TOWARD AN ALGEBRAIC CLASSIFICATION OF MODULE SPECTRA

J. WOLBERT

ABSTRACT. The category of modules over an S-algebra (A_∞ or E_∞ ring spectrum) has many of the good properties of the category of spectra. When the homotopy groups of the S-algebra in question form a sufficiently nice ring, it is possible to see the deviation of the category of modules over an S-algebra from the corresponding algebraic module category. In particular, many algebraic modules are realized as homotopy groups of topological modules over S-algebras. Examples studied include real and complex K-theory, both connective and periodic. Further, Bousfield localization by a smashing spectrum is shown to yield a category of modules over the localized sphere. For periodic K-theory, these methods yield an algebraic criterion to determine when a local spectrum is a module over the K-theory S-algebra, real or complex.

CONTENTS

Research supported in part by NSF graduate fellowship. Email: wolbert@math.uchicago.edu.

1. Introduction

The stable category of spectra is known to have good algebraic properties; this has been expanded and made more precise by Elmendorf, Kriz, Mandell, and May (EKMM) [12]. Any strict commutative (E_∞) ring spectrum is essentially a commutative S-algebra, for which the appropriate diagrams commute on the point-set level. Here, all S-algebras are assumed to be commutative.

Let R be an S-algebra. An R-module is defined in terms of diagrams that commute on the point-set level. The classical definition of module spectrum requires that the diagrams which define the module merely commute up to homotopy; here, such a spectrum will be called a naive module spectrum.

The category of R-modules has all the standard constructions used in stable homotopy theory, yielding many stable homotopy theories. The sphere spectrum S is one example of an S-algebra, and the classical stable homotopy category is equivalent to the derived category of S-modules. The algebra in other examples often involves (known) rings simpler than $\pi_* S$.

Working in the derived category, $\mathscr{D}_R$, of R-modules (weak equivalences inverted), we will not distinguish between an object or map and its weak homotopy type. The derived category is equivalent to the homotopy category of cell modules and cellular maps. There is a smash product of R-modules, $\wedge_R$, which is analogous to the derived tensor product of R_*-modules. In the context of S-modules, $\wedge$ denotes $\wedge_S$, the smash product over S.

Modules over an S-algebra R can be closely related to algebraic modules over the ring R_* of homotopy groups of R. For example, the derived category of a discrete commutative ring is equivalent to the derived category of modules over the Eilenberg-MacLane spectrum of the ring.

Given an R_*-module M_*, M^* denotes the R^* module obtained by regrading: $M^n = M_{-n}$. Note that, for any naive R-module spectrum or R-module X, $\pi_* X$ has a canonical R_*-module structure. *Realization* of an R_*-module refers to finding an R-module with homotopy isomorphic to the given module.

This paper investigates the algebraic classification of modules in these stable homotopy categories of R-modules. The main classification results follow Bousfield [7, 8].

Theorem. Let R be an S-algebra. Then every R_*-module of projective dimension at most two can be realized as the module of homotopy groups of some R-module. Such an R-module is unique up to homotopy if the R_*-module has projective or injective dimension at most one. When M_* is an R_*-module of projective dimension two, there is an equivalence relation finer than homotopy equivalence so that equivalence classes of R-modules with homotopy M_* are in bijective correspondence with the elements of $\mathrm{Ext}^{2,-1}_{R^*}(M^*, M^*)$.

Ext is relevant since the main tool used to prove this theorem is a spectral sequence

$$E_2^{s,t}(M,N) = \mathrm{Ext}_{R^*}^{s,t}(M^*,N^*) \Longrightarrow [M,N]_R^{s+t}$$

converging from algebraic Ext groups to homotopy classes of R-module maps.

In the dimension one case, the spectral sequence reduces to a short exact sequence describing the homotopy classes of maps of R-modules. Complex periodic K-theory, KU, is an example of an S-algebra for which this is true.

Further, when R_* has global dimension at most two and is sufficiently sparse, the category of R-modules can be described completely algebraically. One example is complex connective K-theory, *ku*. This result was first presented in [24].

For real K-theory, the classification is more complex: the relevant ring has infinite cohomological dimension; however, generalizing the concept of ring allows construction of an algebraic category with dimension two. Note that, as *ko*-modules, $ku \simeq ko \wedge C(\eta)$. We also use *kt*, defined here as $ko \wedge C(\eta^2)$, which is a connective version of self-conjugate K-theory [2, 8].

Let *crt* be the category with objects *ko*, *ku*, and *kt* and maps all maps of *ko*-modules. The category of additive functors from *crt* into the category of abelian groups has many of the properties of a module category: We call this the category of united modules or *crt*-modules. Any *ko*-module X gives a *crt*-module $\pi_*^{crt}(X)$ by smashing with S, $C(\eta)$, and $C(\eta^2)$ respectively, and taking homotopy. When the image of a *crt*-module fits into certain exact sequences, it is called *crt*-acyclic. The exactness condition is motivated by the topology and is necessary for realizability.

Theorem. The category of *crt*-acyclic *crt*-modules has enough projectives and all objects have projective dimension at most two. Any *crt*-acyclic *crt*-module can be realized as $\pi_*^{crt}(X)$ for some *ko*-module X. This *ko*-module is unique up to homotopy if the *crt*-module has projective or injective dimension at most one. For a fixed *crt*-module M of projective dimension two, there is an equivalence relation finer than homotopy equivalence so that equivalence classes of *ko*-modules X with $\pi_*^{crt}(X) = M$ are in bijective correspondence with the elements of $\mathrm{Ext}_{crt}^{2,-1}(M,M)$.

Again, the main tool used to prove the theorem is a spectral sequence

$$E_2^{s,t}(M,N) = \mathrm{Ext}_{ko^*}^{s,t}(M^*,N^*) \Longrightarrow [M,N]_{ko}^{s+t}$$

converging from a generalization of Ext for united modules to homotopy classes of maps of *ko*-modules.

A similar result holds for the periodic theory, which has dimension one and is thus simpler. The periodic version of *crt* is called *CRT*.

The algebraic description of $\mathscr{D}_{KO}$ and $\mathscr{D}_{KU}$ given by the above theorems, together with Bousfield's description of the category of K_*-local spectra in terms of objects called *ACRT*-modules, yields an algebraic criterion for when a K_*-local spectrum can be given the structure of a *KO*- or *KU*-module. The notation for localization with respect to periodic K-theory is not ambiguous, since both KO and KU give the same localization functor.

Let U be the right adjoint to the forgetful functor from $ACRT$-modules to CRT-modules. Note that the complexification map $c\colon KO \longrightarrow KU$ is a map of S-algebras: Any KU-module is a KO-module.

Theorem. Let X be a K_*-local spectrum. Then X is equivalent to a KO-module if and only if $K_*^{CRT}(X) \cong U\pi_*^{CRT}(X)$, where $\pi_*^{CRT}(X)$ is a CRT-acyclic CRT-module. Further, X is a KU-module if, in addition, X_* can be given the structure of a KU_*-module.

The criterion shows that any naive module spectrum over these S-algebras is homotopic to a KO- or KU-module respectively.

Corollary. Any naive module spectrum over KO or KU is homotopic to a KO- or KU-module, respectively. Further, the derived category of naive KO- or KU-module spectra is equivalent to the derived category of KO- or KU-modules.

In fact, Bousfield localization at any smashing spectrum is closely related to topological module categories.

Theorem. Given a smashing R-module E, the derived category $\mathscr{D}_R[E^{-1}]$ of E_*-local R-modules is equivalent to the derived category $\mathscr{D}_{R_E}$ of R_E-modules.

In particular, this holds for $R = S$ and $E = KU$.

Returning to ko, we show that any ko_*-module M_* can be realized as the homotopy of some ko-module.

Theorem. Given any ko_*-module M_*, it is possible to construct a crt-acyclic crt-module with ko part M_*. Thus, by the classification theorem for ko-modules, M_* can be realized as the homotopy of some ko-module.

In fact, we can define a general notion of a *united theory*. The result generalizes to any united theory all of whose acyclic objects have dimension at most two.

Theorem. Let R be an S-algebra with a united theory $R\mathscr{F}$. Then given any R_*-module M_* it is possible to construct an acyclic united module M such that the R-part of M is M_*. Thus, by the classification theorem, when the united theory $R\mathscr{F}$ is of dimension at most two, any R_*-module can be realized as the homotopy of some R-module.

In particular, this can also be done for KO.

In order to use these united theories for calculations, it is useful to understand how to compute Ext in these functor categories. The following is a change of rings theorem.

Let ρ and ρ' denote the left and right adjoints to the forgetful functor from *crt*-modules to $[ku, ku]_*^{ko}$-modules. For a *crt*-module M, M^U denotes the *ku*-part of M; M^O, the *ko*-part.

Theorem. For *crt*-acyclic *crt*-modules L and M with $\eta = 0$ in M^O and $A = [ku, ku]_*^{ko}$, there are natural isomorphisms

$$\rho'(M^U) \cong M \cong \rho(M^U),$$

$$\mathrm{Ext}_{crt}^{s,t}(M, L) \cong \mathrm{Ext}_A^{s,t}(M^U, L^U),$$

$$\mathrm{Ext}_{crt}^{s,t}(L, M) \cong \mathrm{Ext}_A^{s,t}(L^U, M^U).$$

Motivation beyond the desire to reduce all homotopy theory to algebra lies in various places. The focus here is on connective topological K-theory: complex K-theory, for example, for its relation to C^* algebras; real K-theory, for example, toward greater understanding of the corresponding Adams spectral sequence.

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2. Bousfield localization and modules over S-algebras

One method of studying stable homotopy theory is to study the category of spectra through the eyes of a homology theory, smashing all objects with a fixed spectrum. A more sophisticated method is to use Bousfield localization. Bousfield localization at a spectrum E constructs a category in which E-homology isomorphisms are inverted. When Bousfield localization coincides with smashing all objects with the localization of the sphere, E is called smashing.

We recall the basic definitions.

For a spectrum E, a map $f\colon A \longrightarrow B$ which induces an isomorphism $E_*A \longrightarrow E_*B$ is called an E_*-*equivalence*. If $E_*A = 0$, then A is E_*-*acyclic*; X is E_*-*local* if $X^*A = 0$ for any E_*-acyclic A, or equivalently, if any E_* equivalence $f\colon A \longrightarrow B$ induces a bijection $f^*\colon [B, X]_* \longrightarrow [A, X]_*$.

The E_*-localization of X, X_E, is the terminal E_*-equivalence out of X. In fact, each spectrum X can be decomposed naturally into E_*-local and E_*-acyclic spectra X_E and ${}_EX$ via the cofibration

$${}_EX \longrightarrow X \xrightarrow{\eta} X_E \longrightarrow \Sigma({}_EX)$$

in the stable category (of S-modules or equivalently of spectra). The E_*-localization functor is sometimes denoted L_E.

There are analogous definitions and localizations in the category of modules over any S-algebra. Bousfield's methods to construct local spectra also generalize to this context.

Note that any R-module X (or even a naive module spectrum over a naive ring spectrum R more generally) is automatically R_*-local since $R_*Y = 0$ implies $X_*Y = 0 = X^*Y$ since any element of X_*Y factors through

$$S \wedge X \wedge Y \longrightarrow R \wedge X \wedge Y$$

via the module structure map [1].

The Bousfield localization of an S-algebra is again an S-algebra [12]. In particular, this is true of the sphere spectrum. If E_*-localization is equivalent to smashing with the E_*-local sphere, then E is called *smashing*. When E is smashing, the derived category of E_*-local spectra is equivalent to that of S_E-modules.

Theorem 1. If E is smashing, then the derived category of E_*-local spectra is equivalent to the derived category of S_E-modules.

This comparison of categories is a nice observation, but not yet very useful for calculations since S_E is usually quite nasty.

The same holds for localization in the category of R-modules for an arbitrary S-algebra R. For an R-module E, we define

$$E_*^R(X) = [R, E \wedge_R X]_*^R = \pi_*(E \wedge_R X).$$

Now, L_E is replaced by L_E^R, which denotes E_*^R-localization. We denote the localization $L_E^R X$ of an R-module X by X_E. E is *smashing* in $\mathscr{D}_R$ if $X_E \simeq X \wedge_R R_E$.

The notation $\mathscr{D}_R[E^{-1}]$ denotes the category of E_*-local R-modules, which is equivalent to the category $\mathscr{D}_R$ after inverting E_*^R-equivalences.

Theorem 2. Given a smashing R-module E, the derived category $\mathscr{D}_R[E^{-1}]$ of E_*^R-local R-modules is equivalent to the derived category $\mathscr{D}_{R_E}$ of R_E-modules.

Proof. Let E be a smashing R-module; the map $X \cong R \wedge_R X \longrightarrow R_E \wedge_R X$ gives the localization functor, so that any E_*^R-local R-module is homotopic to one of the form $R_E \wedge_R X$. Also, $R_E \wedge_R R_E \longrightarrow R_E$ is an isomorphism since the idempotent functor L_E^R is smashing: Localization gives an inverse to the multiplication map.

Next we show that any R_E-module X has $X \simeq R_E \wedge_R X$ as R_E-modules. First note that $L_E^R = L_{R_E}^R$, that is, R_{E*}-isomorphisms are the same as E_*-isomorphisms: Any R_{E*}-isomorphism has an R_{E*}-acyclic cofiber, but any R_{E*}-acyclic has trivial E_*-localization, so it is E_*-acyclic. Thus, X has $X \simeq R_E \wedge_R X$ as R-modules. The forgetful functor and $R_E \wedge_R (-)$ are adjoint functors between $\mathscr{D}_R$ and $\mathscr{D}_{R_E}$; the counit $R_E \wedge_R X \longrightarrow X$ is by definition a map of R_E-modules, giving the desired weak equivalence on the level of R_E-modules.

Further, for E^R_*-local R-modules $W \simeq R_E \wedge_R X$ and $Z \simeq R_E \wedge_R Y$,

$$\begin{aligned}[W,Z]^R_* &\cong [R_E \wedge_R X, R_E \wedge_R Y]^R_* \cong [X, R_E \wedge_R Y]^R_* \\ &\cong [R_E \wedge_R X, R_E \wedge_R Y]^{R_E}_* \cong [W,Z]^{R_E}_*\end{aligned}$$

where the second isomorphism is by $X \simeq R_E \wedge_R X$ and the third follows from freeness. The isomorphisms are all natural.

Therefore, the derived category of E^R_*-local R-modules is equivalent to that of R_E-modules. □

EKMM [12] has generalized this theorem to describe E_*-local R-modules for arbitrary E: The categories $\mathscr{D}_R[E^{-1}]$ and $\mathscr{D}_{R_E}[(R_E \wedge_R E)^{-1}]$ are equivalent.

