

Switching to nonhyperbolic cycles from codim 2 bifurcations of equilibria in ODEs

Yu.A. Kuznetsov^a H.G.E. Meijer^{b,*} W. Govaerts^c B. Sautois^c

^a*Department of Mathematics, Utrecht University, Budapestlaan 6, 3584CD
Utrecht, The Netherlands*

^b*Department of Applied Mathematics, Twente University, P.O. Box 217, 7500
AE, Enschede, The Netherlands*

^c*Department of Applied Mathematics and Computer Science, Ghent University,
Krijgslaan 281-S9, B-9000 Ghent, Belgium*

Abstract

The paper provides full algorithmic details on switching to the continuation of all possible codim 1 cycle bifurcations from generic codim 2 equilibrium bifurcation points in n -dimensional ODEs. We discuss the implementation and the performance of the algorithm in several examples, including an extended Lorenz-84 model and a laser system.

1 Introduction

Consider a system of differential equations depending on two parameters

$$\dot{x} = f(x, \alpha), \quad (x, \alpha) \in \mathbb{R}^n \times \mathbb{R}^2, \quad (1)$$

where f is smooth. In general, there are bifurcation curves in the α -plane, at which the system exhibits codim 1 bifurcations, for example, fold or Hopf bifurcations of equilibrium points. Moreover, generically, one expects points of codim 2 bifurcations, where several curves corresponding to codim 1 bifurcations intersect transversally or tangentially. A codim 2 point is of particular interest if it is not only the origin of some equilibrium bifurcation curves but also of some curves corresponding to bifurcations of periodic orbits (cycles). Such points can be detected by purely local analysis of equilibria and then be used to establish the existence of limit cycle bifurcations and other global

* Corresponding author.

phenomena that could hardly be proved otherwise. That is why codim 2 points are often called the “organizing centers” in applied literature.

The theory of codim 2 bifurcations of equilibria in generic systems (1) is well-developed (see, for example, [1], [12], [18]). There are five well-known codim 2 equilibrium bifurcations: cusp (CP), Bautin (generalized Hopf, GH), double zero (Bodanov-Takens, BT), zero-Hopf (ZH), and double Hopf (HH). It follows from their analysis that branches of nonhyperbolic limit cycles can emanate from GH, ZH, and HH points only. More precisely, a codim 1 bifurcation curve LPC, along which a cycle with a nontrivial multiplier $\mu_1 = 1$ exists, emanates from a generic GH point, while codim 1 bifurcation curves NS, along which a cycle with a pair of multipliers $\mu_{1,2} = e^{\pm i\theta}$ exists, are rooted at generic ZH and HH points. Notice that NS is used to denote both Neimark-Sacker and neutral saddle cycles where $\mu_1\mu_2 = 1$ and that no period-doubling curves can emanate from generic codim 2 equilibrium bifurcations.

Obviously, the application of these theoretical results to realistic models (1) is impossible without numerical tools. The numerical analysis of a codim 2 equilibrium bifurcation includes:

- detection and location of the point in a branch of a codim 1 bifurcation;
- computation of the coefficients of the normal form of the restriction of (1) to the critical center manifold at the bifurcation parameter values and checking the nondegeneracy conditions;
- verification of the transversality of the given family (1) to the codim 2 bifurcation manifold and establishing a correspondence between the unfolding parameters of the normal form and original system parameters α ;
- computing accurate approximations of the codim 1 curves in the α -space and the corresponding singular orbits in the x -space near the bifurcation, sufficient to initialize the numerical continuation of these codim 1 curves using only local information available at the codim 2 point.

While the first two problems were studied in detail (see, [3] and references therein) and have been implemented into the standard bifurcation software CONTENT [19] and MATCONT [6], two last issues received much less attention in the numerical analysis literature, even if bifurcations of nonhyperbolic cycles are concerned. The present paper is aimed at bridging this gap by providing full algorithmic details on switching to all possible codim 1 cycle bifurcations from generic GH, ZH, and HH codim 2 points.

One way to set up a computational switching procedure is to consider a smooth normal form for the codim 2 bifurcation including the parameters $\beta \in \mathbb{R}^2$

$$\dot{w} = G(w, \beta), \quad G : \mathbb{R}^{n_c} \times \mathbb{R}^2 \rightarrow \mathbb{R}^{n_c}. \quad (2)$$

For all codim 2 equilibrium bifurcations these normal forms are known. Sup-

pose that an exact or approximate formula is available that gives the emanating codim 1 bifurcations for the normal form (2). In order to transfer this to the original equation (1) we need a relation

$$\alpha = V(\beta), \quad V : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad (3)$$

between the unfolding parameters β and the given parameters α . In our context, V will be linearly approximated. Moreover, we need a center manifold parametrization

$$x = H(w, \beta), \quad H : \mathbb{R}^{n_c} \times \mathbb{R}^2 \rightarrow \mathbb{R}^n, \quad (4)$$

that incorporates β . Taking (3) and (4) together as $(x, \alpha) = (H(w, \beta), V(\beta))$ yields a center manifold for the suspended system $\dot{x} = f(x, \alpha), \dot{\alpha} = 0$. The invariance condition for the center manifold now turns into a *homological equation*:

$$H_w(w, \beta)G(w, \beta) = f(H(w, \beta), V(\beta)), \quad (5)$$

which we can solve by a recursive procedure based on Fredholm's solvability condition that will give the Taylor coefficients of G and H with respect to w and β . We assume the Taylor series of G to be known as

$$G(w, \beta) = \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu!\mu!} g_{\nu\mu} w^\nu \beta^\mu,$$

and the Taylor series of H and V to be unknown

$$H(w, \beta) = \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu!\mu!} h_{\nu\mu} w^\nu \beta^\mu, \quad V(\beta) = \sum_{|\mu|\geq 1} \frac{1}{\mu!} v_\mu \beta^\mu.$$

Here ν and μ are multi-indices. For $\mu = 0$ this reproduces the critical normal form coefficients first computed in [17], while the coefficients with $|\mu| \geq 1$ yield the necessary data on the parameter dependence.

