Routing into Two Parallel Links: Game-Theoretic Distributed Algorithms

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We study a class of noncooperative networks where N users send traffic to a destination node over two links with given capacities in such a way that a Nash equilibrium is achieved. Under a linear cost structure for the individual users, we obtain several dynamic policy adjustment schemes for the on-line computation of the Nash equilibrium, and study their local convergence properties. These policy adjustment schemes require minimum information on the part of each user regarding the cost/utility functions of the others.

Key Words: Routing; nonzero-sum games; noncooperative equilibria; greedy algorithms

1. INTRODUCTION

We consider a routing problem in networks, in which two parallel links are shared between a number of players. In the context of telecommunication networks, the players could stand for users, who have to decide on what fraction of their traffic to send on each link of the network.

A natural framework within which to analyze this class of problems is that of noncooperative game theory, and an appropriate solution concept is that of Nash equilibrium [2]: a routing policy for the users constitutes a Nash equilibrium if no user can gain by unilaterally deviating from his own policy.

There exists rich literature on the analysis of equilibria in networks, particularly for the case of the infinitesimal user; see, e.g., [6, 10] in the context of road traffic. More recently, the issue of *competitive routing* has been studied, where the network

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is shared by several users with each one having a non-negligible amount of flow. Our starting point here is the work reported in [10, 18, 15, 14, 16] on competitive routing, which has presented the basic optimality concept in routing within a noncooperative framework, and using the Nash equilibrium solution concept. In these cited papers, conditions have been obtained for the existence and uniqueness of a Nash equilibrium. This has enabled, in particular, the design of network management policies that induce efficient equilibria [15].

Nash equilibrium has an obvious operational meaning if all users employ from the beginning their equilibrium policies; in that case it is self-enforcing and is optimal, in the sense that no user has any incentive to deviate from it. The main difficulty with this notion of equilibrium from a practical point of view, however, is that in realistic situations there is no justification for the assumption that the system is initially in equilibrium, nor for the one that there is coordination among the users in reaching an equilibrium. Moreover, the users are generally unable to compute the equilibrium individually, since it is a function of the unknown utilities and other parameters private to each player. Instead, a natural assumption on the behavior of users is that they are likely to behave in a greedy way: each user would update occasionally its own decisions so as to optimize its individual performance, without any coordination with other users. Accordingly, a central objective of this paper is to analyze different types of greedy behavior on the part of the users, and to relate these to the Nash equilibrium.

The need for decentralized distributed individual controls in telecommunication networks has stimulated a substantial amount of research using game theoretic methods, in both routing and flow control [1, 4, 5, 7, 8, 12, 14, 15, 17, 18, 19, 21]. Within the design of flow control, decentralized and distributed greedy protocols have been analyzed. The study of decentralized greedy routing, however, is new, and heretofore only the special case of two links and two players has been studied [18].

In the next section we introduce the model adopted in the paper for the general M link case, and establish the existence of a unique Nash equilibrium. In Section 3, we obtain an explicit expression for the equilibrium under a linear cost on rate of flow for M parallel links. In Section 4 we develop and study stability of various update algorithms, for the case of two links. The concluding remarks of the last section end the paper.

2. THE MODEL

Consider M parallel links between two points — source and destination. There are N users who have to send traffic from the source to the destination. User i requires a total rate of transmission Λ^i, and $\Lambda := \sum_{i=1}^{N} \Lambda^i$ is the total traffic rate sent by all users. The users have to decide on how to distribute their traffic over the M links.

Each link may correspond to a processor, in a context of parallel processors and distributed computing [13]. The traffic then corresponds to a flow of jobs.

We formulate this problem as a noncooperative network, i.e., a network in which the rate at which traffic is sent is determined by selfish users, each having its own utility (or cost). Let

λ_m^i:= the rate of traffic that user i sends over link m,

λ_m:= the total rate of traffic sent over link m, i.e., $\sum_i \lambda_m^i$, and
$\lambda_m^{-i} := \lambda_m - \lambda_m^i$.

As in [15, 18], we assume that the cost per unit of traffic of player i sent over link m is a function of λ_m, which we denote by $f_m(\lambda_m)$. This cost typically represents the expected delay, or a function of the expected delay over the link. Thus, the overall cost for user i (per unit time) of sending at a rate λ_m^i over link m is $\lambda_m^i f_m(\lambda_m)$, and the total cost (over all links) for user i is given by:

$$J_i(\lambda) := \sum_{m=1}^{M} \lambda_m^i f_m(\lambda_m) \,. \tag{1}$$

We assume that the link capacities are large enough so that capacity constraints can be ignored. The collection $\{\lambda_m^i\}_{m,i} =: \lambda$ is called a *multi-policy*, which we say is *feasible* if it satisfies the following two conditions:

- (R1) Positivity: $\lambda_m^i \geq 0, \ \forall i, m$.
- (R2) Rate constraints: $\sum_{m=1}^{M} \lambda_m^i = \Lambda^i, \ \forall i$.