Although Bousfield localization does not require E to be a ring spectrum or an S-algebra, it would be interesting to know, for an S-algebra E, how E-modules and S_E-modules are related. The example of $E = KU$ mentioned in the introduction is discussed in the next section.

3. Categories related to the K-theory spectra

The connective complex K-theory spectrum ku is an S-algebra by infinite loop space technology [18]. As shown in [12], KU is an S-algebra by localization in the category of ku-modules; S_K, the K_*-local sphere, is therefore an S-algebra, as noted above.

The K-local sphere, S_K, is closely related to the (periodic) Image of J spectrum. Localized at a prime p, we have the cofibration (where $r = 3$ if $p = 2$ and r generates the units mod p^2 for an odd prime)

$$J_{(p)} \longrightarrow KO \xrightarrow{\psi^r - 1} KO \longrightarrow \Sigma J_{(p)}$$

and the $K_{(p)*}$-local sphere is the homotopy fiber of the map

$$J_{(p)} \longrightarrow \Sigma^{-1} S\mathbb{Q}$$

which is a rational isomorphism on π_{-1}. Thus it is possible to calculate $\pi_* S_K$ [6].

Since localization of S- or ku-modules with respect to KU is smashing, Theorem 1 shows that each local category $\mathscr{D}_S[KU^{-1}]$ and $\mathscr{D}_{ku}[KU^{-1}]$ is a module category; $\mathscr{D}_{ku}[KU^{-1}]$ is actually the derived category of KU-modules.

We have the diagram of categories

$$\begin{tikzcd} \mathscr{D}_S \arrow[r, "L_{KU}"] \arrow[d, "F"'] & \mathscr{D}_S[KU^{-1}] & \approx & \mathscr{D}_{S_K} \arrow[d, "F"] \\ \mathscr{D}_{ku} \arrow[r, "L_{KU}"] & \mathscr{D}_{ku}[KU^{-1}] & \approx & \mathscr{D}_{KU} \end{tikzcd}$$

where L is a localization functor and F is a free functor. For an S-module X, the free functor to ku-modules is given by $ku \wedge_S X$; similarly, the free functor from S_K-modules to KU-modules is given by $KU \wedge_{S_K} X$, which is equivalent to $KU \wedge_S X$, since KU is K_*-local. It is shown below that the categories $\mathscr{D}_S$, $\mathscr{D}_{S_K}$, $\mathscr{D}_{ku}$, and $\mathscr{D}_{KU}$ are all distinct.

We have three categories to compare: K_*-local spectra (i.e., S_K-modules), K_*-local ku-modules, and KU-modules, or K_*^{ku}-local ku-modules. Note that, as one would expect, not all ku-modules are K_*-local: $L_K ku \simeq S_K \wedge_S ku \not\simeq ku$ by direct calculation.

Further, since they are not rationally periodic (see [6] Corollary 4.4), the homotopy groups of $L_K ku$ are not periodic; not all K_*-local ku-modules are KU-modules. This is in sharp contrast to localization in the category of ku-modules, which is simply the free functor from ku-modules to KU-modules.

To see that not all S_K-modules are ku-modules, we note that no element of $\pi_*(S_K)$ of positive degree can be represented by a map of ku-modules. If S_K were a ku-module, then consider any element

$$\alpha \in [S_K, S_K]_q^{S_K} \cong [ku, S_K]_q^{ku} \cong [S, S_K]_q \subseteq [S, S]_q, \qquad q > 0.$$

The isomorphisms are given by freeness; the inclusion is a theorem relying on the relation of S_K to the image of J. The element α would be represented by a map $S_K \longrightarrow S_K$ and also as a map $S \longrightarrow S$. Since α is of positive degree, $\alpha\colon S \longrightarrow S$ gives the zero map $ku \longrightarrow ku$ after smashing with ku; $1_{ku} \wedge \alpha$ is null. Now the commutative diagram

$$\begin{tikzcd} S \wedge S_K \arrow[r, "\eta\wedge 1"] \arrow[dr, "\cong"'] & ku \wedge S_K \arrow[r, "1\wedge\alpha"] \arrow[d, "\text{action}"] & ku \wedge S_K \arrow[d, "\text{action}"] \\ & S_K \arrow[r, "\alpha"] & S_K \end{tikzcd}$$

would guarantee that α itself be null; but S_{K*} has non-trivial elements in positive degree, so S_K cannot be a ku-module.

A similar result involving real K-theory holds as well.

4. A Spectral Sequence for R-modules

All algebraic maps from the homotopy of a free R-module to that of any R-module can be realized as the homotopy of some map of naive module spectra. The theory of R-modules actually allows the realization of many more maps. The method is to use a spectral sequence to show that maps with certain algebraic properties exist.

EKMM [12] exhibit a spectral sequence (EKMSS)

$$E_2^{s,t}(X, Y) = \mathrm{Ext}_{R^*}^{s,t}(X^*, Y^*) \Longrightarrow \pi_{-(s+t)}(F_R(X, Y)),$$

with differentials $d_r^{s,t} : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$. Here, $\mathrm{Ext}^{s,t}_{R^*}(P,Q) = \mathrm{Ext}^{s,0}_{R^*}(\Sigma^t P, Q)$, $(\Sigma^t L) = L_{*-t}$. The filtration on $\pi_*(F_R(X,Y))$, is given by letting $F^s\pi_*(F_R(X,Y))$ be the image of $\pi_*(F_R(X_s,Y)) \longrightarrow \pi_*(F_R(X,Y))$, where the X_s are constructed from a free resolution of X_*. Thus, the (s,t)-th term of the associated bigraded group of the filtration is

$$E_0^{s,t}\pi_*(F_R(X,Y)) = F^s\pi_{-s-t}(F_R(X,Y))/F^{s+1}\pi_{-s-t}(F_R(X,Y)).$$

One can define $\mathrm{Ext}^*_R(X,Y) = \pi_{-*}(F_R(X,Y))$.

Composition pairings are also discussed in [12]. The pairing

$$F_R(Y,Z) \wedge_R F_R(X,Y) \longrightarrow F_R(X,Z)$$

induces a pairing of spectral sequences of differential R_*-modules that coincides with the algebraic Yoneda pairing on the E_2-level and converges to the pairing induced by composition. This is proven by taking free resolutions of X and Y in the contravariant side of each function spectrum:

$$F_R(Y_s,Z) \wedge_R F_R(X_s,Y) \longrightarrow F_R(Y,Z) \wedge_R F_R(X_s,Y) \longrightarrow F_R(X_s,Z).$$

For $a \in \mathrm{Ext}^{s,t}(Y^*,Z^*)$ and $b \in \mathrm{Ext}^{u,v}(X^*,Y^*)$,

$$d_r(ab) = (d_r a)b + (-1)^{t+s}a(d_r b).$$

The EKMSS is always conditionally convergent; with additional information, it is often strongly convergent, for example, when X has a finite length cellular resolution as an R-module.

This spectral sequence is essential to understanding the difference between R-modules and R_*-modules. When an R_*-module—for example, the homotopy of an R-module—is placed in the E_2 term of this spectral sequence, we regrade it as an R^*-module.

5. Realization of projective and injective modules

5.1. Projective modules. Any projective module P over any ring A is the direct summand of a free module $F \cong P \oplus Q \cong Q \oplus P$. Thus, Eilenberg's swindle gives us a two step free resolution of P:

$$0 \longrightarrow P \oplus Q \oplus F^{\oplus\infty} \longrightarrow P \oplus Q \oplus F^{\oplus\infty} \longrightarrow P \longrightarrow 0,$$

where the map between free modules moves each copy of P to the next P-coordinate to the right.

See [17, 20] for a discussion of modules over graded rings.

When A is the ring of homotopy groups R_* for an S-algebra R, any free module is easily realized as the wedge of copies of R (up to suspensions). The EKMSS (Section 4) shows that any map between free modules can be realized in topology; the cofiber of the realization of this map is the realization of our projective module.

Proposition 3. Let R be an S-algebra. Then any projective R_*-module P_* can be realized as the R_*-module of homotopy groups of some R-module X: $\pi_*(X) = X_* = P_*$. For any other R-module Y, $[X, Y]_*^R \cong \mathrm{Hom}_{R^*}(P^*, Y^*)$.

Proof. The vanishing of higher Ext groups shows that any map from a projective module is uniquely realizable. $\square$

5.2. Injective modules. For a Noetherian ring A, the direct sum of injective modules is injective, and we can decompose any injective module as the direct sum of certain indecomposables.

Recall that the injective hull of a module is the minimal injective extension of the module; it is unique up to non-canonical isomorphism.

Proposition 4. ([15, 16, 20]) Let A be a Noetherian ring, $\mathfrak{p}$ a prime ideal in A. Then the injective hull of $A/\mathfrak{p}$ is indecomposable with respect to direct sum. Further, any indecomposable injective A-module is the injective hull of $A/\mathfrak{p}$ for some prime $\mathfrak{p}$.

It is an easy exercise to show that, for any prime $\mathfrak{p}$ in A, the injective hull of $A/\mathfrak{p}$ is the quotient field of $A/\mathfrak{p}$, taken as an A-module. But [12] allows all such constructions on the spectrum level, so long as all primes are generated by regular sequences. Thus, when the primes of A are generated by regular sequences, all injectives can be realized. Note that this is true without restricting the dimension of the Noetherian ring R_*.

Proposition 5. Let R be an S-algebra such that R_* is a Noetherian ring with all primes generated by regular sequences. Then any injective R_*-module I_* can be realized as the R_*-module of homotopy groups of some R-module Y: $\pi_*(Y) = Y_* = I_*$. For any other R-module X, $[X, Y]_*^R \cong \mathrm{Hom}_{R^*}(X^*, I^*)$.

Proof. The vanishing of higher Ext groups shows that any map into an injective module is uniquely realizable. $\square$

6. Realization of modules of dimension at most one

Given an R_*-module M_* of projective dimension one, there is a projective resolution

$$0 \longrightarrow P_1 \xrightarrow{\alpha} P_0 \longrightarrow M_* \longrightarrow 0$$

and M_* can be realized as the homotopy cofiber of the (unique) realization of α. Given an R_*-module M_*, let $|M_*|$ denote an R-module with homotopy M_*. In this case, the realization $|M_*|$ is unique up to homotopy since the identity map on M_* lifts to a comparison of resolutions

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & M_* \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r] & M_* \arrow[r] & 0. \end{tikzcd}$$

Since the two maps from P_1 to P'_0 are the same in algebra, their realizations in topology are the same in homotopy and we get an equivalence between the two realizations of M_*.

When R_* is a Noetherian ring with all primes generated by regular sequences, we can use the dual construction to realize any module of injective dimension at most one.

Note that, without further hypotheses, we can describe the Hom set between two R-modules of dimension at most one only up to extension:

$$0 \longrightarrow \mathrm{Ext}^{1,-1}_{R^*}(X^*, Y^*) \longrightarrow [X, Y]^R_0 \longrightarrow \mathrm{Hom}_{R^*}(X^*, Y^*) \longrightarrow 0.$$

The difference of any two maps $X \longrightarrow Y$ with the same effect in homotopy is measured by an element of $\mathrm{Ext}^{1,-1}_{R^*}(X^*, Y^*)$. Of course, only one module need be of (projective or injective, depending on the variable) dimension at most one.

Composition is given by the naturality of the EKMSS and the fact that the product in Ext corresponds to the composition product on the associated graded of the Hom sets in the derived module spectrum category.

In the case where R_* is of global dimension at most one, this is an almost complete algebraic description of the category, including the corollary that an R-module is determined by its homotopy groups as an R_*-module. One example, mentioned above, is periodic K-theory.

Let R be an S-algebra with R_* of global dimension at most one and concentrated in even degrees. Since realizations of R_*-modules are unique up to homotopy, any R-module X is the wedge of its even and odd pieces: $X = X_{\mathrm{even}} \vee X_{\mathrm{odd}}$. Let Y be another R-module. By additivity, we can assume X and Y are each concentrated in either even or in odd degrees. The group $[X, Y]^R_*$ depends on the relative parities of X and Y. Using the suspension functor, it suffices to calculate $[X, Y]^R_0$.

If X and Y are both even (or both odd), then note that $\mathrm{Ext}^{1,-1}(X^*, Y^*)$ is zero.

If the parities of X and Y differ, then $\mathrm{Hom}_{R^*}(X^*, Y^*) = 0$ and

$$[X, Y]^R_0 \cong \mathrm{Ext}^{1,-1}_{R^*}(X^*, Y^*).$$

More generally, we have the following theorem.

Theorem 6. For an S-algebra R of global dimension at most one with R_* concentrated in degrees congruent to zero mod k, $k > 1$, each R-module X splits as the wedge of k pieces $X = \bigvee_{j=1}^k X_j$ such that X_{j*} is concentrated in degrees congruent to j mod k. For two R-modules X and Y,

$$[X_i, Y_j]^R_0 \cong \begin{cases} \mathrm{Hom}_{R^*}(X^*_i, Y^*_i) & i = j, \\ \mathrm{Ext}^{1,-1}_{R^*}(X^*_i, Y^*_j) & i+1 = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. There are no non-trivial maps between two modules X and Y if X is concentrated in degrees congruent to m mod k, Y in n mod k, $m \neq n$. Similarly, there are no non-trivial extensions

$$0 \longrightarrow Y_* \longrightarrow M \longrightarrow \Sigma^{-1} X_* \longrightarrow 0$$

when $m + 1 \neq n$. $\square$

7. Classification of R-modules of projective dimension at most two

7.1. Realization. The previous sections showed how to realize any R_*-module of projective dimension at most one as the homotopy of an R-module. When the projective dimension of M_* is two, M_* has a projective resolution

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M_* \longrightarrow 0,$$

which can be split into two short exact sequences

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow K \longrightarrow 0,$$
$$0 \longrightarrow K \longrightarrow P_0 \longrightarrow M_* \longrightarrow 0,$$

where $K = \ker\,(P_0 \longrightarrow M_*)$. Using the EKMSS, any R_*-module of projective dimension two can be realized as the homotopy of some R-module: The modules K and P_0, as well as the map whose cokernel is M_*, can be realized as in the above section. One realization of M_* is the cofiber of the realization of the map $K \longrightarrow P_0$.