To summarize, a bifurcation point is detected within a certain small tolerance. As the prediction depends on the initial point, this translates into small errors of the predicted curve. If we start close enough to the actual new curve, any point will converge to it and in general one expects a convergence cone [15]. If we parametrize the predicted curve by ε , the initial amplitude ε is to be chosen to be within the convergence cone, see also Figure 1.

This procedure is adopted from [3], where it has been applied to the derivation of the asymptotics of the fold and Hopf curves rooted at CP and BT codim 2 points, as well as that for a homoclinic orbit to a saddle emanating at the BT-point. Recently, this technique has been successfully used for switching at codim 2 fixed points of maps to the continuation of nonhyperbolic periodic orbits rooted there [8]. Note that a similar procedure was suggested in [13],

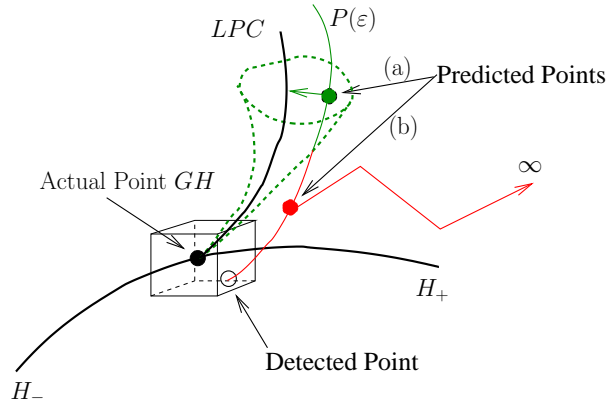


Fig. 1. Sketch of the switch in the case of a GH bifurcation. A predicted point along $P(\varepsilon)$ (a) in the cone will converge to the LPC-curve, outside (b) it will not.

without using the Fredholm condition, and carried through in the ZH-case in [14], where, however, no asymptotics of codim 1 curves were derived. Finally, we point out that the problem of switching to the LPC-curve at the GH bifurcation has been briefly discussed in [3] in a setting without the Taylor expansion in β .

The paper is organized as follows. In Section 2 we revise smooth parameter-dependent normal forms on center manifolds for the considered codim 2 bifurcations, i.e. give $G(w, \beta)$ in GH, ZH, and HH cases, and give the asymptotic expressions of the branches of nonhyperbolic cycles in these normal forms. Then we perform the described above reduction procedure and derive the necessary coefficients $g_{\nu\mu}$, $h_{\nu\mu}$, and v_μ in terms of F and its derivatives. These coefficients are finally used to set up predictors for these branches in the original system (1). An implementation of the resulting formulas in the software MATCONT is discussed at the end of this section. Section 3 presents several applications of the developed technique to known ODE models, an extension of the Lorenz-84 system and a laser model, where we compare the asymptotic formulas for the cycle bifurcations with numerically computed LPC- and NS-branches. A discussion of existing results and open problems in switching to homoclinic branches at ZH and HH bifurcations is given in Section 4.

2 Asymptotics and the Center Manifold

2.1 The ‘new’ curves

The parameter-dependent normal forms are known and can be found in the standard texts, e.g. [18]. As the normal form and the asymptotic expressions are the necessary theoretical ingredient, we present these here.

2.1.1 Generalized Hopf

Near a GH bifurcation the vector field restricted to the center manifold is given by

$$\dot{w} = \lambda(\beta)w + c_1(\beta)w|w|^2 + c_2(\beta_2)w|w|^4 + \mathcal{O}(|w|^6), \quad w \in \mathbb{C}, \quad (6)$$

where $\lambda(0) = i\omega$, and this bifurcation is characterized by $d_1 = \Re(c_1(0)) = 0$ and $d_2 = \Re(c_2(0)) \neq 0$. A curve LPC of fold bifurcation of limit cycles emanates from this point. Let us write $w = \rho e^{i\psi}$, $\lambda(\beta) = i\omega + \beta_1 + ib_1(\beta) + \mathcal{O}(|\beta|^2)$ and $\Re(c_1(\beta)) = \beta_2 + \mathcal{O}(|\beta|^2)$. If we now truncate the normal form to fifth order in w , then the curve LPC is given by

$$\rho = \varepsilon, \beta_1 = d_2\varepsilon^4, \beta_2 = -2d_2\varepsilon^2. \quad (7)$$

2.1.2 Zero-Hopf

Near a ZH bifurcation the vector field restricted to the center manifold is given by

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \beta_1 + f_{200}x^2 + f_{011}|w|^2 + f_{300}x^3 + f_{111}x|w|^2 \\ (i\omega(\beta) + \beta_2)w + g_{110}xw + g_{210}x^2w + g_{021}w|w|^2 \end{pmatrix} + \mathcal{O}(\|(x, w)\|^4), \quad (8)$$

where $(x, w) \in \mathbb{R} \times \mathbb{C}$. An extra Neimark-Sacker (torus) bifurcation of limit cycles (NS) occurs if $\Re(g_{110})f_{011} < 0$.

