A LOCAL RING HAS ONLY FINITELY MANY SEMIDUALIZING COMPLEXES UP TO SHIFT-ISOMORPHISM

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Abstract. A homologically finite complex C over a commutative noetherian ring R is *semidualizing* if $\mathbf{R}\mathrm{Hom}_R(C, C) \simeq R$ in $\mathcal{D}(R)$. We answer a question of Vasconcelos from 1974 by showing that a local ring has only finitely many shift-isomorphism classes of semidualizing complexes. Our proof relies on certain aspects of deformation theory for DG modules over a finite dimensional DG algebra, which we develop.

Contents

1. Introduction

Convention. In this paper, R is a commutative noetherian ring with identity and all R-modules are unital.

This paper is concerned with *semidulizing R-modules*, that is, the finitely generated R-modules C such that $\mathrm{Hom}_R(C, C) \cong R$ and $\mathrm{Ext}^i_R(C, C) = 0$ for $i \geqslant 1$. These modules were introduced, as best we know, by Vasconcelos [34]. They were rediscovered independently by several authors including Foxby [17], Golod [23], and Wakamatsu [38], who all used different terminology for them. Special cases of these modules include Grothendieck's canonical modules over Cohen-Macaulay rings, and duality with respect to a semidualizing module extends Auslander and Bridger's G-dimension [4, 5].

Vasconcelos posed the following in [34, p. 97].

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Question 1.1. If R is local and Cohen-Macaulay, must the set of isomorphism classes of semidualizing R-modules $\mathfrak{S}_0(R)$ be finite?

Christensen and Sather-Wagstaff [12] answer this question affirmatively in the case when R contains a field. Their proof motivates our own techniques, so we describe some aspects of it here. Using standard ideas, they reduce to the case where R is complete with algebraically closed residue field F. Given a maximal R-sequence $\mathbf{x}$, they replace R with the quotient $R/(\mathbf{x})$, which is a finite dimensional algebra over F. The desired result then follows from a deformation-theoretic theorem of Happel [27] which states that, in this context, there are only finitely many R-modules C of a given length r such that $\operatorname{Ext}^1_R(C,C)=0$.

To prove Happel's result, one parametrizes all such modules by an algebraic variety Mod^R_r that is acted on by the general linear group GL^F_r so that the isomorphism class of C is precisely the orbit $\operatorname{GL}^F_r C$. A theorem of Voigt [37] (see also Gabriel [21]) provides an isomorphism between $\operatorname{Ext}^1_R(C,C)$ and the quotient of tangent spaces $\mathsf{T}_C^{\operatorname{Mod}^R_r} / \mathsf{T}_C^{\operatorname{GL}^F_r C}$. Thus, the vanishing $\operatorname{Ext}^1_R(C,C)=0$ implies that the orbit $\operatorname{GL}^F_r C$ is open in Mod^R_r. Since Mod^R_r is quasi-compact, it can only have finitely many open orbits, so R can only have finitely many such modules up to isomorphism.

The main result of this paper, stated next, provides a complete answer to Vasconcelos' question. Note that it does not assume that R is Cohen-Macaulay. Section 5 is devoted to its proof (see 5.3) and some consequences.

Theorem A. *Let R be a local ring. Then the set $\mathfrak{S}_0(R)$ of isomorphism classes of semidualizing R-modules is finite.*

The idea behind our proof is the same as in Christensen and Sather-Wagstaff's proof, with one important difference, pioneered by Avramov: instead of replacing R with $R/(\mathbf{x})$, we use the Koszul complex K on a minimal generating sequence for the maximal ideal of R. More specifically, we replace R with a finite dimensional *DG F-algebra U* that is quasiisomorphic to K. (See Section 2 for background information on DG algebras and DG modules.)

In order to prove versions of the results of Happel and Voigt, we develop certain aspects of deformation theory for DG modules over a finite dimensional DG F-algebra U. This is the subject of Section 4. In short, we parametrize all finite dimensional DG U-modules M with fixed underlying graded F-vector space W by an algebraic variety $\operatorname{Mod}^U(W)$. This variety is acted on by a product $\operatorname{GL}(W)_0$ of general linear groups so that the isomorphism class of M is precisely the orbit $\operatorname{GL}(W)_0\cdot M$. Following Gabriel, we focus on the associated functors of points $\underline{\operatorname{Mod}}^U(W)$, $\underline{\operatorname{GL}}(W)_0$, and $\underline{\operatorname{GL}}(W)_0\cdot M$. (Our notational conventions are spelled out explicitly in Notations 4.1 and 4.5.) Our version of Voigt's result for this context is the following, which we prove in 4.11.

Theorem B. *We work in the setting of Notations 4.1 and 4.5. Given an element $M=(\partial,\mu)\in\operatorname{Mod}^U(W)$, there is an isomorphism of abelian groups*

$$\mathsf{T}_M^{\underline{\operatorname{Mod}}^U(W)} \,/\, \mathsf{T}_M^{\underline{\operatorname{GL}}(W)_0\cdot M} \cong \operatorname{YExt}^1_U(M,M).$$

As a consequence, we deduce that if $\operatorname{Ext}^1_U(M,M)=0$, then the orbit $\underline{\operatorname{GL}}(W)_0\cdot M$ is open in $\underline{\operatorname{Mod}}^U(W)$; see Corollary 4.12. The proof of Theorem A concludes like that of Christensen and Sather-Wagstaff, with a few technical differences.

One technical difference is the following: given DG U-modules M and N, there are (at least) two different modules that one might write as $\mathrm{Ext}^1_U(M,N)$. This is the topic of Section 3. First, there is the derived category version: this is the module $\mathrm{Ext}^1_U(M,N) = \mathrm{H}^1(\mathrm{Hom}_U(F,N))$ where F is a "semi-free resolution" of M. Second, there is the abelian category version: this is the module $\mathrm{YExt}^1_U(M,N)$ that is the set of equivalence classes of exact sequences $0 \to N \to X \to M \to 0$. In general, one has $\mathrm{Ext}^1_U(M,N) \ncong \mathrm{YExt}^1_U(M,N)$. This is problematic as the passage from R to U uses $\mathrm{Ext}^1_U(M,N)$, but Theorem B uses $\mathrm{YExt}^1_U(M,N)$. These are reconciled in the next result. See 3.5 for the proof.

Theorem C. *Let A be a DG R-algebra, and let P, Q be DG A-modules such that Q is graded-projective (e.g., Q is semi-free). Then there is an isomorphism $\mathrm{YExt}^1_A(Q,P) \xrightarrow{\cong} \mathrm{Ext}^1_A(Q,P)$ of abelian groups.*

We actually prove a version of Theorem A for semidualizing complexes over a local ring. We do this in Theorem 5.2. Moreover, we prove versions of these results for certain non-local rings, including all semilocal rings in Theorem 5.11.

2. DG Modules

We assume that the reader is familiar with the category of R-complexes and the derived category $\mathcal{D}(R)$. Standard references for these topics are [6, 8, 10, 16, 22, 28, 35, 36]. For clarity, we include a few definitions and notations.

Definition 2.1. In this paper, complexes of R-modules ("R-complexes" for short) are indexed homologically:

$$M = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots .$$

Sometimes we write (M,∂^M) to specify the differential on M. The degree of an element $m \in M$ is denoted $|m|$. The *infimum*, *supremum*, and *amplitude* of M are

$$\begin{aligned}\inf(M) &:= \inf\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}\\ \sup(M) &:= \sup\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}\\ \mathrm{amp}(M) &:= \sup(M) - \inf(M).\end{aligned}$$

The *tensor product* of two R-complexes M,N is denoted $M \otimes_R N$, and the *Hom complex* is denoted $\mathrm{Hom}_R(M,N)$. A *chain map* $M \to N$ is a cycle of degree 0 in $\mathrm{Hom}_R(M,N)$.

Next we discuss DG algebras, which are treated in, e.g., [1, 2, 6, 8, 29, 30].

Definition 2.2. A *commutative differential graded algebra over R* (*DG R-algebra* for short) is an R-complex A equipped with a chain map $\mu^A \colon A \otimes_R A \to A$ with $ab := \mu^A(a \otimes b)$ that is:

- **associative:** for all $a,b,c \in A$ we have $(ab)c = a(bc)$;
- **unital:** there is an element $1 \in A_0$ such that for all $a \in A$ we have $1a = a$;
- **graded commutative:** for all $a,b \in A$ we have $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ when $|a|$ is odd; and
- **positively graded:** $A_i = 0$ for $i < 0$.

