text{ and } \operatorname{pdim}(\mathcal{I}(G)^\vee) \leq \operatorname{ht}\mathcal{I}(G) \leq \operatorname{pdim}(\mathcal{I}(G)).$$

Proof. It suffices to prove the inequalities involving the regularity, since the bounds on the projective dimension follow from the identities $\operatorname{reg}(\mathcal{I}(G)) = \operatorname{pdim}(R/\mathcal{I}(G)^\vee)$ and $\operatorname{reg}(\mathcal{I}(G)^\vee) = \operatorname{pdim}(R/\mathcal{I}(G))$ (see, for example, [23, Theorem 5.59]). Observe that if $\mathcal{E}'$ is a matching in G then any vertex cover must contain at least a vertex of every edge in $\mathcal{E}'$. Thus, $\alpha'(G) \leq \nu(G) = \operatorname{ht}\mathcal{I}(G)$. It follows from Corollary 6.9 that $\operatorname{reg}(\mathcal{I}(G)) \leq \operatorname{ht}\mathcal{I}(G) + 1$. Since $\nu(G)$ is the least generating degree of $\mathcal{I}(G)^\vee$, we have $\nu(G) \leq \operatorname{reg}(\mathcal{I}(G)^\vee)$ and thus (1) follows. To prove (2) observe that when G is not unmixed, $\operatorname{reg}(\mathcal{I}(G)^\vee)$ is at least the largest generating degree of $\mathcal{I}(G)^\vee$, which is at least $\nu(G) + 1$. □

7. Properly-connected hypergraphs and linear first syzygies

In [13] Fröberg gave a characterization of edge ideals of simple graphs with linear resolutions. In this section, we obtain a partial generalization of Fröberg's result to the class of properly-connected hypergraphs. Specifically, we describe when $\mathcal{I}(\mathcal{H})$ has linear first syzygies.

Let us first recall Fröberg's result. If G is a simple graph, then the **complement of** G, denoted G^c, is the graph whose vertex set is the same as G, but whose edge set is defined by the rule $E \in G^c$ if and only $E \notin G$. Fröberg then showed:

Theorem 7.1. *Let G be a simple graph. Then $\mathcal{I}(G)$ has a linear resolution if and only if G^c is a chordal graph.*

When $\mathcal{H}$ is properly-connected hypergraph, we define the **complement of** $\mathcal{H}$, denoted $\mathcal{H}^c$, as

$$\mathcal{H}^c = \{E \subseteq \mathcal{X} \mid |E| = d \text{ and } E \notin \mathcal{H}\}.$$

So, one might expect Theorem 7.1 generalizes to properly-connected hypergraphs as follows: $\mathcal{I}(\mathcal{H})$ has a linear resolution if and only if $\mathcal{H}^c$ is a triangulated hypergraph. Unfortunately, this is not the case, as shown below, since $\mathcal{H}^c$ need not be properly-connected.

Example 7.2. Let $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$. Let $\mathcal{H} = \mathcal{K}_5^3 \backslash \{x_1x_2x_3, x_3x_4x_5\}$, i.e., $\mathcal{H}$ is the 3-uniform complete hypergraph of order 5 with two edges removed. Then $\mathcal{H}^c = \{x_1x_2x_3, x_3x_4x_5\}$ is not properly-connected since the two edges intersect at x_3, but there is no properly-irredundant chain of length 2 between the two edges. Because $\mathcal{H}^c$ is not even properly-connected, the notion of a triangulated hypergraph is undefined. However, the ideal $\mathcal{I}(\mathcal{H})$ has the linear resolution

$$0 \rightarrow R^4(-5) \rightarrow R^{11}(-4) \rightarrow R^8(-3) \rightarrow \mathcal{I}(\mathcal{H}) \rightarrow 0.$$

We take the first step towards generalizing Theorem 7.1 by asking when $\mathcal{I}(\mathcal{H})$ must have linear first syzygies. Like our previous results, the distance between edges is key.

Definition 7.3. The **edge diameter** of a properly-connected hypergraph $\mathcal{H}$ is

$$\operatorname{diam}(\mathcal{H}) = \max\{\operatorname{dist}_{\mathcal{H}}(E, H) \mid E, H \in \mathcal{H}\},$$

where the diameter is infinite if there exist two edges not connected by any proper chain.

Since $\mathcal{I}(\mathcal{H})$ is a monomial ideal, we know that its first syzygy module is generated by syzygies $S(x^E, x^H)$, for $E, H \in \mathcal{E}$. Moreover, it is clear that $S(x^E, x^H)$ is a linear syzygy if and only if $\operatorname{dist}_{\mathcal{H}}(E, H) = 1$. We shall see that these syzygies generate all of the syzygies on $\mathcal{I}(\mathcal{H})$ if the diameter of $\mathcal{H}$ is small enough. Indeed, a short enough proper chain will give us a way of writing $S(x^E, x^H)$ as a telescoping sum of linear syzygies.