The asymptotic expression is

$$\begin{aligned} \rho = \varepsilon, x = -\frac{f_{111} + 2g_{021}}{2f_{200}}\varepsilon^2, \beta_1 = -f_{011}\varepsilon^2, \\ \beta_2 = \frac{2(\Re(g_{110}) - f_{200})\Re(g_{021}) + \Re(g_{110})f_{111}}{2f_{200}}\varepsilon^2. \end{aligned} \quad (9)$$

This agrees with a formula given in [7].

2.1.3 Double-Hopf

For a HH bifurcation the dynamics on the center manifold is governed by the following normal form:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} (i\omega_1(\beta) + \beta_1)w_1 + f_{2100}w_1|w_1|^2 + f_{1011}w_1|w_2|^2 \\ (i\omega_2(\beta) + \beta_2)w_2 + g_{1110}w_2|w_1|^2 + g_{0021}w_2|w_2|^2 \end{pmatrix} + \mathcal{O}(\|(w_1, w_2)\|^4), \quad (10)$$

where $(w_1, w_2) \in \mathbb{C} \times \mathbb{C}$. Then there are generically two half-lines along which there is a NS bifurcation of limit cycles. In polar coordinates $w_1 = \rho_1 e^{i\psi_1}$, $w_2 =$

$\rho_2 e^{i\psi_2}$ their asymptotics are given as

$$(\rho_1, \rho_2, \beta_1, \beta_2) = (\varepsilon, 0, -\Re(f_{2100})\varepsilon^2, -\Re(g_{1110})\varepsilon^2), \quad (11)$$

$$(\rho_1, \rho_2, \beta_1, \beta_2) = (0, \varepsilon, -\Re(f_{1011})\varepsilon^2, -\Re(g_{0021})\varepsilon^2). \quad (12)$$

2.2 Coefficients of parameter-dependent center manifolds

We assume that the critical normal form coefficients are known (see [17] and [3]) and give here only parameter-related coefficients $h_{\nu\mu}$ from the homological equation. These provide in each case a linear approximation to the parameter transformation (3).

2.2.1 Generalized Hopf

Here we closely follow the idea outlined in [3]. We first expand the eigenvalue and the first Lyapunov coefficient in the original parameters α and collect the equations to obtain the transformation to the unfolding parameters β . Alternatively, one can normalize already in (13) to obtain an orthogonal frame from these equations and obtain scalings from the higher order equations (14). Below we have $\mu = (10), (01)$ as indices and $v_{10} = (1, 0), v_{01} = (0, 1)$ as vectors.

The first two equations (actually four) coming from (5) are

$$\begin{aligned} Ah_{00\mu} &= -J_1 v_\mu, \\ (A - i\omega I_n)h_{10\mu} &= \gamma_{1,\mu} q - A_1(q, v_\mu) - B(q, h_{00\mu}) \end{aligned} \quad (13)$$

The first equation is nonsingular and from the second we find $\gamma_{1,\mu}$ using the

Fredholm alternative. The other systems from (5) are

$$\begin{aligned}
(A - 2i\omega I_n)h_{20\mu} &= 2h_{2000}\gamma_{1,\mu} - [C(q, q, h_{00\mu}) + 2B(q, h_{10\mu}) + B(h_{2000}, h_{00\mu}) \\
&\quad + B_1(q, q, v_\mu) + A_1(h_{2000}, v_\mu)], \\
Ah_{11\mu} &= 2\Re(\gamma_{1,\mu})h_{1100} - [C(q, \bar{q}, h_{00\mu}) + B(h_{1100}, h_{00\mu}) \\
&\quad + B(\bar{q}, h_{10\mu}) + B(q, h_{01\mu}) + B_1(q, \bar{q}, v_\mu) + A_1(h_{1100}, v_\mu)], \\
(A - i\omega I_n)h_{21\mu} &= 2\gamma_{2,\mu}q + h_{2100}(2\gamma_{1,\mu} + \bar{\gamma}_{1,\mu}) + 2h_{10\mu}c_1 \\
&\quad - [D(q, q, \bar{q}, h_{00\mu}) + 2C(q, h_{1100}, h_{00\mu}) + 2C(q, \bar{q}, h_{10\mu}) \\
&\quad + C(q, q, h_{01\mu}) + C(h_{2000}, \bar{q}, h_{00\mu}) + 2B(q, h_{11\mu}) \\
&\quad + 2B(h_{1100}, h_{10\mu}) + B(h_{2000}, h_{01\mu}) + B(h_{2100}, h_{00\mu}) \\
&\quad + B(h_{20\mu}, \bar{q}) + C_1(q, q, \bar{q}, v_\mu) + 2B_1(h_{1100}, q, v_\mu) \\
&\quad + B_1(h_{2000}, \bar{q}, v_\mu) + A_1(h_{2100}, v_\mu)],
\end{aligned} \tag{14}$$

The first two are nonsingular and with the Fredholm alternative we find $\gamma_{2,\mu}$. The parameter transformation (3) is given by

$$\alpha = \left(\Re \begin{pmatrix} \gamma_{1,10} & \gamma_{1,01} \\ \gamma_{2,10} & \gamma_{2,01} \end{pmatrix} \right)^{-1} \beta. \tag{15}$$

2.2.2 Zero-Hopf

This case is also treated in [14], however with only one parameter and for hyperbolic periodic orbits. Thus our computational scheme is different. We list only the necessary equations.