The map μ^A is the *product* on A. Given a DG R-algebra A, the *underlying algebra* is the graded commutative R-algebra $A^\natural = \oplus_{i=0}^{\infty} A_i$. When R is a field and $\mathrm{rank}_R(\oplus_{i\geqslant 0} A_i) < \infty$, we say that A is *finite-dimensional* over R.

A *morphism* of DG R-algebras is a chain map $f\colon A \to B$ between DG R-algebras respecting products and multiplicative identities: $f(aa') = f(a)f(a')$ and $f(1) = 1$.

Fact 2.3. Let A be a DG R-algebra. The fact that the product on A is a chain map says that ∂^A satisfies the *Leibniz rule*: $\partial^A_{|a|+|b|}(ab) = \partial^A_{|a|}(a)b + (-1)^{|a|}a\partial^A_{|b|}(b)$.

The ring R, considered as a complex concentrated in degree 0, is a DG R-algebra. The map $R \to A$ given by $r \mapsto r \cdot 1$ is a morphism of DG R-algebras. It is straightforward to show that the R-module A_0 is an R-algebra. Moreover, the natural map $A_0 \to A$ is a morphism of DG R-algebras. The condition $A_{-1} = 0$ implies that A_0 surjects onto $\mathrm{H}_0(A)$ and that $\mathrm{H}_0(A)$ is an A_0-algebra. Furthermore, the R-module A_i is an A_0-module, and $\mathrm{H}_i(A)$ is an $\mathrm{H}_0(A)$-module for each i.

Definition 2.4. Let A be a DG R-algebra. We say that A is *noetherian* if $\mathrm{H}_0(A)$ is noetherian and the $\mathrm{H}_0(A)$-module $\mathrm{H}_i(A)$ is finitely generated for all $i \geqslant 0$. When R is local, we say that A is *local* if it is noetherian and the ring $\mathrm{H}_0(A)$ is a local R-algebra, that is, $\mathrm{H}_0(A)$ is a local ring whose maximal ideal contains the extension of the maximal ideal of R.

Fact 2.5. Assume that R is local with maximal ideal $\mathfrak{m}$. Let A be a local DG R-algebra, and let $\mathfrak{m}_{\mathrm{H}_0(A)}$ be the maximal ideal of $\mathrm{H}_0(A)$. The composition $A \to \mathrm{H}_0(A) \to \mathrm{H}_0(A)/\mathfrak{m}_{\mathrm{H}_0(A)}$ is a surjective morphism of DG R-algebras with kernel of the form $\mathfrak{m}_A = \cdots \xrightarrow{\partial^A_2} A_1 \xrightarrow{\partial^A_1} \mathfrak{m}_0 \to 0$ for some maximal ideal $\mathfrak{m}_0 \subsetneq A_0$. The quotient $A/\mathfrak{m}_A$ is isomorphic to $\mathrm{H}_0(A)/\mathfrak{m}_{\mathrm{H}_0(A)}$. Since $\mathrm{H}_0(A)$ is a local R-algebra, we have $\mathfrak{m}A_0 \subseteq \mathfrak{m}_0$.

Definition 2.6. Assume that R is local. Given a local DG R-algebra A, the subcomplex $\mathfrak{m}_A$ from Fact 2.5 is the *augmentation ideal* of A.

For this paper, an important example is the next one.

Example 2.7. Given a sequence $\mathbf{a} = a_1, \cdots, a_n \in R$, the Koszul complex $K = K^R(\mathbf{a})$ is a DG R-algebra with product given by the wedge product. If R is local with maximal ideal $\mathfrak{m}$ and $\mathbf{a} \in \mathfrak{m}$, then K is a local DG R-algebra with augmentation ideal $\mathfrak{m}_K = (0 \to R \to \cdots \to R^n \to \mathfrak{m} \to 0)$.

In the passage to DG algebras, we must focus on DG modules, described next.

Definition 2.8. Let A be a DG R-algebra. A *differential graded module over A* (*DG A-module* for short) is an R-complex M with a chain map $\mu^M\colon A \otimes_R M \to M$ such that the rule $am := \mu^M(a \otimes m)$ is associative and unital. The map μ^M is the *scalar multiplication* on M. The *underlying $A^\natural$-module* associated to M is the $A^\natural$-module $M^\natural = \oplus_{i=-\infty}^{\infty} M_i$.

Example 2.9. Consider the ring R as a DG R-algebra. A DG R-module is just an R-complex, and a morphism of DG R-modules is simply a chain map.

Fact 2.10. Let A be a DG R-algebra, and let M be a DG A-module. The fact that the scalar multiplication on M is a chain map says that ∂^M satisfies the *Leibniz rule*: $\partial^A_{|a|+|m|}(am) = \partial^A_{|a|}(a)m + (-1)^{|a|}a\partial^M_{|m|}(m)$. The R-module M_i is an A_0-module, and $\mathrm{H}_i(M)$ is an $\mathrm{H}_0(A)$-module for each i.

Definition 2.11. Let A be a DG R-algebra, and let i be an integer. The ith *suspension* of a DG A-module M is the DG A-module $\Sigma^i M$ defined by $(\Sigma^i M)_n :=$

M_{n-i} and $\partial_n^{\Sigma^i M} := (-1)^i \partial_{n-i}^M$. The scalar multiplication on $\Sigma^i M$ is defined by the formula $\mu^{\Sigma^i M}(a \otimes m) := (-1)^{i|a|}\mu^M(a \otimes m)$.

A *morphism* of DG A-modules is a chain map $f\colon M \to N$ between DG A-modules that respects scalar multiplication: $f(am) = af(m)$. Isomorphisms in the category of DG A-modules are identified by the symbol $\cong$. A *quasiisomorphism* is a morphism $M \to N$ such that each induced map $\mathrm{H}_i(M) \to \mathrm{H}_i(N)$ is an isomorphism; these are identified by the symbol $\simeq$. Two DG A-modules M and N are *quasiisomorphic* is there is a chain of quasiisomorphisms (in alternating directions) from M to N; this equivalence relation is denoted by the symbol $\simeq$. Two DG A-modules M and N are *shift-quasiisomorphic* if there is an integer m such that $M \simeq \Sigma^m N$; this equivalence relation is denoted by the symbol $\sim$.

Notation 2.12. The derived category $\mathcal{D}(A)$ is formed from the category of DG A-modules by formally inverting the quasiisomorphisms; see [29]. Isomorphisms in $\mathcal{D}(A)$ are identified by the symbol $\simeq$, and isomorphisms up to shift in $\mathcal{D}(A)$ are identified by $\sim$.

Definition 2.13. Let A be a DG R-algebra, and let M, N be DG A-modules. The *tensor product* $M \otimes_A N$ is the quotient $(M \otimes_R N)/U$ where U is the subcomplex generated by all elements of the form $(am) \otimes n - (-1)^{|a||m|} m \otimes (an)$. Given an element $m \otimes n \in M \otimes_R N$, we denote the image in $M \otimes_A N$ as $m \otimes n$.

Fact 2.14. Let A be a DG R-algebra, and let M, N be DG A-modules. The tensor product $M \otimes_A N$ is a DG A-module via the scalar multiplication

$$a(m \otimes n) := (am) \otimes n = (-1)^{|a||m|} m \otimes (an).$$

Definition 2.15. Let A be a DG R-algebra. A DG A-module M is *bounded below* if $M_n = 0$ for all $n \ll 0$; it is *bounded* if $M_n = 0$ when $|n| \gg 0$; it is *degree-wise finite* if M_i is finitely generated over A_0 for each i; it is *homologically bounded below* if the total homology module $\mathrm{H}(M)$ is bounded below; it is *homologically bounded* if $\mathrm{H}(M)$ is bounded; it is *homologically degree-wise finite* if each $\mathrm{H}_0(A)$-module $\mathrm{H}_n(M)$ is finitely generated; and it is *homologically finite* if it is homologically both bounded and degree-wise finite. The full subcategory of $\mathcal{D}(A)$ whose objects are the homologically bounded below DG A-modules is denoted $\mathcal{D}_+(A)$.

Here we discuss one type of resolution for DG modules. See Section 3 for a discussion of other kinds of resolutions.