Theorem 7.4. *Suppose that $\mathcal{H}$ is a d-uniform properly-connected hypergraph. Then $\mathcal{I}(\mathcal{H})$ has linear first syzygies if and only if $\operatorname{diam}(\mathcal{H}) \leq d$.*

Proof. Assume first that $\operatorname{diam}(\mathcal{H}) \leq d$. It follows from the Taylor resolution that the first syzygy module of $\mathcal{I}(\mathcal{H})$ is generated by syzygies $S(x^E, x^H)$, where $E, H \in \mathcal{E}$. We shall show that $S(x^E, x^H)$ is generated by linear syzygies. Let $t = \operatorname{dist}_{\mathcal{H}}(E, H)$. Then, since $\operatorname{diam}(\mathcal{H}) \leq d$, we have $t \leq d$. If $(E_0, \ldots, E_t)$ is the proper irredundant chain, then by Lemma 4.9 we can write $E = E_0 = \{z_1, \ldots, z_d\}$, $E_i = \{y_1, \ldots, y_i, z_{i+1}, \ldots, z_d\}$ where $y_i \notin E_j$ for $j < i$, and $E_t = H$.

It can be seen that $S(x^E, x^H)$ is given by the equality $y_1 \cdots y_t x^{E_0} - z_1 \cdots z_t x^{E_t} = 0$. Furthermore,

$$y_1 \ldots y_t x^{E_0} - z_1 \ldots z_t x^{E_t} = \sum_{k=0}^{t-1} \left(\prod_{i=1}^{k} z_i \prod_{j=k+2}^{t} y_j \right) (y_{k+1} x^{E_k} - z_{k+1} x^{E_{k+1}}).$$

Thus, $S(x^E, x^H)$ is generated by linear syzygies.

Conversely, suppose that $\mathcal{I}(\mathcal{H})$ has linear first syzygies, that is, $\beta_{1,j}(\mathcal{I}(\mathcal{H})) = 0$ for $j \neq d+1$. If $\operatorname{diam}(\mathcal{H}) \geq d+1$, then this implies that there exists at least two edges E, H with $\operatorname{dist}_{\mathcal{H}}(E, H) \geq d+1$, i.e., $\{E, H\}$ is a set of pairwise $(d+1)$-disjoint edges of $\mathcal{H}$. By Lemma 6.5 this implies that $\beta_{1,2d}(\mathcal{I}(\mathcal{H})) \neq 0$. But this contradicts the fact that $\mathcal{I}(\mathcal{H})$ has linear first syzygies. □

Example 7.5. If $\operatorname{diam}(\mathcal{H}) \leq d$ is small, $\mathcal{I}(\mathcal{H})$ may still have nonlinear second syzygies. For example, if $G = C_5$ is the 5-cycle, then $\operatorname{diam}(G) = 2$. However $\mathcal{I}(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$ has nonlinear second syzygies since $\beta_{2,5}(\mathcal{I}(G)) = 1$, as shown in Example 6.13.

Interestingly, if $\mathcal{H}$ is triangulated, knowing that $\mathcal{I}(\mathcal{H})$ that linear first syzygies is enough to know that the entire resolution of $\mathcal{I}(\mathcal{H})$ is linear.

Corollary 7.6. *Suppose that $\mathcal{H}$ is a d-uniform properly-connected hypergraph that is also triangulated. Then the following are equivalent:*

(a) *$\mathcal{I}(\mathcal{H})$ has a linear resolution.*
(b) *$\mathcal{I}(\mathcal{H})$ has linear first syzygies.*
(c) $\operatorname{diam}(\mathcal{H}) \leq d$.

Proof. The implication (a) $\Rightarrow$ (b) is immediate, and (b) $\Rightarrow$ (c) is a consequence of Theorem 7.4. To show that (c) $\Rightarrow$ (a), the bound on $\operatorname{diam}(\mathcal{H})$ implies that $\mathcal{H}$ cannot have two or more pairwise $(d+1)$-disjoint edges (otherwise $\operatorname{diam}(\mathcal{H}) > d$). By Theorem 6.11 this implies that $\operatorname{reg}(\mathcal{I}(\mathcal{H})) = (d-1)+1 = d$. Since $\mathcal{I}(\mathcal{H})$ is generated in degree d, this forces $\mathcal{I}(\mathcal{H})$ to have a linear resolution. □

Restricted to simple graphs, Corollary 7.6 gives the following result.

Corollary 7.7. *Suppose that G is a chordal graph. Then the following are equivalent:*

(a) *$\mathcal{I}(G)$ has a linear resolution.*
(b) *$\mathcal{I}(G)$ has linear first syzygies.*
(c) $\operatorname{diam}(G) \leq 2$.

Acknowledgements

The authors would especially like to thank Jessica Sidman who made some contributions to this paper in its preliminary stages, and with whom we had many useful discussions. The authors would like to thank J. Herzog and X. Zheng for stimulating discussions on the regularity of edge ideals. Part of this research was carried out while the second author visited the first at Tulane University. The authors acknowledge the support from Louisiana Board of Regents for this visit. The second author also thanks Tulane University for its hospitality during his visit. The second author further acknowledges the research support received by NSERC. The computer algebra system `CoCoA` [4] was used to generate examples.

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