

# Geometric inequalities for black holes

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## Abstract

It is well known that the three parameters that characterize the Kerr black hole (mass, angular momentum and horizon area) satisfy several important inequalities. Remarkably, some of these inequalities remain valid also for dynamical black holes. This kind of inequalities play an important role in the characterization of the gravitational collapse. They are closely related with the cosmic censorship conjecture. In this article recent results in this subject are reviewed.

## 1 Geometric inequalities in General Relativity

A classical example of a geometric inequality is the isoperimetric inequality for closed plane curves given by

$$L^2 \geq 4\pi A \quad (= \text{circle}), \quad (1)$$

where  $A$  is the area enclosed by a curve  $C$  of length  $L$ . In (1) equality holds if and only if  $C$  is a circle, see figure 1. For a review on this subject see [68]. The inequality (1) applies to complicated geometric objects (i.e. arbitrary closed planar curves). The equality in (1) is achieved only for an object of “optimal shape” (i.e. the circle) which is described by few parameters (in this case only one: the radius). Moreover, this object has a variational characterization: the circle is uniquely characterized by the property that among all simple closed plane curves of given length  $L$ , the circle of circumference  $L$  encloses the maximum area.

General Relativity is a geometric theory, hence it is not surprising that geometric inequalities appear naturally in it. Many of these inequalities are similar in spirit as the isoperimetric inequality (1). In particular, all the geometric inequalities discussed in this article will have the same structure as (1): the inequality applies for a rich class of objects and the equality only applies for

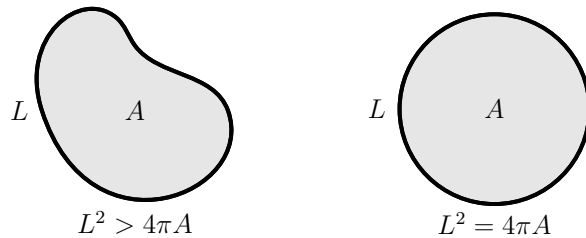


Figure 1: The isoperimetric inequality. On the left an arbitrary curve, where the strict inequality holds. On the right the circle, where the equality holds.

an object of “optimal shape” (always indicated in parenthesis as in (1)). This object, like the circle, can be described by few parameters and it has also a variational characterization.

However, General Relativity is also a physical theory. It is often the case that the quantities involved have a clear physical interpretation and the expected behavior of the gravitational and matter fields often suggests geometric inequalities which can be highly non-trivial from the mathematical point of view. The interplay between physics and geometry gives to geometric inequalities in General Relativity their distinguished character. These inequalities relate quantities that have both a physical interpretation and a geometrical definition.

The plan of this article follows this interplay between physics and mathematics. In section 2 we present the physical motivations for the black holes geometric inequalities. In section 3 we summarize some theorems where these inequalities have been recently proved. Finally, in section 4 we list relevant open problems and we also describe recent results on geometric inequalities for bodies.

## 2 Physical picture

An important example of a geometric inequality is the positive mass theorem. Let  $m$  be the total ADM mass on an asymptotically flat complete initial data such that the dominant energy condition is satisfied. Then we have

$$0 \leq m \quad (= \text{Minkowski}). \quad (2)$$

The mass  $m$  is a pure geometrical quantity [9][13][20]. However, from the geometrical mass definition, without the physical picture, it would be very hard even to conjecture the inequality (2). In fact the proof of the positive mass theorem turns out to be very subtle [71][72][82].

A key assumption in the positive mass theorem is that the matter fields should satisfy an energy condition. This condition is expected to hold for all physically realistic matter. This kind of general properties which do not depend

very much on the details of the model are not easy to find for a macroscopic object. And hence it is difficult to obtain simple and general geometric inequalities among the parameters that characterize ordinary macroscopic objects. Black holes represent a unique class of very simple macroscopic objects and hence they are natural candidates for geometrical inequalities. Nevertheless, in section 4 we will present also a geometric inequality valid for ordinary bodies.

The black hole uniqueness theorem ensures that stationary black holes in vacuum are characterized by the Kerr exact solution of Einstein equations<sup>1</sup>. For simplicity we will not consider the electromagnetic field in this article, however most of the results presented here can be generalized to include that case.

It is somehow remarkable that the same family of solutions of Einstein equations that describe the unique stationary black hole (i.e. the Kerr metric) also describe naked singularities. In effect, the Kerr metric depends on two parameters: the mass  $m$  and the angular momentum  $J$ . This metric is a solution of Einstein vacuum equations for any choice of the parameters  $m$  and  $J$ . However, it represents a black hole if and only if the following remarkably inequality holds

$$\sqrt{|J|} \leq m. \quad (3)$$

Otherwise the spacetime contains a naked singularity. Figure 2 shows the parameter space of the Kerr solution. Extreme black holes are defined by the equality in (3). These black holes lie at the boundary between naked singularities and black holes. For most of the inequalities discussed in this article, extreme black holes play the role of the circle in the isoperimetric inequality (1): they reach the equality and they represent objects of “optimal shape”.

The area of the horizon of the Kerr black hole is given by the simple but very important formula

$$A = 8\pi \left( m^2 + \sqrt{m^4 - J^2} \right). \quad (4)$$

From equation (4) we deduce that the following three geometric inequalities hold for a Kerr black hole

$$\sqrt{\frac{A}{16\pi}} \leq m \quad (= \text{Schwarzschild}), \quad (5)$$

$$\sqrt{|J|} \leq m \quad (= \text{Extreme Kerr}), \quad (6)$$

$$8\pi|J| \leq A \quad (= \text{Extreme Kerr}). \quad (7)$$

As expected from the discussion above, the inequality (6) is needed to define the black hole horizon area in (4): if (6) does not hold, then the expression (4) is not a real number. We have listed this inequality again here to emphasize its connection with the other two in the following discussion. Inequalities (5) and (7) follow from (6) and (4). Note that these inequalities relate the three relevant parameters of the Kerr black hole  $(m, J, A)$ .

