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# Optimization of structures with uncertain constraints based on convex model and satisfaction degree of interval

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**Abstract**

An optimization method for uncertain structures is suggested based on convex model and a satisfaction degree of interval. In the investigated problem, the uncertainty only exists in constraints. Convex model is used to describe the uncertainty in which the intervals of the uncertain parameters are only needed, not necessarily to know the precise probability distributions. A satisfaction degree of interval which represents the possibility that one interval is smaller than another is employed to deal with the uncertain constraints. Based on a predetermined satisfaction degree level, the uncertain constraints are transformed to deterministic ones, and the transformed optimization problem can be solved by traditional optimization methods. For complex structural problems that the optimization model cannot be expressed in an explicit form, the interval analysis method is adopted to calculate the intervals of the constraints efficiently, and whereby eliminate the optimization nesting. Two numerical examples have been presented to demonstrate the efficiency of the suggested method.


*Keywords:* Uncertainty; Convex model; Satisfaction degree of interval; Interval analysis; Structural optimization

## 1. Introduction

Uncertainty in material properties, geometric dimensions, loads and other parameters are always unavoidable in engineering structural problems. To obtain a reliable design, the effects of the various uncertain factors should be considered. The probability models are widely used to describe the uncertainty and whereby many probability optimization methods have been developed. The reference [1] may be the first one to use probability optimization, and this work has been followed by numerous other applications. However, the amount of the information available for the uncertain parameters is often not enough to accurately define the probability distributions. In addition, Ben-Haim and Elishakoff [2] have indicated that even small deviations from the real probability distributions may cause large errors in the reliability analysis.

In recent years, another method named convex model [2] is developed to deal with the uncertainty in terms of the convex domains, which range from one-dimensional uniformly bounded lines to multi-dimensional boxes or ellipsoids. Convex model forms a convex set of functions or vectors, and each element of the set represents a possible realization of an uncertain event. The size of the convex set reflects the variability of the events and is described by a particular shape, i.e. multi-dimensional box or ellipsoid, and thus the clustering of the uncertain events is defined [3]. The algebraic definition of the convexity should be satisfied for convex sets. Using convex model, only the knowledge of the bounds of the uncertain parameters is required, instead of the precise probability distributions.

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Thus the uncertainty analysis will become more convenient and economical. Detailed theory on convex model can refer to the references e.g. [2,4]. Ben-Haim [5] gave a comparison of the probability method and convex model. Elishakoff [6,7] employed an "uncertain triangle" to represent the three methods used to describe uncertainty, namely probability method, fuzzy sets and convex model. Furthermore, convex model has been applied to many aspects of engineering mechanics, such as non-linear buckling of a column with uncertain initial imperfections [8], stability analysis of the elastic bars on uncertain foundations [9], bound analysis of a beam [10] and uncertain analysis in structural number determination in flexible pavement design [11] etc.

Applying convex model to the optimization of uncertain structures has not been widely studied. Elishakoff et al. [12] optimized a truss with uncertain but bounded loads. Convex model is implemented on the uncertain constraints through an anti-optimization process, which seeks for the worst condition of the constraints caused by the uncertain loads. The anti-optimization technique based on convex model was further developed by Lombardi et al. [13] for the study of composite materials and Barbieri et al. [14] for the shape and size optimization of trusses. Lombardi [15] proposed a two step optimization method, where the anti-optimization was solved only once for all constraints before starting the optimization and it took a great computational saving. Ganzerli and Pantelides [16] proposed a superposition method of convex model to eliminate the anti-optimization process, and thus the optimization efficiency was improved greatly. Au et al. [17] proposed a novel method of robust design using convex model. Gurav et al. [18] provided an enhanced anti-optimization technique, which incorporated design sensitivities with database technique and was further modified to use parallel computing in order to promote the computational efficiency. In these mentioned methods (being termed as "worst-case method" in the following text), the uncertain optimization can be divided into two processes, namely the main optimization and the anti-optimization. In anti-optimization, a worst condition case of the constraints will be searched in the domain of the uncertain parameters, and based on it the constraints are judged whether they are satisfied or not. The worst-case method gives an approach to deal with the uncertain constraints, however, it requires the constraints to be satisfied for all of the possible values of the uncertain parameters. Thus it is a relatively conservative method, and will make the treatment of the uncertain constraints excessively strict and hence often lead to an unexpected design objective. In probability optimization, we generally expect the constraints to be satisfied with a predetermined confidence level [19]. Through changing the confidence level, the satisfying extent of the constraints under the uncertain parameters can be adjusted. Here this concept of the satisfying extent of constraints may be extended into the structural optimization using convex model, and develop a more general approach based on the expected constraint satisfying level instead of the worst condition. Thus the restricting degree of the uncertain constraints can be relaxed at certain extent according to the practical problem, and whereby the design objective can be always improved.

In this paper, an optimization method is suggested to deal with the uncertain structures. Only the uncertain constraints are considered in this method. The uncertainty in structures is modeled by convex model in which the intervals of the uncertain parameters are only required. The satisfaction degree of interval is extended to the structural optimization with uncertainty, and it is used to compare the interval of a constraint caused by the uncertainty with the allowable value. Based on a satisfaction degree level, the uncertain constraint is changed to a deterministic one. Through adjusting the satisfaction degree level, the feasible field of the design vector can be changed. The worst-case method is just a special case of the constraint satisfaction degree method. The penalty function method is adopted to deal with the transformed deterministic constraints, and the intergeneration projection genetic algorithm (IP-GA) [20,21] is employed as optimization operator. The interval analysis method is used to obtain the bounds of the constraints at each optimization iterate efficiently for complex structural problems, thus the optimization nesting can be avoided. The presented method is applied to two numerical examples, namely a 10-bar truss and a 25-bar truss, and it is proven effective.

## 2. Formulation of the algorithm

A general structural optimization problem can be formulated as follows:

$$\min_{\mathbf{X}} f(\mathbf{X})$$

subject to

$$\begin{aligned} & g_i(\mathbf{X}) \leqslant b_i, \quad i = 1, 2, \ldots, l, \\ & \mathbf{X} \in \Omega^n, \end{aligned} \tag{1}$$

where $f$ is the objective function and $\mathbf{X}$ is an $n$-dimensional design vector. $g_i$ is the $i$th constraint function and $l$ is the number of the constraints. $b_i$ is an allowable value of the $i$th constraint. In practical structural problems, $f$ and $g$ may depend on some parameters with uncertainty. In our study, the uncertainty is limited in the constraints and Eq. (1) becomes:

$$\min_{\mathbf{X}} f(\mathbf{X})$$

subject to

$$\begin{aligned} & g_i(\mathbf{X}, \mathbf{U}) \leqslant b_i, \quad i = 1, 2, \ldots, l, \\ & \mathbf{X} \in \Omega^n, \end{aligned} \tag{2}$$

where $\mathbf{U}$ is an $m$-dimensional uncertain vector. Using convex model to describe the uncertainty, the intervals of the uncertain parameters are only needed. Then the convex domain $c_{\mathbf{U}}$ of the uncertainty can be expressed as follows:

$$c_{\mathbf{U}} = \mathbf{U}^I = U_i^I = \{U_i^{\mathrm{L}} \leqslant U_i \leqslant U_i^{\mathrm{R}}\}, \quad i = 1, 2, \ldots, m. \tag{3}$$

The superscripts $I$, $L$ and $R$ denote interval, lower and upper bounds of interval, respectively. The convex domain is a bounded set which limits the variation of the uncertain parameters. Thus for each optimization iterate of $\mathbf{X}$, the possible values of the constraint $g_i$ will form an interval provided that $g_i$ is a continuous function of $\mathbf{U}$. In the following sections, a satisfaction degree of interval will be introduced and subsequently used to deal with the uncertain constraints.