$$\begin{aligned}
\text{(a)} \quad A[h_{00010}, h_{00001}] &= [q_1, 0] - J_1[v_{10}, v_{01}], \\
\text{(b)} \quad A[h_{10010}, h_{10001}] &= [h_{20000}, 0] - A_1(q_1, [v_{10}, v_{01}]) \\
&\quad - B(q_1, [h_{00010}, h_{00001}]) \\
\text{(c)} \quad (A - i\omega I_n)[h_{01010}, h_{01001}] &= [h_{11000}, q_2] - A_1(q_2, [v_{10}, v_{01}]) \\
&\quad - B(q_2, [h_{00010}, h_{00001}])
\end{aligned} \tag{16}$$

In contrast to the other cases, here the first system is already singular. Taking the inner-product with the adjoint null-vector we obtain the new orthogonal frame

$$\begin{aligned}
\gamma &= (\gamma_1, \gamma_2) = p_1^T J_1, \quad s_1^T = \gamma / \|\gamma\|^2, \quad s_2^T = (-\gamma_2, \gamma_1), \\
v_{10} &= s_1 + \delta_1 s_2, \quad v_{01} = \delta_2 s_2.
\end{aligned} \tag{17}$$

Polynomial terms in the normal form (8) like $\beta_1 x$ are also resonant, but they can be eliminated by hypernormalization. After solving (16.a) with a bordered matrix, see [11], still a multiple of q_1 may be added to h_{00010} . We use this to perform hypernormalization. Let us write

$$r_1 = -A^{INV} \begin{pmatrix} q_1 - J_1 s_1 \\ 0 \end{pmatrix}, \quad r_2 = -A^{INV} \begin{pmatrix} -J_1 s_2 \\ 0 \end{pmatrix},$$

where A^{INV} indicates the use of the bordered matrix, then we can write

$$h_{00010} = r_1 + \delta_1 r_2 + \delta_3 q_1, \quad h_{00001} = \delta_2 r_2 + \delta_4 q_1,$$

for some δ 's. Then by applying the Fredholm alternative to (16.b,c) we can solve for all δ 's at once.

$$\begin{aligned} LL \begin{pmatrix} \delta_1 \\ \delta_3 \end{pmatrix} &= - \begin{pmatrix} \langle p_1, A_1(q_1, r_1) + B(q_1, r_1) \rangle \\ \langle p_2, A_1(q_2, r_1) + B(q_2, r_1) \rangle \end{pmatrix} \\ \Re(LL) \begin{pmatrix} \delta_2 \\ \delta_4 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \quad (18)$$

where

$$LL = \begin{pmatrix} \langle p_1, A_1(q_1, r_2) + B(q_1, r_2) \rangle & 2f_{200} \\ \langle p_2, A_1(q_2, r_2) + B(q_2, r_2) \rangle & g_{110} \end{pmatrix}.$$

2.2.3 Double Hopf

Although high-dimensional, this case can be treated in a relatively simple manner. Using the same notation as for the generalized Hopf from (5) we get

$$\begin{aligned} Ah_{0000\mu} &= -J_1 v_\mu, \\ (A - i\omega_1 I_n)h_{1000\mu} &= \gamma_{1,\mu} q_1 - A_1(q_1, v_\mu) - B(q_1, h_{0000\mu}), \\ (A - i\omega_2 I_n)h_{0010\mu} &= \gamma_{2,\mu} q_2 - A_1(q_2, v_\mu) - B(q_2, h_{0000\mu}). \end{aligned} \quad (19)$$

As the first equation is non-singular, formal substitution of h_{000010} and h_{000001} and the Fredholm alternative leads to the same transformation (15) from unfolding to the system parameters.

2.3 Implementation of the Predictors

We have implemented our switching routines in MATCONT [6]. For the continuation of LPC and NS curves it uses a minimally augmented defining system [16], i.e. we need to supply an approximation of the limit cycle, the period and the parameters. The parameters follow from applying the inverse transformation to (3). There is always one dynamic variable ψ giving a free phase shift along the bifurcating limit cycle with a period $\frac{2\pi}{\omega_1(\varepsilon)}$. For the initial cycle we make an equidistant mesh $\psi = 2n\pi/N, n = 0 \dots N$ where $N + 1$ is the number of mesh points. Let q denote the eigenvector corresponding to the eigenvalue $i\omega_1$, then points on the limit cycle are given by $x_0 + \varepsilon(qe^{i\psi} + \bar{q}e^{-i\psi})$. Similarly, terms as $\varepsilon^2 h_{20} e^{2i\psi}$ and $\varepsilon^2 h_{0010}$ are included. An internal routine of MATCONT then adapts this limit cycle on an equidistant mesh to a mesh defined at the non-equidistant collocation points.

For the NS curves the system is augmented with the real part k of the multiplier. In this case the normal forms (8),(10) also define a second rotation with frequency $\omega_2(\varepsilon)$ and we have $k = \cos\left(\frac{2\pi\omega_2(\varepsilon)}{\omega_1(\varepsilon)}\right)$.

MATCONT uses Moore-Penrose continuation for which also a tangent vector to the bifurcation curve is needed. This tangent vector is easily obtained by differentiating the predictor w.r.t. ε .

Below we list some case-specific details.

2.3.1 Generalized Hopf

The period is given by $T = 2\pi/\omega + (2d_2 b_{1,2} - \Im(c_1(0)))\varepsilon^2$, with $b_{1,2} = \frac{\partial b_1}{\partial \beta_2}$. The parameters are given by $\alpha = \alpha_0 + V(0, -2d_2\varepsilon^2)^T$.