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<sup>1</sup>It is worth mention that important aspects of the black hole uniqueness problem remain still open, see recent review article [25] and reference therein.

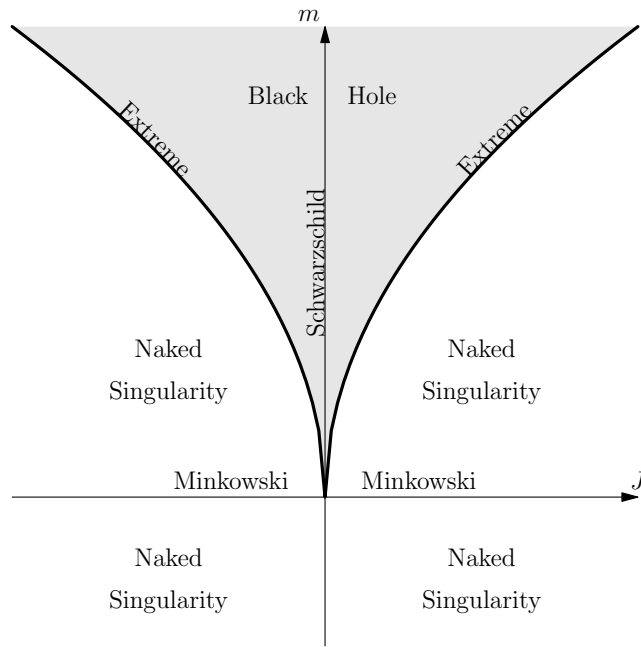


Figure 2: A point in this graph is a Kerr solution with parameters  $m$  and  $J$ . The horizontal axis where  $m = 0$  is Minkowski space. The Schwarzschild solution is given by the vertical axis where  $J = 0$ . In the gray region the parameters satisfy the inequality (3) and hence the Kerr solution describe a black hole. The boundary of this region is given by the equality in (3), these solutions are called extreme black holes. In the white region, excluding the horizon axis, the Kerr solution contains a naked singularity. That includes also the negative mass region.

Let us discuss the physical meaning of the inequalities (5), (6) and (7). In the inequality (5), the difference

$$m - \sqrt{\frac{A}{16\pi}}, \quad (8)$$

represents the rotational energy of the Kerr black hole. This is the maximum amount of energy that can be extracted from the black hole by the Penrose process [19]. When the difference (8) is zero, the black hole has no angular momentum and hence it is the Schwarzschild black hole.

From Newtonian considerations, we can interpret the inequality (6) as follows [80]. In a collapse the gravitational attraction ( $\approx m^2/r^2$ ) at the horizon ( $r \approx m$ ) dominates over the centrifugal repulsive forces ( $\approx J^2/mr^3$ ).

Finally, concerning the inequality (7), the black hole temperature is given by the following formula

$$\kappa = \frac{1}{4m} \left( 1 - \frac{(8\pi J)^2}{A^2} \right). \quad (9)$$

The temperature is positive if and only if the inequality (7) holds. Moreover the temperature is zero if and only if the equality in (7) holds and hence the black hole is extreme.

There exists another relevant geometrical inequality which can be deduced from the formula (4)

$$8\pi \left( m^2 - \sqrt{m^4 - J^2} \right) \leq A \quad (= \text{Extreme Kerr}). \quad (10)$$

Remarkably, as it was pointed out in [59] for the case of the electric charge and in [38] for the present case of angular momentum, the inequality (10) can be deduced purely from the inequalities (6) and (7) (i.e. without using the equality (4)) by simple algebra. Namely

$$m^2 = \sqrt{m^4 - J^2} + J^2, \quad (11)$$

$$\leq |J| + \sqrt{m^4 - J^2}, \quad (12)$$

$$\leq \frac{A}{8\pi} + \sqrt{m^4 - J^2}, \quad (13)$$

where in the line (12) we have used (6) and in line (13) we have used (7). In that sense, the inequalities (5), (6) and (7) are more fundamental than (10). However, the inequality (10) is important by itself since it related with the Penrose inequality with angular momentum, see [59] [38].

We have seen that for stationary black holes the inequalities (5), (6) and (7) are straightforward consequences of the area formula (4).

However, black holes are in general non stationary, see figure 3. Astrophysical phenomena like the formation of a black hole by gravitational collapse or a binary black hole collision are highly dynamical. For such systems, the black

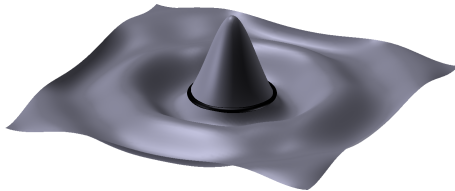


Figure 3: Schematic representation of an initial data for a non-stationary black hole. The black ring represents a trapped surface. Outside and inside the trapped surface the gravitational field is highly dynamical.

hole can not be characterized by few parameters as in the stationary case. In fact, even stationary but non-vacuum black holes have a complicated structure (for example black holes surrounded by a rotating ring of matter, see the numerical studies in [7]). Remarkably, inequalities (5), (6) and (7) extend (under appropriate assumptions) to the fully dynamical regime. Moreover, these inequalities are deeply connected with properties of the global evolution of Einstein equations, in particular with the cosmic censorship conjecture.

To discuss the physical arguments that support these inequalities in the dynamical regime it is convenient to start with the inequality (7). For a dynamical black hole, the physical quantities that are well defined are the total ADM mass  $m$  of the spacetime and the area  $A$  of the black hole horizon. The total mass  $m$  of the spacetime measures the sum of the black hole mass and the mass of the gravitational waves surrounding it. In the stationary case, the mass of the black hole is equal to the total mass of the spacetime, but this is no longer true for a dynamical black hole. The mass  $m$  is a global quantity, it carries information on the whole spacetime. In contrast, the area of the horizon  $A$  is a quasi-local quantity, it carries information on a bounded region of the spacetime.