### 2.1. Satisfaction degree of interval

The satisfaction degree of interval represents the possibility that one interval is larger or smaller than another, and it is used to compare intervals. For intervals $A$ and $B$, the reference [22] gives a definition of the satisfaction degree $p(A \leqslant B)$ as follows:

$$p(A \leqslant B) = \frac{\max(0, \mathrm{len}(A) + \mathrm{len}(B) - \max(0, A^{\mathrm{R}} - B^{\mathrm{L}}))}{\mathrm{len}(A) + \mathrm{len}(B)}, \tag{4}$$

where 'len' denotes the length of an interval:

$$\mathrm{len}(A) = A^{\mathrm{R}} - A^{\mathrm{L}}, \quad \mathrm{len}(B) = B^{\mathrm{R}} - B^{\mathrm{L}}. \tag{5}$$

$p(A \leqslant B)$ is actually a fuzzy definition of the possibility that interval $A$ is smaller than interval $B$ (or $B$ is larger than $A$). For intervals $A$ and $B$, there are a total of six positional cases as shown in Fig. 1, and based on these Eq. (4) can be rewritten as follows:

$$p(A \leqslant B) = \begin{cases} 0, & \text{Case } 1, \\ \frac{B^{\mathrm{R}} - A^{\mathrm{L}}}{\mathrm{len}(A) + \mathrm{len}(B)}, & \text{Cases } 2,\ 3,\ 4 \text{ and } 5, \\ 1, & \text{Case } 6. \end{cases} \tag{6}$$

It can be found that $p(A \leqslant B)$ is equal to 0 for case 1 as $A$ is always larger than $B$. For case 6, $p(A \leqslant B)$ is equal to 1 as $A$ is always smaller than $B$. For cases 2–5, under the fixed len($A$) and len($B$), the value of $p(A \leqslant B)$ will become larger with the increasing of $B^{\mathrm{R}}$ or decreasing of $A^{\mathrm{L}}$. It is because that interval $B$ will move toward the right side of the coordinate axis as a whole or $A$ toward the left side under the fixed lengths, and intuitively $A$ has a larger possibility to be smaller than $B$ and hence $p(A \leqslant B)$ becomes larger. As a result, $p(A \leqslant B)$ has the following properties:

1. $0 \leqslant p(A \leqslant B) \leqslant 1$.
2. If $p(A \leqslant B) = p(B \leqslant A)$, then $p(A \leqslant B) = p(B \leqslant A) = \frac{1}{2}$ and $A = B$.
3. If $A^{\mathrm{R}} \leqslant B^{\mathrm{L}}$, then $p(A \leqslant B) = 1$, and it represents that $A$ is absolutely less than $B$.
4. If $A^{\mathrm{L}} \geqslant B^{\mathrm{R}}$, then $p(A \leqslant B) = 0$, and it represents that $A$ is absolutely larger than $B$.
5. If $p(A \leqslant B) = q$, then $p(A \geqslant B) = 1 - q$.

When the interval $B$ is degenerated into a real number $b$, the satisfaction degree $p(A \leqslant b)$ has the following form:

$$p(A \leqslant b) = \frac{\max(0, \mathrm{len}(A) - \max(0, A^{\mathrm{R}} - b))}{\mathrm{len}(A)}. \tag{7}$$

Fig. 2 is a geometrical description of Eq. (7). If $A^{\mathrm{R}} \leqslant b$ and $A^{\mathrm{L}} \geqslant b$, $p(A \leqslant b) = 1$ and $p(A \leqslant b) = 0$, respectively; if $A^{\mathrm{L}} \leqslant b \leqslant A^{\mathrm{R}}$, $p(A \leqslant b)$ behaves a linear relation with respect to $b$.

### 2.2. Treatment of the uncertain constraints

In probability optimization, generally the constraints are made to be satisfied with a certain predetermined confidence level and the uncertain constraints are transformed to the deterministic constraints [19]. Similarly, we can make the constraints in Eq. (2) satisfied with certain satisfaction degree level:

$$p(C_i \leqslant b_i) \geqslant \lambda_i, \quad i = 1, 2, \ldots, l, \tag{8}$$

$$C_i = [g_i^{\mathrm{L}}(\mathbf{X}), g_i^{\mathrm{R}}(\mathbf{X})], \tag{9}$$

where $\lambda_i$ is a predetermined satisfaction degree level of the $i$th constraint. $C_i$ is an interval of the $i$th constraint at $\mathbf{X}$ which is caused by the uncertainty, and $g_i^{\mathrm{L}}(\mathbf{X})$ and $g_i^{\mathrm{R}}(\mathbf{X})$ are the lower and upper bounds of this interval, respectively:

$$g_i^{\mathrm{L}}(\mathbf{X}) = \min_{\mathbf{U} \in c_{\mathbf{u}}} g_i(\mathbf{X}, \mathbf{U}), \quad g_i^{\mathrm{R}}(\mathbf{X}) = \max_{\mathbf{U} \in c_{\mathbf{U}}} g_i(\mathbf{X}, \mathbf{U}). \tag{10}$$

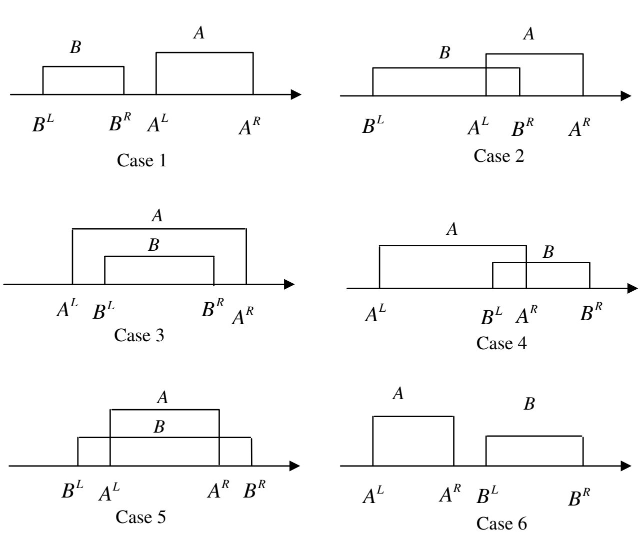


Fig. 1. Six positional relations between intervals $A$ and $B$.

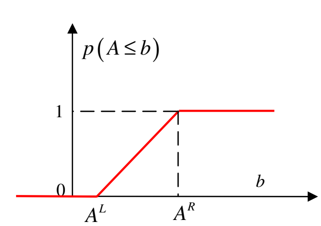


Fig. 2. Satisfaction degree $p(A \leqslant b)$.

The satisfaction degree $p(C_i \leqslant b_i)$ can be calculated by Eq. (7). Through Eq. (10), the uncertain vector $\mathbf{U}$ is eliminated, and the transformed constraints Eq. (8) become deterministic. $\lambda_i$ can be adjusted to control the feasible field of $\mathbf{X}$. When $\lambda_i$ is larger, the inequality constraints Eq. (8) will be restricted more strictly and the feasible field of $\mathbf{X}$ will become smaller. When $\lambda_i$ reaches 1.0, the treatment of the uncertain constraints is most conservative. It requires the constraints to be satisfied for all of the possible combinations of the uncertain parameters, and this is actually the worst-case method adopted by the current publications [12–17]. When $\lambda_i$ is 0, Eq. (8) is absolutely satisfied and it is actually a non-constraint treatment. Thus the present method gives a more general form to deal with the uncertain constraints in Eq. (2), and the worst-case method is just its special case.