The optimality concept we adopt is that of Nash equilibrium, i.e., we seek a feasible multi-policy $\lambda = \{\lambda_m^i\}_{m,i}$ such that no player (user) i can benefit by deviating unilaterally from his policy $\lambda^i = \{\lambda_m^i\}_m$ to another feasible multi-policy. In other words, for every player i,

$$J^i(\lambda) = J^i(\lambda^1, \dots, \lambda^{i-1}, \lambda^i, \lambda^{i+1}, \dots, \lambda^M) = \min_{\tilde{\lambda}^i} J^i(\lambda^1, \dots, \lambda^{i-1}, \tilde{\lambda}^i, \lambda^{i+1}, \dots, \lambda^M)$$

where the minimum is taken over all policies $\tilde{\lambda}^i$ that lead to a feasible multi-policy.

We recall the following result from [18] for the unconstrained case.

PROPOSITION 2.1. *Assume that $f_m : \mathbb{R} \to [0, \infty]$, $m = 1, ..., M$, are continuous and convex increasing. Then the game has a unique Nash equilibrium.*

In [15, 18], a further special case was considered, where the cost functions have the structure: $f_m(\lambda_m) = 1/(C_m - \lambda_m)$, which corresponds to the expected delay of an M/M/1 queue (i.e., an infinite buffer queue in which arrivals occur according to a Poisson process with rate λ_m, service times have independent exponential distribution with parameter $1/C_m$, and arrivals and service times are mutually independent). For this special structure, it is possible to compute explicitly the corresponding Nash equilibrium solution.

In this paper we specialize to the following particular (but different) cost function, which will also allow us to compute and characterize the Nash equilibrium:

$$f_m(\lambda_m) = a_m \lambda_m \,. \tag{2}$$

In the context of telecommunication systems, this cost would represent the expected waiting time of a packet in a light traffic regime. Indeed, under a large variety of statistical conditions, the expected waiting time of a packet in link m is proportional to $\lambda_m/(C_m - \lambda_m)$. For example, in an M/G/1 queue (i.e., a queue with an infinite buffer, where arrivals follow a Poisson process with parameter λ_m, where service times are independent with general distribution, and where all interarrival and service times are independent), the expected value of the waiting time W_m is given by

$$E[W_m] = \frac{\beta_m^{(2)}}{2(1-\lambda_m\beta_m^{(1)})}\,\lambda_m\,,$$

where $\beta_m^{(1)}$ and $\beta_m^{(2)}$ are, respectively, the first and second moments of the service time in the queue. The parameter C_m introduced earlier can then be identified with $1/\beta_m^{(1)}$, since the number of packets that the server can handle per unit time is on the average $1/\beta_m^{(1)}$. Now, under the light traffic regime, i.e., with $\lambda_m << C_m$, the above expression for EW_m can be approximated by $a_m\lambda_m$, where $a_m = \beta_m^{(2)}$.

A special application for this model arises in road traffic. The setting is appropriate if a user is viewed as a company that ships goods using many vehicles. The company then has to decide what fraction of its traffic to route on each link.

3. COMPUTATION OF NASH EQUILIBRIUM

In this section we obtain an explicit expression for the Nash equilibrium when f_m is given by (2), and further show that the equilibrium is unique. The main result is given in the following theorem.

THEOREM 3.1. *For the N-player M-link game with player costs given by (1), where f_m is structured as (2), there exists a unique Nash equilibrium given by*

$$\lambda_l^i = \Big(\frac{1}{a_l} \Big/ \sum_{m=1}^{M}\frac{1}{a_m}\Big)\,\Lambda^i, \qquad i=1,...,N;\; l=1,...,M. \tag{3}$$

Proof: In order to avoid using the rate constraints for user i, we can assume that he has only $M-1$ decision variables: $(\lambda_1^i,...,\lambda_{M-1}^i)$; the value of λ_M^i is then determined from the relationship: $\lambda_M^i = \Lambda^i - \sum_{k=1}^{M-1}\lambda_k^i$. In terms of these decision variables, the cost (1) for user i, with f_m given by (2), can be rewritten as:

$$J_i(\lambda) = \sum_{m=1}^{M-1}\lambda_m^i a_m\lambda_m + \left(\Lambda^i - \sum_{k=1}^{M-1}\lambda_k^i\right) a_M\lambda_M. \tag{4}$$

In the computation of the Nash equilibrium, we first ignore the positivity constraints (R1), by assuming the existence of a Nash equilibrium solution with strictly positive flows, and then show that such a solution indeed exists. From the uniqueness result of [18], it then follows that this is indeed the *unique* equilibrium point.