It is well known that the energy of the gravitational field cannot be represented by a local quantity (i.e. a scalar field). The best one can hope is to obtain a quasi-local expression. The same applies to the angular momentum. In general, it is difficult to find physically relevant quasi-local quantities like mass and angular momentum (see the review article [77]). However, in axial symmetry, there is a well defined notion of quasi-local angular momentum: the Komar integral of the axial Killing vector. Moreover, the angular momentum is conserved in vacuum. That is, axially symmetric gravitational waves do not carry angular momentum.

Then, for axially symmetric dynamical black holes we have two well defined quasi-local quantities: the area of the horizon  $A$  and the angular momentum  $J$ . Note that the inequality (7) relates only quasi-local quantities.

Using  $A$  and  $J$  we can define the quasi-local mass for a dynamical black hole by the Kerr formula (4), that is

$$m_{bh} = \sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}}. \quad (14)$$

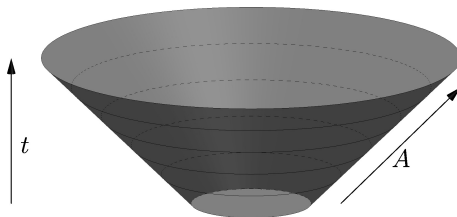


Figure 4: The area theorem. The horizon area of a dynamical black hole increase with time.

This is, in principle, just a definition. Since  $m_{bh}$  is given by the Kerr formula (4) it automatically satisfies the inequalities (6) and (5). However, the relevant question is: does  $m_{bh}$  describes the quasi-local mass of a non-stationary black hole? This question is closed related to the validity of the inequality (7) in the dynamical regime. In order to answer it let us analyze the evolution of  $m_{bh}$ .

For a dynamical black hole, by the area theorem, we know that the horizon area  $A$  increase with time, see figure 4. In general, the quasi-local mass of the black hole is not expected to be a monotonically increasing quantity. Energy can be extracted from a rotating black hole by the Penrose process. However, if we assume axial symmetry then the angular momentum will be conserved at the quasi-local level. On physical grounds, one would expect that in this situation the quasi-local mass of the black hole should increase with the area, since there is no mechanism at the classical level to extract mass from the black hole. In effect, the Penrose process involves an interchange of angular momentum between the black hole and the exterior. But the angular momentum transfer is forbidden in axial symmetry. Then, both the area  $A$  and the quasi-local mass  $m_{bh}$  should monotonically increase with time in axial symmetry.

Let us take a time derivative of  $m_{bh}$ . To analyze this, it is illustrative to write down the complete differential, namely the first law of thermodynamics

$$\delta m_{bh} = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J, \quad (15)$$

where

$$\kappa = \frac{1}{4m_{bh}} \left( 1 - \frac{(8\pi J)^2}{A^2} \right), \quad \Omega_H = \frac{4\pi J}{A m_{bh}}. \quad (16)$$

In equation (15) we have followed the standard notation for the formulation of the first law; we emphasize, however, that in our context this equation is a trivial consequence of (14). In axial symmetry  $\delta J = 0$  and hence we obtain

$$\delta m_{bh} = \frac{\kappa}{8\pi} \delta A. \quad (17)$$

By the area theorem we have

$$\delta A \geq 0. \quad (18)$$

Then  $\delta m_{bh} \geq 0$  if and only if  $\kappa \geq 0$ , that is  $\delta m_{bh} \geq 0$  if and only if the inequality (7) holds. Then, it is natural to conjecture that this inequality should be satisfied for any axially symmetric black hole. If the horizon violates (7), then in the evolution the area will increase but the mass  $m_{bh}$  will decrease. This will indicate that the quantity  $m_{bh}$  does not have the desired physical meaning. Also, a rigidity statement is expected. Namely, the equality in (7) is reached only by the extreme Kerr black hole where  $\kappa = 0$ .

This inequality provides a remarkable quasi-local measure of how far a dynamical black hole is from the extreme case, namely an ‘extremality criteria’ in the spirit of [17], although restricted only to axial symmetry. In the article [31] it has been conjectured that, within axially symmetry, to prove the stability of a nearly extreme black hole is perhaps simpler than a Schwarzschild black hole. It is possible that this quasi-local extremality criteria will have relevant applications in this context. Note also that the inequality (7) allows to define, at least formally, the positive temperature of a dynamical black hole  $\kappa$  by the formula (16) (see Refs. [11] [10] for a related discussion of the first law in dynamical horizons). If inequality (7) holds, then  $m_{bh}$  defines a non-trivial quantity that increase monotonically with time, like the black hole area  $A$ .

It is important to emphasize that the physical arguments presented above in support of (7) are certainly weaker in comparison with the ones behind the Penrose inequalities that support the inequalities (5) and (6) that we will discuss below. A counter example of any of these inequality will prove that the standard picture of the gravitational collapse is wrong. On the other hand, a counter example of (7) will just prove that the quasi-local mass (15) is not appropriate to describe the evolution of a non-stationary black hole. One can imagine other expressions for quasi-local mass, may be more involved, in axial symmetry. On the contrary, reversing the argument, a proof of (7) will certainly suggest that the mass (15) has physical meaning for non-stationary black holes as a natural quasi-local mass (at least in axial symmetry). Also, the inequality (7) provide a non trivial control of the size of a black hole valid at any time.

In a seminal article Penrose [70] proposed a remarkably physical argument that connects global properties of the gravitational collapse with geometric inequalities on the initial conditions. That argument lead to the well known Penrose inequality (5) for dynamical black holes (without any symmetry assumption). In the following we review this argument imposing axial symmetry, where angular momentum is conserved. And, more important, we include a relevant new ingredient: we assume that the inequality (7) holds.