Note that for a ε^4 -approximation also seventh order derivatives would be needed; this follows from Remark 3.3.2 in [21]. Therefore we restrict to $\mathcal{O}(\varepsilon^3)$ in the implementation.

2.3.2 Zero-Hopf

In the continuation we also need to provide the period and the multiplier. Approximating formulas are defined as follows where x, β_1, β_2 are as in (9)

$$\begin{aligned} T &= 2\pi/\omega(0) - (\omega_1\beta_1 + \omega_2\beta_2 + \Im(g_{110})x) - \Im(g_{021})\varepsilon^2, \\ k &= 1 - (4\pi\Re(g_{110})f_{011})(\varepsilon/\omega_0)^2. \end{aligned} \tag{20}$$

2.3.3 Double Hopf

Approximating formulas for the period and the multiplier on one branch are given by

$$\begin{aligned} T &= \frac{2\pi}{\omega_1 + d\omega_1 \varepsilon^2}, & k &= \cos(T(\omega_2 + d\omega_2 \varepsilon^2)), \\ (d\omega_1, d\omega_2) &= -\Im(\gamma_1 \gamma_2)^T (\Re(\gamma_1 \gamma_2)^T)^{-1} \Re(f_{2100}, g_{1110})^T + \Im(f_{2100}, g_{1110}). \end{aligned} \quad (21)$$

and similarly for the other branch.

3 Examples

3.1 New curves in an extension of the Lorenz-84 model

The first example is an extended version of the Lorenz-84 model. A bifurcation analysis of this model was presented in [22,23]. In this system X models the intensity of a baroclinic wave and Y and Z the sine and cosine coefficients of the wave. This model may be extended with a variable U to study the influence of external parameters such as temperature and the model then shows several limit cycle bifurcations [20]. It has the form:

$$\begin{cases} \dot{X} = -Y^2 - Z^2 - \alpha X + \alpha F - \gamma U^2 \\ \dot{Y} = XY - \beta XZ - Y + G \\ \dot{Z} = \beta XY + XZ - Z \\ \dot{U} = -\delta U + \gamma UX + T \end{cases} \quad (22)$$

The parameters F and T are varied while we fix $\alpha = .25, \beta = 1, G = .25, \delta = 1.04, \gamma = .987$. The bifurcation diagram displays one fold bifurcation and two Hopf bifurcation curves, see Figure 2. We find all codim 2 points of equilibria, in particular GH, ZH and HH .

We have applied our switching routines to all three emanating curves, since the NS bifurcation from ZH is a neutral saddle. The predictions in parameter space are shown in Figure 2 next to the numerically continued curves. The predicted points were used as a starting point for the continuation of these limit cycle bifurcations, which shows that our approach works. Another numerical check is provided by inspecting the tangent vector, which we provide together with a first point. When we find a second point on the curve by continuation and adapt the defining system, we will obtain a more precise

Label	F	T	Normal Form coefficients
GH	2.3763601	.050197432	$d_2 = 0.1558012$
HH	2.5332211	.026273943	$p_{11}p_{22} = -1, \theta = -3.648550, \delta = -1.052987$ $\Theta = 1230.630, \Delta = -210.861$
ZH	1.2834193	.000126541	$s = 1, \theta = 0.3715145, E = -1$

Table 1

Parameter values of F and T at the bifurcation points in Figure 2 together with normal coefficients (scaled, see [18]).

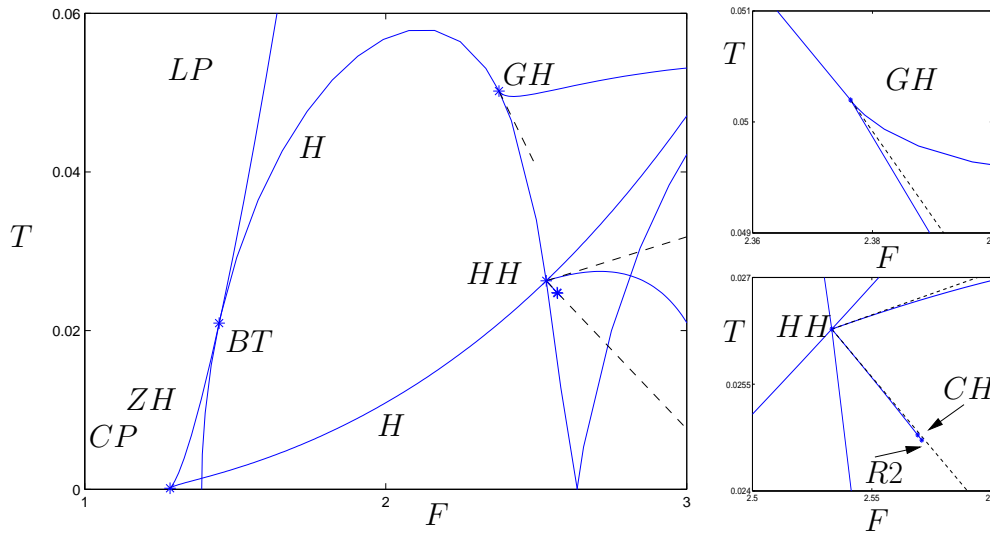


Fig. 2. Bifurcation diagram of the Extended Lorenz-84 model. Symbols denote LP Limit Point, H Hopf, LPC Limit Point of Cycles, NS Neimark-Sacker, GH Generalized Hopf, HH Double Hopf, ZH Zero-Hopf, BT =Bogdanov-Takens. Dashed lines show the predicted new curves; (a) Zoom near the GH point, (b) Zoom near the HH point.

tangent vector. For a small continuation step, this tangent vector and the predicted one should be close. Indeed, for the examples reported here, the first digits always coincided.