Now, when R_* has global dimension at most 2, $\mathrm{Ext}^{s,t}_{R^*}(M^*, N^*)$ vanishes for $s > 2$. Thus the only possible non-trivial differentials are

$$d_2 : \mathrm{Ext}^{0,t}_{R^*}(M^*, N^*) \longrightarrow \mathrm{Ext}^{2,t-1}_{R^*}(M^*, N^*),$$

the spectral sequence collapses at E_3, and a map $\theta\colon M_* \longrightarrow N_*$ of R_*-modules yields an obstruction $d_2(\theta) \in \mathrm{Ext}^{2,-1}_{R^*}(M^*, N^*)$ which vanishes if and only if θ can be realized as the effect in homotopy of a map $X \longrightarrow Y$ of R-modules with $\pi_* X \cong M_*$ and $\pi_* Y = N_*$.

7.2. The Difference Between Two Realizations of an R_*-module. A realization of an R_*-module M_* consists of an R-module X and an isomorphism

$$\alpha\colon \pi_*(X) = X_* \xrightarrow{\simeq} M_*.$$

Following Bousfield, two realizations (X, α) and (Y, β) are said to be *strictly equivalent* if there is an equivalence $f : X \simeq Y$ with $\beta f_* = \alpha$. Then $D(\alpha, \beta) = \beta d_2(\beta^{-1}\alpha)\alpha^{-1}$ defines the *difference* in $\mathrm{Ext}^{2,-1}_{R^*}(M^*, M^*)$. This difference satisfies:

1. $D(\alpha, \alpha) = 0$;
2. $D(\alpha, \beta) = 0$ iff α is strictly equivalent to β;
3. $D(\alpha, \gamma) = D(\alpha, \beta) + D(\beta, \gamma)$;
4. $D(\alpha, \beta) = -D(\beta, \alpha)$;
5. $D(g\alpha, g\beta) = gD(\alpha, \beta)g^{-1}$ for each $g \in \mathrm{Aut}\, M_*$.

Call the collection of all strict equivalence classes of realizations of M_* $\mathscr{R}(M_*)$. Adapting the methods of Bousfield [8], $\mathscr{R}(M_*)$ can be determined algebraically.

Theorem 7. For each R_*-module M_* and realization $\alpha\colon X_* \xrightarrow{\sim} M_*$, the difference function gives a bijection $D(\alpha,-)\colon \mathscr{R}(M_*) \xrightarrow{\sim} \mathrm{Ext}^{2,-1}_{R^*}(M^*, M^*)$.

Proof. Injectivity of the difference function follows from the above properties; only surjectivity needs proof. Let

$$0 \longrightarrow K \longrightarrow P_0 \xrightarrow{\varepsilon} M_* \longrightarrow 0$$

be a short exact sequence, where P_0 is projective and K is the kernel of ε. The map ε can be chosen so that X is the cofiber of the realization of the map $K \longrightarrow P_0$. Identify X_* with M_* by the isomorphism α.

Any element $u \in \mathrm{Ext}^{2,-1}_{R^*}(M^*, M^*) = \mathrm{Ext}^{1,-1}_{R^*}(K, M^*)$ lifts to some $\overline{u} \in \mathrm{Ext}^{1,-1}_{R^*}(K, P_0)$, since K has projective dimension at most 1; the EKMSS then yields a (unique) map $\overline{u}\colon \mid K \mid \longrightarrow \mid P_0 \mid$, where $\mid J \mid$ is a realization of the R_*-module J. If the cofibration

$$\Sigma^{-1}X \longrightarrow \mid K \mid \xrightarrow{f} \mid P_0 \mid \longrightarrow X$$

gives the realization α, then

$$\Sigma^{-1}Y \longrightarrow \mid K \mid \xrightarrow{f+\overline{u}} \mid P_0 \mid \longrightarrow Y$$

yields a realization $\beta\colon Y_* \xrightarrow{\sim} M_*$ with $D(\alpha,\beta) = u$. $\square$

This strict equivalence of realizations is a finer equivalence than homotopy type; $\mathrm{Aut}(M_*)$ acts on $\mathscr{R}(M_*)$ by composition. There is, however, a forgetful bijection from the orbit set $\mathscr{R}(M_*)/\mathrm{Aut}(M_*)$ to the set of realizations of M_*.

A *crossed homomorphism* $d\colon G \longrightarrow B$ for a group G acting on an abelian group B is a function with $d(gh) = d(g) + g \cdot d(h)$; the associated crossed homomorphism action $G \times B \longrightarrow B$ carries (g, b) to $d(g) + g \cdot b$. Now, $\mathrm{Aut}(M_*)$ acts on $\mathrm{Ext}^{2,-1}_{R^*}(M^*, M^*)$ by $g \cdot u = gug^{-1}$; given a realization α of M^*, we have a crossed homomorphism $d\colon \mathrm{Aut}(M^*) \longrightarrow \mathrm{Ext}^{2,-1}_{R^*}(M^*, M^*)$ given by $d(g) = D(\alpha, g\alpha)$. The associated crossed homomorphism action of $\mathrm{Aut}(M_*)$ on the Ext group corresponds to the composition action of $\mathrm{Aut}(M_*)$ on $\mathscr{R}(M_*)$ by the bijection of the above theorem.

Thus, we have the following theorem.

Theorem 8. The homotopy types of R-modules with homotopy M_* are in bijective correspondence with the elements of $\mathrm{Ext}^{2,-1}_{R^*}(M^*, M^*)/\mathrm{Aut}(M_*)$.

This classification is *not* purely algebraic, however, since in general we lack an algebraic description of the d_2 differential of the EKMSS. One possible approach toward a more algebraic classification is to use Toda brackets. Alternatively, given a functorial realization of R-modules with zero differential in the EKMSS, the methods of [7] give an easier way to construct a category equivalent to the derived category of R-modules. We describe these next.

7.3. Sparse Graded Rings. Let R be an S-algebra of global dimension at most two with R_* concentrated in degrees congruent to zero mod k, $k \geq 2$. Examples of such S-algebras are ku and $ko_{(p)}$ for an odd prime p; since η is null after inverting 2, $\pi_* ko_{(p)} = \mathbb{Z}[u]$, $|u| = 4$. Any R_*-module M_* is then the homotopy of a wedge $X = \bigvee_{i=1}^{m} X_i$ with the homotopy of X_i concentrated in degrees congruent to i mod m. Call this the *wedge realization* of M_*. By the classification of R-modules, each wedge summand is unique up to homotopy, since any N_* concentrated in degrees congruent to i mod m can be resolved by modules concentrated in the same degrees mod m.

The analysis of [7] applies to this situation; the category of R_*-modules has a completely algebraic structure. The key is to note that there is a functorial realization of any R_*-module as an R-module for which the differentials in the EKMSS between any two such R-modules are all zero: Realize any R_*-modules M_*, N_* as wedges X and Y as above. Since the wedge realization is homotopy unique, there is a right inverse to the map $[X, Y]_0 \longrightarrow \mathrm{Hom}_{R^*}(M^*, N^*)$ given by the natural realizations of maps between the wedge summands of like degree together with the zero map between non-matching summands. Thus, the map of Hom sets is onto, and the differential is necessarily zero.

The next step is to construct a *Bousfield k-invariant* to measure the difference between a given R-module and these realizations with trivial EKMSS differential. Let X be an R-module. Define

$$k_X \in E_2^{2,-1}(X, X) \cong \mathrm{Ext}_{R^*}^{2,-1}(X^*, X^*)$$

as follows: Let X' be an R-module with homotopy isomorphic to X_*, equivalent to the wedge realization above. Choose an isomorphism $\alpha \colon X'_* \longrightarrow X_*$. Define $k_X = (d_2\alpha)\alpha^{-1}$; $\alpha \in E_2(X', X)$, $\alpha^{-1} \in E_2(X, X')$. The element k_X is independent of the choice of X' and α, for if X'' is another choice of X', with β an isomorphism from X'' to X, choose an isomorphism $\gamma \colon X''_* \longrightarrow X_*$ such that $\beta = \alpha\gamma$. Since X'' and X' are equivalent to wedge realizations, $d_2(\gamma) = 0$, with $d_2(\beta)\beta^{-1} = d_2(\alpha)\gamma\gamma^{-1}\alpha^{-1} = d_2(\alpha)\alpha^{-1}$. Thus, k_X is well-defined.

The differential in the EKMSS can now be expressed algebraically.

Proposition 9. Let R be an S-algebra of global dimension at most two with R_* concentrated in degrees congruent to zero mod m, $m \geq 2$. For R-modules X and Y, the EKMSS differential

$$d_2 \colon E_2^{0,t}(X, Y) \longrightarrow E_2^{2,t-1}(X, Y)$$

is given by $d_2 f = k_Y f + (-1)^{t-1} f k_X$ for each $f \in E_2^{0,t}(X, Y)$.

Proof. The proof is exactly as in [7], Proposition 8.10. Let X' denote the wedge realization of X_*. For isomorphisms $\alpha \colon X'_* \longrightarrow X_*$ and $\beta \colon Y'_* \longrightarrow Y_*$, consider α and

β as elements of the E_2 term: $\alpha \in E_2^{0,0}(X', X)$, $\alpha^{-1} \in E_2^{0,0}(X, X')$, $\beta \in E_2^{0,0}(Y', Y)$. Choose $f' \in E_2^{0,t}(X', Y')$ such that $\beta f' \alpha^{-1} = f \in E_2^{0,t}(X, Y)$. Since $d_2 f' = 0$,

$$\begin{aligned} d_2 f &= d_2(\beta f' \alpha^{-1}) = (d_2\beta) f' \alpha^{-1} + (-1)^t \beta f' (d_2 \alpha^{-1}) \\ &= (d_2\beta)\beta^{-1}\beta f' \alpha^{-1} + (-1)^t \beta f' \alpha^{-1} \alpha (d_2\alpha^{-1}) \\ &= k_Y f + (-1)^t f \alpha (d_2 \alpha^{-1}). \end{aligned}$$

Now,

$$\alpha(d_2\alpha^{-1}) = d_2(\alpha\alpha^{-1}) - (d_2\alpha)\alpha^{-1} = d_2(1) - k_X = -k_X$$

and this proves the result. □

This allows us to prove the following characterization of maps in the R-module category.

Corollary 10. For R-modules X and Y, a homomorphism f of degree t from X_* to Y_* is the homotopy of a map of R-modules if and only if $k_Y f = (-1)^t f k_X$.

Now let $\mathbb{M}\mathrm{k}$ denote the category of pairs (M_*, k), with M_* an R_*-module and k in $\mathrm{Ext}^{2,-1}_{R^*}(M^*, M^*)$; morphisms f from (M_*, k) to (N_*, k') satisfy $k'f = fk$ in $\mathrm{Ext}^{2,-1}_{R^*}(M^*, N^*)$.

Theorem 11. Let R be an S-algebra of global dimension at most two, with R concentrated in degrees congruent to zero mod m, $m \geq 2$. Then for any (M_*, k) in $\mathbb{M}\mathrm{k}$, there is an R-module Y such that $(M_*, k) = (Y_*, k_Y)$. Thus, the homotopy types of R-modules correspond to isomorphism types in $\mathbb{M}\mathrm{k}$.

Proof. This is essentially the same as [7], Theorem 9.1. We already have a full additive functor from R-modules to $\mathbb{M}\mathrm{k}$, as noted above. The wedge realization provides a realization of any R_*-module with zero k-invariant. That any k-invariant can be obtained can be seen by lifting Ext elements, as in the proof of the more general classification above. Again, let X' denote the wedge realization of X_*. Given (M_*, k) in $\mathbb{M}\mathrm{k}$, let

$$0 \longrightarrow P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon} M_* \longrightarrow 0$$

be a projective resolution of M_*. Let $K = \ker \varepsilon$; the exact sequence splits into two short exact sequences

$$0 \longrightarrow P_2 \xrightarrow{d_1} P_1 \xrightarrow{a} K \longrightarrow 0, \qquad 0 \longrightarrow K \xrightarrow{b} P_0 \xrightarrow{\varepsilon} M_* \longrightarrow 0.$$

Given any $\xi : K' \to P_0'$ such that $\xi_* = 0$, the homotopy cofiber of $b + \xi$ gives a realization Y_ξ of M_*. It suffices to choose ξ so that $k_{Y_\xi} = k$; Y_ξ is then the desired realization of (M_*, k).

Any element $u \in \mathrm{Ext}^{2,-1}_{R^*}(M^*, M^*) = \mathrm{Ext}^{1,-1}_{R^*}(K, M^*)$ lifts to some $\overline{u} \in \mathrm{Ext}^{1,-1}_{R^*}(K, P_0)$, the group which classifies maps from $|K|$ to $|P_0|$ which are zero on homotopy groups. Any such $\overline{u}$ is equivalent to a map ξ as above, since K has projective dimension at most one. If b is the map which gives the wedge realization Y' of M_*, then $b + \overline{u}$ yields the desired Y_ξ. □

It is possible to construct additive bigraded categories of R-modules and of R_*-modules paired with k-invariants. These categories are additively equivalent, as in [7].

Note that this analysis requires only a natural realization of R_*-modules as R-modules with zero differential in the EKMSS, which is a weaker condition than sparseness.

8. Modules over additive categories

In order to use algebra to classify modules over the real K-theory S-algebras *ko* and *KO*, it is necessary to generalize the algebraic concept of a module to that of a functor over an appropriate generalization of a ring.

A ring can be considered [19] as a suitably structured category with only one object. Much homological algebra generalizes to "modules" over a small additive category $\mathscr{C}$, specifically, additive functors from $\mathscr{C}$ to abelian groups; the morphisms are the natural transformations. One can consider the Hom sets in $\mathscr{C}$ as giving additive operations on the modules. The category $\mathscr{Ab}^{\mathscr{C}}$ of additive functors from $\mathscr{C}$ to $\mathscr{Ab}$ is actually equivalent to a category of modules over an honest ring; we will, however, keep our additive category while also considering our functors naively as sets of modules with operations.

Since our functor category is abelian, we have kernels and cokernels and can define exact sequences and resolutions. Although the definition of Ext is well-known for any abelian category, this section outlines the exposition of [19] in order to have an explicit description of the derived functors.

As an example, a graded ring $R = \{R_n\}_{n\in\mathbb{Z}}$ is an additive category $\mathscr{C}$ with object set $\mathbb{Z}$ and morphisms $\mathscr{C}(m,n) = R_{n-m}$; composition is given by multiplication in the ring. Note that it only makes sense to add elements of the same degree, as is standard for topologists. Then, $\mathscr{Ab}^{\mathscr{C}}$ is the category of graded R-modules.

8.1. The Yoneda lemma. Given a functor $F\colon \mathscr{C} \longrightarrow \mathscr{Ab}$, and an object X of $\mathscr{C}$, the isomorphism

$$\mathrm{Nat}(\mathscr{C}(X,-), F) \cong FX$$

sends each natural transformation to the image of the identity map on X. This isomorphism is natural in X and F. In terms of $\mathscr{Ab}^{\mathscr{C}}$, we have the isomorphism $\mathscr{Ab}^{\mathscr{C}}(\mathscr{C}(X,-), F(-)) \cong F(X)$.