We will assume that the following statements hold in a gravitational collapse:

- (i) Gravitational collapse results in a black hole (weak cosmic censorship).
- (ii) The spacetime settles down to a stationary final state. We will further assume that at some finite time all the matter have fallen into the black hole and hence the exterior region is vacuum.

Conjectures (i) and (ii) constitute the standard picture of the gravitational collapse. Relevant examples where this picture is confirmed (and where the



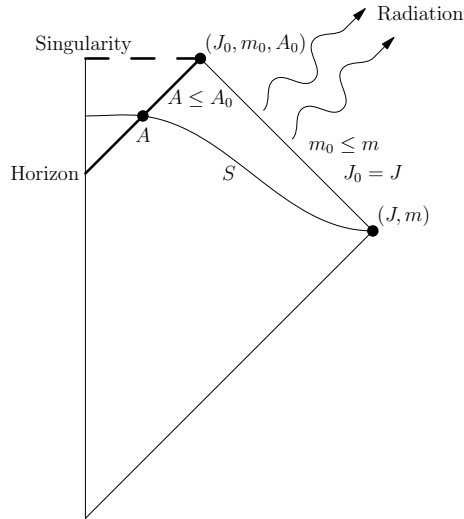


Figure 5: The Penrose diagram of a gravitational collapse. The initial Cauchy surface is denoted by  $S$ . The area  $A$  increase along the event horizon. The mass  $m$  decrease along null infinity. We have assumed axial symmetry and hence the angular momentum remains constant along null infinity  $J = J_0$ .

role of angular momentum is analyzed) are the collapse of neutron stars studied numerically in [12] [48].

The black hole uniqueness theorem implies that the final stationary state postulated in (ii) is given by the Kerr black hole. Let us denote by  $m_0, J_0, A_0$ , respectively, the mass, angular momentum and horizon area of the remainder Kerr black hole. Penrose argument runs as follows. Take a Cauchy surface  $S$  in the spacetime such that the collapse has already occurred. This is shown in figure 5. Let  $\Sigma$  denotes the intersection of the event horizon with the Cauchy surface  $S$  and let  $A$  be its area. Let  $(m, J)$  be the total mass and angular momentum at spacelike infinity. These quantities can be computed from the initial surface  $S$ . By the black hole area theorem we have that the area of the black hole increase with time and hence

$$A_0 \geq A. \quad (19)$$

Since gravitational waves carry positive energy, the total mass of the spacetime should be bigger than the final mass of the remainder Kerr black hole

$$m \geq m_0. \quad (20)$$

The difference  $m - m_0$  is the total amount of gravitational radiation emitted by the system.

To relate the initial angular momentum  $J$  with the final angular momentum  $J_0$  is much more complicated. Angular momentum is in general non-conserved.

There exists no simple relation between the total angular momentum  $J$  of the initial conditions and the angular momentum  $J_0$  of the final black hole. For example, a system can have  $J = 0$  initially, but collapse to a black hole with final angular momentum  $J_0 \neq 0$ . We can imagine that on the initial conditions there are two parts with opposite angular momentum, one of them falls in to the black hole and the other escape to infinity. Axially symmetric vacuum spacetimes constitute a remarkable exception because the angular momentum is conserved. In that case we have

$$J = J_0. \quad (21)$$

For a discussion of this conservation law in detail see [34] and reference therein.

We have assumed that the inequality (7) holds, then by the discussion above we have that the quasi-local mass  $m_{bh}$  increase with time, that is

$$m_{bh} \leq m_0. \quad (22)$$

We emphasize that this inequality is highly non-trivial. The quantity  $m_{bh}$  is computed on the initial surface  $S$ , in contrast to compute  $m_0$  we need to know the whole spacetime. Using (22) and (20) we finally obtain

$$\sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}} = m_{bh} \leq m. \quad (23)$$

This inequality has the natural interpretation that the mass of the black hole  $m_{bh}$  should always be smaller than the total mass of the spacetime  $m$ . The inequality (23) represents a generalization of the Penrose inequality with angular momentum. This inequality implies

$$\sqrt{|J|} \leq m. \quad (24)$$

In fact, the inequality (24) can be deduced directly by the same heuristic argument without using the area theorem. It depends only on the following assumptions

- Gravitational waves carry positive energy.
- Angular momentum is conserved in axial symmetry.
- In a gravitational collapse the spacetime settles down to a final Kerr black hole.

Let us summarize the discussion of this section. For an axially symmetric, dynamical black hole, the following two geometrical inequalities are expected

$$8\pi|J| \leq A \quad (= \text{Extreme Kerr horizon}), \quad (25)$$

$$\sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}} \leq m \quad (= \text{Kerr black hole}). \quad (26)$$

The inequality (25) is quasi-local and the inequality (26) is global. The global inequality (26) implies the following two inequalities

$$\sqrt{\frac{A}{16\pi}} \leq m \quad (= \text{Schwarzschild}), \quad (27)$$

$$\sqrt{|J|} \leq m. \quad (= \text{extreme Kerr black hole}). \quad (28)$$

That is:

*The three geometrical inequalities (5), (6) and (7) valid for the Kerr black holes are expected to hold also for axially symmetric, dynamical black holes.*

The Penrose inequality (27) is valid also without the axial symmetry assumption. It is important to emphasize that all the quantities involved in the geometrical inequalities above can be calculated on the initial surface. For simplicity, we have avoided the distinction between event horizon and apparent horizons (defined in terms of trapped surfaces) to calculate the area  $A$ . This point is important for the Penrose inequality (see the discussion in [65]) but not for the other inequalities which are the main subject of this review. In particular the horizon area  $A$  in (25) is the area of an appropriated defined trapped surface.