With the positivity constraint thus ignored, the best response of user i can be obtained by computing the partial derivative of J_i with respect to λ_l^i, $l=1,...,M-1$, and setting it equal to *zero.* (Indeed, the best response of player i corresponds to the minimum cost for that player for given strategies of the other players. Since the cost is convex, this minimum is indeed obtained at the value of $\{\lambda_l^i\}_l$ for which the partial derivative of J_i with respect to λ_l^i is zero.)

$$\frac{\partial J_i(\lambda)}{\partial\lambda_l^i} = a_l\lambda_l + a_l\lambda_l^i - a_M\lambda_M - a_M\lambda_M^i = 0, \quad l=1,...,M-1. \tag{5}$$

The best response $\{\lambda_l^{i^*}\}_l$ for player i, for fixed strategies $\{\lambda_l^j\}_{l,j}$, $j \neq i$, is thus obtained by solving

$$a_l(\lambda_l^{-i} + \lambda_l^{i^*}) + a_l\lambda_l^{i^*} - a_M\Big(\Lambda - \sum_{k=1}^{M-1}(\lambda_k^{-i} + \lambda_k^{i^*})\Big) - a_M\Big(\Lambda^i - \sum_{k=1}^{M-1}\lambda_k^{i^*}\Big) = 0.$$

This yields the following expression for the minimizing rate $\lambda_l^{i^*}$, $l = 1, ..., M-1$:

$$\lambda_l^{i^*} = \frac{1}{2(a_l + a_M)}\left[-(a_l + a_M)\lambda_l^{-i} + a_M\left(\Lambda - \sum_{\substack{k=1\\k\neq l}}^{M-1}(\lambda_k^{-i} + \lambda_k^{i^*}) + \Lambda^i - \sum_{\substack{k=1\\k\neq l}}^{M-1}\lambda_k^{i^*}\right)\right] \tag{6}$$

Since $\lambda_l^* = \lambda_l$ is a necessary condition for $\{\lambda_l^i\}_{l,i}$ to be in equilibrium, we obtain by taking the sum, $\sum_{i=1}^N \lambda_l^{i^*} = \lambda_l^* = \lambda_l$,

$$\begin{aligned}\lambda_l &= \frac{1}{2(a_l + a_M)}\left[-(a_l + a_M)\lambda_l(N-1) + a_M\left(N\Lambda - N\sum_{\substack{k=1\\k\neq l}}^{M-1}\lambda_k + \Lambda - \sum_{\substack{k=1\\k\neq l}}^{M-1}\lambda_k\right)\right]\\ &= \frac{1}{2(a_l + a_M)}\big[-(a_l + a_M)\lambda_l(N-1) + a_M(N+1)(\Lambda - \sum_{\substack{k=1\\k\neq l}}^{M-1}\lambda_k)\big]\end{aligned}$$

We thus arrive at the following necessary condition:

$$\lambda_l = \frac{a_M}{a_l + a_M}(\Lambda - \sum_{\substack{k=1\\k\neq l}}^{M-1}\lambda_k) = \frac{a_M}{a_l + a_M}(\lambda_M + \lambda_l)\,, \tag{7}$$

and solving for λ_l, we have: $\lambda_l = (a_M/a_l)\lambda_M$, or equivalently, $\lambda_l = k/a_l$, for some constant k. This constant can be obtained from the rate constraint, i.e., $\Lambda = \sum_{l=1}^M \lambda_l = k\sum_{l=1}^M(1/a_l)$, which leads to

$$\lambda_l = \left(\frac{1}{a_l}\bigg/\sum_{l=1}^M\frac{1}{a_l}\right)\Lambda. \tag{8}$$

By substituting this into (5) and equating the resulting expression to *zero*, we obtain $\lambda_l^i a_l = \lambda_M^i a_M$. Since $\sum_{l=1}^M \lambda_l^i = \Lambda^i$, this finally leads to

$$\lambda_l^i = \left(\frac{1}{a_l}\bigg/\sum_{l=1}^M\frac{1}{a_l}\right)\Lambda^i. \tag{9}$$

◇

Remark. We observe from the theorem above that the total amount of traffic at equilibrium, given by (8), is *not a function of the number of users.* In particular, this is also the socially optimal solution, i.e., it is the routing policy obtained when there is only one user whose total requirement for transmission rate equals Λ (this can be seen from our analysis by setting $\Lambda^j = 0$ for all users $j \neq i$). This implies that the Nash equilibrium is *efficient* (this means that it coincides with the solution to a global optimization problem in which the cost to be minimized is the sum of costs of all users). It is thus also a Pareto optimal solution (a Pareto optimal solution is a multi-strategy for which there does not exist a dominating one, i.e. we cannot find another multi-strategy that performs at least as good for all players, and strictly better for at least one player. This is a natural concept in the context of cooperation between users). Thus, even if there were cooperation, the users would still use the same equilibrium policy. ◇