Definition 2.16. Let A be a DG R-algebra, and let L be a DG A-module. A subset E of L is called a *semibasis* if it is a basis of the underlying $A^\natural$-module $L^\natural$. If L is bounded below, then L is called *semi-free* if it has a semibasis.[1] A *semi-free resolution* of a DG A-module M is a quasiisomorphism $F \xrightarrow{\simeq} M$ of DG A-modules such that F is semi-free. Given a semi-free resolution $F \xrightarrow{\simeq} M$ and a DG A-module N, set $M \otimes_A^{\mathbf{L}} N := F \otimes_A N$ and $\mathrm{Tor}_i^A(M, N) := \mathrm{H}_i(M \otimes_A^{\mathbf{L}} N)$.

[1]As is noted in [8], when L is not bounded below, the definition of "semi-free" is significantly more technical. Using the more general notion, one can define the up-coming derived functors $M \otimes_A^{\mathbf{L}} N$, $\mathrm{Tor}_i^A(M, N)$, $\mathbf{R}\mathrm{Hom}_A(M, N)$, and $\mathrm{Ext}_A^i(M, N)$ for any pair of DG A-modules, with no boundedness assumptions. However, our results do not require this level of generality, so we focus only on this case. Furthermore, for $M \otimes_A^{\mathbf{L}} N$ and $\mathrm{Tor}_i^A(M, N)$, one only needs semi-flat resolutions, and for $\mathbf{R}\mathrm{Hom}_A(M, N)$ and $\mathrm{Ext}_A^i(M, N)$, one only needs semi-projective resolutions. Consult [8, Sections 2.8 and 2.10] for a discussion of these notions and the relations between them.

Assume that R and A are local. A *minimal semi-free resolution* of M is a semi-free resolution $F \xrightarrow{\simeq} M$ such that F is *minimal*, i.e., each (equivalently, some) semibasis of F is finite in each degree and the differential on $(A/\mathfrak{m}_A)\otimes_A F$ is 0.[2]

Fact 2.17. Let A be a DG R-algebra, and let M be a DG A-module that is homologically bounded below. Then M has a semi-free resolution over A by [8, Theorem 2.7.4.2]. For each DG A-module N, the complex $M\otimes^{\mathbf{L}}_A N$ is well-defined (up to isomorphism) in $\mathcal{D}(A)$; hence the modules $\operatorname{Tor}_i^A(M,N)$ are well-defined over $\operatorname{H}_0(A)$ and over R. Given a semi-free resolution $G \xrightarrow{\simeq} N$, one has $M\otimes^{\mathbf{L}}_A N \simeq M\otimes_R G$.

Assume that A is noetherian, and let j be an integer. Assume that M is homologically degree-wise finite and $\operatorname{H}_i(M)=0$ for $i<j$. Then M has a semi-free resolution $F \xrightarrow{\simeq} M$ such that $F^\natural \cong \oplus_{i=j}^{\infty}\Sigma^i(A^\natural)^{\beta_i}$ for some integers β_i, and so $F_i=0$ for all $i<j$; see [2, Proposition 1]. In particular, homologically finite DG A-modules admit such "degree-wise finite, bounded below" semi-free resolutions.

Assume that R and A are local with $k=A/\mathfrak{m}_A$. Then M has a minimal semi-free resolution $F \xrightarrow{\simeq} M$ such that $F_i=0$ for all $i<j$; see [2, Proposition 2]. In particular, homologically finite DG A-modules admit minimal semi-free resolutions. Moreover, the condition $\partial^{k\otimes_A F}=0$ shows that $\beta_i=\operatorname{rank}_k(\operatorname{Tor}_i^A(M,k))$ for all i.

Definition 2.18. Let A be a DG R-algebra, and let M,N be DG A-modules. Given an integer i, a *DG A-module homomorphism of degree i* is an element $f\in\operatorname{Hom}_R(M,N)_i$ such that $f(am)=(-1)^{i|a|}af(m)$ for all $a\in A$ and $m\in M$. The graded submodule of $\operatorname{Hom}_R(M,N)$ consisting of all DG A-module homomorphisms $M\to N$ is denoted $\operatorname{Hom}_A(M,N)$.

Given a semi-free resolution $F \xrightarrow{\simeq} M$, set $\mathbf{R}\operatorname{Hom}_A(M,N):=\operatorname{Hom}_A(F,N)$ and $\operatorname{Ext}^i_A(M,N):=\operatorname{H}_{-i}(\mathbf{R}\operatorname{Hom}_A(M,N))$ for each integer i.

Fact 2.19. Let A be a DG R-algebra, and let M,N be DG A-modules. The complex $\operatorname{Hom}_A(M,N)$ is a DG A-module via the action

$$(af)(m):=a(f(m))=(-1)^{|a||f|}f(am).$$

For each $a\in A$ the multiplication map $\mu^{M,a}\colon M\to M$ given by $m\mapsto am$ is a homomorphism of degree $|a|$.

Assume that M is homologically bounded below. The complex $\mathbf{R}\operatorname{Hom}_A(M,N)$ is independent of the choice of semi-free resolution of M, and we have an isomorphism $\mathbf{R}\operatorname{Hom}_A(M,N)\simeq\mathbf{R}\operatorname{Hom}_A(M',N')$ in $\mathcal{D}(A)$ whenever $M\simeq M'$ and $N\simeq N'$; see [6, Propositions 1.3.1–1.3.3].

In the passage from R to U in our proof of Theorem A, we use Christensen and Sather-Wagstaff's notion of semidualizing DG U-modules from [13], defined next.

Definition 2.20. Let A be a DG R-algebra, and let M be a DG A-module. The *homothety morphism* $X^A_M\colon A\to\operatorname{Hom}_A(M,M)$ is given by $X^A_M(a):=\mu^{M,a}$, i.e., $X^A_M(a)(m)=am$. When M is homologically bounded below, this induces a *homothety morphism* $\chi^A_M\colon A\to\mathbf{R}\operatorname{Hom}_A(M,M)$.

Assume that A is noetherian. Then M is a *semidualizing* DG A-module if M is homologically finite and the homothety morphism $\chi^A_M\colon A\to\mathbf{R}\operatorname{Hom}_A(M,M)$ is a

[2]Note that our definition of minimality differs from the definition found in [8, (2.12.1)]. However, the definitions are often equivalent, and we do not use any technical aspects of the definition from [8, (2.12.1)] in this paper.

quasiisomorphism. Let $\mathfrak{S}(A)$ denote the set of shift-isomorphism classes in $\mathcal{D}(A)$ of semidualizing DG A-modules, that is, the set of equivalence classes of semidualizing DG A-modules under the relation $\sim$ from Notation 2.12.

The following base-change results are used in the passage from R to U in our proof of Theorem A.

Remark 2.21. Let $A \to B$ be a morphism of DG R-algebras, and let M and N be DG A-modules. The "base changed" complex $B \otimes_A M$ has the structure of a DG B-module by the action $b(b' \otimes m) := (bb') \otimes m$. This structure is compatible with the DG A-module structure on $B \otimes_A M$ via restriction of scalars. Furthermore, this induces a well-defined operation $\mathcal{D}_+(A) \to \mathcal{D}_+(B)$ given by $M \mapsto B \otimes^{\mathbf{L}}_A M$.

Given $f \in \operatorname{Hom}_A(M,N)_i$, define $B \otimes_A f \in \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)_i$ by the formula $(B \otimes_A f)(b \otimes m) := (-1)^{i|b|} b \otimes f(m)$. This yields a morphism of DG A-modules $\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)$ given by $f \mapsto B \otimes_A f$. When M is homologically bounded below, this provides a well-defined morphism $\mathbf{R}\operatorname{Hom}_A(M,N) \to \mathbf{R}\operatorname{Hom}_B(B \otimes^{\mathbf{L}}_A M, B \otimes^{\mathbf{L}}_A N)$ in $\mathcal{D}(A)$.

The next lemma is essentially from [30] and [32].

Lemma 2.22. *Let $\varphi\colon A \to B$ be a quasiisomorphism of noetherian DG R-algebras, that is, a morphism of DG R-algebras that is also a quasiisomorphism.*

(a) *The base change functor $B \otimes^{\mathbf{L}}_A -$ induces an equivalence of derived categories $\mathcal{D}_+(A) \to \mathcal{D}_+(B)$ whose quasi-inverse is given by restriction of scalars.*

(b) *For each DG A-module $X \in \mathcal{D}_+(A)$, one has $X \simeq B \otimes^{\mathbf{L}}_A X$ in $\mathcal{D}(A)$, and thus*

$$\inf(B \otimes^{\mathbf{L}}_A X) = \inf(X)$$
$$\sup(B \otimes^{\mathbf{L}}_A X) = \sup(X)$$
$$\operatorname{amp}(B \otimes^{\mathbf{L}}_A X) = \operatorname{amp}(X).$$

(c) *The equivalence from part* (a) *induces a bijection from $\mathfrak{S}(A)$ to $\mathfrak{S}(B)$.*

Proof. (a) See, e.g., [30, 7.6 Example].