Through the above treatment, the uncertain optimization problem Eq. (2) can be formulated as a deterministic optimization problem:

$$\min_{\mathbf{X}} f(\mathbf{X})$$

subject to

$$\begin{aligned} &p(C_i \leqslant b_i) \geqslant \lambda_i, \quad i = 1, 2, \ldots, l, \\ &C_i = [g_i^{\mathrm{L}}(\mathbf{X}), g_i^{\mathrm{R}}(\mathbf{X})] = \left[\min_{\mathbf{U} \in c_{\mathbf{u}}} g_i(\mathbf{X}, \mathbf{U}), \max_{\mathbf{U} \in c_{\mathbf{u}}} g_i(\mathbf{X}, \mathbf{U})\right], \\ &\quad \mathbf{X} \in \Omega^n. \end{aligned} \tag{11}$$

Obviously, Eq. (11) can be solved in traditional deterministic way.

*2.3. Computational procedure*

Using the penalty function method [23], Eq. (11) can be changed as a non-constraint optimization problem in terms of a penalty function $\tilde{f}$:

$$\min_{\mathbf{X}} \tilde{f} = f(\mathbf{X}) + \sigma \sum_{i=1}^{l} \varphi(p(C_i \leqslant b_i) - \lambda_i), \tag{12}$$

where $\sigma$ is a penalty factor which is usually specified as a large value, and $\varphi$ is a function which has the following form:

$$\varphi(p(C_i \leqslant b_i) - \lambda_i) = (\max(0, -(p(C_i \leqslant b_i) - \lambda_i)))^2. \tag{13}$$

In this paper, the intergeneration projection genetic algorithm (IP-GA) is adopted as optimization operator. IP-GA combines the micro GA with the intergeneration projection operator, and has a fine global convergence performance [20,21]. The optimization flowchart is shown in Fig. 3. In the optimization process, many trial design vectors are generated, and for each one the objective function and the intervals of the uncertain constraints are calculated. Then the constraint satisfaction degrees and the penalty function are also obtained. The generation number is employed as stopping criterion, and the optimization circle is repeated until the stopping criterion is satisfied. GA is a kind of global optimization method, and it needs only the information of functional values and hence the derivative information is not required. GA is also a robust method as its solutions can be ensured to be better and better with the generations. Generally, the number of iterative generations is used as stopping criterion for GA. In theory, GA

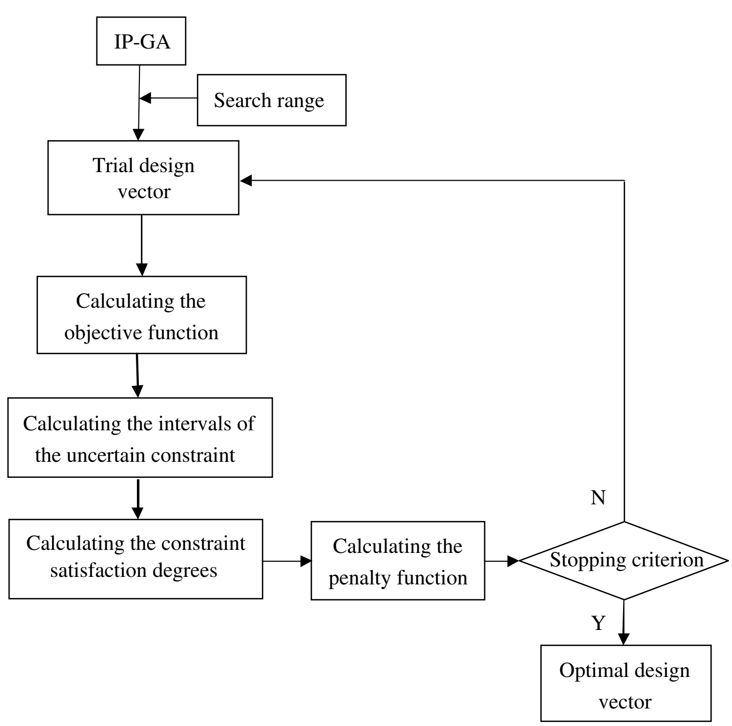


Fig. 3. Optimization flowchart based on IP-GA.

can search an enough accurate global optimum as long as it is given an enough number of generations. However, excessive generations sometimes will lead to unacceptable computation cost. Thus an appropriate generation number needs to be specified according to the practical engineering problem when using GA.

If the optimization model can be expressed in an explicit form of the uncertain parameters and furthermore it is linear, the intervals of the uncertain constraints can be obtained explicitly at each iterate. The optimization process in this case will be analyzed in the following numerical example of 10-bar truss. For many practical complex structures, the optimization model is always based on finite element method (FEM) and hence implicit. If two optimization processes defined by Eq. (10) are employed to calculate the bounds of each constraint, the optimization nesting will be caused inevitably and the optimization efficiency will be very low. In this paper, the interval analysis method [24,25] will be adopted to calculate the intervals of the constraints at each iterate very quickly, and whereby a much higher efficiency can be achieved than the nesting optimization. The optimization process in this case will be analyzed in the following numerical example of 25-truss.

## 3. Numerical examples and discussion

### 3.1. 10-bar aluminium truss

A well-known 10-bar aluminium truss [12,15–17] as shown in Fig. 4 is investigated. The cross-sectional areas $A_j$, $j = 1, 2, \ldots, 10$ of the bars are optimized to obtain a minimum weight subject to the stress and displacement constraints. The length $L$ of the horizontal and vertical bars is 9.144 m. The Young's modulus $E$ of the truss is 68,948 MPa and the density $\rho$ is 2768 kg/m$^3$. The maximum allowable stress of bar 9 in tension or compression is 517.11 MPa. The other bars have a same allowable stress in tension or compression which is 172.37 MPa. A maximum vertical displacement constraint with 0.1270 m is applied on joint 2. Joint 4 is subjected to a vertical load $F_1$, and joint 2 is subjected to a horizontal load $F_3$ and a vertical load $F_2$. In this numerical example, only the loads are uncertain and the uncertainty domain is

$$c_U = \left\{F_1^L \leqslant F_1 \leqslant F_1^R, F_2^L \leqslant F_2 \leqslant F_2^R, F_3^L \leqslant F_3 \leqslant F_3^R\right\}. \tag{14}$$

The nominal values of the loads are: $F_1 = F_2 = 444.8$ kN, $F_3 = 1779.2$ kN.