8.2. Projective and injective functors. An object (i.e. functor) F of $\mathscr{Ab}^{\mathscr{C}}$ is called *projective* if $\mathscr{Ab}^{\mathscr{C}}(F,-)$ preserves epis. This is equivalent to requiring that, for a projective F and any epi natural transformation $A \twoheadrightarrow B$, any natural transformation

$F \longrightarrow B$ lifts to one $F \longrightarrow A$; thus, $\mathrm{Nat}(F, A) \twoheadrightarrow \mathrm{Nat}(F, B)$. Dually, G is *injective* if $\mathscr{Ab}^{\mathscr{C}}(-, G)$ preserves monics.

8.3. Free objects in $\mathscr{Ab}^{\mathscr{C}}$. For X in $\mathscr{C}$, M in $\mathscr{Ab}^{\mathscr{C}}$, consider M as the disjoint union of the various $M^X = M(X)$; an element x of M^X is then considered to be an element of M. For an index set I, $\{x_i\}_{i\in I} \subseteq M$ is a *family of generators* for M if for every $x \in M$ there are objects X_i of $\mathscr{C}$ and $\lambda_i \in \mathscr{C}(X_i, X)$, where only finitely many λ_i are non-zero, such that

$$x = \sum_{i\in I} M^{\lambda_i}(x_i).$$

That is, any element of an abelian group in the image of the functor M is the sum of the images of the x_i under maps which are in the image of M. Equivalently, the natural transformation

$$\bigoplus_{i\in I} \mathscr{C}(X_i, -) \longrightarrow M$$

taking 1_{X_i} to x_i should be an epi in $\mathscr{Ab}^{\mathscr{C}}$; when this natural transformation is iso, the family $\{x_i\}$ is called a *basis* for M. This is true exactly when the λ_i above are unique. In this case, M is called *free*. Thus, the free objects in $\mathscr{Ab}^{\mathscr{C}}$ are exactly the sums of representable functors from $\mathscr{C}$ to $\mathscr{Ab}$. When the objects of $\mathscr{C}$ are graded, then all sums of suspensions of representable functors are free.

8.4. A Hom functor. For an additive category $\mathscr{C}$, we have the *symbolic Hom functor*

$$\mathrm{Hom}_{\mathscr{C}} \colon (\mathscr{Ab}^{\mathscr{C}})^{op} \otimes \mathscr{Ab}^{\mathscr{C}} \longrightarrow \mathscr{Ab}.$$

This is, up to natural equivalence, the unique limit preserving functor satisfying

$$\mathrm{Hom}_{\mathscr{C}}(\mathscr{C}(X, -), F(-)) = F(X) = \mathscr{Ab}^{\mathscr{C}}(\mathscr{C}(X, -), F(-)),$$

the last isomorphism by the Yoneda lemma. The Kan extension theorem [19] then gives that $\mathrm{Hom}_{\mathscr{C}}(G, F) = \mathscr{Ab}^{\mathscr{C}}(G, F)$ for any F and G.

This Hom functor has a left adjoint

$$\otimes_{\mathscr{C}} \colon \mathscr{Ab}^{\mathscr{C}} \otimes \mathscr{Ab}^{\mathscr{C}^{op}} \longrightarrow \mathscr{Ab}$$

which satisfies

$$F \otimes_{\mathscr{C}} \mathscr{C}(-, X) = F(X).$$

That is, $(-) \otimes_{\mathscr{C}} G$ is a left adjoint to $\mathrm{Hom}_{\mathscr{C}}(G, -)$.

8.5. Derived functors. For F and G in the functor category $\mathscr{A}b^{\mathscr{C}}$, define

$$\mathrm{Ext}^n_{\mathscr{C}}(G, F) = H^n(\mathrm{Hom}_{\mathscr{C}}(X, F))$$

where X is a projective resolution of G. The standard properties of Ext hold, and since coproducts in $\mathscr{A}b$ preserve monics, Ext can be computed by simultaneously resolving both variables, with F resolved by an acyclic right complex.

Similarly, $\mathrm{Tor}^{\mathscr{C}}_n(F, G) = H_n(F \otimes_{\mathscr{C}} Y)$, where Y is a projective resolution of G in $\mathscr{A}b^{\mathscr{C}^{op}}$, gives the derived functors of $\otimes_{\mathscr{C}}$.

9. United homology theories

Bousfield, in [8, 9], defines an additive category CRT (which he calls $\mathrm{Alg}(CRT)$) and the corresponding category of CRT-modules. Bousfield's method generalizes to other united homology theories such as the one below, as well as to united cohomology theories.

9.1. United homology theories. Let R be an S-algebra and $\mathscr{F}$ some collection of finite spectra. Define the category $R\mathscr{F}$ to be the category with objects

$$\{R \wedge F_\alpha | F_\alpha \in \mathscr{F}\}$$

and morphisms all homotopy classes of R-module maps between any pair of objects. $R\mathscr{F}$ is an additive category which yields two united homology theories: First, let $M = R^{\mathscr{F}}_*(X)$ denote the united theory on spectra X given by the functors from $R\mathscr{F}$ to abelian groups with

$$M^F = M(R \wedge F) = \pi_*(R \wedge F \wedge X).$$

Second, there is a united theory on R-modules X given by the functors $N = \pi^{\mathscr{F}}_*(X)$ from $R\mathscr{F}$ to abelian groups with

$$N^F = N(R \wedge F) = \pi_*(X \wedge_R (R \wedge F)) \cong \pi_*(X \wedge F).$$

This has a naive variant. If R is merely a ring spectrum, then the maps in $R\mathscr{F}$ should be all maps in the derived category of naive module spectra. The theory $R^{\mathscr{F}}_*(X)$ is as defined above; $N = \pi^{\mathscr{F}}_*(X)$ is defined by $N^F = \pi_*(X \wedge F)$.

The first type of united theory is used by Bousfield [8] to classify K_*-local spectra. The second type is used below to classify both KO-modules and ko-modules.

Since $[R \wedge F, R \wedge F']^R_* \cong [S, R \wedge F' \wedge DF]_*$, the Spanier-Whitehead duals of the finite spectra in $\mathscr{F}$ yield the representable functors (free objects). Thus the free objects under such a theory are given by the united homology of all suspensions of the Spanier-Whitehead duals of the finite spectra in $\mathscr{F}$. Note that the objects of $R\mathscr{F}$ need only be semi-finite as R-modules when R is an S-algebra, that is, $D^2_R F \simeq F$ for F in $R\mathscr{F}$.

A united module, or $R\mathscr{F}$-module, M is called *$\mathscr{F}$-acyclic* if it takes cofibrations of R-modules in $R\mathscr{F}$ to long exact sequences.

9.2. Connective united K-theory. This is described in more detail in the next section. The obstacle to using the same techniques as for *ku* to classify *ko*-modules is precisely that ko_* has infinite global dimension. By killing nilpotent elements, we obtain a theory $R\mathscr{F} = crt$ with global dimension 2 and it suffices to construct an appropriate spectral sequence in order to obtain the desired classification. An appropriate choice of finite spectra is S, $C(\eta)$, and $C(\eta^2)$, yielding *ko*, *ku*, and *kt*.

One can check that using only S and $C(\eta)$ over *ko* yields an algebraic category with infinite homological dimension; adding $C(\eta^2)$, or *kt*, eliminates this difficulty.

9.3. Periodic united K-theory. The inspiration for studying connective united K-theory came from Bousfield's use of a periodic version, *CRT*, in [8]; the categories *crt* and *CRT* use the same finite spectra S, $C(\eta)$, and $C(\eta^2)$. The periodic theory has global dimension one. Bousfield adjusts the category of modules by adding Adams operations to classify K_*-local spectra. The united category without operations classifies *KO*-modules.

10. The operation algebra for connective united K-theory

10.1. Anderson cofibrations with spheres. To calculate the algebra of operations, we use the cofibration sequences of [2, 8] (η denotes the Hopf map)

$$\Sigma S \xrightarrow{\eta} S \xrightarrow{c} C(\eta) \xrightarrow{r} \Sigma^2 S,$$

$$\Sigma^2 S \xrightarrow{\eta^2} S \xrightarrow{\varepsilon} C(\eta^2) \xrightarrow{k} \Sigma^3 S,$$

and

$$C(\eta^2) \xrightarrow{h} C(\eta) \xrightarrow{\varphi} \Sigma^2 C(\eta) \xrightarrow{j} \Sigma C(\eta^2),$$

which, when smashed with *ko*, give cofibration sequences relating *ko*, *ku*, and *kt*.

First, describe and fix names for maps among S, $C(\eta)$, and $C(\eta^2)$:

- $\eta\colon S^1 \longrightarrow S$ is the Hopf map, the non-zero element of $\pi_1 S$;
- $c\colon S \longrightarrow C(\eta)$, $\varepsilon\colon S \longrightarrow C(\eta^2)$ are given by inclusion of the zero cell;
- $h\colon C(\eta^2) \longrightarrow C(\eta)$ is any map of degree one on the zero cells;
- $j\colon C(\eta) \longrightarrow \Sigma^{-1}C(\eta^2)$ is degree one on the 2-cells;
- $k\colon \Sigma^{-1}C(\eta^2) \longrightarrow S^2$ is degree one on the 2-cells;
- $r\colon C(\eta) \longrightarrow S^2$ is top cell projection;
- $\psi_1 = \psi\colon C(\eta) \longrightarrow C(\eta)$ and $\psi_2 = \psi\colon C(\eta^2) \longrightarrow C(\eta^2)$ are degree one on 0-cells and degree -1 on top cells;
- $\varphi\colon C(\eta) \longrightarrow \Sigma^2 C(\eta)$ kills the zero cell and sends the 2-cell to the bottom cell of $\Sigma^2 C(\eta)$ by a degree one map, as $C(\eta) \xrightarrow{1-\psi} S^2 \xrightarrow{\varepsilon} \Sigma^2 C(\eta)$.

Note that the definitions of h and r imply that the diagrams

$$\begin{tikzcd} S \arrow[r, "\varepsilon"] \arrow[dr, "c"'] & C(\eta^2) \arrow[d, "h"] \\ & C(\eta) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} C(\eta) \arrow[r, "j"] \arrow[dr, "r"'] & \Sigma^{-1}C(\eta^2) \arrow[d, "k"] \\ & S^2 \end{tikzcd}$$

commute.

10.2. Anderson cofibrations of *ko*-modules. The maps above induce, upon smashing with the identity on ko,

- $\eta\colon \Sigma ko \longrightarrow ko$;
- $c\colon ko \longrightarrow ku$ and $\varepsilon\colon ko \longrightarrow kt$, unit maps;
- $\zeta\colon kt \longrightarrow ku$, with $c = \zeta\varepsilon$;
- $\gamma\colon ku \longrightarrow \Sigma^{-1}kt$, from j;
- $\tau\colon \Sigma^{-1}kt \longrightarrow \Sigma^2 ko$, from k;
- $r\colon ku \longrightarrow \Sigma^2 ko$;
- $\psi_U = \psi\colon ku \longrightarrow ku$ and $\psi_T = \psi\colon kt \longrightarrow kt$;
- $\varphi\colon ku \longrightarrow \Sigma^2 ku$.

Note that some of the degrees of maps differ from those of the same name in Bousfield's periodic version [8]; the difference comes from the lack of an inverse to the Bott element. To simplify the number of generating operations, note that $r = \tau\gamma$ and $c = \zeta\varepsilon$. The map c is complexification; r is realification.

The following cofibrations are obtained:

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{r} \Sigma^2 ko,$$

$$\Sigma^2 ko \xrightarrow{\eta^2} ko \xrightarrow{\varepsilon} kt \xrightarrow{\Sigma\tau} \Sigma^3 ko,$$

$$kt \xrightarrow{\zeta} ku \xrightarrow{\varphi} \Sigma^2 ku \xrightarrow{\Sigma^2\gamma} \Sigma kt.$$

Here, ku is merely shorthand for $ko \wedge C(\eta)$ and similarly for kt, however it is possible to prove that these are equivalent as ko-modules to the usual ku and kt by the classification below (Theorem 17). There are other proofs of this fact as well, for example, using the homology of ko and ku [Bruner, personal correspondence].

10.3. Determining the operations. Recall the ring structures of ko_*, ku_*, and kt_*:

$$ko_* = \mathbb{Z}[\eta, \omega, \beta_O]/(2\eta, \eta^3, \eta\omega, \omega^2 - 4\beta_O),\ |\eta| = 1,\ |\omega| = 4,\ |\beta_O| = 8;$$

$$ku_* = \mathbb{Z}[\beta_U],\ |\beta_U| = 2; \quad \text{and}$$

$$kt_* = \mathbb{Z}[\eta, \xi, \beta_T]/(2\eta, \eta^2, \eta\xi, \xi^2),\ |\eta| = 1,\ |\xi| = 3,\ |\beta_T| = 4.$$

In the world of ko-modules, $X_* = [ko, X]_*^{ko}$, so these rings give all maps from ko to each of the three objects. Explicitly, for $n \geq 0$ (other groups are all zero),

$$
\begin{array}{ll}
[ko, ko]_{8n}^{ko} = \langle \beta_O^n \rangle = \mathbb{Z} & [ko, ko]_{8n+1}^{ko} = \langle \beta_O^n \eta \rangle = \mathbb{Z}/2 \\
[ko, ko]_{8n+2}^{ko} = \langle \beta_O^n \eta^2 \rangle = \mathbb{Z}/2 & [ko, ko]_{8n+4}^{ko} = \langle \beta_O^n \omega \rangle = \mathbb{Z} \\
[ko, ko]_{8n+i}^{ko} = 0, \ i \in \{3, 5, 6, 7\} & \\
[ko, ku]_{2n}^{ko} = \langle \beta_U^n \rangle = \mathbb{Z} & [ko, ku]_{2n+1}^{ko} = 0 \\
[ko, kt]_{4n}^{ko} = \langle \beta_T^n \rangle = \mathbb{Z} & [ko, kt]_{4n+1}^{ko} = \langle \beta_T^n \eta \rangle = \mathbb{Z}/2 \\
[ko, kt]_{4n+2}^{ko} = 0 & [ko, kt]_{4n+3}^{ko} = \langle \beta_T^n \xi \rangle = \mathbb{Z}
\end{array}
$$

Except where noted, the maps given below are proven to be generators by looking at the action on homotopy groups.