4. THE TWO-LINK CASE: CONVERGENCE TO EQUILIBRIUM

The equilibrium has the property that once it is reached, the users will continue to use the same policy, and the system will remain in that equilibrium. A crucial question is the dynamics of reaching that equilibrium. Toward addressing this question, we shall make the behavioral assumption that users occasionally update their policies in a *greedy* way, i.e., they use their best response to the current policy of other users. We study the convergence of such schemes for the two-link case, that is with $M = 2$.

For each fixed policy of user j, the best response of user i, $i \neq j$, is computed through (6), with $M = 2$. Since the sum $\lambda_1^i + \lambda_2^i = \Lambda^i$ is fixed for each i, it will suffice to focus on only the best response λ_1^i, of user i, and for convenience we suppress in the following the subscript $m = 1$. Let $\lambda^i(E)$ be the equilibrium solution for user i, as given in Theorem 3.1, and introduce $\overline{\lambda}^i := \lambda^i - \lambda^i(E)$, $\overline{\lambda}^{i^*} := \lambda^{i^*} - \lambda^i(E)$. In terms of this notation, we obtain from (6) and Theorem 3.1:

$$\overline{\lambda}^{i^*} = \sum_{k=1}^{N} (\nabla A)_{ik} \overline{\lambda}^k, \tag{10}$$

where ∇A is an $N \times N$ matrix whose entries are given by

$$(\nabla A)_{ik} = \frac{\partial \lambda^i}{\partial \lambda^k} = \begin{cases} -1/2 & \text{if } i \neq k \\ 0 & \text{if } i = k. \end{cases} \tag{11}$$

Remark. The optimal response requires from each user to know only the sum of the flows on each link, and not their individual values — which makes implementation easier. Indeed, it suffices for a user to know the cost that he accrues on each link and his own flows, to be able to compute the sum of flows of other users on each link. ◇

Remark. In the algorithms that follow, we do not take into account the flow positivity constraints in the computation of the optimal responses. Thus, if we start far enough from the equilibrium point, then the computed flows could become negative. The following stability results should therefore be interpreted only locally, and the analysis becomes valid once the flows stir away from the boundaries at zero.

4.1. Round-Robin response

We first consider a *round-robin* adjustment scheme where the users update their policies (by using the optimal response (6) or (10)) sequentially, in the order $1, 2, \ldots, N$, $1, 2, \ldots, N$, etc.

Denote by $\overline{\lambda}^i(n)$ the value of $\overline{\lambda}^i$ at the nth iteration for Player i. We note that it is a function of all $\overline{\lambda}^j$'s for $j \neq i$, where for $j < i$ these have already been updated n times, and for $j > i$ these have only been updated $n-1$ times.

More precisely, we can express the updates in the form:

$$\begin{aligned}
\overline{\lambda}^1(n+1) &= g_1(\overline{\lambda}^2(n), \overline{\lambda}^3(n) ..., \overline{\lambda}^N(n)), \\
\overline{\lambda}^2(n+1) &= g_2(\overline{\lambda}^1(n+1), \overline{\lambda}^3(n), ..., \overline{\lambda}^N(n)), \\
&\vdots \\
\overline{\lambda}^N(n+1) &= g_N(\overline{\lambda}^1(n+1), ..., \overline{\lambda}^{N-1}(n+1)).
\end{aligned}$$

Thus, using (10) we have

$$\begin{aligned}
\overline{\lambda}(n+1) &= -\frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\overline{\lambda}(n) - \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}\overline{\lambda}(n+1) \\
&=: -\frac{1}{2}\mathcal{A}\overline{\lambda}(n) - \frac{1}{2}\mathcal{B}\overline{\lambda}(n+1). \qquad (12)
\end{aligned}$$

This provides an implicit recursive updating formula for the deviation of the policies from their equilibrium values at the end of the $n+1st$ update cycle. Solving for $\overline{\lambda}(n+1)$ from (12), we have

$$\overline{\lambda}(n+1) = -\mathcal{C}^{-1}\mathcal{A}\overline{\lambda}(n)\,, \qquad (13)$$

provided that $\mathcal{C}$ is invertible, where $\mathcal{C} := (2\mathcal{I} + \mathcal{B})$, and $\mathcal{I}$ denotes the identity matrix. We now show that this scheme is convergent.