According to the equilibrium and compatibility equations, the axial forces $N_j$, $j = 1, 2, \ldots, 10$ in the bars can be achieved explicitly [17]:

$$N_1 = F_2 - \frac{\sqrt{2}}{2}N_8, \quad N_2 = -\frac{\sqrt{2}}{2}N_{10} \tag{15}$$

$$N_3 = -F_1 - 2F_2 + F_3 - \frac{\sqrt{2}}{2}N_8,$$

$$N_4 = -F_2 + F_3 - \frac{\sqrt{2}}{2}N_{10}, \tag{16}$$

$$N_5 = -F_2 - \frac{\sqrt{2}}{2}N_8 - \frac{\sqrt{2}}{2}N_{10}, \quad N_6 = -\frac{\sqrt{2}}{2}N_{10}, \tag{17}$$

$$N_7 = \sqrt{2}(F_1 + F_2) + N_8, \quad N_8 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}, \tag{18}$$

$$N_9 = \sqrt{2}F_2 + N_{10}, \quad N_{10} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}, \tag{19}$$

$$a_{11} = \left(\frac{1}{A_1} + \frac{1}{A_3} + \frac{1}{A_5} + \frac{2\sqrt{2}}{A_7} + \frac{2\sqrt{2}}{A_8}\right)\frac{L}{2E}, \tag{20}$$

$$a_{12} = a_{21} = \frac{L}{2A_5E}, \tag{21}$$

$$a_{22} = \left(\frac{1}{A_2} + \frac{1}{A_4} + \frac{1}{A_6} + \frac{2\sqrt{2}}{A_9} + \frac{2\sqrt{2}}{A_{10}}\right)\frac{L}{2E}, \tag{22}$$

$$b_1 = \left(\frac{F_2}{A_1} - \frac{F_1 + 2F_2 - F_3}{A_3} - \frac{F_2}{A_5} - \frac{2\sqrt{2}(F_1 + F_2)}{A_7}\right)\frac{\sqrt{2}L}{2E}, \tag{23}$$

$$b_2 = \left(\frac{\sqrt{2}(F_3 - F_2)}{A_4} - \frac{\sqrt{2}F_2}{A_5} - \frac{4F_2}{A_9}\right)\frac{L}{2E}. \tag{24}$$

The vertical displacement $\delta_2$ of joint 2 can be calculated through the following equation:

$$\delta_2 = \left(\sum_{i=1}^{6}\frac{N_i^0 N_i}{A_i} + \sqrt{2}\sum_{i=7}^{10}\frac{N_i^0 N_i}{A_i}\right)\frac{L}{E}, \tag{25}$$

where $N_i^0$ can be obtained from Eqs. (15)–(19) with a substitution $F_1 = F_3 = 0$ and $F_2 = 1$.

The optimization model can be formulated as follows according to Eq. (11):

$$\min_{\mathbf{A}} W(\mathbf{A}) = \sum_{i=1}^{10}(\rho L_i A_i) = \rho L\left(\sum_{i=1}^{6}A_i + \sqrt{2}\sum_{i=7}^{10}A_i\right)$$

subject to

$$\begin{aligned} &p(\sigma_i^I(\mathbf{A}) \leqslant \sigma_{i,\text{allow}}) \geqslant \lambda_i, \quad i = 1, 2, \ldots, 10, \\ &p(\delta_2^I(\mathbf{A}) \leqslant \delta_{2,\text{allow}}) \geqslant \lambda_{11}, \quad \mathbf{A} \in \Omega^{10}, \end{aligned} \tag{26}$$

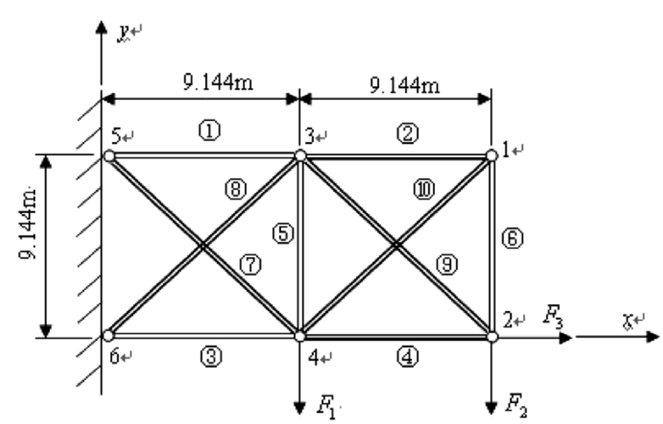


Fig. 4. A 10-bar aluminium truss [12].

where

$$\sigma_i^I(\mathbf{A}) = [\sigma_i^{\mathrm{L}}(\mathbf{A}), \sigma_i^{\mathrm{R}}(\mathbf{A})], \quad \delta_2^I(\mathbf{A}) = [\delta_2^{\mathrm{L}}(\mathbf{A}), \delta_2^{\mathrm{R}}(\mathbf{A})], \tag{27}$$

where $W$ denotes the weight of the truss. The bounds of the stress interval $\sigma_i^I(\mathbf{A})$ and the displacement interval $\delta_2^I(\mathbf{A})$ for a specific design vector $\mathbf{A}$ can be obtained:

$$\sigma_i^{\mathrm{L}}(\mathbf{A}) = \min_{\mathbf{F}\in c_{\mathbf{U}}} \frac{|N_i(\mathbf{F},\mathbf{A})|}{A_i}, \quad \sigma_i^{\mathrm{R}}(\mathbf{A}) = \max_{\mathbf{F}\in c_{\mathbf{U}}} \frac{|N_i(\mathbf{F},\mathbf{A})|}{A_i}, \quad i = 1,2,\ldots,10, \tag{28}$$

$$\delta_2^{\mathrm{L}}(\mathbf{A}) = \min_{\mathbf{F}\in c_{\mathbf{U}}} \delta_2(\mathbf{F},\mathbf{A}), \quad \delta_2^{\mathrm{R}}(\mathbf{A}) = \max_{\mathbf{F}\in c_{\mathbf{U}}} \delta_2(\mathbf{F},\mathbf{A}). \tag{29}$$

Because $N_i$ and $\delta_2$ are both linear functions of the load vector $\mathbf{F}$, their extreme points can be found at the vertices of the convex set $c_{\mathbf{U}}$ [2]. Thus the bounds of $N_i$ and $\delta_2$ can be achieved only through comparing the eight combinations of the bounds of the uncertain loads. Therefore Eqs. (28) and (29) can be rewritten as follows:

$$\left.\begin{array}{l} \sigma_i^{\mathrm{L}}(\mathbf{A}) = \min \frac{|N_i(\mathbf{F},\mathbf{A})|}{A_i}, \ \sigma_i^{\mathrm{R}}(\mathbf{A}) = \max \frac{|N_i(\mathbf{F},\mathbf{A})|}{A_i} \\ \delta_2^{\mathrm{L}}(\mathbf{A}) = \min \delta(\mathbf{F},\mathbf{A}), \ \delta_2^{\mathrm{R}}(\mathbf{A}) = \max \delta(\mathbf{F},\mathbf{A}) \end{array}\right\} F_j = F_j^{\mathrm{L}}, F_j^{\mathrm{R}}, \quad i = 1,2,\ldots,10, \ j = 1,2,3. \tag{30}$$

Thus only through eight evaluations of the constraint function, the interval of each constraint can be obtained at a specific design vector.