Finally we present some measure of the error of the switching routines as a function of the initial amplitude ε , see Figure 3 and its caption. Interestingly this Figure represents the idea of Figure 1. Using a small initial amplitude ε may not work due to a numerical error in the calculated codim 2 point, on the other hand ε must not be taken too large for the approximation to remain valid.

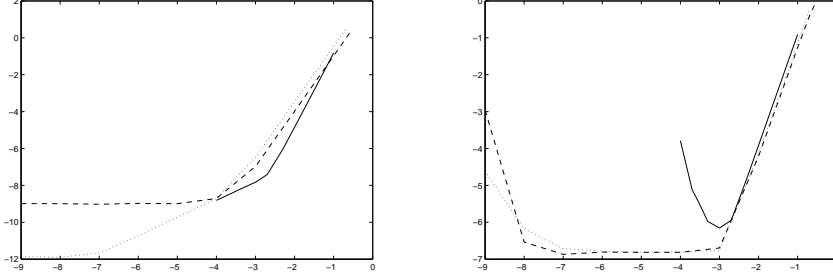


Fig. 3. Error measures:(a) The residual R of the first Newton-step, (b) The distance between the predicted and the first corrected point. For this we left out the tangent vector in the continuation. MATCONT then tries to correct the first point immediately instead of starting the continuation. We have taken 20 mesh and 4 collocation points and $\varepsilon \in [10^{-7}, .2]$. Data shown if predictor converged.

3.2 Switching in a Laser model

In [24] a single-mode inversionless laser with a three-level phaser was studied and shown to operate in various modes. These modes are “off” (non-lasing), continuous waves, periodic, quasi-periodic and chaotic lasing. In particular the boundary of the region of chaos seems to be defined by several limit cycle bifurcations born from several codim 2 equilibrium bifurcations. Thus we want to start with our routines such boundary computations without first doing simulations and limit cycle continuations in this 9-dimensional system.

The model is 9-dimensional system given by 3 real and 3 complex equations:

$$\begin{cases} \dot{\Omega}_l = -\frac{\gamma_{cav}}{2}\Omega_l - g\Im(\sigma_{ab}), \\ \dot{\rho}_{aa} = R_a - \frac{i}{2}(\Omega_l(\sigma_{ab} - \sigma_{ab}^*) + \Omega_p(\sigma_{ac} - \sigma_{ac}^*)), \\ \dot{\rho}_{bb} = R_b + \frac{i}{2}\Omega_l(\sigma_{ab} - \sigma_{ab}^*), \\ \dot{\sigma}_{ab} = -(\gamma_1 + i\Delta_l)\sigma_{ab} - \frac{i}{2}(\Omega_l(\rho_{aa} - \rho_{bb}) - \Omega_p\sigma_{cb}), \\ \dot{\sigma}_{ac} = -(\gamma_2 + i\Delta_p)\sigma_{ac} - \frac{i}{2}(\Omega_p(2\rho_{aa} + \rho_{bb} - 1) - \Omega_l\sigma_{cb}^*), \\ \dot{\sigma}_{cb} = -(\gamma_3 + i(\Delta_l - \Delta_p))\sigma_{cb} - \frac{i}{2}(\Omega_l\sigma_{ac}^* - \Omega_p\sigma_{ab}), \end{cases} \quad (23)$$

with $R_a = -.505\rho_{aa} - .405\rho_{bb} + .45$, $R_b = .0495\rho_{aa} - .0505\rho_{bb} + .0055$ and $\Delta_l := \Delta_{cav} + g\Re(\sigma_{ab})/\Omega_l$. The parameters are fixed at $\gamma_1 = .05$, $\gamma_2 = .25525$, $\gamma_3 = .25025$, $\gamma_{cav} = .03$, $g = 100$, $\Delta_p = 0$ while Δ_{cav} and Ω_p are varied to study several detuning effects. For more details, see [24].

We have reproduced a part from the bifurcation diagram which corresponds to continuous wave and periodically pulsating solutions, i.e. with $\Omega_l \neq 0$, see Figure 4. As the system has \mathbb{Z}_2 -symmetry the same bifurcations are found for $\Delta_{cav} \rightarrow -\Delta_{cav}$. For clarity of the figure we do not display these here. We list