Since $ku_*(\eta) = 0$ and the duals of $C(\eta)$ and $C(\eta^2)$ are $DC(\eta) = \Sigma^{-2}C(\eta)$ and $DC(\eta^2) = \Sigma^{-3}C(\eta^2)$, the cofibrations in 10.2 give

$$
[ku, ku]_n^{ko} = ku_{n+2}(C(\eta)) = ku_{n+2} \oplus ku_n = \begin{cases} \mathbb{Z} = \langle \varphi \rangle & n = -2, \\ \mathbb{Z} \oplus \mathbb{Z} = \langle \beta_U^k, \beta_U^k \psi \rangle & n = 2k, \ k \geq 0, \\ 0 & \text{otherwise.} \end{cases}
$$

Similarly, since $kt_{n+2}(C(\eta)) \cong ku_{n+2}(C(\eta^2)) = ku_{n+2} \oplus ku_{n-1}$

$$
[ku, kt]_n^{ko} \cong \begin{cases} \mathbb{Z} = \langle \varepsilon r \beta_U^k \rangle & n = 2(k-1), \ k \geq 0, \\ \mathbb{Z} = \langle \gamma \beta_U^k \rangle & n = 2k+1, \ k \geq 0, \\ 0 & \text{otherwise;} \end{cases}
$$

$$
[kt, ku]_n^{ko} = ku_{n+3}(C(\eta^2)) = \begin{cases} \mathbb{Z} = \langle \beta_U^k \zeta \varepsilon \tau \rangle & n = 2k-3, \ k \geq 0, \\ \mathbb{Z} = \langle \beta_U^k \zeta \rangle & n = 2k, \ k \geq 0, \\ 0 & \text{otherwise;} \end{cases}
$$

and since $kt_*(\eta^2) = 0$, $kt_{n+3}(C(\eta^2)) = kt_{n+3} \oplus kt_n$,

$$
[kt, kt]_n^{ko} \cong \begin{cases} \mathbb{Z} = \langle \varepsilon \tau \rangle & n = -3, \\ \mathbb{Z}/2 = \langle \beta_T^k \eta \varepsilon \tau \rangle & n = 4k-2, \ k \geq 0, \\ \mathbb{Z} \oplus \mathbb{Z} = \langle \beta_T^k, \beta_T^k \psi_T \rangle & n = 4k, \ k \geq 0, \\ \mathbb{Z} \oplus \mathbb{Z}/2 = \langle \beta_T^{k+1} \varepsilon \tau, \beta_T^k \eta \rangle & n = 4k+1, \ k \geq 0, \\ \mathbb{Z} = \langle \beta_T^k \xi \rangle & n = 4k+3, \ k \geq 0, \\ 0 & \text{otherwise.} \end{cases}
$$

For $[kt, kt]^{ko}_{4k-2}$, note that it suffices to prove that $\eta\varepsilon\tau$ is essential. But this is clear: replace S-cells in $\eta\varepsilon k\colon C(\eta^2) \longrightarrow \Sigma^2 C(\eta^2)$ with ko-cells; one sees the composition is not null.

10.4. Relations between operations. Note that $\omega = \tau\beta_T\gamma\beta_U\zeta\varepsilon$ and $\xi = \gamma\beta_U\zeta$, further reducing our generating maps.

The effect of the various maps from ko on homotopy is clear from the above ring structure; for example, $\eta f = f\eta$ for any map f (when the composition makes sense).

Composing these maps with those in 10.2, we obtain the following relations:

$$
\begin{array}{lll}
\psi_U\psi_U = 1, & \psi_T\psi_T = 1, & \psi_T\varepsilon = \varepsilon, \\
\psi_U\beta_U = -\beta_U\psi_U, & \psi_T\beta_T = \beta_T\psi_T, & \psi_T\xi = -\xi, \\
\psi_T\eta = \eta, & \psi_U\zeta = \zeta\psi_T = \zeta, & \beta_U\varphi = 1 - \psi_U \\
\psi_T\gamma = -\gamma\psi_U = \gamma, & \tau\psi_T = -\tau, & \varepsilon\beta_O = \beta_T^2\varepsilon, \\
\zeta\beta_T = \beta_U^2\zeta, & \beta_T\gamma = \gamma\beta_U^2, & \tau\beta_T^2 = \beta_O\tau, \\
\tau\varepsilon = \eta, & \tau\beta_T\varepsilon = 0, & \zeta\gamma = 0, \\
\gamma\beta_U\zeta = \eta, & \beta_T\varepsilon\tau = \varepsilon\tau\beta_T + \eta\beta_T, & \zeta\varepsilon\tau\gamma\beta_U = 1 + \psi_U, \\
\varepsilon\tau\gamma\beta_U\zeta = 1 + \psi_T, & \tau\gamma\beta_U\zeta\varepsilon = 2, & \gamma\beta_U\zeta\varepsilon\tau = 1 - \psi_T,
\end{array}
$$

with additional relations involving other elements of ko_*, ku_*, and kt_*. These relations are more than sufficient, however, to prove the desired results. The proof that the *crt*-module category has projective dimension two relies on a reduction to the complex part of the module, regarded as a module over ku_*. Knowing some of the relations in the operation algebra is useful, but it is not necessary to know all of them to use the united theory.

10.5. Operations on united modules. Rephrasing the terminology of functors in terms of modules, the algebra of operations on a triple $M = \{M^O_*, M^U_*, M^T_*\}$ of $\mathbb{Z}$-graded abelian groups which form a module over connective united K-theory is generated by homomorphisms

$$
\begin{array}{lll}
\beta_O\colon M^O_* \longrightarrow M^O_{*+8}, & \beta_U\colon M^U_* \longrightarrow M^U_{*+2}, & \beta_T\colon M^T_* \longrightarrow M^T_{*+4}, \\
\eta\colon M^O_* \longrightarrow M^O_{*+1}, & \psi_U\colon M^U_* \xrightarrow{\simeq} M^U_*, & \psi_T\colon M^T_* \xrightarrow{\simeq} M^T_*, \\
\eta\colon M^T_* \longrightarrow M^T_{*+1}, & \varepsilon\colon M^O_* \longrightarrow M^T_*, & \zeta\colon M^T_* \longrightarrow M^U_*, \\
\gamma\colon M^U_* \longrightarrow M^T_{*+1}, & \tau\colon M^T_* \longrightarrow M^O_{*-3}, & \varphi\colon M^U_* \longrightarrow M^U_{*-2},
\end{array}
$$

satisfying appropriate relations as in 10.4.

Note that unlike *CRT*, objects in *crt* need not have periodicity; β_O, β_U, and β_T need not be isomorphisms.

Define ΣM by $(\Sigma M)^X_* = \Sigma(M^X) = M^X_{*-1}$ for X one of O, U, or T. A *crt*-module M is *crt*-acyclic if its operations give rise to long exact sequences analogous to those of 10.2

$$M^O_{n-1} \xrightarrow{\eta} M^O_n \xrightarrow{c} M^U_n \xrightarrow{r} M^O_{n-2},$$

$$M^O_{n-2} \xrightarrow{\eta^2} M^O_n \xrightarrow{\varepsilon} M^T_n \xrightarrow{\Sigma\tau} M^O_{n-3},$$

and

$$M^T_n \xrightarrow{\zeta} M^U_n \xrightarrow{\varphi} M^U_{n-2} \xrightarrow{\Sigma^2\gamma} M^T_{n-1}.$$

Note that $\pi^{crt}_* X$ is *crt*-acyclic for any *ko*-module X. A *CRT*-module is *CRT*-acyclic if the same sequences are exact (see [8]).

11. A spectral sequence for united homology

The object of this section is to construct a spectral sequence analogous to that of [12]. Let R be an S-algebra; $R\mathscr{F}$, a united homology theory over R; M and N, R-modules. This spectral sequence

$$E_2 = \mathrm{Ext}^{s,t}_{\mathscr{F}}(\pi^{\mathscr{F}}_*(M), \pi^{\mathscr{F}}_*(N)) \Longrightarrow E^{s,t}_0 \pi_*(F_R(M,N))$$

converges from Ext over the united theory to the R_*-module of homotopy classes of R-module maps from M to N. Notation, in fact most of the exposition, follows theirs.

When the united theory is small, the algebraic category is abelian, so that derived functors such as Ext are well known. We present an explicit description here which is useful for understanding the realization of united modules and maps between them.

11.1. Realizing Hom-sets of free objects. As noted above, the monogenic free objects for united $\mathscr{F}$-theory are suspensions of $\pi^{\mathscr{F}}_*(R \wedge DF)$ for $F \in \mathscr{F}$, where D denotes the Spanier-Whitehead dual. Now, the functor

$$\mathrm{Hom}_{\mathscr{F}} \colon (\mathscr{A}b^{\mathscr{F}})^{op} \otimes \mathscr{A}b^{\mathscr{F}} \longrightarrow \mathscr{A}b$$

satisfies the Yoneda relation $\mathrm{Hom}_{\mathscr{F}}(\mathscr{F}(R \wedge F, -), M(-)) = M^F$. Thus, for any functor M realizing the united $\mathscr{F}$-theory of an R-module X,

$$\mathrm{Hom}_{\mathscr{F}}(R_*(-), M(-)) = M^S = X_* = \pi_* F_R(R, X), \text{ and}$$
$$\mathrm{Hom}_{\mathscr{F}}((R \wedge DF)_*(-), M(-)) = M^F = \pi_*(X \wedge F) = \pi_* F_R(R \wedge DF, X).$$

It is thus clear that, for an R-module X and a representable functor M, any element of the Hom set $\mathrm{Hom}_{\mathscr{F}}(M, \pi^{\mathscr{F}}_*(X))$ is realizable as a map of R-modules. The $R \wedge DF$ become the analogues of spheres in a new cellular theory for R-modules.

11.2. Constructing the spectral sequence. Given a (cell) R-module M, choose a free resolution of $\pi_*^{\mathscr{F}} M$ by $\mathscr{F}$-modules

$$\cdots \longrightarrow F_s \xrightarrow{d_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{\epsilon} \pi_*^{\mathscr{F}} M \longrightarrow 0.$$

Set $Q_0 = \ker \epsilon$, $Q_s = \ker d_s$ for $s \geq 1$. Let K_s denote the wedge of (de)suspensions of $F_\alpha \in \mathscr{F}$ so that $FK_s = R \wedge K_s$ yields F_s under $\pi_*^{\mathscr{F}}$. Set $M_0 = M$. Inductively construct cofibrations of R-modules

$$FK_s \xrightarrow{k_s} M_s \xrightarrow{i_s} M_{s+1} \xrightarrow{j_{s+1}} \Sigma FK_s$$

for $s \geq 0$ such that

(1) k_0 realizes ϵ on $\pi_*^{\mathscr{F}}$,
(2) $\pi_*^{\mathscr{F}} M_s = \Sigma^s Q_{s-1}$, $s \geq 1$,
(3) k_s realizes $\Sigma^s d_s \colon \Sigma^s F_s \longrightarrow \Sigma^s Q_{s-1}$ on $\pi_*^{\mathscr{F}}$ for $s \geq 1$ (thus $\pi_*^{\mathscr{F}} k_s$ is surjective),
(4) i_s induces the zero homomorphism on $\pi_*^{\mathscr{F}}$ for $s \geq 0$,
(5) j_{s+1} realizes the inclusion $\Sigma^{s+1} Q_s \longrightarrow \Sigma^{s+1} F_s$ on $\pi_*^{\mathscr{F}}$ for $s \geq 0$.

Note that (iii) gives (iv), (v), and the $s+1$ case of (ii).

Obtain an exact couple by defining

$$D_1^{s,t} := \pi_{-s-t}(F_R(M_s, N)), \quad E_1^{s,t} := \pi_{-s-t}(F_R(FK_s, N)).$$

The cofibrations obtained inductively above induce maps

$$\begin{aligned} i := (i_s)^* \colon D_1^{s,t} &\longrightarrow D_1^{s-1,t+1} \\ j := (k_s)^* \colon D_1^{s,t} &\longrightarrow E_1^{s,t} \\ k := (j_{s+1})^* \colon E_1^{s,t} &\longrightarrow D_1^{s+1,t}. \end{aligned}$$

Since $E_1^{s,t} = \pi_{-s-t}(F_R(FK_s, N)) \cong \mathrm{Hom}_{\mathscr{F}}{}^t(F_s, \pi_*^{\mathscr{F}} N)$ and $d_1 = jk = \mathrm{Hom}_{\mathscr{F}}(d, 1)$,

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{F}}^{s,t}(\pi_*^{\mathscr{F}} M, \pi_*^{\mathscr{F}} N).$$

Note that the edge homomorphism is induced by the map $FK_0 \longrightarrow M$.

All that remains is to prove appropriate convergence.

11.3. Determining convergence. Let $i^{0,s} \colon F_R(M_s, N) \longrightarrow F_R(M, N)$ be the map induced by the iterate $M \longrightarrow M_s$; filter $\pi_*(F_R(M, N))$ by

$$F^s(\pi_*(F_R(M, N))) := \mathrm{Image}((i^{0,s})_* \colon \pi_*(F_R(M_s, N)) \longrightarrow \pi_*(F_R(M, N)).$$

Then the (s,t)-th term of the associated bigraded group of $\pi_*(F_R(M, N))$ with this filtration is given by

$$E_0^{s,t} \pi_*(F_R(M, N)) = F^s \pi_{-s-t}(F_R(M, N)) / F^{s+1} \pi_{-s-t}(F_R(M, N)).$$

The group $E_\infty^{s,t}$ is the subquotient of $E_1^{s,t}$ given by $Z_\infty^{s,t}/B_\infty^{s,t}$, with $B_\infty^{s,t} = j(\ker(i^{0,s})_*)$. The definition of the spectral sequence implies that the additive relation $(i^{0,s})_* \circ j^{-1}$ induces an isomorphism

$$E_\infty^{s,t} \cong E_0^{s,t}\pi_*(F_R(M,N)).$$

Note that (iv) in 11.2 above implies Tel M_s is trivial, so the homotopy limit

$$\operatorname{holim} F_R(M_s, N) \simeq F_R(\operatorname{Tel} M_s, N)$$

is also trivial. The Lim^1 exact sequence for computing $\pi_*(\operatorname{holim} F_R(M_s,N))$ then gives that

$$\operatorname{Lim} \pi_*(F_R(M_s,N)) = 0, \text{ and } \operatorname{Lim}^1 \pi_*(F_R(M_s,N)) = 0.$$

Thus, the spectral sequence is conditionally convergent in Boardman's sense [5]. When the category of $\mathscr{F}$-modules has finite projective dimension, the united homology spectral sequence converges strongly. For example, the category of *crt*-acyclic *crt*-modules has projective dimension two; the united *crt*-homology spectral sequence converges strongly.