A counter example of the global inequality (26) will imply that cosmic censorship is not true. Conversely a proof of it gives indirect evidence of the validity of censorship, since it is very hard to understand why this highly non-trivial inequality should hold unless censorship can be thought of as providing the underlying physical reason behind it.

The inequalities (5), (6) and (7) can be divided into two groups:

1.  $\sqrt{\frac{A}{16\pi}} \leq m$ : the area appears as lower bound.
2.  $\sqrt{|J|} \leq m$  and  $8\pi|J| \leq A$ : the angular momentum appears as lower bound and the area appears as upper bound.

The mathematical methods used to study these two groups are, up to now, very different. This review is mainly concerned with the second group.

Finally, we mention that for the Kerr black hole there exists a remarkable equality of the form  $(8\pi J)^2 = A^+ A^-$ , where  $A^+$  and  $A^-$  denote the areas of event and Cauchy horizon (see figure 6). This equality has been proved for general stationary spacetimes in the following series of articles [5] [50] [4]. It has recently received considerable attention in the string community (see [27] and [79] and references therein). The key property used in these studies is that the product of horizon areas is independent of the mass of the black hole. It is interesting to note that there exists, up to now, no generalization of this kind of equality (or a related inequality) to the dynamical regime.

### 3 Theorems

The Penrose inequality

$$\sqrt{\frac{A}{16\pi}} \leq m \quad (= \text{Schwarzschild}), \quad (29)$$

has been intensively studied. It is a very relevant geometric inequality for black holes since it is valid without any symmetry assumption. For a comprehensive review on this subject see [65]. The most important results concerning this inequality are the proofs of Huisken-Ilmanen [55] and Bray [18] for the Riemannian case. The general case remains open. Also, there is up to now no result concerning the Penrose inequality with angular momentum (26) discussed in the previous section.

In the following we present a sample of the main results concerning inequalities (28) and (25) that have been recently proved.

For the global inequality (28) we have the following theorem.

**Theorem 3.1** *Consider an axially symmetric, vacuum, asymptotically flat and maximal initial data set with two asymptotic ends. Let  $m$  and  $J$  denote the total mass and angular momentum at one of the ends. Then, the following inequality holds*

$$\sqrt{|J|} \leq m \quad (= \text{Extreme Kerr}). \quad (30)$$

For the precise definitions, fall off conditions and assumptions on the initial data we refer to original articles cited below.

The first proof of the global inequality (30) was provided in a series of articles [30], [29], [28] which end up in the global proof given in [32]. The proof is based on a variational characterization of the extreme Kerr initial data. In [21] and [23] the result was generalized and the proof simplified. In [24] [26] the charge was included. In [74] relevant improvements on the rigidity statements were made. In particular in that article it was proved the first rigidity result including charge and a measure of the distance to extreme Kerr black hole was introduced. In [85] the result was proved with the maximal condition replaced by a small trace assumption for the second fundamental form of the initial data. Related results concerning the force between black holes were proved in [44]. Finally, the mass formula and the variational techniques involved in the proof of the inequality (30) were very recently used to study the linear stability of the extreme Kerr black hole [36].

Under the hypothesis of theorem 3.1 (namely, vacuum and axial symmetry) the angular momentum is defined as conserved quasi-local integral. In particular, if the topology of the manifold is trivial (i.e.  $\mathbb{R}^3$ ), then the angular momentum is zero and hence theorem 3.1 reduces to the positive mass theorem. In order to have non-zero angular momentum we need to allow non-trivial topologies, for example manifolds with two asymptotic ends as it is the case in theorem 3.1. An important initial data set that satisfies the hypothesis of the theorem is provided by an slice  $t = \text{constant}$  in the Kerr black hole in the

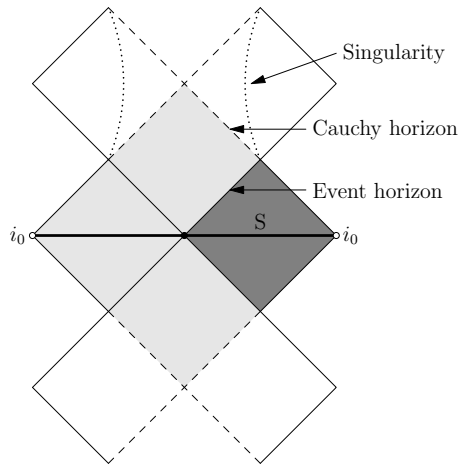


Figure 6: Conformal diagram of the non-extreme Kerr black hole. The points  $i_0$  represent spacelike infinity. The surface  $S$  have two identical asymptotically flat ends  $i_0$ .

standard Boyer-Lindquist coordinates, see figures 6 and 7. The non-extreme initial data have a different geometry as the extreme initial data. The former are asymptotically flat at both ends. In contrast, extreme initial data, which reach the equality in (30), have one asymptotically flat end and one cylindrical end, see figure 8. That geometry represents the “optimal shape” with respect to the inequality (30). Figure 8 is the analog of figure 1 for the geometrical inequality (30).

Regarding the quasi-local inequality (25) we have the following result.

**Theorem 3.2** *Given an axisymmetric closed marginally trapped and stable surface  $\Sigma$ , in a spacetime with non-negative cosmological constant and fulfilling the dominant energy condition, it holds the inequality*

$$8\pi|J| \leq A \quad (= \text{Extreme Kerr throat}), \quad (31)$$

where  $A$  and  $J$  are the area and angular momentum of  $\Sigma$ .

This is a pure spacetime and local result. That is, there is no mention of a three-dimensional initial hypersurface where the two-dimension surface  $\Sigma$  is embedded. Axisymmetry is only imposed on  $\Sigma$ . Moreover, this theorem does not assume vacuum. The matter fields can have also angular momentum and it can be transferred to the black hole, however the inequality (31) remains true even for that case. It is important to note that the angular momentum that appears in (31) is the gravitational one (i.e. the Komar integral). In fact this inequality is non-trivial even for the Kerr-Newman black hole, see the discussion in [34].