(b) The equivalence from part (a) implies that the map $X \to B \otimes^{\mathbf{L}}_A X$ is a quasiisomorphism, and the displayed equalities follow directly.

(c) Let X be a homologically bounded below DG A-module. We show that X is a semidualizing DG A-module if and only if $B \otimes^{\mathbf{L}}_A X$ is a semidualizing DG B-module. Since the maps $X \to B \otimes^{\mathbf{L}}_A X$ and $A \to B$ are quasiisomorphisms, it follows that X is homologially finite over A if and only if $B \otimes^{\mathbf{L}}_A X$ is homologially finite over B. It remains to show that the homothety morphism $\chi^A_X\colon A \to \mathbf{R}\operatorname{Hom}_A(X,X)$ is an isomorphism in $\mathcal{D}(A)$ if and only if $\chi^B_{B \otimes^{\mathbf{L}}_A X}\colon B \to \mathbf{R}\operatorname{Hom}_B(B \otimes^{\mathbf{L}}_A X, B \otimes^{\mathbf{L}}_A X)$ is an isomorphism in $\mathcal{D}(B)$. It is routine to show that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\chi^A_X} & \mathbf{R}\operatorname{Hom}_A(X,X) \\ {\scriptstyle \varphi}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \omega} \\ B & \xrightarrow[\chi^B_{B \otimes^{\mathbf{L}}_A X}]{} & \mathbf{R}\operatorname{Hom}_B(B \otimes^{\mathbf{L}}_A X, B \otimes^{\mathbf{L}}_A X) \end{array}$$

where ω is the morphism from Remark 2.21. As ω is an isomorphism by [32, Proposition 2.1], the desired equivalence follows. $\square$

Definition 2.23. Let A be a DG R-algebra, and let M be a DG A-module. Given an integer n, the *nth soft left truncation of* M is the complex

$$\tau(M)_{(\leqslant n)} := \cdots \to 0 \to M_n/\operatorname{Im}(\partial^M_{n+1}) \to M_{n-1} \to M_{n-2} \to \cdots$$

with differential induced by ∂^M.

Remark 2.24. Let A be a DG R-algebra, and let M be a DG A-module. Fix an integer n. Then the truncation $\tau(M)_{(\leqslant n)}$ is a DG A-module with the obvious scalar multiplication, and the natural chain map $M \to \tau(M)_{(\leqslant n)}$ is a morphism of DG A-modules. This morphism is a quasiisomorphism if and only if $n \geqslant \sup(M)$. See [8, (4.1)].

Definition 2.25. Let A be a local DG R-algebra, and let M be a homologically finite DG A-module. For each integer i, the ith *Betti and Bass numbers* are

$$\beta_i^A(M) := \operatorname{rank}_k(\operatorname{Tor}_i^A(k, M)) \qquad \mu_A^i(M) := \operatorname{rank}_k(\operatorname{Ext}_A^i(k, M))$$

respectively, where $k = A/\mathfrak{m}_A$. The *Poincaré and Bass series* of M are the formal Laurent series

$$P_A^M(t) := \sum_{i \in \mathbb{Z}} \beta_i^A(M) t^i \qquad I_M^A(t) := \sum_{i \in \mathbb{Z}} \mu_A^i(M) t^i.$$

3. DG Ext and Yoneda Ext

Given a DG R-algebra A, and DG A-modules M and N such that M is homologically bounded below, we have the DG-Ext module $\operatorname{Ext}_A^1(M, N)$ from Definition 2.18. In general, this module does not parametrize the extensions $0 \to N \to L \to M \to 0$; see Example 3.2. To parametrize such extensions, we need "Yoneda Ext", which we describe next. The main point of this section is to prove Theorem C which is important for our solution of Vasconcelos' question.

Definition 3.1. Let A be a DG R-algebra. The category of DG A-modules described in Definition 2.11 is an abelian category; see, e.g., [29, Introduction]. So, given DG A-modules L, M, the *Yoneda Ext group* $\operatorname{YExt}_A^1(L, M)$, defined as the set of equivalence classes of exact sequences $0 \to M \to X \to L \to 0$ of DG A-modules, is a well-defined abelian group under the Baer sum; see, e.g., [39, (3.4.6)].

The next example shows that $\operatorname{YExt}_A^1(M, N)$ and $\operatorname{Ext}_A^1(M, N)$ are distinct.

Example 3.2. Let $R = k[\![X]\!]$, and consider the following exact sequence of DG R-modules, i.e., exact sequence of R-complexes:

$$0 \longrightarrow \underline{R} \longrightarrow \underline{R} \longrightarrow \underline{k} \longrightarrow 0$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{X} & R & \longrightarrow & k & \longrightarrow & 0 \\
 & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{X} & R & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

This sequence does not split over R (it is not even degree-wise split) so it gives a non-trivial class in $\mathrm{YExt}^1_R(\underline{k}, \underline{R})$, and we conclude that $\mathrm{YExt}^1_R(\underline{k}, \underline{R}) \neq 0$. On the other hand, $\underline{k}$ is homologically trivial, so we have $\mathrm{Ext}^1_R(\underline{k}, \underline{R}) = 0$ since 0 is a semi-free resolution of $\underline{k}$.

In preparation for the proof of Theorem C, we require two more items.

Definition 3.3. Let A be a DG R-algebra. A DG A-module Q is *graded-projective* if $\mathrm{Hom}_A(Q, -)$ preserves surjective morphisms, that is, if $Q^\natural$ is a projective graded $R^\natural$-module; see [8, Theorem 2.8.3.1].

Remark 3.4. If Q is semi-free, then $Q^\natural \cong \oplus_i \Sigma^i (R^\natural)^{(\beta_i)}$ is a free (hence projective) graded $R^\natural$-module, so Q is graded-projective.

3.5 (Proof of Theorem C)**.** Let $\zeta \in \mathrm{YExt}^1_A(Q, P)$ be represented by the sequence

$$0 \to P \to X \to Q \to 0. \tag{3.5.1}$$

Since Q is graded-projective, the sequence (3.5.1) graded-splits (see [8, (2.8.3.1)]), that is, this sequence is isomorphic to one of the form

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & {\scriptstyle \partial^P_{i+1}}\downarrow & & {\scriptstyle \partial^X_{i+1}}\downarrow & & {\scriptstyle \partial^Q_{i+1}}\downarrow & & \\
0 & \longrightarrow & P_i & \xrightarrow{\epsilon_i} & P_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \longrightarrow & 0 \\
 & & {\scriptstyle \partial^P_{i}}\downarrow & & {\scriptstyle \partial^X_{i}}\downarrow & & {\scriptstyle \partial^Q_{i}}\downarrow & & \\
0 & \longrightarrow & P_{i-1} & \xrightarrow{\epsilon_{i-1}} & P_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \longrightarrow & 0 \\
 & & {\scriptstyle \partial^P_{i-1}}\downarrow & & {\scriptstyle \partial^X_{i-1}}\downarrow & & {\scriptstyle \partial^Q_{i-1}}\downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{3.5.2}$$

where ϵ_j is the natural inclusion and π_j is the natural surjection for each j. Since this diagram comes from a graded-splitting of (3.5.1), the scalar multiplication on the middle column of (3.5.2) is the natural one $a\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \left[\begin{smallmatrix} ap \\ aq \end{smallmatrix}\right]$.[3]

The fact that (3.5.2) commutes implies that ∂_i^X has a specific form:

$$\partial_i^X = \begin{bmatrix} \partial_i^P & \lambda_i \\ 0 & \partial_i^Q \end{bmatrix}. \tag{3.5.3}$$

Here, we have $\lambda_i \colon Q_i \to P_{i-1}$, that is, $\lambda \in \operatorname{Hom}_R(Q,P)_{-1}$. Since the maps in the sequence (3.5.2) are morphisms of DG A-modules, it follows that λ is a cycle in $\operatorname{Hom}_A(Q,P)_{-1}$. Thus, λ represents a homology class in $\operatorname{Ext}^1_A(Q,P)$, and we define $\Psi\colon \operatorname{YExt}^1_A(Q,P) \to \operatorname{Ext}^1_A(Q,P)$ by the formula $\Psi(\zeta) := \lambda$.