The stopping criterion is set as 1000 generations for IP-GA. The uncertainty level is 10% off from the nominal values of the loads. The search range $\Omega^{10}$ is specified as $\{0.645\ \mathrm{cm}^2 \leqslant A_i \leqslant 96.8\ \mathrm{cm}^2,\ i = 1,2,\ldots,10\}$. All of the constraints are set as the same satisfaction degree level. The optimization results under the satisfaction degree levels 1.0, 0.8, 0.6, 0.4 and 0.2 are listed in Tables 1–5, respectively. It is found that the minimum weight of the truss decreases with decreasing of the satisfaction degree level. The relation between the minimum weight and the satisfaction degree level is shown in Fig. 5, and it can be seen that they behave an approximate linear relation. For satisfaction degree level 1.0, the design weight of the truss has a maximum value 886.19 kg, and this is the most conservative and costly design. For satisfaction degree level 0.2, it reaches a minimum value 678.17 kg. Thus from the view point of the manufacturing cost, the small satisfaction degree level is expected. However, small satisfaction degree level means large possibility of violating the constraints. For satisfaction degree level 0.0, the possibility is largest as it has degenerated into a non-constraint optimization problem. From Tables 1–5, it also can be found that the satisfaction degrees of the constraints at the optimum also decrease with decreasing of the satisfaction degree level,

Table 1
Optimization results under the satisfaction degree level 1.0 (10-bar)

| Bar no. | Area (cm$^2$) | Stress interval (MPa) | Satisfaction degree |
|---|---|---|---|
| 1 | 29.08 | [138.57, 170.81] | 1.00 |
| 2 | 0.65 | [72.12, 105.75] | 1.00 |
| 3 | 44.08 | [31.95, 172.16] | 1.00 |
| 4 | 90.75 | [123.29,172.07] | 1.00 |
| 5 | 29.04 | [134.11, 164.83] | 1.00 |
| 6 | 0.65 | [72.12, 105.75] | 1.00 |
| 7 | 79.85 | [141.01, 172.34] | 1.00 |
| 8 | 0.69 | [53.66, 152.42] | 1.00 |
| 9 | 29.04 | [192.43, 235.20] | 1.00 |
| 10 | 0.65 | [102.00, 150.00] | 1.00 |

Interval of the displacement $\delta_2$ is [5.56 cm, 12.64 cm], satisfaction degree is 1.00.
Weight of the truss is 886.19 kg.

Table 2
Optimization results under the satisfaction degree level 0.8 (10-bar)

| Bar no. | Area (cm$^2$) | Stress interval (MPa) | Satisfaction degree |
|---|---|---|---|
| 1 | 29.08 | [141.37, 179.05] | 0.82 |
| 2 | 1.92 | [112.27, 156.56] | 1.00 |
| 3 | 38.73 | [42.56, 198.03] | 0.84 |
| 4 | 86.72 | [131.69, 182.03] | 0.81 |
| 5 | 24.65 | [143.29, 179.61] | 0.80 |
| 6 | 2.09 | [103.17, 143.87] | 1.00 |
| 7 | 75.82 | [145.82, 178.22] | 0.82 |
| 8 | 2.91 | [52.29, 152.70] | 1.00 |
| 9 | 21.11 | [252.62, 308.75] | 1.00 |
| 10 | 2.37 | [128.63, 179.39] | 0.86 |

Interval of the displacement $\delta_2$ is [6.52 cm, 14.24 cm], satisfaction degree is 0.80.
Weight of the truss is 829.18 kg.

Table 3
Optimization results under the satisfaction degree level 0.6 (10-bar)

| Bar no. | Area (cm$^2$) | Stress interval (MPa) | Satisfaction degree |
|---|---|---|---|
| 1 | 29.04 | [138.39, 171.07] | 1.00 |
| 2 | 0.65 | [121.53, 170.87] | 1.00 |
| 3 | 32.27 | [43.68, 234.82] | 0.67 |
| 4 | 80.66 | [139.24, 193.99] | 0.61 |
| 5 | 29.04 | [133.04, 163.67] | 1.00 |
| 6 | 0.65 | [121.53, 170.87] | 1.00 |
| 7 | 74.08 | [152.08, 185.88] | 0.60 |
| 8 | 0.67 | [33.26, 159.07] | 1.00 |
| 9 | 19.36 | [286.32, 349.95] | 1.00 |
| 10 | 0.82 | [135.33, 190.27] | 0.67 |

Interval of the displacement $\delta_2$ is [6.33 cm, 15.41 cm], satisfaction degree is 0.70.
Weight of the truss is 775.88 kg.

Table 4
Optimization results under the satisfaction degree level 0.4 (10-bar)

| Bar no. | Area (cm$^2$) | Stress interval (MPa) | Satisfaction degree |
|---|---|---|---|
| 1 | 25.49 | [156.96, 194.87] | 0.41 |
| 2 | 0.65 | [135.57, 191.03] | 0.66 |
| 3 | 22.58 | [62.33, 334.70] | 0.40 |
| 4 | 76.62 | [146.74, 204.33] | 0.45 |
| 5 | 24.50 | [157.58, 194.08] | 0.41 |
| 6 | 0.62 | [135.57, 191.03] | 0.66 |
| 7 | 71.38 | [158.02, 193.13] | 0.41 |
| 8 | 0.65 | [7.15, 159.69] | 1.00 |
| 9 | 19.39 | [285.03, 348.37] | 1.00 |
| 10 | 0.97 | [127.48, 179.62] | 0.86 |

Interval of the displacement $\delta_2$ is [4.25 cm, 15.67 cm], satisfaction degree is 0.74.
Weight of the truss is 711.59 kg.

Table 5
Optimization results under the satisfaction degree level 0.2 (10-bar)

| Bar no. | Area (cm$^2$) | Stress interval (MPa) | Satisfaction degree |
|---|---|---|---|
| 1 | 24.35 | [163.85, 203.94] | 0.21 |
| 2 | 0.65 | [108.65, 154.47] | 1.00 |
| 3 | 19.36 | [72.71, 390.00] | 0.31 |
| 4 | 72.59 | [154.57, 215.45] | 0.29 |
| 5 | 24.20 | [160.44, 197.75] | 0.32 |
| 6 | 0.65 | [108.21, 153.85] | 1.00 |
| 7 | 68.55 | [164.63, 201.21] | 0.21 |
| 8 | 0.65 | [9.84, 159.60] | 1.00 |
| 9 | 19.36 | [286.94, 350.70] | 1.00 |
| 10 | 0.65 | [152.43, 216.71] | 0.31 |

Interval of the displacement $\delta_2$ is [2.99 cm, 15.72 cm], satisfaction degree is 0.76.
Weight of the truss is 678.17 kg.

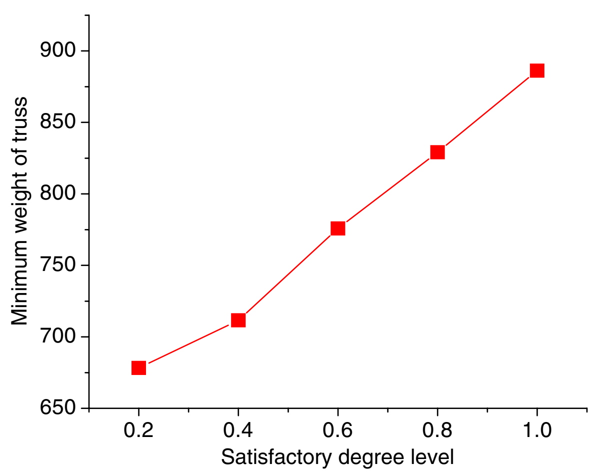


Fig. 5. Relation of the satisfaction degree level and the minimum weight of the truss.

and their minimums are 1.00, 0.80, 0.60, 0.40 and 0.21, respectively. Thus a tradeoff between the design objective and the risk of violating the constraints should be made through adjusting the satisfaction degree level of the constraints.