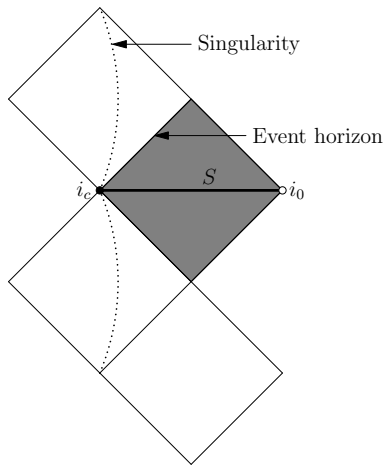


Figure 7: Conformal diagram of the extreme Kerr black hole. The point  $i_0$  represents spacelike infinity, the point  $i_c$  represent the cylindrical end. The surface  $S$  has one asymptotically flat end  $i_0$  and one cylindrical end  $i_c$ .

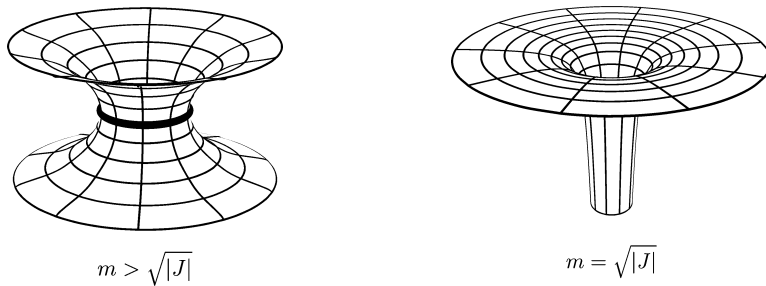


Figure 8: On the left, an the initial data with two asymptotically flat ends, like the non-extreme Kerr black holes. For these data the strict inequality holds. On the right, the data of extreme Kerr black hole, with one asymptotically flat and one cylindrical end. For this data the equality holds.

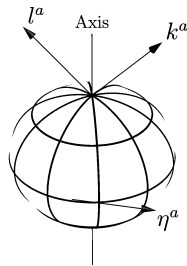


Figure 9: Axially symmetric two-surface. The axial Killing vector  $\eta$  is tangent to the surface. The null vectors  $\ell^a$  and  $k^a$  are normal to  $\mathcal{S}$

Theorem 3.2 has the following history. The quasi-local inequality (31) was first conjectured to hold in stationary spacetimes surrounded by matter in [8]. In that article the extreme limit of this inequality was analyzed and also numerical evidences for the validity in the stationary case was presented (using the numerical method and code developed in [7]). In a series of articles [51] [52] the inequality (31) (including also the electromagnetic charge) was proved for that class of stationary black holes. See also the review article [6].

In the dynamical regime, the inequality (31) was conjectured to hold in [33] based on the heuristic argument mentioned in section 2. In that article also the main relevant techniques for its proof were introduced, namely the mass functional on the surface and its connections with the area. A proof (but with technical restrictions) was obtained in [1] [43]. The first general and pure quasi-local result was proven in [40], where the relevant role of the stability condition for minimal surfaces was pointed out. The generalization to trapped surfaces and non-vacuum has been proved in [57]. The electromagnetic charge was included in [45] and [46]. This inequality has been extended to higher dimensions in [53] and [69]. In [84] [83] and [41] it has been also extended to Einstein-Maxwell dilaton gravity. In [47] related inequalities that involve the shape of the black hole were proved.

To describe the concept of stable trapped surface (this condition was first introduced in [2]) used in theorem 3.2 let us consider an axially symmetric closed two-surface  $\Sigma$  with the topology of a two-sphere. The surface  $\Sigma$  is embedded in the spacetime. Let  $\ell^a$  and  $k^a$  be null vectors spanning the normal plane to  $\Sigma$  and normalized as  $\ell^a k_a = -1$ , see figure 9. The expansion is defined by  $\theta^{(\ell)} = \nabla_a \ell^a$ , where  $\nabla$  is the spacetime connection. The surface  $\Sigma$  is marginally trapped if  $\theta^{(\ell)} = 0$ . Given a closed marginally trapped surface  $\Sigma$  we will refer to it as spacetime stably outermost if there exists an outgoing ( $-k^a$ -oriented) vector  $X^a = \gamma \ell^a - \psi k^a$ , with  $\gamma \geq 0$  and  $\psi > 0$ , such that the variation of  $\theta^{(\ell)}$  with respect to  $X^a$  fulfills the condition

$$\delta_X \theta^{(\ell)} \geq 0. \quad (32)$$

Here  $\delta$  denotes a variation operator associated with a deformation of the surface  $\Sigma$  (c.f. for example [16] [2])). For maximal initial data the stability condition

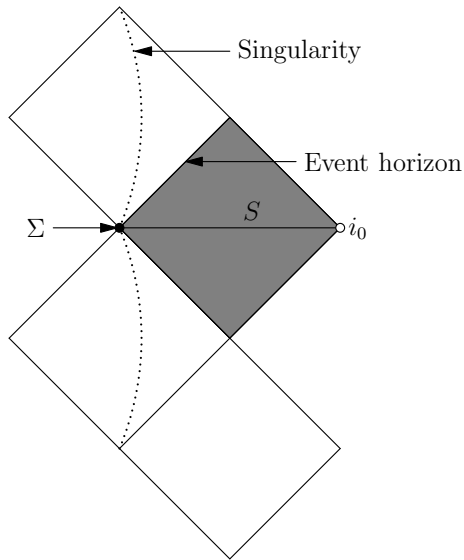


Figure 10: Location of the extreme Kerr throat surface  $\Sigma$  in the spacetime.

(32) is closely related with the stability condition for minimal surfaces (see [37], [57]). The stability of a minimal surface is the requirement that the area is a local minimum.