We show that Ψ is well-defined. Let ζ be represented by another exact sequence

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial^P_{i+1}} & & \downarrow{\scriptstyle \partial^{X'}_{i+1}} & & \downarrow{\scriptstyle \partial^Q_{i+1}} & & \\
0 & \longrightarrow & P_i & \xrightarrow{\epsilon_i} & P_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^P_{i}} & & \downarrow{\scriptstyle \partial^{X'}_{i}} & & \downarrow{\scriptstyle \partial^Q_{i}} & & \\
0 & \longrightarrow & P_{i-1} & \xrightarrow{\epsilon_{i-1}} & P_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^P_{i-1}} & & \downarrow{\scriptstyle \partial^{X'}_{i-1}} & & \downarrow{\scriptstyle \partial^Q_{i-1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{3.5.4}$$

where

$$\partial_i^{X'} = \begin{bmatrix} \partial_i^P & \lambda'_i \\ 0 & \partial_i^Q \end{bmatrix}. \tag{3.5.5}$$

We need to show that $\lambda - \lambda' \in \operatorname{Im}(\partial_0^{\operatorname{Hom}_A(Q,P)})$. The sequences (3.5.2) and (3.5.4) are equivalent in $\operatorname{YExt}^1_R(Q,P)$, so for each i there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_i & \xrightarrow{\epsilon_i} & P_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \longrightarrow & 0 \\
 & & \downarrow{=} & & \left[\begin{smallmatrix} u_i & v_i \\ w_i & x_i \end{smallmatrix}\right]\downarrow{\cong} & & \downarrow{=} & & \\
0 & \longrightarrow & P_i & \xrightarrow{\epsilon_i} & P_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \longrightarrow & 0
\end{array} \tag{3.5.6}$$

where the middle vertical arrow is a DG A-module isomorphism, and such that the following diagram commutes

$$\begin{array}{ccc}
P_i \oplus Q_i & \xrightarrow[\cong]{\left[\begin{smallmatrix} u_i & v_i \\ w_i & x_i \end{smallmatrix}\right]} & P_i \oplus Q_i \\
\left[\begin{smallmatrix} \partial_i^P & \lambda_i \\ 0 & \partial_i^Q \end{smallmatrix}\right]\downarrow & & \downarrow\left[\begin{smallmatrix} \partial_i^P & \lambda'_i \\ 0 & \partial_i^Q \end{smallmatrix}\right] \\
P_{i-1} \oplus Q_{i-1} & \xrightarrow[\cong]{\left[\begin{smallmatrix} u_{i-1} & v_{i-1} \\ w_{i-1} & x_{i-1} \end{smallmatrix}\right]} & P_{i-1} \oplus Q_{i-1}.
\end{array} \tag{3.5.7}$$

The fact that diagram (3.5.6) commutes implies that $u_i = \operatorname{id}_{P_i}$, $x_i = \operatorname{id}_{Q_i}$, and $w_i = 0$. Also, the fact that the middle vertical arrow in diagram (3.5.6) describes a DG A-module morphism implies that the sequence $v_i\colon Q_i \to P_i$ respects scalar

[3]Given R-modules M and N, we write elements of $M \oplus N$ as column vectors $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$ with $m \in M$ and $n \in N$. This permits us to use matrix notation for homomorphisms between such modules.

multiplication, i.e., we have $v \in \mathrm{Hom}_A(Q,P)_0$. The fact that diagram (3.5.7) commutes implies that $\lambda_i - \lambda'_i = \partial^P_i v_i - v_{i-1}\partial^Q_i$. We conclude that $\lambda - \lambda' = \partial_0^{\mathrm{Hom}_A(Q,P)}(v) \in \mathrm{Im}(\partial_0^{\mathrm{Hom}_A(Q,P)})$, so Ψ is well-defined.

Next we show that Ψ is additive. Let $\zeta, \zeta' \in \mathrm{YExt}^1_A(Q,P)$ be represented by exact sequences $0 \to P \xrightarrow{\epsilon} X \xrightarrow{\pi} Q \to 0$ and $0 \to P \xrightarrow{\epsilon'} X' \xrightarrow{\pi'} Q \to 0$ respectively, where $X_i = P_i \oplus Q_i = X'_i$ and the differentials ∂^X and $\partial^{X'}$ are described as in (3.5.3) and (3.5.5), respectively. We need to show that the Baer sum $\zeta + \zeta'$ is represented by an exact sequence $0 \to P \xrightarrow{\widetilde{\epsilon}} \widetilde{X} \xrightarrow{\widetilde{\pi}} Q \to 0$ respectively, where $\widetilde{X}_i = P_i \oplus Q_i$ and $\partial^{\widetilde{X}}_i = \left[\begin{smallmatrix} \partial^P_i & \lambda_i + \lambda'_i \\ 0 & \partial^Q_i \end{smallmatrix}\right]$, with scalar multiplication $a\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \left[\begin{smallmatrix} ap \\ aq \end{smallmatrix}\right]$. Note that it is straightforward to show that the sequence $\widetilde{X}$ defined in this way is a DG A-module, and the natural maps $P \xrightarrow{\widetilde{\epsilon}} \widetilde{X} \xrightarrow{\widetilde{\pi}} Q$ are DG-linear, using the analogous properties for X and X'.

We construct the Baer sum in two steps. The first step is to construct the pull-back diagram

$$\begin{array}{ccc} X'' & \xrightarrow{\pi''} & X' \\ {\scriptstyle \pi'''}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{\pi} & Q. \end{array}$$

The DG module X'' is a submodule of the direct sum $X \oplus X'$, so each X''_i is the submodule of

$$X \oplus X' = X_i \oplus X'_i = P_i \oplus Q_i \oplus P_i \oplus Q_i$$

consisting of all vectors $\left[\begin{smallmatrix} x \\ x' \end{smallmatrix}\right]$ such that $\pi'_i(x') = \pi_i(x)$, that is, all vectors of the form $[p \quad q \quad p' \quad q']^T$ such that $q = q'$. In other words, we have

$$P_i \oplus Q_i \oplus P_i \xrightarrow{\cong} X''_i \tag{3.5.8}$$

where the isomorphism is given by $[p \quad q \quad p']^T \mapsto [p \quad q \quad p' \quad q]^T$. The differential on $X \oplus X'$ is the natural diagonal map. So, under the isomorphism (3.5.8), the differential on X'' has the form

$$X''_i \cong P_i \oplus Q_i \oplus P_i \xrightarrow{\partial^{X''}_i = \left[\begin{smallmatrix} \partial^P_i & \lambda_i & 0 \\ 0 & \partial^Q_i & 0 \\ 0 & \lambda'_i & \partial^P_i \end{smallmatrix}\right]} P_{i-1} \oplus Q_{i-1} \oplus P_{i-1} \cong X''_{i-1}.$$

The second step is to construct $\widetilde{X}$, which is the cokernel of the morphism $\gamma\colon P \to X''$ given by $p \mapsto \left[\begin{smallmatrix} -p \\ 0 \\ p \end{smallmatrix}\right]$. In other words, since γ is injective, the complex $\widetilde{X}$ is determined by the exact sequence $0 \to P \xrightarrow{\gamma} X'' \xrightarrow{\tau} \widetilde{X} \to 0$. It is straightforward to show that the following diagram describes such an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_i & \xrightarrow{\left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right]} & P_i \oplus Q_i \oplus P_i & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right]} & P_i \oplus Q_i & \longrightarrow & 0 \\ & & {\scriptstyle \partial^P_i}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix} \partial^P_i & \lambda_i & 0 \\ 0 & \partial^Q_i & 0 \\ 0 & \lambda'_i & \partial^P_i \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} \partial^P_i & \lambda_i + \lambda'_i \\ 0 & \partial^Q_i \end{smallmatrix}\right]} & & \\ 0 & \longrightarrow & P_{i-1} & \xrightarrow{\left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right]} & P_{i-1} \oplus Q_{i-1} \oplus P_{i-1} & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right]} & P_{i-1} \oplus Q_{i-1} & \longrightarrow & 0. \end{array}$$

By inspecting the right-most column of this diagram, we see that $\widetilde{X}$ has the desired form. Furthermore, checking the module structures at each step of the construction, we see that the scalar multiplication on $\widetilde{X}$ is the natural one $a\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \left[\begin{smallmatrix} ap \\ aq \end{smallmatrix}\right]$.