### 3.2. 25-bar steel truss

A 25-bar steel truss as shown in Fig. 6 is studied. The cross-sectional areas of the bars are optimized to achieve the minimum volume of the truss subject to several displacement constraints. The Young's modulus of the truss is 199949.2 MPa and the Poisson's ratio is 0.3. The length $L$ of each horizontal or vertical bar is 15.24 m. Joint 12 is hinge-supported, and joints 6, 8 and 10 are roller-supported. Joints 7, 9 and 10 are subjected to the vertical loads $F_3$, $F_2$ and $F_1$, respectively. Joint 1 is subjected to a horizontal load $F_4$. The bars (1)–(4) have a same cross-sectional area denoted by $A_1$; the bars (16)–(25) have a same cross-sectional area denoted by $A_2$; the bars (11)–(15) have a same cross-sectional area denoted by $A_3$, and the bars (5)–(10) denoted by $A_4$. The horizontal displacement of joint 6 is denoted by $d_1$ and its allowable maximum is 23 mm. The vertical displacements of joints 7, 9 and 11 are denoted by $d_2$, $d_3$ and $d_4$, and their allowable maximums are 47 mm, 40 mm and 48 mm, respectively. Only the four loads denoted by a vector $\mathbf{F}$ are uncertain, and their nominal values are: $F_1 = F_3 = 1779.2$ kN, $F_2 = 2224$ kN and $F_4 = 1334.4$ kN. The convex set of the uncertainty can be expressed as follows:

$$c_{\mathbf{U}} = \mathbf{F}^I = \{F_1^{\mathrm{L}} \leqslant F_1 \leqslant F_1^{\mathrm{R}}, F_2^{\mathrm{L}} \leqslant F_2 \leqslant F_2^{\mathrm{R}}, F_3^{\mathrm{L}} \leqslant F_3 \leqslant F_3^{\mathrm{R}}, F_4^{\mathrm{L}} \leqslant F_4 \leqslant F_4^{\mathrm{R}}\}. \tag{31}$$

An optimization model can be formulated based on Eq. (11):

$$\min_{\mathbf{A}} V(\mathbf{A}) = \sum_{i=1}^{25}(L_i A_i) = L(4A_1 + 10\sqrt{2}A_2 + 5A_3 + 6A_4)$$

subject to

$$p(d_i^I(\mathbf{A}) \leqslant d_{i,\mathrm{allow}}) \geqslant \lambda_i, \quad i = 1,2,3,4, \ \mathbf{A} \in \Omega^4, \tag{32}$$

where $V$ is the volume of truss. For a specific $\mathbf{A}$, the values of $d_i(\mathbf{A},\mathbf{F})$ form an interval $d_i^I(\mathbf{A})$ as $\mathbf{F}$ is uncertain, and its bounds can be written:

$$d_i^{\mathrm{L}}(\mathbf{A}) = \min_{\mathbf{F}\in c_{\mathbf{U}}} d_i(\mathbf{A},\mathbf{F}), \quad d_i^{\mathrm{R}}(\mathbf{A}) = \max_{\mathbf{F}\in c_{\mathbf{U}}} d_i(\mathbf{A},\mathbf{F}), \quad i = 1,2,3,4. \tag{33}$$

Following is the interval analysis method [24–26] which is used to solve Eq. (33) with high efficiency.

Based on the interval mathematics [27], the interval vector $\mathbf{F}^I$ can be rewritten in the following form:

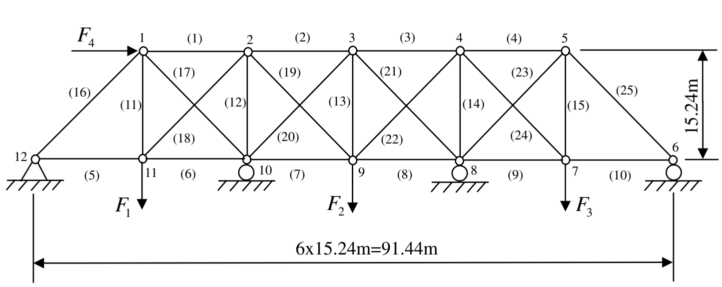


Fig. 6. A 25-bar steel truss [17].

$$\mathbf{F}^I = [\mathbf{F}^{\mathrm{L}}, \mathbf{F}^{\mathrm{R}}] = [\mathbf{F}^c - \mathbf{F}^w, \mathbf{F}^c + \mathbf{F}^w] = \mathbf{F}^c + [-1,1]\mathbf{F}^w, \quad i = 1,2,3,4, \tag{34}$$

where $\mathbf{F}^c$ and $\mathbf{F}^w$ denote the midpoint vector and the radius vector of $\mathbf{F}^I$, respectively:

$$\mathbf{F}^c = \frac{\mathbf{F}^{\mathrm{L}} + \mathbf{F}^{\mathrm{R}}}{2}, \quad F_i^c = \frac{F_i^{\mathrm{L}} + F_i^{\mathrm{R}}}{2}, \quad i = 1,2,3,4, \tag{35}$$

$$\mathbf{F}^w = \frac{\mathbf{F}^{\mathrm{R}} - \mathbf{F}^{\mathrm{L}}}{2}, \quad F_i^w = \frac{F_i^{\mathrm{R}} - F_i^{\mathrm{L}}}{2}, \quad i = 1,2,3,4. \tag{36}$$

Based on Eq. (34), the uncertain vector $\mathbf{F}$ can be rewritten in the following form:

$$\mathbf{F} = \mathbf{F}^c + \delta\mathbf{F}, \tag{37}$$

where

$$\delta\mathbf{F} \in [-1,1]\mathbf{F}^w, \quad \delta F_i \in [-1,1]F_i^w, \quad i = 1,2,3,4. \tag{38}$$

The constraint functions can be expanded at $\mathbf{F}^c$ through the first-order Taylor expansion:

$$d_i(\mathbf{A},\mathbf{F}) = d_i(\mathbf{A},\mathbf{F}^c + \delta\mathbf{F}) \approx d_i(\mathbf{A},\mathbf{F}^c) + \sum_{j=1}^{4} \frac{\partial d_i(\mathbf{A},\mathbf{F}^c)}{\partial F_j}\delta F_j, \quad i = 1,2,3,4. \tag{39}$$

Because $\delta\mathbf{F}$ belongs to an interval vector defined by Eq. (38), the interval of $d_i$ caused by the uncertainty can be obtained through the natural interval extension [24]:

$$d_i^I(\mathbf{A}) = d_i(\mathbf{A},\mathbf{F}^c) + \sum_{j=1}^{4} \frac{\partial d_i(\mathbf{A},\mathbf{F}^c)}{\partial F_j}[-1,1]F_j^w, \quad i = 1,2,3,4. \tag{40}$$

Thus the bounds of $d_i$ can be obtained through the following explicit expressions:

$$d_i^{\mathrm{L}}(\mathbf{A}) = d_i(\mathbf{A},\mathbf{F}^c) - \sum_{j=1}^{4} \left|\frac{\partial d_i(\mathbf{A},\mathbf{F}^c)}{\partial F_j}\right| F_j^w, \quad i = 1,2,3,4, \tag{41}$$

$$d_i^{\mathrm{R}}(\mathbf{A}) = d_i(\mathbf{A},\mathbf{F}^c) + \sum_{j=1}^{4} \left|\frac{\partial d_i(\mathbf{A},\mathbf{F}^c)}{\partial F_j}\right| F_j^w, \quad i = 1,2,3,4. \tag{42}$$