The extreme throat geometry, with angular momentum  $J$ , was defined in [33] (see also [1] and [40]). This concept captures the local geometry near the horizon of an extreme Kerr black hole. The extreme throat is the asymptotic limit in the cylindrical end of an extreme Kerr black hole, see figure 10 and 11. Both the intrinsic and extrinsic geometry of this surface are fixed. For example, it has an intrinsic metric given by

$$|J| \left( (1 + \cos^2 \theta) d\theta^2 + \frac{4 \sin^2 \theta}{(1 + \cos^2 \theta)} d\phi^2 \right). \quad (33)$$

It is an oblate spheroid with respect to the axis of rotation (see figure 12, on the right).

The extreme Kerr throat achieves the equality in (31), this surface has the “optimal shape” with respect to this inequality. It also has a variational characterization. Figure 12 is the analog of figures 1 and 8 for inequality (31).

The results in theorem 3.2 have been used in a recent non-existence proof of stationary black hole binaries [67] [66] [22].

The rigidity statement in theorem 3.2 (namely that the equality in (31) implies that the surface is an extreme Kerr throat) has been proved in a different context: for extreme isolated horizons and near-horizon geometries of extremal black holes in [56], [62] and [60], see also the review article [61] and references therein.



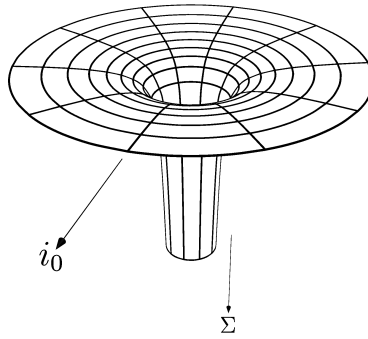


Figure 11: Location of the extreme Kerr throat surface  $\Sigma$  on the initial data.

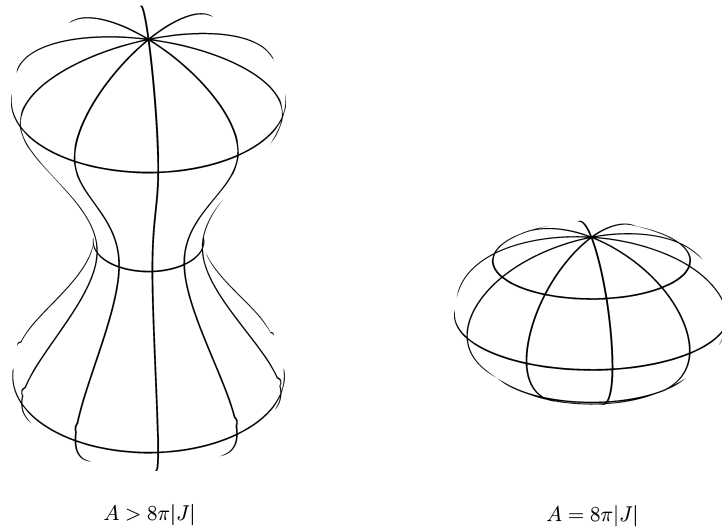


Figure 12: On the left, an arbitrary axially symmetric stable two surface. For this kind of surface the strict inequality holds. On the right, the extreme throat sphere, where the equality holds.

## 4 Open problems and recent results on bodies

In this final section I would like to present the main open problems regarding the black holes geometrical inequalities discussed in the previous sections. My aim is to present open problems which are relevant (and probably involve the discovery of new techniques) and at the same time they appear feasible to solve. For more details see the review article [34]. The open problem mentioned there regarding the inclusion of the electric charge in the quasi-local inequality (31) have been solved [45] [46].

For the global inequality (30) there are two main open problems, which involve generalizations of the assumptions in theorem 3.1:

- Remove the maximal condition.
- Generalization for asymptotic flat manifolds with multiple ends.

Concerning the maximal condition, as we mention above, in a recent article [85] this assumption have been replaced by a small trace condition. See also the discussion in [34]. The most relevant open problem is the second one. The physical heuristic argument presented in section 2 applies to that case and hence there little doubt that the inequality holds. This problem is related with the uniqueness of the Kerr black hole with degenerate and disconnected horizons. It is probably a hard problem. There are very interesting partial results in [23] and also numerical evidences in [39].

Probably the most important open problem for geometrical inequalities for axially symmetric black holes is the following:

- Prove the Penrose inequality with angular momentum (23).

We mention in section 2 that there is a clear physical connection between the global inequality (30) and the Penrose inequality with angular momentum in axial symmetry (23). However, the techniques used to prove the inequality (30) are very different than the one used to prove the classical Penrose inequality (29) (see the discussion in [34]).

For the quasi-local inequality (31) the two main problems are the following:

- A generalization of the inequality (31) without axial symmetry.
- A generalization of the inequality (31) for ordinary bodies.

The problem of finding versions of inequality (31) without any symmetry assumption, in contrast with the other open problems presented above, is not a well-defined mathematical problem since there is no unique notion of quasi-local angular momentum in the general case. However, exploring the scope of the inequality in regions close to axial symmetry (in some appropriate sense) can perhaps provide such a notion. From the physical point of view, we do not see any reason why this inequality should only hold in axial symmetry. Note that the global inequality (30) only holds in axial symmetry. This is clear from the

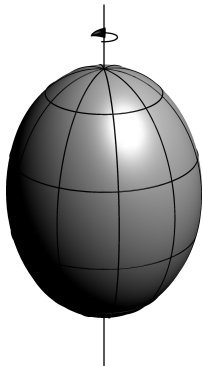


Figure 13: Axially symmetric rotating body.

physical point of view (see the discussion in [34]) and in [54] highly non-trivial counter examples have been constructed.