Next, we show that Ψ is injective. Suppose that $\zeta \in \operatorname{Ker}(\Psi)$ is represented by the displays (3.5.1)–(3.5.3). The condition $\Psi(\zeta) = 0$ says that $\lambda \in \operatorname{Im}(\partial_0^{\operatorname{Hom}_A(Q,P)})$, so there is an element $s \in \operatorname{Hom}_A(Q,P)_0$ such that $\zeta = \partial_0^{\operatorname{Hom}_A(Q,P)}(s)$. This says that for each i we have $\lambda_i = \partial_i^P s_i - s_{i-1}\partial_i^Q$. From this, it is straightforward to show that the following diagram commutes:

$$\begin{array}{ccc}
P_i \oplus Q_i & \xrightarrow[\cong]{\left[\begin{smallmatrix} 1 & s_i \\ 0 & 1 \end{smallmatrix}\right]} & P_i \oplus Q_i \\
{\scriptstyle \left[\begin{smallmatrix} \partial_i^P & \lambda_i \\ 0 & \partial_i^Q \end{smallmatrix}\right]}\big\downarrow & & \big\downarrow{\scriptstyle \left[\begin{smallmatrix} \partial_i^P & 0 \\ 0 & \partial_i^Q \end{smallmatrix}\right]} \\
P_{i-1} \oplus Q_{i-1} & \xrightarrow[\cong]{\left[\begin{smallmatrix} 1 & s_{i-1} \\ 0 & 1 \end{smallmatrix}\right]} & P_{i-1} \oplus Q_{i-1}.
\end{array}$$

From the fact that s is A-linear, it follows that the maps $\left[\begin{smallmatrix} 1 & s_i \\ 0 & 1 \end{smallmatrix}\right]$ describe an A-linear isomorphism $X \xrightarrow{\cong} P \oplus Q$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{\epsilon} & X & \xrightarrow{\pi} & Q & \longrightarrow & 0 \\
 & & {\scriptstyle =}\big\downarrow & & {\scriptstyle \cong}\big\downarrow & & {\scriptstyle =}\big\downarrow & & \\
0 & \longrightarrow & P & \xrightarrow{\epsilon} & P \oplus Q & \xrightarrow{\pi} & Q & \longrightarrow & 0.
\end{array}$$

In other words, the sequence (3.5.1) splits, so we have $\zeta = 0$, and Ψ is injective.

Finally, we show that Ψ is surjective. For this, let $\xi \in \operatorname{Ext}_A^1(Q,P)$ be represented by $\lambda \in \operatorname{Ker}(\partial_{-1}^{\operatorname{Hom}_A(Q,P)})$. Using the fact that λ is A-linear such that $\partial_{-1}^{\operatorname{Hom}_A(Q,P)}(\lambda) = 0$, one checks directly that the displays (3.5.2)–(3.5.3) describe an exact sequence of DG A-module homomorphisms of the form (3.5.1) whose image under Ψ is ξ. $\square$

To describe higher Yoneda Ext groups, we need another variant of the notion of projectivity for DG modules.

Definition 3.6. Let A be a DG R-algebra. Projective objects in the category of DG A-modules are called *categorically projective* DG A-modules.

Remark 3.7. Let A be a DG R-algebra. Our definition of "categorically projective" is equivalent to the one given in [8, Section 2.8.1], because of [8, Theorem 2.8.7.1]. Furthermore, the category of DG A-modules has enough projectives by [8, Corollary 2.7.5.4 and Theorem 2.8.7.1(iv)]. Thus, given DG A-modules L and M, for each $i \geqslant 0$ we have a well-defined Yoneda Ext group $\operatorname{YExt}_A^i(L,M)$, defined in terms of a resolution of L by categorically projective DG A-modules:

$$\cdots \to Q_1 \to Q_0 \to L \to 0.$$

A standard result shows that when $i = 1$, this definition of Yoneda Ext is equivalent to the one given in Definition 3.1.

Corollary 3.8. *Let A be a DG R-algebra, and let P, Q be DG A-modules such that Q is graded-projective (e.g., Q is semi-free). Then there is an isomorphism $\operatorname{YExt}_A^i(Q,P) \cong \operatorname{Ext}_A^i(Q,P)$ of abelian groups for all $i \geqslant 1$.*

Proof. Using Theorem C, we need only justify the isomorphism $\mathrm{YExt}^i_A(Q,P) \cong \mathrm{Ext}^i_A(Q,P)$ for $i \geqslant 2$. Let

$$L^+_\bullet = \cdots \xrightarrow{\partial^L_2} L_1 \xrightarrow{\partial^L_1} L_0 \xrightarrow{\pi} Q \to 0$$

be a resolution of Q by categorically projective DG A-modules. Since each L_j is categorically projective, we have $\mathrm{YExt}^i_A(L_j,-) = 0$ for all $i \geqslant 1$. From [8, Theorem 2.8.7.1] we conclude that $L_j \simeq 0$ for each j, so we have $\mathrm{Ext}^i_A(L_j,-) = 0$ for all i. Set $Q_i = \operatorname{Im}\partial^L_i$ for each $i \geqslant 1$. Each L_i is graded-projective by [8, Theorems 2.8.6.1 and 2.8.7.1], so the fact that Q is graded-projective implies that each Q_i is graded-projective.

Now, a straightforward dimension-shifting argument to explain the first and third isomorphisms in the following display for $i \geqslant 2$:

$$\mathrm{YExt}^i_A(Q,P) \cong \mathrm{YExt}^1_A(Q_{i-1},P) \cong \mathrm{Ext}^1_A(Q_{i-1},P) \cong \mathrm{Ext}^i_A(Q,P).$$

The second isomorphism is from Theorem C since each Q_i is graded-projective. □

The next example shows that one can have $\mathrm{YExt}^0_A(Q,P) \not\cong \mathrm{Ext}^0_A(Q,P)$, even when Q is semi-free.

Example 3.9. Continue with the assumptions and notation of Example 3.2, and set $Q = P = \underline{R}$. It is straightforward to show that the morphisms $\underline{R} \to \underline{R}$ are precisely given by multiplication by fixed elements of R, so we have the first step in the next display:

$$\mathrm{YExt}^0_A(\underline{R},\underline{R}) \cong R \neq 0 = \mathrm{Ext}^0_A(\underline{R},\underline{R}).$$

The third step follows from the condition $\underline{R} \simeq 0$.

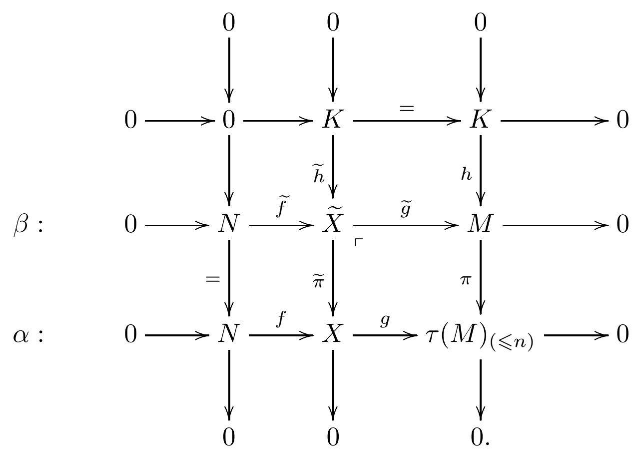
In our proof of Theorem A, we need to know when YExt respects truncations.