THEOREM 4.1. *The Round-Robin update algorithm converges to the unique Nash equilibrium given in Theorem 3.1.*

Proof: From (13) we see that a necessary and sufficient condition for convergence of the Round Robin scheme is that all eigenvalues of $-\mathcal{C}^{-1}\mathcal{A}$ be in the interior of the unit disk. The eigenvalues of $-\mathcal{C}^{-1}\mathcal{A}$ are the zeros z of

$$\det(\mathcal{C}^{-1}\mathcal{A} + \mathcal{I}z) = 0 \qquad (14)$$

or equivalently, of $\det \mathcal{M}(z) = 0$ where

$$\mathcal{M}(z) := \mathcal{A} + 2\mathcal{I}z + \mathcal{B}z = \begin{bmatrix} 2z & 1 & 1 & \cdots & 1 \\ z & 2z & 1 & \cdots & 1 \\ z & z & 2z & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z & z & z & \cdots & 2z \end{bmatrix}$$

Simplification of this matrix (by multiplying rows by constants and adding rows), yields:

$$\begin{bmatrix} z & (1-2z) & 0 & \cdots & 0 & 0 \\ 0 & z & (1-2z) & \cdots & 0 & 0 \\ 0 & 0 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & (1-2z) \\ -(1-2z) & 0 & 0 & \cdots & 0 & -1 \end{bmatrix} =: \mathcal{M}'(z)$$

Details of this simplification, for a generic row i is as follows:

(i) For $i < N$, we have replaced row i by the difference between row i and row $i+1$ in matrix $\mathcal{M}(z)$.

(ii) For $i = N$, we have first multiplied this row by $-1/z$ and then added to it row 1.

It is easy to see that $z = 1$ is not a solution of $\det \mathcal{M}(z) = 0$, but it is a solution of $\det \mathcal{M}'(z) = 0$. On the other hand, $z = 0$ is a solution of $\det \mathcal{M}(z) = 0$, but not a solution of $\det \mathcal{M}'(z) = 0$. The reason for this is that in step (ii) we have multiplied row N by $-1/z$, and in doing so we have cancelled a zero at $z = 0$ from the characteristic polynomial of the determinant of $\mathcal{M}(z)$. In this process, this zero was replaced by a new one at $z = 1$. For all other rows and for all other values of z, the multiplying factor is nonzero, and hence the remaining set of solutions of $\det \mathcal{M}'(z) = 0$ is the same as the remaining set of solutions of $\det \mathcal{M}(z) = 0$.

Thus it remains to show that the zeros of $\det(\mathcal{M}'(z)) = -z^{N-1} + (1-2z)^N(-1)^N$ are all in the interior of the unit disk.

Consider first $|z| > 1$. Since $|1-2z| > |z|$,

$$|\det(\mathcal{M}'(z))| \geq |(1-2z)^N(-1)^N| - |z^{N-1}| > |z|^N - |z|^{N-1} = (|z|-1)|z|^{N-1} > 0.$$

It then remains to check the case $|z| = 1$ with $z \neq 1$. Toward this end, write z as $z = z_x + iz_y$ where z_x and z_y are respectively the real and imaginary parts of z. We have (since $z_x < 1$) $|1-2z| = \sqrt{(1-2z_x)^2 + 2z_y{}^2} = \sqrt{5-4z_x} > 1$. Hence

$$|\det(\mathcal{M}'(z))| \geq |(1-2z)^N(-1)^N| - |z^{N-1}| = |(1-2z)|^N - 1 > 0.$$

We conclude that all zeros of $\mathcal{M}(z)$ are in the interior of the unit disk, which implies that the Round Robin update scheme converges to the unique Nash equilibrium. $\diamond$

4.2. Pairwise simultaneous adjustment

We consider in this subsection a Round Robin scheme in which the users update their policies in pairs, i.e., first players 1 and 2 update, then players 3 and 4 update, and so on. For this scheme to make sense, we take the number of players to be even. We call this scheme *Round Robin in blocks of two*, which also converges as shown below.

THEOREM 4.2. *The Round Robin in blocks of two converges to the unique Nash equilibrium given in Theorem 3.1.*

Proof: With the notation of the previous subsection, we obtain the implicit update rule as $\lambda(n+1) = -\frac{1}{2}\mathcal{A}\lambda(n) - \frac{1}{2}\mathcal{B}\lambda(n+1)$ where

$$\mathcal{A} := \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \text{ and } \mathcal{B} := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 \end{bmatrix}.$$