Exactly the same words used in [12] yield the naturality of this spectral sequence.

11.4. A Yoneda pairing. Consider the pairing

$$F_R(M,N) \wedge_R F_R(L,M) \longrightarrow F_R(L,N).$$

The maps $M \longrightarrow M_s$ induce compatible pairings

$$F_R(M_s,N) \wedge_R F_R(L_r,M) \longrightarrow F_R(M,N) \wedge_R F_R(L_r,M) \longrightarrow F_R(L_r,N),$$

where the L_r are constructed from L in the same manner as the M_s are constructed from M. As usual, these pairings induce the required pairing of spectral sequences, giving a Yoneda product on Ext at E_2 and the product induced by composition at E_∞.

11.5. Naive module spectra. Since much of this theory relies only on free objects and extended module spectra, and in particular since the definition of united homology theory only requires a ring spectrum and not an S-algebra, one might hope for this spectral sequence to generalize to that wider setting. This, however, is unlikely: the point-set level category is necessary to construct the free resolutions of the R-modules; since the homotopy cofiber or fiber of a map of naive module spectra cannot be given an obvious module structure, we cannot make resolutions in the naive module spectrum category.

11.6. A dual spectral sequence. For an S-algebra R, a united theory $R\mathscr{F}$ has a dual category $DR\mathscr{F}$ found by taking the Spanier-Whitehead dual in the category of R-modules of every object and morphism in $R\mathscr{F}$. Note that this satisfies all the properties of $R\mathscr{F}^{op}$: $DR\mathscr{F}$ is equivalent to the opposite category of $R\mathscr{F}$. Thus, we can understand the tensor product as a functor

$$\otimes_{\mathscr{F}} : \mathscr{A}b^{R\mathscr{F}} \otimes \mathscr{A}b^{DR\mathscr{F}} \longrightarrow \mathscr{A}b$$

with derived functor $\mathrm{Tor}(M, N)$ defined on a pair of functors M covariant on $R\mathscr{F}$ and N covariant on $DR\mathscr{F}$.

Let $M_*\mathscr{F}$ denote $\pi_*^{\mathscr{F}}(M)$ and $N^*\mathscr{F}$, the cohomological version obtained from the modules $[R \wedge F, N]$, $F \in \mathscr{F}$. Thus, $N^*\mathscr{F} \cong N_*D\mathscr{F}$.

To construct the spectral sequence, let M be a right (cell) R-module, N a left (cell) R-module. The exact couple defined by

$$D^1_{s,t} := \pi_{s+t+1}(M_{s+1} \wedge_R N), \quad E^1_{s,t} := \pi_{s+t}(FK_s \wedge_R N)$$

yields a spectral sequence with $E^2_{s,t} = \mathrm{Tor}^{\mathscr{F}}_{s,t}(M_*\mathscr{F}, N^*\mathscr{F})$, converging to $E^\infty_{s,t} = E_0^{s,t}\pi_*(M \wedge_R N)$. The filtration on $\pi_*(M \wedge_R N)$ is given by the resolution of M.

Alternately, one can resolve in the N variable.

The E^2 term can be identified by noting that $E^1_{s,t} \cong F_s \otimes_{\mathscr{F}} N$ by construction. Convergence and functoriality are proven as for the Ext spectral sequence. The Tor spectral sequence always converges strongly: see [12].

12. Classification of modules over real connective K-theory

For an S-algebra R with homotopy groups forming a Noetherian ring of dimension at most two, the R-modules are classified by their homotopy groups and a difference in Ext, as seen above. For other ring spectra, the same method can be used once an algebraic category with sufficient structure and global dimension at most two is built [9]. As noted above, additional free objects provided by a united homology theory can lower the dimension of the algebraic category.

This classification is carried out here for *ko*.

12.1. Homological dimension of *crt*-modules. The following theorems rely on work of Bousfield [8] regarding abelian groups with involution, and essentially parallel his similar theorems for the periodic case. In the category of *crt*-modules, all projectives are free and all *crt*-acyclic objects have projective dimension at most two.

Theorem 12. Given a *crt*-module M, the following are equivalent:

1. M has projective dimension (as a *crt*-module) at most 2,
2. M has finite projective dimension, and
3. M is *crt*-acyclic.

This theorem relies on the following determining property of projectives.

Theorem 13. Given a *crt*-module M, the following are equivalent:

(1) M is projective (as a *crt*-module),
(2) M is *crt*-acyclic and M^U is free over $\mathbb{Z}[\beta_U] = [ko, ku]_*^{ko}$, and
(3) M is free.

Proof of Theorem 12. Since η is nilpotent, equivalences under *ko*-, *ku*-, and *kt*-homology are all the same, and the projective dimension of a *crt*-module is bounded by the ku_*-projective dimension of its U-part. Thus, Theorem 12 follows from 13 and the fact that whenever two *crt*-modules in a short exact sequence are *crt*-acyclic, the third is as well. □

Proof of Theorem 13. (i) $\Longrightarrow$(ii): Given M projective, there is another projective P such that $M \oplus P \cong F$, where F is free, hence the π_*^{crt} of a wedge of suspensions of *ko*, *ku*, and *kt*. Since free objects are all *crt*-acyclic, so are their direct summands. Now, M^U is a direct summand of F^U, hence projective over $ku_* \subseteq [ku, ku]_*^{ko}$ since F^U is. If M^U is finitely generated over ku_*, then it is free. If not, note that ku_* is commutative and Noetherian and has no non-zero idempotents; by [3] this implies that any non-finitely generated projective is also free.

(ii) $\Longrightarrow$(iii): Given M satisfying (2), we want to decompose M as the direct sum of free objects. By Proposition 3.7 of [8] (Lemma 14 below), each M_n^U decomposes as

$$M_n^U = G_n \oplus \psi G_n \oplus i^+ H_n \oplus i^- I_n$$

for G_n, H_n, I_n free abelian, where ψ interchanges summands on $G \oplus \psi G$, $\psi = 1$ on $i^+ G$, and $\psi = -1$ on $i^- G$.

For a of degree n, let $FU(a)$ denote the monogenic free *crt*-module generated by a isomorphic to $\Sigma^n \pi_*^{crt}(Dku)$. Similarly, $FO \cong \pi_*^{crt}(Dko)$ and $FT \cong \pi_*^{crt}(Dkt)$.

First, we find a direct summand of M which is isomorphic to the direct sum of copies of FU. Select elements $\{a_{\alpha,n}\}$ with $a_{\alpha,n} = a_\alpha$ in $i^+ H_n \subseteq M_n^U$, a_α not in the image of β_U, and such that $\{a_{\alpha,n}\}$ is a $\mathbb{Z}$-basis for the set of elements x in $i^+ H_n$ with $\beta_U x$ in $G_{n+2} \oplus \psi G_{n+2}$. That such a basis is possible (disjoint from image β_U) is shown in Lemma 15. Let K be the kernel of the map

$$\bigoplus_\alpha \Sigma^n FU(a_{\alpha,n}) \longrightarrow M.$$

$K^U = 0$, since each element a generates a copy of $\mathbb{Z}[\beta_U]a \oplus \Sigma^2 \mathbb{Z}[\beta_U]b$ in M^U with ψ-action given by $\psi^2 b = \psi(\beta_U a - b) = b$, isomorphic to the ψ-action on FU. Since K is a submodule of a direct sum of copies of FU, $\eta K^O = 0$, which gives $K^O = 0$. Thus, $K = 0$ and the map is monic. In fact (see Lemma 15), the free module surjects onto a direct summand of M^U, so the cokernel N also satisfies (2). Further, $N^U = (N^U)^+ \oplus (N^U)^-$ as graded abelian groups, where

$$G^+ = \{x \in G \colon \psi x = x\}$$

and

$$G^- = \{x \in G \colon \psi x = -x\}.$$

Next, we find a direct summand of N which is the direct sum of copies of FO.

Note that $\eta^2 = r\beta_U^2 c$ by the relations in 10.2 and 10.5. Thus, η^2 factors as the composition

$$N_n^O \xrightarrow{c} (N_n^U)^+ \xrightarrow{r\beta^2} r\beta_U^2(N_n^U)^+ \subseteq N_{n+2}^O.$$

The identities $cr\beta_U = 1+\psi$ and $\beta_U\psi = -\psi\beta_U$ give $cr\beta_U^2(N_n^U)^+ = (1+\psi)(N_{n+2}^U)^- = 0$, so $r\beta_U^2(N_n^U)^+$ is a $\mathbb{Z}/2$-module (i.e., is contained in ηN^O).

Choose a $\mathbb{Z}$-basis $\{b_\beta\} \cup \{c_\gamma\} \cup \{d_\delta\} \cup \{e_\epsilon\}$ for N^U by extending a basis for the image of β_U to the whole of N^U such that the b's project to a basis for $r\beta_U^2(N^U)^+/\eta^2 N^O$, the c's project to a basis for $\eta^2 N^O$, the d's are trivial in $r\beta_U^2(N^U)^+$, and the e's are in $(N^U)^-$.

For any $c_{\gamma,n}$ not in the image of β_U, define $m_{\gamma,n}$ as follows: Since $2(N^U)^+ \subseteq c(N^O)$ and $\eta^2 = r\beta_U^2 c$, there is an $m_{\gamma,n} \in N_n^O$ with $c(m_\gamma) - c_\gamma \in \operatorname{Span}\{d_\delta\}_\delta$, that is, $\eta^2 m_\gamma = r\beta_U^2 c_\gamma$. Then $c_{\gamma,n}$ can be replaced by $c(m_{\gamma,n})$. The construction of the m_γ implies that

$$\bigoplus_\gamma \Sigma^n FO(m_{\gamma,n}) \longrightarrow N$$

is monic with crt-acyclic cokernel P such that P^U is free over $\mathbb{Z}[\beta_U]$, $P^U = (P^U)^+ \oplus (P^U)^-$ as graded abelian groups, and $\eta^2 P^O = 0$. The map is monic since the m_γ hit elements which are non-zero under η^2; everything else is forced by $\beta_O \neq 0$ on N^O, since N^U is free over $\mathbb{Z}[\beta_U]$. Also, the map from the free module generated by the m_γ hits generators $c(m_\gamma)$ in a basis for N^U, so the cokernel has U-part free over $\mathbb{Z}[\beta_U]$. Since $\eta^2 N^O \subseteq r\beta_U^2(N^U)^+$, all of $\eta^2 N^O$ is hit; $\eta^2 P^O = 0$.

Lastly, we see that P is a direct sum of copies of FT by using the exact sequence

$$\cdots \longrightarrow P_{*-1}^U \xrightarrow{\gamma} P_*^T \xrightarrow{\zeta} P_*^U \xrightarrow{\varphi} P_{*-2}^U \xrightarrow{\gamma} P_{*-1}^T \longrightarrow \cdots.$$

Since β_U is injective on P^U, $0 = 1 - \psi_U = \beta_U\varphi$ implies $\varphi = 0$. So, since P^U is free over $\mathbb{Z}[\beta_U]$, the exact sequence

$$0 \longrightarrow (P_{*-1}^U)^+ \oplus (P_{*-1}^U)^-/2 \xrightarrow{\gamma} P_*^T \xrightarrow{\zeta} (P_*^U)^+ \longrightarrow 0$$

reduces mod torsion to

$$0 \longrightarrow (P_{*-1}^U)^+ \xrightarrow{\gamma} P_*^T/\mathrm{tors} \xrightarrow{\zeta} (P_*^U)^+ \longrightarrow 0,$$

with $\gamma(P_{*-1}^U)^+ = (P_*^T/\mathrm{tors})^-$ (to see this, look at the relations between ψ_T, ψ_U, and γ). The exact sequence for η^2 gives the short exact sequence

$$0 \longrightarrow P_*^O/\mathrm{tors} \xrightarrow{\varepsilon} P_*^T/\mathrm{tors} \xrightarrow{\tau} P_{*-3}^O/\mathrm{tors} \longrightarrow 0,$$

since

$$\mathrm{tors}(P_{*-3}^O) = \ker(c \colon P_{*-3}^O \longrightarrow P_{*-3}^U) = \eta P_{*-4}^O$$

and

$$\mathrm{tors}(P_*^T) = \gamma(P_{*-1}^U)^- = \gamma\zeta P_{*-1}^T = \eta P_{*-1}^T$$

by the above short exact sequence and the effect of ζ. Thus, since $\psi_T\varepsilon = \varepsilon$, $\varepsilon(P_*^O/\,\mathrm{tors}) = (P_*^T/\,\mathrm{tors})^+$.

Note that $P_*^T/\,\mathrm{tors}$ is free abelian in each degree, so that we have decompositions

$$P_n^T/\,\mathrm{tors} = (G_n \oplus \psi G_n) \oplus i^+H_n \oplus i^-I_n.$$

Since any torsion is killed by η, there is an epimorphism

$$P_n^T \twoheadrightarrow \eta\varepsilon\tau\beta_T(P_n^T).$$

Now, $\eta\varepsilon\tau\beta_T\varepsilon = \eta\varepsilon\eta = 0$, so η kills $\varepsilon(P_*^O/\,\mathrm{tors}) = (P_*^T/\,\mathrm{tors})$; η also kills $\gamma_*(\beta_U^{2n}) = \beta_T^{n-1}\xi$ and $\gamma_*(\beta_U^{2n-1}) = \beta_T^n\eta$, hence $\eta\varepsilon\tau\beta_T\gamma = 0$ and $\eta\gamma(P_{*-1}^U)^+ = \eta(P_*^T/\,\mathrm{tors})^- = 0$. Lastly, $\tau\psi = -\tau$ and $\psi\beta_T = \beta_T\psi$ imply that $\eta\varepsilon\tau\beta_T\psi = \eta\varepsilon\tau\beta_T$. Thus, the restriction to G_n is a surjection $G_n \twoheadrightarrow \eta\varepsilon\tau\beta_T(P_n^T)$.

Choose a $\mathbb{Z}$-basis $\{r_\rho\} \cup \{s_\sigma\} \cup \{t_\tau\}$ of P_*^T such that $\{r_\rho\} \cup \{s_\sigma\}$ gives a basis for $\eta\varepsilon\tau\beta_T P^T$, extending a basis $\{s_\sigma\}$ for the sum of the images of β_T and ξ, and satisfying $\{t_\tau\} \longrightarrow 0$ in $\eta\varepsilon\tau\beta_T P^T$. Lift each element $r_{\rho,n}$ to $\tilde{r}_\rho$ in P_n^T, giving a monic map

$$\bigoplus_\rho \Sigma^n FT(\tilde{r}_{\rho,n}) \longrightarrow P$$

with *crt*-acyclic cokernel Q; again, Q^U is free over $\mathbb{Z}[\beta_U]$ (use the short exact sequence relating $(P_*^U)^+$ and $P_*^T/\,\mathrm{tors}$), $Q^U = (Q^U)^+ \oplus (Q^U)^-$, $\eta^2(Q^O) = 0$ (true for P, FT), and further, $\eta\varepsilon\tau(Q^T) = 0$, since any element in the image of $\eta\varepsilon\tau$ is in the image of the FT's. Note that there is no need to kill the s_σ; they are hit by linear combinations of the $\beta_T^?\xi^? r_\rho$'s.