Proposition 3.10. *Let A be a DG R-algebra, and let M and N be DG A-modules. Assume that n is an integer such that $N_i = 0$ for all $i > n$. Then the natural map $\mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)},N) \to \mathrm{YExt}^1_A(M,N)$ induced by the morphism $\pi\colon M \to \tau(M)_{(\leqslant n)}$ is a monomorphism.*

Proof. Let Υ denote the map $\mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)},N) \to \mathrm{YExt}^1_A(M,N)$ induced by π. Let $\alpha \in \mathrm{Ker}(\Upsilon) \subseteq \mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)},N)$ be represented by the exact sequence

$$0 \to N \xrightarrow{f} X \xrightarrow{g} \tau(M)_{(\leqslant n)} \to 0. \qquad (3.10.1)$$

Note that, since $N_i = 0 = (\tau(M)_{(\leqslant n)})_i$ for all $i > n$, we have $X_i = 0$ for all $i > n$. Then $0 = \Upsilon([\alpha]) = [\beta]$ where β comes from the following pull-back diagram:



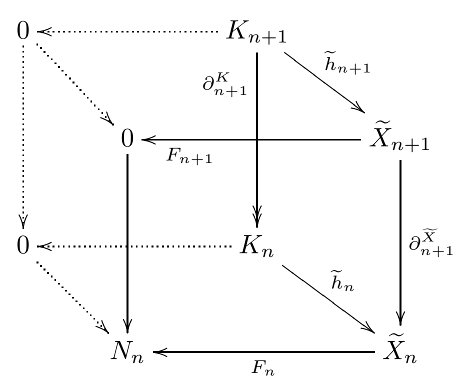
(3.10.2)

The middle row β of this diagram is split exact since $[\beta] = 0$, so there is a morphism $F\colon \widetilde{X} \to N$ of DG A-modules such that $F \circ \widetilde{f} = \mathrm{id}_N$. Note that K has the form

$$K = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} \mathrm{Im}(\partial^M_{n+1}) \to 0 \tag{3.10.3}$$

because of the right-most column of the diagram.

We claim that $F \circ \widetilde{h} = 0$. It suffices to check this degree-wise. When $i > n$, we have $N_i = 0$, so $F_i = 0$, and $F_i \circ \widetilde{h}_i = 0$. When $i < n$, the display (3.10.3) shows that $K_i = 0$, so $\widetilde{h}_i = 0$, and $F_i \circ \widetilde{h}_i = 0$. For $i = n$, we first note that the display (3.10.3) shows that ∂^K_{n+1} is surjective. In the following diagram, the faces with solid arrows commute because $\widetilde{h}$ and F are morphisms:



Since ∂^K_{n+1} is surjective, a simple diagram chase shows that $F_n \circ \widetilde{h}_n = 0$. This establishes the claim.

To conclude the proof, note that the previous claim shows that the map $K \to 0$ is a left-splitting of the top row of diagram (3.10.2) that is compatible with the left-splitting F of the middle row. It is then straightforward to show that F induces a morphism $\overline{F}\colon X \to N$ of DG A-modules that left-splits the bottom row of diagram (3.10.2). Since this row represents $\alpha \in \mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)}, N)$, we conclude that $[\alpha] = 0$, so Υ is a monomorphism. $\square$

The next example shows that the monomorphism from Proposition 3.10 may not be an isomorphism.

Example 3.11. Continue with the assumptions and notation of Example 3.2. The following diagram describes a non-zero element of $\mathrm{YExt}^1_R(M,N)$:

$$0 \longrightarrow N \longrightarrow \underline{R} \longrightarrow M \longrightarrow 0$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{1} & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & R & \xrightarrow{X} & R & \xrightarrow{\pi} & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

It is straightforward to show that $\tau(M)_{(\leqslant 0)} = 0$, so we have

$$0 = \mathrm{YExt}^1_A(\tau(M)_{(\leqslant 0)}, N) \hookrightarrow \mathrm{YExt}^1_A(M,N) \neq 0$$

so this map is not an isomorphism.

Proposition 3.12. *Let A be a DG R-algebra, and let C be a graded-projective (e.g., semi-free) DG A-module such that $\mathrm{Ext}^1_R(C,C) = 0$. For $n \geqslant \sup(C)$, one has*

$$\mathrm{YExt}^1_A(C,C) = 0 = \mathrm{YExt}^1_A(\tau(C)_{(\leqslant n)}, \tau(C)_{(\leqslant n)}).$$

Proof. From Theorem C, we have $\mathrm{YExt}^1_A(C,C) \cong \mathrm{Ext}^1_A(C,C) = 0$. For the remainder of the proof, assume without loss of generality that $\sup(C) < \infty$. Another application of Theorem C explains the first step in the next display:

$$\mathrm{YExt}^1_A(C, \tau(C)_{(\leqslant n)}) \cong \mathrm{Ext}^1_A(C, \tau(C)_{(\leqslant n)}) \cong \mathrm{Ext}^1_A(C,C) = 0.$$

The second step comes from the assumption $n \geqslant \sup(C)$ which guarantees that the natural map $C \to \tau(C)_{(\leqslant n)}$ is a quasiisomorphism. Proposition 3.10 implies that $\mathrm{YExt}^1_A(\tau(C)_{(\leqslant n)}, \tau(C)_{(\leqslant n)})$ is isomorphic to a subgroup of $\mathrm{YExt}^1_A(C, \tau(C)_{(\leqslant n)}) = 0$, so we have $\mathrm{YExt}^1_A(\tau(C)_{(\leqslant n)}, \tau(C)_{(\leqslant n)}) = 0$, as desired. □

4. Some Deformation Theory for DG Modules

The ideas for this section are from [3, 21, 27].

Notation 4.1. Let F be an algebraically closed field, and let

$$U := (0 \to U_q \xrightarrow{\partial^U_q} U_{q-1} \xrightarrow{\partial^U_{q-1}} \cdots \xrightarrow{\partial^U_1} U_0 \to 0)$$

be a finite-dimensional DG F-algebra. Let $\dim_F(U_i) = n_i$ for $i = 0, \ldots, q$. Let

$$W := \bigoplus_{i=0}^{s} W_i$$

be a graded F-vector space with $r_i := \dim_F(W_i)$ for $i = 0, \ldots, s$.

A DG U-module structure on W consists of two pieces of data. First, we need a differential ∂. Second, once the differential ∂ has been chosen, we need a scalar multiplication μ. Let $\mathrm{Mod}^U(W)$ denote the set of all ordered pairs (∂,μ) making W into a DG U-module. Let $\mathrm{End}_F(W)_0$ denote the set of F-linear endomorphisms of W that are homogeneous of degree 0. Let $\mathrm{GL}(W)_0$ denote the set of F-linear automorphisms of W that are homogeneous of degree 0, that is, the invertible elements of $\mathrm{End}_F(W)_0$.

Let $F[\epsilon] := F\epsilon \oplus F$ be the algebra of dual numbers, where $\epsilon^2 = 0$. For our convenience, we write elements of $F[\epsilon]$ as column vectors: $a\epsilon + b = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. We identify $U[\epsilon] := F[\epsilon] \otimes_F U$ with $U\epsilon \oplus U \cong U \oplus U$, and $W[\epsilon] := F[\epsilon] \otimes_F W$ with $W\epsilon \oplus W \cong W \oplus W$. Using this protocol, we have $\partial_i^{U[\epsilon]} = \left[\begin{smallmatrix} \partial_i^U & 0 \\ 0 & \partial_i^U \end{smallmatrix}\right]$.

We next describe geometric structures on the sets $\mathrm{Mod}^U(W)$ and $\mathrm{GL}(W)_0$.

Remark 4.2. We work in the setting of Notation 4.1.

A differential ∂ on W is an element of the graded vector space $\mathrm{Hom}_F(W,W)_{-1}$ such that $\partial\partial = 0$. The vector space $\mathrm{Hom}_F(W_i, W_{i-1})$ has dimension $r_i r_{i-1}$, so the map ∂ corresponds to an element of the affine space $\mathbb{A}_F^d$ where $d := \sum_i r_i r_{i-1}$. The vanishing condition $\partial\partial = 0$ is equivalent to the entries of the matrices representing ∂ satisfying certain fixed homogeneous quadratic polynomial equations over F. Hence, the set of all differentials on W is a Zariski-closed subset of $\mathbb{A}_F^d$.

Once the differential ∂ has been chosen, a scalar multiplication μ is in particular a cycle in $\mathrm{Hom}_F(U \otimes_F W, W)_0$. For all i,j, the vector space $\mathrm{Hom}_F(U_i \otimes_F W_j, W_{i+j})$ has dimension $n_i r_j r_{i+j}$, so the map μ corresponds to an element of the affine space $\mathbb{A}_F^{d'}$ where $d' := \sum_c \sum_i n_i r_{c-i} r_c$. The condition that μ be an associative, unital cycle is equivalent to the entries of the matrices representing ∂ and μ satisfying certain fixed polynomials over F. Thus, the set $\mathrm{Mod}^U(W)$ is a Zariski-closed subset of $\mathbb{A}_F^d \times \mathbb{A}_F^{d'} \cong \mathbb{A}_F^{d+d'}$.

Remark 4.3. We work in the setting of Notation 4.1.