Finally, concerning the second problem there have been recently some results in [35]. Consider a rotating body  $U$  with angular momentum  $J(U)$ , see figure 13. Let  $\mathcal{R}(U)$  be a measure (with units of length) of the size of the body.

In [35], the following universal inequality for all bodies is conjectured

$$\mathcal{R}^2(U) \gtrsim \frac{G}{c^3} |J(U)|, \quad (34)$$

where  $G$  is the gravitational constant and  $c$  the speed of light. The symbol  $\gtrsim$  is intended as an order of magnitude, the precise universal (i.e. independent of the body) constant will depend on the definition of  $\mathcal{R}$ . We have reintroduced in (34) the fundamental constants in order to make more transparent the discussion below.

The arguments in support of the inequality (34) are based in the following three physical principles:

- (i) The speed of light  $c$  is the maximum speed.
- (ii) For bodies which are not contained in a black hole the following inequality holds

$$\mathcal{R}(U) \gtrsim \frac{G}{c^2} m(U), \quad (35)$$

where  $m(U)$  is the mass of the body.

- (iii) The inequality (34) holds for black holes.

Let us discuss these assumptions. Item (i) is clear. Item (ii) is called the *trapped surface conjecture* [75]. Essentially, it says that if the reverse inequality as in (35) holds then a trapped surface should enclose  $U$ . That is: if matter is enclosed in a sufficiently small region, then the system should collapse to a black hole. This is related with the *hoop conjecture* [78] (see also [81] [42] [64] ).

The trapped surface conjecture has been proved in spherical symmetry [15] [14] [58] and also for a relevant class of non-spherical initial data [63]. The general case remains open but it is expected that some version of this conjecture should hold.

Concerning item (iii), the area  $A$  is a measure of the size of a trapped surface, hence the inequality (31) represents a version of (34) for axially symmetric black holes. If we include the physical constants, this inequality has the form

$$A \geq 8\pi \frac{G}{c^3} |J|. \quad (36)$$

In fact the inequality (36) was the inspiration for the inequality (34). A possible generalization of (36) for bodies is to take the area  $A(\partial U)$  of the boundary  $\partial U$  of the body  $U$  as measure of size. But unfortunately the area of the boundary is not a good measure of the size of a body in the presence of curvature. In particular, an inequality of the form  $A(\partial U) \geq Gc^{-3}|J(U)|$  does not hold for bodies. The counter example is essentially given by a rotating torus in the weak field limit, with large major radius and small minor radius. The details of this calculation will be presented in [3].

Using the three physical principles (i), (ii) and (iii) in [35] it is argued that the inequality (34) should hold. One of the main difficulties in the study of inequalities of the form (34) is the very definition of the measure of size. In fact, despite the intensive research on the subject, there is no known universal measure of size such that the trapped surface conjecture (or, more general, the hoop conjecture) holds (see the interesting discussions in [64] [49] [76] [47]). However, the remarkable point is that in order to find an appropriate measure of size  $\mathcal{R}$  such that (34) holds it is not necessary to prove first (3), and hence we do not need to find the relevant measure of mass  $m(U)$  for the trapped surface conjecture. In [35] a size measure is proposed and for that measure the following version of the inequality (34) has been proved for constant density bodies. This theorem is a consequence of the Schoen-Yau theorem [73].

**Theorem 4.1** *Consider a maximal, axially symmetric, initial data set that satisfy the dominant energy condition. Let  $U$  be an open set on the data. Assume that the energy density is constant on  $U$ . Then the following inequality holds*

$$\mathcal{R}^2(U) \geq \frac{24}{\pi^3} \frac{G}{c^3} |J(U)|. \quad (37)$$

The definition of the radius  $\mathcal{R}$  in (37) is as follows. Let  $\mathcal{R}_{SY}(U)$  be the Schoen-Yau radius defined in [73]. This radius is expressed in terms of the largest torus that can be embedded in  $U$ . See figure 14.

Consider a region  $U$  with a Killing vector  $\eta^i$  with norm  $\lambda$ , we define the radius  $\mathcal{R}$  by

$$\mathcal{R}(U) = \frac{2}{\pi} \frac{(\int_U \lambda)^{1/2}}{\mathcal{R}_{SY}(U)}. \quad (38)$$

The definition of the radius (38) is, no doubt, very involved. It is not expected to be the optimal size measure for a body. It should be considered, together with

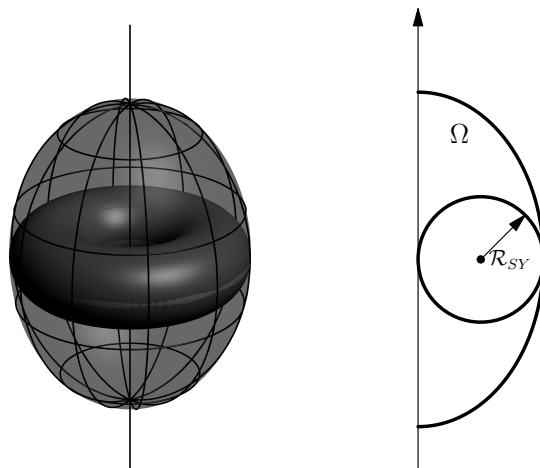


Figure 14: On the left, the Schoen-Yau  $\mathcal{R}_{SY}$  radius for a body is defined in terms of the biggest embedded torus. On the right, the same torus is showed on the plane orthogonal to the axial Killing vector. On that plane the torus is a circle, and the radius  $\mathcal{R}_{SY}$  is related to the radius of the biggest embedded circle.

theorem 4.1, as an example where the conjecture (34) can be proved with the current available mathematical techniques. For examples and further discussion on this radius we refer to [35].

This article is based on the longer review article [34], we refer to that article for more details. The two main differences with respect to [34] are the following. First, several new results appeared after the publication of [34]. These results have been included here. Second, the physical arguments in section 2 have been significantly improved and clarified, based on the discussion in [38].

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