The explicit update formula is again (13) with the new definitions for $\mathcal{A}$ and $\mathcal{B}$. Let again $\mathcal{C} := (2\mathcal{I} + \mathcal{B})$, with the above definitions of $\mathcal{A}$ and $\mathcal{B}$. As before, the eigenvalues of interest are the zeros of $\det(\mathcal{C}^{-1}\mathcal{A} + \mathcal{I}z) = 0$, which are equivalent to the zeros of $\det(\mathcal{M}(z)) = 0$ where

$$\mathcal{M}(z) := \mathcal{A} + 2\mathcal{I}z + \mathcal{B}z = \begin{bmatrix} 2z & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2z & 1 & 1 & \cdots & 1 & 1 \\ z & z & 2z & 1 & \cdots & 1 & 1 \\ z & z & 1 & 2z & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z & z & z & z & \cdots & 2z & 1 \\ z & z & z & z & \cdots & 1 & 2z \end{bmatrix} \tag{15}$$

We can simplify this matrix by replacing row i by the difference between row i and row $i+1$, $i = 1, ..., N-1$, and obtain

$$\mathcal{M}'(z) := \begin{bmatrix} 2z-1 & 1-2z & 0 & 0 & \cdots & 0 & 0 \\ 1-z & z & 1-2z & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2z-1 & 1-2z & \cdots & 0 & 0 \\ 0 & 0 & 1-z & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2z-1 & 1-2z \\ z & z & z & z & \cdots & 1 & 2z \end{bmatrix}$$

The characteristic polynomial $P_n(z)$ of this matrix is given by

$$(x-1)^n[(x-1)^{n-2}(x^2+1) + 2[(x-1)^{n-3} + (x-1)^{n-4} + \cdots + (x-1) + 1]] \tag{16}$$

where $x = 2z$, and $n = N/2$. For $\lambda \neq 1$, it can be simplified to:

$$\frac{(2z-1)^n}{1-z}[1 - z(2z-1)^n] \tag{17}$$

This step is obtained by induction. Indeed, it holds for $n = 1$, for which we obtain $P_1(z) = (2z-1)(2z+1)$. Assume that it holds for some n.

Note that the characteristic polynomial for the matrix without the first column and the last row is $(1-x)^{2n}$. $P_{n+1}(z)$ can be written as $P_{n+1} = (x-1)[Minor^1 + Minor^2]$, where by the induction hypothesis we have:

$$Minor^1 = z(P_n + (1-x)^{2n}), \qquad Minor^2 = (1-z)P_n + z[(1-x)^{2n}].$$

Thus,

$$\begin{aligned} P_{n+1} &= (x-1)[2z(1-x)^{2n} + \frac{(x-1)^n}{1-z}(1-z(x-1)^n) \\ &= \frac{(x-1)^{n+1}}{1-z}[2z(x-1)^n(1-z) + 1 - z(x-1)^n] \\ &= \frac{(x-1)^{n+1}}{1-z}[(x-1)^n(z-2z^2)+1] = \frac{(x-1)^{n+1}}{1-z}[1-z(x-1)^{n+1}]\,. \end{aligned}$$

This establishes by induction the form (17) of the characteristic polynomial $P_n(z)$ for $z \neq 1$.

Now, we prove that the roots of the polynomial $P_n(x)$ are in the interior of the unit circle:

If $|z| > 1$, then $|x| > 2$, so that $|x-1| > 1$ and $|x-1|^n > 1$. Hence $|1-z(x-1)^n| > 0$, which implies that $P_n(z) \neq 0$.

By a simple verification in (16), $z = 1$ is not a zero.

If $|z| = 1, z \neq 1$, we have (with the notation of the previous subsection)

$$|z| = \sqrt{z_x^2 + z_y^2} \Rightarrow |x-1| = \sqrt{5 - 4z_x} > 1 \Rightarrow |1 - z(x-1)^n| > 0.$$

This implies that z is not a zero of $P_n(z)$.

We thus conclude that all zeros of the characteristic polynomial $P_n(x)$ are in the interior of the unit disk, and hence the update scheme converges to the unique Nash equilibrium. ◇

4.3. Random greedy algorithms

Consider now the following **Random Polling** algorithm. There is a fixed integer $K < N$, such that at each update time n, K players are chosen at random to update. We shall first show that this algorithm converges to the Nash equilibrium when $K = 1$. Later we shall show that for $K > 3$ the random polling is unstable.