Here, FEM is used to calculate the structural displacement responses. The truss element is employed to create the FEM mesh. Each bar is an element and there are a total of 25 elements. Assembling all of the elemental stiffness matrixes, the FEM governing equation of the truss can be achieved as follows:

$$\mathbf{K}(\mathbf{A})\mathbf{d}_g(\mathbf{A},\mathbf{F}) = \mathbf{F}_g(\mathbf{F}), \tag{43}$$

where $\mathbf{K}$, $\mathbf{d}_g$ and $\mathbf{F}_g$ are global stiffness matrix, displacement vector and load vector, respectively. $\mathbf{d}_g$ and $\mathbf{F}_g$ are used to distinguish from aforementioned $\mathbf{d}$ and $\mathbf{F}$ as $\mathbf{d}$ and $\mathbf{F}$ are just some components of $\mathbf{d}_g$ and $\mathbf{F}_g$. Differentiating both sides of Eq. (43) with respect to the uncertain parameter at $\mathbf{F}^c$ yields:

$$\mathbf{K}(\mathbf{A})\frac{\partial \mathbf{d}_g(\mathbf{A},\mathbf{F}^c)}{\partial F_j} = \frac{\partial \mathbf{F}_g(\mathbf{F}^c)}{\partial F_j}, \quad j = 1,2,3,4. \tag{44}$$

Through one FEM computation, $\mathbf{d}_g(\mathbf{A},\mathbf{F}^c)$ can be obtained for a specific $\mathbf{A}$. Then through four FEM computations formulated by Eq. (44), the derivatives $\frac{\partial \mathbf{d}_g(\mathbf{A},\mathbf{F}^c)}{\partial F_j}$, $j = 1,2,3,4$ can be also obtained. As a result, the bounds of the constraints at $\mathbf{A}$ can be calculated only through five FEM computations based on Eqs. (41) and (42). Thus the time-consuming optimization processes $\min_{\mathbf{F}\in c_{\mathrm{U}}} d_i(\mathbf{A},\mathbf{F})$ and $\max_{\mathbf{F}\in c_{\mathrm{U}}} d_i(\mathbf{A},\mathbf{F})$ in Eq. (33) can be avoided, and the optimization efficiency will be improved greatly.

A generation number 300 is used as stopping criterion for IP-GA. An uncertainty level of 10% off from the nominal values of the uncertain loads is considered. The search range $\Omega^4$ is specified as $\{1\,\mathrm{cm}^2 \leqslant A_i \leqslant 100\,\mathrm{cm}^2, i = 1,2,3,4\}$. All of the constraints are set as the same satisfaction degree level, and the optimization results under the satisfaction degree levels 1.0, 0.8, 0.6, 0.4 and 0.2 are listed in Tables 6–10. It is found that the design volumes of the truss are 2.58 $\mathrm{mm}^3$, 2.32 $\mathrm{mm}^3$, 2.23 $\mathrm{mm}^3$, 2.14 $\mathrm{mm}^3$ and 2.13 $\mathrm{mm}^3$ with these five cases, respectively. For satisfaction degree level 1.0, the design volume is largest and hence most costly. With decreasing of the satisfaction degree level, the design volume and the satisfaction degrees of the constraints at the optimum also decrease. For example, the intervals of the displacement constraints at the optimum are [19.70, 24.89], [38.90, 51.12], [32.06, 44.85], and [40.46, 52.99], respectively, when the satisfaction degree level is 0.6. They all have the possibilities to violate the allowable maximum displacements 23 mm, 47 mm, 40 mm and 48 mm, respectively. Furthermore with

Table 6
Optimization results under the satisfaction degree level 1.0 (25-bar)

| Optimal areas ($\mathrm{mm}^2$) | Displacement intervals (mm) | Satisfaction degree |
|---|---|---|
| $A_1$: 719.4 | $d_1$: [17.82, 22.39] | 1.00 |
| $A_2$: 6293.5 | $d_2$: [34.40, 44.76] | 1.00 |
| $A_3$: 5374.2 | $d_3$: [28.78, 39.46] | 1.00 |
| $A_4$: 8848.4 | $d_4$: [35.66, 46.26] | 1.00 |

Volume of the truss is 2.58 $\mathrm{m}^3$.

Table 7
Optimization results under the satisfaction degree level 0.8 (25-bar)

| Optimal areas ($\mathrm{mm}^2$) | Displacement intervals (mm) | Satisfaction degree |
|---|---|---|
| $A_1$: 409.7 | $d_1$: [18.83, 23.91] | 0.82 |
| $A_2$: 5064.5 | $d_2$: [36.49, 48.36] | 0.89 |
| $A_3$: 6293.5 | $d_3$: [29.49, 42.04] | 0.84 |
| $A_4$: 8316.1 | $d_4$: [38.18, 50.41] | 0.80 |

Volume of the truss is 2.32 $\mathrm{m}^3$.

Table 8
Optimization results under the satisfaction degree level 0.6 (25-bar)

| Optimal areas ($\mathrm{mm}^2$) | Displacement intervals (mm) | Satisfaction degree |
|---|---|---|
| $A_1$: 487.1 | $d_1$: [19.70, 24.89] | 0.64 |
| $A_2$: 5122.6 | $d_2$: [38.90, 51.12] | 0.66 |
| $A_3$: 5248.4 | $d_3$: [32.06, 44.85] | 0.62 |
| $A_4$: 7996.8 | $d_4$: [40.46, 52.99] | 0.60 |

Volume of the truss is 2.23 $\mathrm{m}^3$.

Table 9
Optimization results under the satisfaction degree level 0.4 (25-bar)

| Optimal areas (mm$^2$) | Displacement intervals (mm) | Satisfaction degree |
|---|---|---|
| $A_1$: 196.8 | $d_1$: [20.48, 26.27] | 0.44 |
| $A_2$: 5054.8 | $d_2$: [39.69, 53.53] | 0.53 |
| $A_3$: 5054.8 | $d_3$: [30.37, 45.44] | 0.64 |
| $A_4$: 7532.3 | $d_4$: [42.23, 56.63] | 0.40 |

Volume of the truss is 2.14 m$^3$.

Table 10
Optimization results under the satisfaction degree level 0.2 (25-bar)

| Optimal areas (mm$^2$) | Displacement intervals (mm) | Satisfaction degree |
|---|---|---|
| $A_1$: 100.0 | $d_1$: [20.36, 26.29] | 0.45 |
| $A_2$: 5054.8 | $d_2$: [39.59, 54.12] | 0.51 |
| $A_3$: 5054.8 | $d_3$: [29.10, 45.26] | 0.68 |
| $A_4$: 7532.3 | $d_4$: [42.55, 57.72] | 0.36 |

Volume of the truss is 2.13 m$^3$.

decreasing of the satisfaction degree level, these possibilities will become larger. Here the design objective and the risk are also contradictive, namely a fine design objective is usually at the price of a big risk that the constraints have the possibility to be violated.

In this numerical example, the maximum generations of IP-GA are 300 and each generation contains 5 individuals, thus a total of 1500 individuals are needed. Additionally, calculating each individual requires 5 FEM computations, thus the total number of the FEM computations for optimization based on the interval analysis method is 7500. On the other hand, if we also use IP-GA with the maximum generations 300 and population size 5 to solve Eq. (33), the optimization nesting will be caused inevitably. For each optimization iterate of **A**, 8 times of IP-GA optimization processes will be called. For the whole uncertain optimization, the total number of the FEM computations will reach $1.8 \times 10^7$. As a result, applying the interval analysis method to the uncertain structural optimization can improve the optimization efficiency exponentially, and it can make it possible to analyze many complex structural problems.