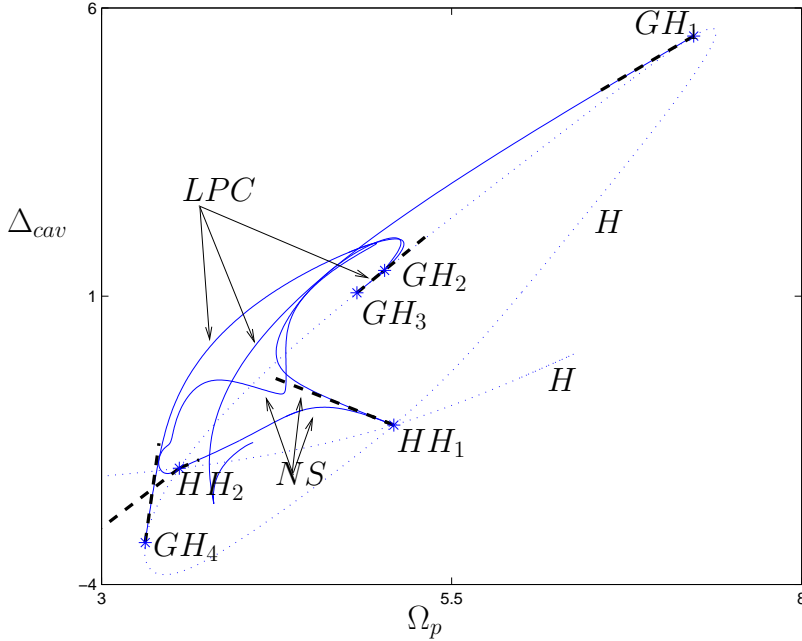


Fig. 4. Bifurcation diagram of the inversionless laser. Hopf curves(denoted by H) are dotted, Limit cycle bifurcations are denoted by LPC Limit Point of Cycles, NS Neimark-Sacker. Dashed lines show the predicted new curves.

the codim 2 points in Table 3.2. The normal form coefficients of HH_1 confirm the claim of [24] that the most complicated type was encountered; only the 3-torus is (un)stable. This is also confirmed when we continue the Neimark-Sacker bifurcations. For HH_2 the NS curves are not in the same quadrant defined by the Hopf curves, while they are for HH_1 . All cycle bifurcations were computed with 20 mesh points and 4 collocation points and the initial amplitude was set to $\varepsilon = .001$, which worked immediately in all cases. Let us remark that one LPC curve connects GH_2 and GH_3 points and stays close to the Hopf curve. Similarly, a NS curve starts at HH_1 , becomes neutral between two 1:2 resonances and ends at HH_2 . It would have taken much more effort to find this feature otherwise.

4 Discussion

This paper contributes to the bifurcation analysis of codim 2 singularities of equilibria in multidimensional ODEs by providing explicit predictors for branches of nonhyperbolic cycles emanating from these bifurcations. We have tested it on several examples with good results. We believe that this work will further facilitate automated analysis of nonlinear systems. However, we like to mention that we also tried the double Hopf point in a model for the lateral

Label	Ω_p	Δ_{cav}	Normal Form coefficients
GH_1	7.228819	5.511455	$d_2 = -46.49852$
GH_2	5.021574	1.446387	$d_2 = 3.813132$
GH_3	4.824066	1.059367	$d_2 = 195.1119$
GH_4	3.312120	-3.273568	$d_2 = -6.468468$
HH_1	5.087299	-1.2362053	$p_{11}p_{22} = -1, \theta = -.07194543, \delta = -13.91412$ $\Theta = .9595389, \Delta = -2602.275$
HH_2	3.555848	-1.983857	$p_{11}p_{22} = 1, \theta = -.1179924, \delta = -26.59452$ $\Theta = -10.81042, \Delta = -2713.608$

Table 2

Parameter values of Ω_p and Δ_{cav} at the codim 2 points together with normal coefficients (scaled, see [18]).

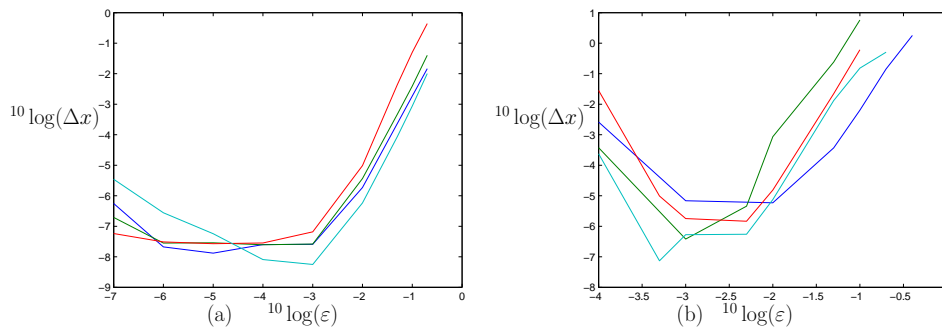


Fig. 5. Error measures along the eight curves: $10 \log$ of the distance between the predicted and the first corrected point versus $10 \log(\varepsilon)$. (a) Along the Neimark-Sacker curves (b) Along the LPC curves. This Figure again resembles the idea of Figure 1.

pyloric neuron [9,10]. Although we were able to switch to one branch and continue it without any problem, the Jacobian of the defining system along the second branch was numerically singular. In this model with multiple time scales probably a special numerical scheme is necessary.

It is well known that branches of orbits homoclinic to hyperbolic equilibria are also rooted at **BT**, **ZH**, and **HH** codim 2 bifurcation points. The **BT** case has been treated in [2] (see also [3], where the computational setting is most close to the present paper). The corresponding predictor for the homoclinic branch is implemented in **MATCONT**. The problem of providing predictors for homoclinic branches rooted at **ZH** and **HH** points is more challenging. Some important results in this direction are obtained in [4,7,5], where the systems reduced to the center manifold were considered. However, a complete set of formulas suitable for switching to homoclinic curves in these cases is still not available. For instance, in the **ZH** case the normal form (8) exhibits

homoclinic bifurcations of saddle-focus equilibria in the parameter plane along a bifurcation curve with the linear approximation

$$\beta_{2,hom} = \frac{\Re(g_{110})\beta_1}{f_{200}(2f_{200} - 3\Re(g_{110}))} \left[\Re(g_{210}) - \frac{3\Re(g_{110})}{2f_{200}}f_{300} + \frac{(f_{200} - \Re(g_{110}))}{f_{011}}f_{111} - \frac{2(f_{200} - \Re(g_{110}))^2}{f_{011}\Re(g_{110})}\Re(g_{021}) \right],$$

provided that $\Re(g_{110})f_{011} < 0$ and $\Re(g_{110})f_{200} < 0$. Application of the above reduction to the parameter-dependent center manifolds in the ZH case yields an approximation to the bifurcation curve in the parameter plane. Now the challenge is to construct a suitable initial solution in state space. On this work in progress will be reported elsewhere.