THEOREM 4.3. *The Random Polling algorithm with $K = 1$ converges to the Nash equilibrium with probability 1.*

Proof: We shall show that $\lim_{n\to\infty} \overline{\lambda}(n) = 0$ with probability one. Let $\mathcal{A}(i)$ denote the matrix that corresponds to the update of player i, i.e., the matrix given by $\mathcal{A}(i)_{(j,j)} = 1$ and $\mathcal{A}(i)_{i,j} = -1/2$ for all $j \neq i$, and all other elements are 0. For example,

$$\mathcal{A}(1) = \begin{bmatrix} 0 & -1/2 & -1/2 & \cdots & -1/2 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Let $\psi(n)$ be the (random) user that updates at step n. Then $\overline{\lambda}(n)$ has the form

$$\overline{\lambda}(n) = \mathcal{A}(\psi(n))\mathcal{A}(\psi(n-1))...\mathcal{A}(\psi(1))\overline{\lambda}(0).$$

In order to prove the Theorem, we introduce the function $Z : \mathbb{R}^N \to \mathbb{R}_+$: $Z(x) := \max_{Y \subset \{1,...,N\}} \left| \sum_{j \in Y} x_j \right|$, where $x = (x_1, ..., x_N) \in \mathbb{R}^N$. We shall first show that for any $i = 1, ..., N$, and any $x \in \mathbb{R}^N$,

$$Z(\mathcal{A}(i)x) \leq Z(x). \tag{18}$$

Consider some $Y \subset \{1, ..., N\}$. Assume that $i \notin Y$. Then $[\mathcal{A}(i)x]_j = x_j$ for all $j \neq i$. Hence $\left|\sum_{j \in Y} [\mathcal{A}(i)x]_j\right| = \left|\sum_{j \in Y} x_j\right|$ which implies

$$\left| \sum_{j \in Y} [\mathcal{A}(i)x]_j \right| \leq \max_{Y' \subset \{1,...,N\}} \left| \sum_{j \in Y'} x_j \right|. \tag{19}$$

Next, assume that $i \in Y$. Since $[\mathcal{A}(i)x]_i = -\frac{1}{2}\sum_{j \neq i} x_j$, we have

$$\begin{aligned} \sum_{j \in Y} [\mathcal{A}(i)x]_j &= \sum_{j \in Y, j \neq i} [\mathcal{A}(i)x]_j + [\mathcal{A}(i)x]_i \\ &= \sum_{j \in Y, j \neq i} x_j - \frac{1}{2}\sum_{j \neq i} x_j = \frac{1}{2}\sum_{j \in Y, j \neq i} x_j - \frac{1}{2}\sum_{j \notin Y} x_j \end{aligned}$$

This again implies (19), which establishes (18).

Since $\psi(n)$ are i.i.d., and for every j, the probability that $\psi(n) = j$ is strictly positive, it follows that the round-robin sequence $(1, 2, ..., N)$ appears in the sequence $\psi(n)$ infinitely often with probability 1. Moreover, for any integer V, the sequence $(1, 2, ..., N)^V$ appears infinitely often with probability 1. Let $B = \prod_{j=1}^{N} \mathcal{A}(j)$. We have established in Theorem 4.1 that $B^n \to 0$. This implies, in particular, that the L_1 norm of B^n converges to 0. It is then easy to show that for any $\epsilon > 0$ there exists some integer V such that $Z(B^V x) \leq \epsilon Z(x)$, for any $x \in \mathbb{R}^N$. This, together with (18), implies that with probability 1, $\lim_{n \to \infty} Z(\overline{\lambda}(n)) = 0$, and hence $\lim_{n \to \infty} \overline{\lambda}(n) = 0$. ◇

Remark. We see from the above proof that convergence to the Nash equilibrium actually holds under more general probabilistic assumptions. In fact, the convergence is obtained along each sample path for which the sequence B^V appears infinitely often. ◇

Next, we show that if we relax the condition $K = 1$ in the random polling algorithm, then this could give rise to instability. To see this, consider a sequence of i.i.d. random vectors $\{X_n\}$, $n = 1, 2, ...$, where $X_n = (X_1^1, \dots, X_n^N)$, with X_n^i taking the values 0 or 1, with respective probabilities $1 - p$ and p. (Note that for a fixed n, we allow the different components of X_n to be dependent.) Introduce the *Random Greedy algorithm* as follows: player i updates at time n if and only if $X_n^i = 1$. Consider the symmetric case where $p := EX_n^i$ does not depend on i or n. Then it can be shown that if $p \geq 4/(N+1)$, the Random Greedy algorithm does not converge. Indeed, it follows from (10) that

$$\overline{\lambda}^i(n+1) = -\frac{X_n^i}{2} \sum_{k \neq i} \overline{\lambda}^k(n) + (1 - X_n^i)\overline{\lambda}^i(n).$$

Letting $L^i := E\overline{\lambda}^i$, we obtain $L^i(n+1) = -\frac{p}{2} \sum_{k \neq i} L^k(n) + (1-p)L^i(n)$. Furthermore letting $L(n) := \sum_{i=1}^N L^i$, we have

$$L(n+1) = \left[-\frac{p}{2}(N-1) + (1-p)\right] L(n) = \left[-\frac{p}{2}(N+1) + 1\right] L(n).$$

If the term in the square brackets is smaller than or equal to -1, then $L(n)$ does not converge. This condition is equivalent to $p \geq 4/(N+1)$. This implies, in particular, that the parallel update algorithm, in which all users update simultaneously ($p = 1$), does not converge if $N > 3$. Moreover, the random polling algorithm does not converge if $K > 3$. This phenomenon may be related to the well known fact that different conditions for convergence may apply to the Gauss-Seidel and Jacobi iterative schemes; see e.g., [2, 3].