Generally, the determination of satisfaction degree level is concerned with two factors: the optimization problem and the attitude of the decision maker. For some practical engineering problems which care for the design objective more than the other factors, a relatively small satisfaction degree level can be selected; for some problems in which the reliability and security are most important, then a relatively large satisfaction degree level should be specified. In addition, the attitude of the decision maker also influences the selection of satisfaction degree level. An optimistic decision maker usually uses a small satisfaction degree level, while a pessimistic decision maker intends to use a relatively large one.

## 4. Conclusion

This paper suggests a new uncertain structural optimization method based on convex model and constraint satisfaction degree. For many practical engineering problems which lack of information of the uncertain parameters, convex model method is a convenient and effective selection for the uncertainty description. Satisfaction degree of interval provides a general way to deal with the uncertain constraints, and the prevailed worst-case method is just a special case of this method. The worst-case method is a relatively conservative approach that requires the constraints satisfied for all of the possible combinations of the uncertain parameters. However, the presented method allows the constraints to have certain possibility to be violated, and whereby improve the objective function. The optimization results of two numerical examples indicate that the design objective of a structure is always at the price of a constraint risk. The decision maker can make a tradeoff between the design objective and risk according to the actual problem and his own experience. Thus the presented method is a more flexible approach, and it leaves the engineer a larger decision space through adjusting the satisfaction degree level. In addition, the interval analysis method is employed to calculate the bounds of the uncertain constraints based on FEM very efficiently, and the optimization nesting can be eliminated. Combining convex model, constraint satisfaction degree with the interval analysis method will construct a strong uncertain optimization tool, and this will make it possible to deal with many complex structures with uncertainty. On the other hand, it should be noticed that the intervals of the uncertain parameters should be relatively small when using the interval analysis method, as the first-order Taylor approximation will be acceptable only in the near neighborhood of the expansion point. Fortunately, this condition can be often satisfied as the uncertainty always behaves a small disturbance around the nominal values of the parameters in practical structural problems.

The presented method can be also easily extended to the problems with uncertain objective function. The uncertain objective function can be expected to be smaller or larger than a specific value or interval which can be determined according to our requirement to the practical structure. Then using the satisfaction degree of interval, the uncertain objective function can be also transferred into a deterministic objective function which can be solved by traditional optimization methods.

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## References

[1] H.H. Hilton, M. Feigen, Minimum weight analysis based on structural reliability, J. Aero. Sci. 27 (1960) 641–652.
[2] Y. Ben-Haim, I. Elishakoff, Convex Models of Uncertainties in Applied Mechanics, Elsevier Science Publisher, Amsterdam, 1990.

[3] Chris P. Pantelides, Sara Ganzerli, Comparison of fuzzy set and convex model theories in structural design, Mech. Syst. Signal. Pr. 15 (3) (2001) 499–511.

[4] D.M. Frangopol, R.B. Corotis, Reliability-based structural system optimization: state-of-the-art versus state-of-the-practice, in: F.Y. Cheng (Ed.), Analysis and Computation, ASCE, New York, 1996, pp. 67–76.

[5] Y. Ben-Haim, A non-probabilistic concept of reliability, Struct. Saf. 14 (1994) 227–295.

[6] I. Elishakoff, Essay on uncertainties in elastic and viscoelastic structures: from A.M. Fraudenthal's criticism to modern convex modeling, Comput. Struct. 56 (6) (1995) 871–895.

[7] I. Elishakoff, An idea on the uncertainty triangle, Editors Rattle Space, The Shock and Vibration Digest 22 (10) (1995) 1.

[8] I. Elishakoff, G.Q. Cai Jr., J.H. Starnes, Non-linear buckling of a column with initial imperfection via stochastic and non-stochastic convex models, Int. J. Non-Linear Mech. (29) (1994) 71–82.

[9] C.P. Pantelides, Stability of elastic bars on uncertain foundations using a convex model, Int. J. Solids Struct. 33 (9) (1996) 1257–1269.

[10] L. Fryba, N. Yoshikawa, Bounds analysis of a beam based on the convex model of uncertain foundation, J. Sound Vib. 212 (3) (1998) 547–557.

[11] Nii O. Attoh-Okine, Uncertain analysis in structural number determination in flexible pavement design – a convex model approach, Constr. Build. Mater. 16 (2002) 67–71.

[12] I. Elishakoff, R.T. Haftka, J. Fang, Structural design under bounded uncertainty – optimization with anti-optimization, Comput. Struct. 53 (6) (1994) 1401–1405.

[13] M. Lombardi, C. Cinquini, R. Contro, R.T. Haftka, Anti-optimization technique for designing composite structures, in: N. Olhoff, G.I.N. Rozvany (Eds.), Proceedings First World Congress of Structural and Multidisciplinary Optimization, Goslar, Germany, 1995.

[14] E. Barbieri, C. Cinquini, M. Lombardi, Shape/size optimization of truss structures using non-probabilistic description of uncertainties, in: S. Hernandez, C.A. Brebbia (Eds.), Proceedings OPTI' 97, Computer Aided Optimum Design of Structures V, Computational Mechanics and Publications, Southampton, UK, 1997, pp. 163–172.

[15] M. Lombardi, Optimization of uncertain structures using non-probabilistic models, Comput. Struct. 67 (1998) 99–103.

[16] S. Ganzerli, C.P. Pantelides, Optimum structural design via convex model superposition, Comput. Struct. 74 (2000) 639–647.

[17] F.T.K. Au, Y.S. Cheng, L.G. Tham, G.W. Zeng, Robust design of structures using convex models, Comput. Struct. 81 (2003) 2611–2619.

[18] S.P. Gurav, J.F.L. Goosen, F. vanKeulen, Bounded-But-Unknown uncertainty optimization using design sensitivities and parallel computing: application to MEMS, Comput. Struct. 83 (2005) 1134–1149.

[19] B.D. Liu, R.Q. Zhao, G. Wang, Uncertain Programming with Applications, Tsinghua University Press, Beijing, 2003.

[20] Y.G. Xu, G.R. Liu, Z.P. Wu, A novel hybrid genetic algorithm using local optimizer based on heuristic pattern move, Appl. Artif. Intell. 15 (7) (2001) 601–631.

[21] G.R. Liu, X. Han, Computational Inverse Techniques in Nondestructive Evaluation, CRC Press, Florida, 2003.

[22] X.W. Liu, Q.L. Da, A satisfactory solution for interval number linear programming, J. Syst. Engrg. 14 (1999) 123–128, China.

[23] B.L. Chen, Optimization Theories and Algorithms, Tsinghua University Press, Beijing, China, 2002.

[24] Z.P. Qiu, Convex Method Based on Non-probabilistic Set-theory and Its Application, National Defence Industry Press, China, 2005.

[25] Z.P. Qiu, X.J. Wang, Comparison of dynamic response of structures with uncertain non-probabilistic interval analysis method and probabilistic approach, Int. J. Solids Struct. 40 (2003) 5423–5439.

[26] Y.T. Zhou, C. Jiang, X. Han, Interval and subinterval analysis methods of the structures and their error estimations, Int. J. Comput. Methods 3 (2) (2006) 229–244.

[27] R.E. Moore, Methods and Applications of Interval Analysis, Prentice-Hall Inc., London, 1979.