Another direction for future research is a problem of switching to secondary cycle bifurcations at codim 2 bifurcations of cycles in (1). Here a generalization of the periodic normalization technique from [16] to critical codim 2 cases and its extension to parameter-dependent systems in the spirit of [8] are required.

Acknowledgement

The authors want to thank S. Wiezcorek for bringing up and his assistance with the laser model.

References

- [1] V.I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer-Verlag, New York, Heidelberg, Berlin, 1983.
- [2] W.-J. Beyn. Numerical analysis of homoclinic orbits emanating from a Takens-Bogdanov point. *IMA J. Numer. Anal.*, 14:381–410, 1994.
- [3] W.-J. Beyn, A. Champneys, E. Doedel, W. Govaerts, Yu.A. Kuznetsov, and B. Sandstede. Numerical continuation, and computation of normal forms. In B. Fiedler, editor, *Handbook of Dynamical Systems, Vol. 2*, pages 149–219. Elsevier Science, Amsterdam, 2002.
- [4] H. W. Broer and G. Vegter. Subordinate Šil’nikov bifurcations near some singularities of vector fields having low codimension. *Ergodic Theory Dynam. Systems*, 4:509–525, 1984.
- [5] A. R. Champneys and V. Kirk. The entwined wiggling of homoclinic curves emerging from saddle-node/Hopf instabilities. *Phys. D*, 195:77–105, 2004.

- [6] A. Dhooge, W. Govaerts, and Yu.A. Kuznetsov. MATCONT:A MATLAB package for numerical bifurcation analysis of ODEs. *ACM Trans. Math. Software*, 29:141–164, 2003.
- [7] P. Gaspard. Local birth of homoclinic chaos. *Phys. D*, 62:94–122, 1993.
- [8] R.K. Ghaziani, W. Govaerts, Yu.A. Kuznetsov, and H.G.E. Meijer. Numerical methods for two-parameter local bifurcation analysis of maps. *SIAM J. Sci. Comput.*, 29:2644–2667, 2007.
- [9] W. Govaerts, J. Guckenheimer, and A. Khibnik. Defining functions for multiple Hopf bifurcations. *SIAM J. Numer. Anal.*, 34(3):1269–1288, 1997.
- [10] W. Govaerts, Yu. A. Kuznetsov, and B. Sijnave. Numerical methods for the generalized Hopf bifurcation. *SIAM J. Numer. Anal.*, 38(1):329–346, 2000.
- [11] W.J.F. Govaerts. *Numerical Methods for Bifurcations of Dynamical Equilibria*. SIAM, Philadelphia, 2000.
- [12] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. Springer-Verlag, New York, 1983.
- [13] M. Ipsen, F. Hynne, and P. G. Sørensen. Systematic derivation of amplitude equations and normal forms for dynamical systems. *Chaos*, 8:834–852, 1998.
- [14] M. Ipsen, F. Hynne, and P.G. Sørensen. Amplitude equations for reaction-diffusion systems with a hopf bifurcation and slow real modes. *Phys. D*, 136:66–92, 2000.
- [15] A.D. Jepson and D.W. Decker. Convergence cones near bifurcation. *SIAM J. Numer. Anal.*, 23:959–975, 1986.
- [16] Yu. A. Kuznetsov, W. Govaerts, E. J. Doedel, and A. Dhooge. Numerical periodic normalization for codim 1 bifurcations of limit cycles. *SIAM J. Numer. Anal.*, 43:1407–1435, 2005.
- [17] Yu.A. Kuznetsov. Numerical normalization techniques for all codim 2 bifurcations of equilibria in ODEs. *SIAM J. Numer. Anal.*, 36:1104–1124, 1999.
- [18] Yu.A. Kuznetsov. *Elements of Applied Bifurcation Theory*. Springer Verlag, Berlin, 2004. Third Edition.
- [19] Yu.A. Kuznetsov and V.V. Levitin. CONTENT: A multiplatform environment for analyzing dynamical systems. (<ftp.cwi.nl/pub/CONTENT>), 1995–1997.
- [20] Yu.A. Kuznetsov, H.G.E. Meijer, and L. van Veen. The fold-flip bifurcation. *Int. J. Bif. Chaos*, 14:2253–2282, 2004.
- [21] H.G.E. Meijer. *Codimension 2 Bifurcations of Iterated Maps*. PhD thesis, Utrecht University, Netherlands, 2006.
- [22] A. Shil’nikov, G. Nicolis, and C. Nicolis. Bifurcation and predictability analysis of a low-order atmospheric circulation model. *Int. J. Bif. Chaos*, 5:1701–1711, 1995.

- [23] Lennaert van Veen. Baroclinic flow and the Lorenz-84 model. *Int. J. Bif. Chaos*, 13:2117–2139, 2003.
- [24] S. Wiczorek and W.W. Chow. Self-induced chaos in a single-mode inversionless laser. *Phys. Rev. Lett.*, 97:113903, 2006.