Lemma 16, the analogue of [8] Proposition 3.11 for *crt*-acyclics, then gives that $Q = 0$, so M is a free *crt*-module. $\square$

12.2. Technical lemmas used in the proof of Theorem 13. First we define the abelian category *Inv* of involution modules, as in [8], to be the category of $\mathbb{Z}\{1,\psi\}$ modules where $\psi^2 = 1$; this is the category of abelian groups with involution. For an abelian group G, $G \oplus \psi G$ denotes the involution module in which ψ interchanges summands; i^+G denotes G with involution $\psi = 1$; i^-G has involution $\psi = -1$. For an involution module M, $M^+ = \{x \in M | \psi x = x\}$ and $M^- = \{x \in M | \psi x = -x\}$.

Lemma 14. Any object M of *Inv* which is free as an abelian group is isomorphic to $(G \oplus \psi G) \oplus i^+H \oplus i^-I$ for free abelian groups G, H, and I.

Proof. This is Proposition 3.7 of [8]. $\square$

Lemma 15. Given a *crt*-module M such that M^U is free over $\mathbb{Z}[\beta_U]$, with decomposition as abelian groups

$$M_n^U = G_n \oplus \psi G_n \oplus i^+ H_n \oplus i^- I_n,$$

any element in $G_n \oplus \psi G_n$ must be in a submodule generated under ψ and β_U by a generator of a copy of $ku_*(C(\eta))$ taken as a $[ku, ku]_*^{ko}$-module. Such a generator is necessarily in $i^+ H_n$ for some n. Further, the image of $ku_*(C(\eta))$ in M^U is a direct summand over $\mathbb{Z}[\beta_U, \psi]$.

Proof of Lemma 15. We want to prove that any element of $G_n \oplus \psi G_n$ must be generated under ψ and β_U by a generator of a copy of $ku_*(C(\eta))$.

Let $\mathbb{Z}[\beta_U]x$ be a $\mathbb{Z}[\beta_U]$-free submodule and direct summand of M^U generated by x and closed under ψ. Note that the united operation φ requires $\psi x = x$, so that x can be in the kernel of φ.

For a module with $M^U = \mathbb{Z}[\beta_U]x \oplus \mathbb{Z}[\beta_U]y$ as $\mathbb{Z}[\beta_U]$-modules and indecomposable over $\mathbb{Z}[\beta_U, \psi]$, x and y must have different degrees of the same parity, since any group $\mathbb{Z}e_1 \oplus \psi\mathbb{Z}e_2$ has an element $e_1 - e_2$ in the image of β_U. Let $|y| > |x|$ and suppose $|x| = 0$, so $|y| \geq 2$. To rule out $|y| \geq 4$, use the η exact sequence and $\eta^3 = 0$ to show that M^O is (-2)-connected, since M^U is (-1)-connected. Then use the exact sequence with φ to see that M_{-1}^T includes torsion $\mathbb{Z}/2$ (coker φ); the ε exact sequence shows then that $M_{-1}^O = \mathbb{Z}/2$. The η exact sequence now forces $M_{-1}^U = \mathbb{Z}/2$, a contradiction.

This covers all possible cases, since $\psi^2 = 1$ and $\psi\beta_U = -\beta_U\psi$. $\square$

Let $A = [ku, ku]_*^{ko}$. For any *crt*-module M, M^U is an A-module. Call an A-module *φ-acyclic* if the complex

$$\cdots \longrightarrow M_{*+2}^U \xrightarrow{\varphi} M_*^U \xrightarrow{\varphi} M_{*-2}^U \longrightarrow \cdots$$

given by $\varphi \in A_{-2}$ is exact.

Lemma 16. Let M be a *crt*-acyclic *crt*-module. The following are equivalent:

(1) M^U is φ-acyclic,
(2) $\eta = 0$ in M^O, and
(3) $\eta^2 = 0$ in M^O and $\eta\varepsilon\tau = 0$ in M^T.

Proof. This is the *crt* analogue of Proposition 3.11 of [8].

(i) implies (ii): Note that $\varphi = cr$. Now, an element $x \in M^O$ is in the image of r if and only if $\eta x = 0$. Also, for any $x \in M^O$, $\varphi(cx) = 0$; by φ-acyclicity, there is an element $y \in \varphi^{-1}(cx)$. Suppose x is not in the image of r. Then $ry \neq x$, but $cry = \varphi y = cx$, so $x - ry = \eta z$ ($\ker c = \operatorname{im} \eta$). Since $ry \neq x$, $\eta z \neq 0$ and z is not in the image of r. Thus, we can do for $z = x_1$ what we just did for $x = x_0$, obtaining a sequence $x_n \in M^O$ such that $\eta x_n \neq 0$ and $x_{n-1} - \eta x_n$ is in the image of r. Since

$\eta r = 0$, $\eta x_0 = \eta^2 x_1 = \eta^3 x_2 = 0$, so we must have had x in the image of r (kernel of η) after all: $\eta = 0$ in M^O.

(ii) implies (i): If $\eta = 0$, then

$$\begin{aligned}
\ker \varphi &= \ker(cr) = \ker(r) \cup r^{-1}(\ker(c) \cap \operatorname{im}(r)) \\
&= \ker(r) \cup r^{-1}(\operatorname{im}(\eta) \cap \ker(\eta)) = \ker(r) \\
&= \operatorname{im}(c) = c(\ker(\eta)) = c(\operatorname{im}(r)) = \operatorname{im}(cr) = \operatorname{im}(\varphi).
\end{aligned}$$

Thus, M^U is φ-acyclic.

(ii) implies (iii): Now, $\eta = 0$ in M^O implies $\eta^2 = 0$ in M^O. Since the composition

$$M^T_* \xrightarrow{\tau} M^O_{*-3} \xrightarrow{\varepsilon} M^T_{*-3} \xrightarrow{\eta} M^T_{*-2}$$

equals the composition

$$M^T_* \xrightarrow{\tau} M^O_{*-3} \xrightarrow{\eta} M^O_{*-2} \xrightarrow{\varepsilon} M^T_{*-2},$$

we also get $\eta\varepsilon\tau = 0$ in M^T.

(iii) implies (ii): If $\eta^2 = 0$ in M^O, then

$$0 \longrightarrow M^O_* \xrightarrow{\varepsilon} M^T_* \xrightarrow{\tau} M^O_{*-3} \longrightarrow 0$$

is exact, but $\eta\varepsilon\tau = 0$ in M^T implies $\eta\tau = 0$

$$M^T_* \overset{\tau}{\twoheadrightarrow} M^O_{*-3} \xrightarrow{\eta} M^O_{*-2} \overset{\varepsilon}{\hookrightarrow} M^T_{*-2}$$

and τ (which surjects onto M^O) maps into the kernel of η, that is, $\eta = 0$ in M^O. $\square$

12.3. Classifying *ko*-modules. As in the dimension two case for rings above (Theorem 7), all *crt*-modules can be realized as π^{crt}_* of some *ko*-module, again using projective resolutions.

The analysis is completed after verifying that the classification of *ko*-modules with the same $\pi^{crt}_*\ M$ is given by a quotient of $\mathrm{Ext}^{2,-1}_{crt}(M, M)$, exactly as in [8] and in Theorem 7.

Theorem 17. The category of *crt*-acyclic *crt*-modules has enough projectives and all objects have projective dimension at most two. Any *crt*-acyclic *crt*-module can be realized as $\pi^{crt}_*(X)$ for some *ko*-module X. This *ko*-module is unique up to homotopy if the *crt*-module has projective or injective dimension at most one. For a fixed *crt*-module M of projective dimension two, there is an equivalence relation finer than homotopy equivalence so that equivalence classes of *ko*-modules X with $\pi^{crt}_*(X) = M$ are in bijective correspondence with the elements of $\mathrm{Ext}^{2,-1}_{crt}(M, M)$.

13. Modules over periodic K-theory and K_*-local spectra

A simpler classification results when $\mathscr{F}$-modules have $\mathscr{F}$-projective dimension at most one. One example is Bousfield's *CRT*-theory [8], noted above. This classification yields an algebraic criterion for when a K_*-local spectrum can be given the structure of a *KO*-module or *KU*-module.

13.1. Classifying *KO*-modules. Since Bousfield proves that all *CRT*-acyclic *CRT*-modules have projective dimension at most one and that all projectives are free (hence easily realized), we obtain a classification of *CRT*-modules using the united *CRT*-homology spectral sequence. A *KO*-module M is determined by $\pi_*^{CRT}(M)$ and the group of *KO*-module maps between two *KO*-modules M and N is given by the short exact sequence

$$0 \to \mathrm{Ext}^{1,-1}_{CRT}(\pi_*^{CRT}(M), \pi_*^{CRT}(N)) \to [M,N]_0^{KO} \to \mathrm{Hom}_{CRT}(\pi_*^{CRT}(M), \pi_*^{CRT}(N)) \to 0.$$

Note that we have now classified R-modules where R is real or complex K-theory, either connective or periodic.

13.2. Local spectra with module structures. If X is a K_*-local spectrum (or S-module), it would be good to know when $X \simeq M$ as spectra for some *KO*- or *KU*-module M. Note that X_* for a spectrum X has only the structure of an S_*-module; here, we need consider it only as a graded abelian group.

A *CRT*-module enriched with Adams operations is called an *ACRT*-module [8]. These *ACRT*-modules classify K_*-local spectra by taking $K_*^{CRT}(X) \cong \pi_*^{CRT}(KO \wedge X)$ as *CRT*-modules for X K_*-local, together with Adams operations induced by those on *KO*.

Note that $\pi_*^{CRT}(X)$ is not in general a *CRT*-module, though the acyclicity condition always holds. If X has the structure of a naive *KO*-module spectrum, however, then $\pi_*^{CRT}(X)$ is a *CRT*-module.

Let U be the right adjoint to the forgetful functor from *ACRT*-modules to *CRT*-modules. Note that the complexification map $c\colon KO \longrightarrow KU$ is a map of S-algebras: Any *KU*-module is a *KO*-module.

Theorem 18. Let X be a K_*-local spectrum or S-module. Then X is equivalent to a *KO*-module if and only if $K_*^{CRT}(X) \cong U\pi_*^{CRT}(X)$ as *ACRT*-modules, where $\pi_*^{CRT}(X)$ is a *CRT*-acyclic *CRT*-module. Further, X is a *KU*-module if and only if, in addition, X_* can be given the structure of a KU_*-module.

Proof. Necessity is obvious.

If $\pi_*^{CRT}(X)$ is *CRT*-acyclic, then by the classification theorem for *KO*-modules, there is a *KO*-module Y, unique up to homotopy, such that $\pi_*^{CRT}(X) \cong \pi_*^{CRT}(Y)$.

For X and Y to have the same homotopy type as spectra, it suffices to check their K_*-local type, since both are K_*-local. The question is now answered by the analysis in [8] relating the categories of CRT-modules and CRT-modules with Adams operations, or $ACRT$-modules.

Bousfield constructs a right adjoint U to the forgetful functor from $ACRT$-modules to CRT-modules. Since U is a right adjoint, it preserves injectives; in fact, this is how he shows that the category of $ACRT$-modules has enough injectives. Further, given a naive KO-module spectrum Y, $K_*^{CRT}(Y) \cong U\pi_*^{CRT}(Y)$ as $ACRT$-modules. Thus, X is a KO-module if $K_*^{CRT}(X) \cong U\pi_*^{CRT}(Y)$. In particular, although the category of $ACRT$-modules has global injective dimension two, since U preserves injectives, our prospective KO-module X must have $K_*^{CRT}(X)$ with injective dimension at most one as an $ACRT$-module.

In order for a K_*-local spectrum X to be a KU-module, X must be a KO-module by neglect of structure and X_* must have a KU_*-module structure; in fact, since the Bott element must be an isomorphism, any KU_*-module structures on X_* are isomorphic. Let Y_* denote X_* with a KU_*-module structure. Y_* has an injective resolution

$$0 \longrightarrow Y_* \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$$

of KU_*-modules, which we realize (homotopy uniquely) as a cofibration

$$Y \longrightarrow |I_0| \longrightarrow |I_1| \longrightarrow \Sigma Y$$

of KU-modules; $\pi_*(Y) \cong Y_*$.

By neglect of structure, Y is a KO-module, hence X and Y have the same homotopy type if they satify the condition $K_*^{CRT}(X) \cong U\pi_*^{CRT}(Y)$. In this case, X is homotopy equivalent to the KU-module Y. $\square$

Note that this shows readily that any naive KU-module spectrum is equivalent to a KU-module; thus, any naive KU-module spectrum can be given the structure of a KU-module. Further, since any map of naive module spectra is a map between modules of dimension at most one, it can be realized as a strict map. Thus, the homotopy category of naive module spectra over KU (weak equivalences inverted) is equivalent to the derived category of KU-modules.

Corollary 19. The derived category of KU-modules is equivalent to the derived category of naive KU-module spectra. The same is true for KO.

This answers a question of Mark Hovey [personal communication] of when a K_*-local spectrum has the structure of a KU-module spectrum.

14. Realizing modules of dimension higher than two

Given an S-algebra R with a united theory $R\mathscr{F}$ (the *crt* and CRT theories, for example), and given any R_*-module M_* it is possible to construct an $R\mathscr{F}$-module with M_* as its value at $R \wedge S$ (9.1). For the sake of clarity, the exposition here will focus on the example of *ko*.

Theorem 20. Given any ko_*-module M_*, it is possible to construct a *crt*-acyclic *crt*-module with *ko* part M_*. Thus, by the classification theorem for *ko*-modules, M_* can be realized as the homotopy of some *ko*-module.

The general case is stated at the end of this section. The method of proof will be clear from the example of *ko*; essentially, it is to build a complex of copies of $\pi_*^{crt}(ko)$ from a free resolution of the module, then to see that the homology of the complex is concentrated in degree zero and is our desired *crt*-module.