In order to avoid the above instability in the case of parallel update, some restrictions have to be imposed. This motivates us to introduce in the next section a forgetting factor that smooths the variations due to best responses of players.

4.4. Parallel update with a forgetting factor (relaxation parameter)

We have seen above that if $N > 3$ and all players update simultaneously at each step ($K = N$), then there is no convergence to the Nash equilibrium. The instability seems to be due to the fact that when all players change their strategies in the same manner, then this leads to growing oscillations. We consider in this subsection a possible dampening of these oscillations by allowing only a partial update; we consider a natural scenario where a player does not completely change his strategy from one step to the next, but rather uses a new strategy obtained as a convex combination of the calculated optimal response and the previously used action. In other words, the deviation vector $\overline{\lambda}_{n+1}$ at the $n + 1th$ step is given by

$$\overline{\lambda}_{n+1} = (\gamma \nabla A + (1 - \gamma)\mathcal{I})\overline{\lambda}_n,$$

where ∇A is as defined by (11), $\mathcal{I}$ is the identity matrix, and $\gamma > 0$ is called the *forgetting factor.*

THEOREM 4.4. *Consider the parallel update algorithm with forgetting factor γ. Then the system is stable if and only if $0 < \gamma < 4/(N+1)$.*

Proof: We have stability if and only if all eigenvalues of $\gamma\nabla A + (1-\gamma)\mathcal{I}$ are in the interior of the unit disk, or equivalently, the zeros of the determinant of

$$Q(z) = \gamma\nabla A + (1-\gamma-z)\mathcal{I} = \begin{bmatrix} 1-\gamma-z & -\gamma/2 & -\gamma/2 & \cdots & -\gamma/2 \\ -\gamma/2 & 1-\gamma-z & -\gamma/2 & \cdots & -\gamma/2 \\ -\gamma/2 & -\gamma/2 & 1-\gamma-z & \cdots & -\gamma/2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma/2 & -\gamma/2 & -\gamma/2 & \cdots & 1-\gamma-z \end{bmatrix}$$

are within the interior of the unit disk. The zeros of $Q(z)$ are the same as those of the matrix $Q'(z)$, obtained by replacing row i, $i = 2, ..., N$, by the difference between row i and row $i-1$:

$$Q'(z) = \begin{bmatrix} 1-\gamma-z & -\gamma/2 & -\gamma/2 & \cdots & -\gamma/2 & -\gamma/2 \\ -w & w & 0 & \cdots & 0 & 0 \\ 0 & -w & w & \cdots & 0 & 0 \\ 0 & 0 & -w & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & -w & w \end{bmatrix} \quad \text{where } w = 1 - \frac{\gamma}{2} - z.$$

The determinant of $Q'(z)$ is easily computed and is given by

$$det[Q'(z)] = \left(1 - \gamma - z - \frac{(N-1)\gamma}{2}\right)(1 - \frac{\gamma}{2} - z)^{N-1}.$$

Its zeros are $(1-\gamma/2)$ and $(1-(N+1)\gamma/2)$, and this completes the proof. ◇

5. CONCLUSIONS

We have studied in this paper the problem of static competitive routing, first for M parallel links and subsequently in more detail for the case of two parallel links, and with linear link holding costs. In the two-link case, we have focused on the stability of the equilibrium, i.e., on the question of whether equilibrium is actually reached if players start initially at some other arbitrary point.

Our analysis suggests that under some scenarios where users update their actions independently and in a completely greedy way, the unique equilibrium may be unstable. In particular, if all users update simultaneously, oscillations could occur and the routing decisions diverge if the number of players is larger than 3. On the other hand if memory is added and updated actions use a convex combination of previous actions and the new greedy best response, then stability can be achieved.

Our approach is related to the so called "Cournot Adjustment" scheme [9] in economics. The round-robin scheme is known as the *alternating-move* Cournot dynamic; it is closely related to the Gauss-Seidel iteration for the solution of linear equations; see, e.g., [2, 3]. An alternative way of adding memory to the best response would be to compute best responses to the *time averaged actions* of other players, and not just to their latest strategy. This approach, known as "fictitious play" [9] will be the subject of future study.

We plan to study in the future the stability of equilibria in more complex topologies, in particular for those where it is known that there exists a unique equilibrium.

Furthermore, we intend to consider other cost functions, such as those whose linearized versions contain bias terms, and those that are player-dependent.

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