14.1. The construction. Let M_* denote a (graded) ko_*-module. Take a free resolution $F_\bullet^O \longrightarrow M_* \longrightarrow 0$ of M_*; each F_n^O is a free, graded ko_*-module. Let F_n^U be a free graded ku_*-module on generators corresponding to those of F_n^O. Now construct a bicomplex

$$\begin{tikzcd}
\cdots \arrow[r] & \Sigma F_n^O \arrow[r, "\eta"] \arrow[d, "d_n"'] & F_n^O \arrow[r, "c"] \arrow[d, "d_n"'] & F_n^U \arrow[r, "r"] \arrow[d, "d_n^U"'] & \Sigma^2 F_n^O \arrow[r] \arrow[d, "d_n"'] & \cdots \\
\cdots \arrow[r] & \Sigma F_{n-1}^O \arrow[r, "\eta"] \arrow[d, "d_{n-1}"'] & F_{n-1}^O \arrow[r, "c"] \arrow[d, "d_{n-1}"'] & F_{n-1}^U \arrow[r, "r"] \arrow[d, "d_{n-1}^U"'] & \Sigma^2 F_{n-1}^O \arrow[r] \arrow[d, "d_{n-1}"'] & \cdots \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \Sigma F_0^O \arrow[r, "\eta"] & F_0^O \arrow[r, "c"] & F_0^U \arrow[r, "r"] & \Sigma^2 F_0^O \arrow[r] & \cdots
\end{tikzcd}$$

where each X_k has an additional internal grading. Suspension affects only this internal grading: $\Sigma X_{k,*} = X_{k,*-1}$. The maps η, c, r are induced by the maps

$$\Sigma ko_* \xrightarrow{\eta} ko_* \xrightarrow{c} ku_* \xrightarrow{r} \Sigma^2 ko_*;$$

and the differentials of $F_\bullet^U$ are given by the following method: $F_n^O \longrightarrow F_{n-1}^O$ has a unique realization as a map of *ko*-modules; smash with the identity on $C(\eta)$.

This is equivalent to defining $F_\bullet^U$ as $F_\bullet^O \otimes_{ko_*} ku_*$, since the modules in $F_\bullet^O$ are all free over ko_*. Thus, we can construct a complex F of free *crt*-modules as

$$F = F_\bullet^O \otimes_{ko_*} \pi_*^{crt}(ko),$$

where

$$F_\bullet^X = F_\bullet^O \otimes_{ko_*} \pi_*(kX).$$

We obtain diagrams similar to the one above corresponding to the other long exact sequences defining *crt*-acyclicity. The rows are exact complexes by construction. Since free modules are flat, each complex $F_\bullet^X$ is a resolution. Define a *crt*-module M by $M^O = M_*$, $M^U = H_\bullet(F_\bullet^U)$, and $M^T = H_\bullet(F_\bullet^T)$, operations induced by those on the resolutions $F_\bullet^X$.

Now M can be realized in topology (since *crt*-theory has dimension two) if and only if M is *crt*-acyclic. To see that M is *crt*-acyclic, recall that any bicomplex $\mathbb{F}$ has two spectral sequences [13] (where ∂, induced by η, denotes the horizontal differential and d, the vertical)

$$E_2^{p,q} = H_p(H_{q-p}(\mathbb{F}, d), \partial), \qquad \text{and} \qquad E_2'^{p,q} = H_p(H_{q-p}(\mathbb{F}, \partial), d)$$

each converging to the homology of the associated total complex (appropriately filtered). In our case, the spectral sequences are half-plane spectral sequences. Since they are spectral sequences from a bicomplex, they automatically converge conditionally [5]. Further, only finitely many differentials are non-zero since each spectral sequence has differentials which go from the groups in the free resolutions downward (in the above diagram) eventually into the lower half-plane, which is all zero.

In this case, the one spectral sequence converges to zero and the other, since it is a single line, converges from the homology of the complex

$$\cdots \longrightarrow \Sigma M^O \longrightarrow M^O \longrightarrow M^U \longrightarrow \Sigma^2 M^O \longrightarrow \cdots$$

to zero, forcing the complex in question to be exact. Similar analysis yields exactness for the M^O-M^T and M^U-M^T complexes. Thus, M is *crt*-acyclic and can be realized as $\pi_*^{crt}(X)$ for some *ko*-module X.

The details of the generalization of this construction are in the section below. The guarantee that any $\mathscr{F}$-module can be realized holds only for theories of dimension at most two; in such a case, the analysis realizing and classifying such modules is exactly as in the *crt* case.

14.2. General united theories. Let $R\mathscr{F}$ be a united theory for an S-algebra R, with X an R-module in $R\mathscr{F}$. Set $A = [X, X]_*^R$. Note that $\pi_*^{\mathscr{F}}(DX)$ consists of A-modules and A-module homomorphisms. Given any A-module M_* with A-free resolution

$$F_\bullet^X \longrightarrow M_* \longrightarrow 0,$$

we can form $F = F_\bullet^X \otimes_A \pi_*^{\mathscr{F}}(DX)$, which is a complex of $R\mathscr{F}$-modules such that F^X is the complex $F_\bullet^X$. The acyclicity conditions are proven as in the *crt*-case above, and we define the $R\mathscr{F}$-module M by $M^Y = H_\bullet(F_\bullet^Y)$ for Y in $R\mathscr{F}$, operations induced by those on F.

Thus we obtain an $\mathscr{F}$-acyclic $R\mathscr{F}$-module M.

When M has projective dimension at most two, we can use the techniques of earlier sections to realize M as $\pi_*^{\mathscr{F}}(Z)$ for an R-module Z.

This finishes the proof of Theorem 20 and its generalization:

Theorem 21. Let R be an S-algebra with a united theory $R\mathscr{F}$, X an R-module in $R\mathscr{F}$. Let $A = [X, X]_*^R$. Then given any A-module M_* it is possible to construct an $\mathscr{F}$-acyclic united module M such that $M^X = M_*$. Thus, by the techniques preliminary to the classification theorem, when the united theory $R\mathscr{F}$ is of dimension at most two, any A-module can be realized as the homotopy of $DX \wedge_R Z$ for some R-module Z. In particular, any R_*-module is the homotopy of some R-module.

In particular, any ko_*-module is the homotopy of some ko-module, and the same is true for KO.

Note also that any $[ku, ku]_*^{ko}$-module can be realized as $\pi_*(ku \wedge_{ko} Z)$ for some ko-module Z, since ku is self-dual up to suspension as a ko-module.

15. A change of rings isomorphism

Let $A = [ku, ku]_*^{ko}$. Recall the definition of φ-acyclic preceding Lemma 16.

15.1. Universal functors between A-modules and *crt*-modules. The forgetful functor $M \mapsto M^U$ from *crt*-modules to (graded) A-modules has both a left adjoint and a right adjoint.

Lemma 22. The left adjoint ρ' to the forgetful functor from *crt*-modules to (graded) A-modules is given by

$$\begin{aligned} (\rho' M)_*^U &= M_*, \\ (\rho' M)_*^O &= N_*, \text{ and} \\ (\rho' M)_*^T &= N_* \oplus N_{*+1}, \end{aligned}$$

where N_* is the double desuspension of the cokernel of φ. The operations on $\rho' M$ are given by $\eta = 0$, $\psi_T[x, y] = [x, -y]$, $\beta_O[x] = [\beta_U^4 x]$, $\beta_T[x, y] = [\beta_U^2 x, \beta_U^2 y]$, $\varepsilon[x] = [x, 0]$, $\zeta[x, y] = [x + \psi x]$, $\gamma x = [0, x]$, and $\tau[x, y] = [y]$.

The right adjoint ρ to the forgetful functor is given by

$$\begin{aligned} (\rho M)_*^U &= M_*, \\ (\rho M)_*^O &= L_*, \text{ and} \\ (\rho M)_*^T &= L_* \oplus L_{*+1}, \end{aligned}$$

where L_* is the kernel of φ. The operations on ρM are given by $\eta = 0$, $\psi_T(x, y) = (x, -y)$, $\beta_O(x) = \beta_U^4 x$, $\beta_T(x, y) = (\beta_U^2 x, \beta_U^2 y)$, $\varepsilon(x) = (x, 0)$, $\zeta(x, y) = x$, $\gamma x = (0, x + \psi x)$, and $\tau(x, y) = y$.

Proof. The adjunctions are verified by using the operation sequences, noting that maps must commute with all *crt*-operations: Given an A-module map $\alpha\colon M_* \longrightarrow X_*^U$, where X is a *crt*-module, there should be a unique map $\alpha^{crt}\colon \rho'(M) \longrightarrow X$ with $\alpha^U = \alpha$. The definition of ρ' and the *crt*-operations require that $\alpha^U = \alpha$, α^T is determined by α^O (because of the O-T and U-T sequences), and $\alpha^O[y] = r\alpha(y)$, which is well-defined since $[y] = [z]$ if and only if $\varphi(y - z) = 0$, and $r\varphi = 0$. Similarly for $\alpha'\colon X_*^U \longrightarrow M_*$, we need a unique map $\alpha'^{crt}\colon X \longrightarrow \rho(M)$. This time, $\alpha^O(x) = \alpha c(x) \in \rho(M) = \ker \varphi$ since α is a map of A-modules, so $\varphi\alpha c(x) = \alpha\varphi c(x) = \alpha(0)$. $\square$

Lemma 23. If M is a projective A-module, then $\rho' M$ is a projective *crt*-module. If M is an injective A-module, then ρM is an injective *crt*-module.

Lemma 24. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of A-modules. If N is φ-acyclic, then $0 \longrightarrow \rho' L \longrightarrow \rho' M \longrightarrow \rho' N \longrightarrow 0$ is an exact sequence of *crt*-modules. If L is φ-acyclic, then $0 \longrightarrow \rho L \longrightarrow \rho M \longrightarrow \rho N \longrightarrow 0$ is an exact sequence of *crt*-modules.

Proof. Any right adjoint preserves limits, hence is left exact and preserves monics; dually for any left adjoint and epis. Thus, it suffices to check the exactness of $0 \longrightarrow \rho' L \longrightarrow \rho' M$ and $\rho M \longrightarrow \rho N \longrightarrow 0$ where N and L are φ-acyclic. But this is clear from the definitions of ρ and ρ' and the fact that A-module maps commute with φ. $\square$

Lemma 25. For any φ-acyclic A-module M, $\rho' M$ and ρM are *crt*-acyclic.

Proof. Again, by direct calculation from the definitions of the adjoint functors. $\square$

Theorem 26. For *crt*-acyclic *crt*-modules L and M with $\eta = 0$ in M^O, there are natural isomorphisms

$$\rho'(M^U) \cong M \cong \rho(M^U),$$

$$\operatorname{Ext}^{s,t}_{crt}(M, L) \cong \operatorname{Ext}^{s,t}_{A}(M^U, L^U),$$

$$\operatorname{Ext}^{s,t}_{crt}(L, M) \cong \operatorname{Ext}^{s,t}_{A}(L^U, M^U).$$

Proof. We use the fact that M^U is φ-acyclic (Lemma 16), the lemmas above, and the long exact sequences which link the *crt*-operations. Since $\eta = 0$, we have the isomorphisms

$$\rho'(M^U)^O = \operatorname{coker} \varphi = M^U / \operatorname{im} \varphi = M^U / \ker(r) \xrightarrow{r} \Sigma^2 M^O$$

and $M^O \cong \operatorname{im}(c) \cong \ker \varphi = \rho(M^U)^O$. Given these isomorphisms and that, by adjointness, Hom_{crt} is determined by Hom_A, the isomorphism between U-pieces lifts to a *crt*-isomorphism.

Taking a free or injective A-resolution of M, depending on whether M is in the contravariant or covariant variable, we obtain the desired change of rings isomorphisms. $\square$

16. Future Directions

16.1. C^*-algebras and connective K-theory. Module spectra (in the classical, naive sense) over connective K-theory which are of the form $ku \wedge X$ for a compact space X are closely related to C^*-algebras. For X and Y compact spaces,

$$[\mathscr{K} \otimes CX, \mathscr{K} \otimes CY] \cong [X, ku \wedge Y]^{Stable} = [ku \wedge X, ku \wedge Y]^{ku},$$

where $\mathscr{K}$ denotes compact operators. and CZ is the space of all maps $Z \longrightarrow \mathbb{C}$. Dădărlat [11] and Blackadar [4] give more details on this relation between connective K-theory and C^*-algebras. Segal [22] has given a more geometric construction of the connective K-theory spectrum.

Thus, it would be good to know which module spectra over ku are actually of the form $ku \wedge X$, at least for X a finite CW-complex.

16.2. Extended modules. The ko Adams spectral sequence is useful for its quick convergence. This leads one to a desire to understand ko-modules better, for example, which ko_*-modules can occur as the ko-homology of a space or spectrum.

Note that, since ku and kt are finite ko-modules, the result above on finitely generated R_*-modules generalizes to finitely generated crt-modules.

In a different vein, a theorem of Jung and Stolz [23] states that a manifold admits a positive scalar curvature metric if and only if the image in ko-theory of the spin bordism class given by the classifying map $M \longrightarrow B\pi_1 M$ of the universal cover lies in a certain subgroup of $ko_n B\pi_1 M$ ($n = \dim M \geq 5$). Thus, the study of $ko \wedge B\pi$ would be interesting. The periodic case reduces to representation theory. While there is no general description of $H\mathbb{Z} \wedge B\pi$, connective K-theory may be a middle ground between this and the periodic K-theory of $B\pi$.

16.3. Questions. Among other interesting questions is one suggested by Neil Strickland: when is a module over MU_* the homotopy of an MU-module? Here we have realized all modules of projective dimension at most two, but the question remains for all higher and infinite MU_*-dimensional modules.

Perhaps more tractable would be to investigate other rings of finite homological dimension first. This might require a better understanding of the relation between the associated graded of the Hom groups from the EKMSS and the actual Hom groups.

Along another tack is the investigation of other united homology theories. It would be interesting to find situations apart from K-theory where these theories are useful.

The work of T.-Y. Lin included results about modules over S_*. One hope would be that we could determine S_*-injectives or obtain more information about maps between 2-cell complexes. Since S_* is not concentrated in even (or otherwise sparse) degrees, the algebra over this uncalculated ring is likely to be difficult to approach, but it should be possible to see part of the picture.

The categories $\mathscr{D}_R$ of R-modules give alternate worlds of homotopy theory. As this paper shows, these worlds are often simpler than the usual stable category. It would be interesting to investigate, for general S-algebras R, whether there is a choice of R such that the every Bousfield class (R-modules with the same localization functor) has a complement in the algebra of Bousfield classes of R-modules. Also possibly interesting would be analogues of the chromatic filtration and whether the telescope conjecture might be true over some choice of R. This might aid in determining the deviation of the telescope conjecture from the truth.

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Department of Mathematics, University of Chicago, Chicago, IL 60637, USA