An element $\alpha \in \mathrm{GL}(W)_0$ is an element of the graded vector space $\mathrm{Hom}_F(W,W)_0$ with a multiplicative inverse. The vector space $\mathrm{Hom}_F(W_i, W_i)$ has dimension r_i^2, so the map α corresponds to an element of the affine space $\mathbb{A}_F^e$ where $e := \sum_i r_i^2$. The invertibility of α is equivalent to the invertibility of each "block" $\alpha_i \in \mathrm{Hom}_F(W_i, W_i)$, which is an open condition. Thus, the set $\mathrm{GL}(W)_0$ is a Zariski-open subset of $\mathbb{A}_F^e$, so it is smooth over F.

Alternately, one can view $\mathrm{GL}(W)_0$ as the product $\mathrm{GL}(W_0) \times \cdots \times \mathrm{GL}(W_s)$. Since each $\mathrm{GL}(W_i)$ is an algebraic group smooth over F, it follows that $\mathrm{GL}(W)_0$ is also an algebraic group that is smooth over F.

Next, we describe an action of $\mathrm{GL}(W)_0$ on $\mathrm{Mod}^U(W)$.

Remark 4.4. We work in the setting of Notation 4.1.

Let $\alpha \in \mathrm{GL}(W)_0$. For every $(\partial,\mu) \in \mathrm{Mod}^U(W)$, we define $\alpha \cdot (\partial,\mu) := (\widetilde{\partial}, \widetilde{\mu})$, where $\widetilde{\partial} := \alpha \circ \partial \circ \alpha^{-1}$ and $\widetilde{\mu} := \alpha \circ \mu \circ (U \otimes_F \alpha^{-1})$. It is straightforward to show that the ordered pair $(\widetilde{\partial}, \widetilde{\mu})$ describes a DG U-module structure for W, that is, we have $\alpha \cdot (\partial,\mu) := (\widetilde{\partial}, \widetilde{\mu}) \in \mathrm{Mod}^U(W)$. From the definition of $\alpha \cdot (\partial,\mu)$, it follows readily that this describes a $\mathrm{GL}(W)_0$-action on $\mathrm{Mod}^U(W)$.

It is straightforward to show that the map α gives a DG U-module isomorphism $(W,\partial,\mu) \xrightarrow{\cong} (W, \widetilde{\partial}, \widetilde{\mu})$. Conversely, given another element $(\partial', \mu') \in \mathrm{Mod}^U(W)$, if there is a DG U-module isomorphism $\beta\colon (W,\partial,\mu) \xrightarrow{\cong} (W,\partial',\mu')$, then $\beta \in \mathrm{GL}(W)_0$ and $(\partial',\mu') = \beta \cdot (\partial,\mu)$. In other words, the orbits in $\mathrm{Mod}^U(W)$ under the action of $\mathrm{GL}(W)_0$ are the isomorphism classes of DG U-module structures on W.

Note that the maps defining the action of $\mathrm{GL}(W)_0$ on $\mathrm{Mod}^U(W)$ are regular, that is, determined by polynomial functions. This is because the inversion map $\alpha \mapsto \alpha^{-1}$ on $\mathrm{GL}(W)_0$ is regular, as is the multiplication of matrices corresponding to the compositions defining $\widetilde{\partial}$ and $\widetilde{\mu}$.

Notation 4.5. We work in the setting of Notation 4.1.

The set $\mathrm{Mod}^U(W)$ is the set of F-rational points of a scheme $\underline{\mathrm{Mod}}^U(W)$ over F, which we describe using the functorial point of view, following [14, 15]: for each commutative F-algebra S, we have[4]

$$\underline{\mathrm{Mod}}^U(W)(S) := \{\text{DG } S \otimes_F U\text{-module structures on } S \otimes_F W\}.$$

Sometimes we write $\mathrm{Mod}^{S\otimes_F U}(S \otimes_F W)$ in place of $\underline{\mathrm{Mod}}^U(W)(S)$. Similarly, $\mathrm{GL}(W)_0$ is the set of F-rational points of a scheme $\underline{\mathrm{GL}}(W)_0$ over F: for each commutative F-algebra S, we have

$$\underline{\mathrm{GL}}(W)_0(S) := \{\text{homogeneous } S\text{-linear automorphisms of } S \otimes_F W \text{ of degree } 0\}.$$

The fact that $\mathrm{Mod}^U(W)$ and $\mathrm{GL}(W)_0$ are the sets of F-rational points of these schemes means that $\mathrm{Mod}^U(W) = \underline{\mathrm{Mod}}^U(W)(F)$ and $\mathrm{GL}(W)_0 = \underline{\mathrm{GL}}(W)_0(F)$.

Fix a commutative F-algebra S. As in Remark 4.4, the group $\underline{\mathrm{GL}}(W)_0(S)$ acts on $\underline{\mathrm{Mod}}^U(W)(S)$: for each $\alpha \in \underline{\mathrm{GL}}(W)_0(S)$ and $(\partial,\mu) \in \underline{\mathrm{Mod}}^U(W)(S)$, define $\alpha\cdot(\partial,\mu) := (\widetilde{\partial},\widetilde{\mu})$, where $\widetilde{\partial} := \alpha\circ\partial\circ\alpha^{-1}$ and $\widetilde{\mu} := \alpha\circ\mu\circ((S\otimes_F U)\otimes_S \alpha^{-1})$. Again, the orbits in $\underline{\mathrm{Mod}}^U(W)(S)$ under the action of $\underline{\mathrm{GL}}(W)_0(S)$ are the isomorphism classes of DG $S \otimes_F U$-module structures on $S \otimes_F W$.

Let $M = (\partial,\mu) \in \mathrm{Mod}^U(W)$. The orbit of M under $\underline{\mathrm{GL}}(W)_0$ is the subscheme $\underline{\mathrm{GL}}(W)_0 \cdot M$ of $\underline{\mathrm{Mod}}^U(W)$ defined as

$$(\underline{\mathrm{GL}}(W)_0 \cdot M)(S) := \underline{\mathrm{GL}}(W)_0(S) \cdot (S \otimes_F M)$$

which is the DG isomorphism class of $S \otimes_F M$ over $S \otimes_F U$. Let $\varrho\colon \underline{\mathrm{GL}}(W)_0 \to \underline{\mathrm{GL}}(W)_0 \cdot M$ denote the following natural map: for each commutative F-algebra S and each $\alpha \in \underline{\mathrm{GL}}(W)_0(S)$ we have $\varrho(\alpha) := \alpha \cdot (S \otimes_F M)$.

Given a scheme $\underline{X}$ over F and a point $x \in \underline{X}(F)$, let $\mathsf{T}^{\underline{X}}_x$ denote the Zariski tangent space to $\underline{X}$ at x.

Remark 4.6. We work in the setting of Notations 4.1 and 4.5. Let $M \in \mathrm{Mod}^U(W)$. From [14, II, §5, 3], we know that the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$, equipped with its natural reduced subscheme structure, is locally closed in $\underline{\mathrm{Mod}}^U(W)$, and the map ϱ is regular and faithfully flat. Also, [14, II, §5, 2.6] tells us that $\underline{\mathrm{GL}}(W)_0$ is smooth.

Lemma 4.7. *We work in the setting of Notations 4.1 and 4.5. Let $M \in \mathrm{Mod}^U(W)$. The map $\varrho\colon \underline{\mathrm{GL}}(W)_0 \to \underline{\mathrm{GL}}(W)_0 \cdot M$ and the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$ are smooth.*

Proof. We begin by showing that the fibre $\mathrm{Stab}(M)$ of $\varrho(F)$ over M is smooth over F. Since F is algebraically closed, it suffices to show that $\mathrm{Stab}(M)$ is regular. Since $\mathrm{GL}(W)_0$ is regular, to show that $\mathrm{Stab}(M) \subseteq \mathrm{GL}(W)_0$ is regular it suffices to show

[4]Technically, the inputs for this functor should be taken from the category of affine schemes over $\mathrm{Spec}(F)$, but the equivalence between this category and the category of commutative F-algebras makes this equivalent to our approach.

that $\operatorname{Stab}(M)$ is defined by linear equations. To find these linear equations, note that the stabilizer condition $\alpha \cdot M = M$ is equivalent to the conditions $\partial = \alpha \circ \partial \circ \alpha^{-1}$ and $\mu = \alpha \circ \mu \circ (U \otimes_F \alpha^{-1})$, that is, $\partial \circ \alpha = \alpha \circ \partial$ and $\mu \circ (U \otimes_F \alpha) = \alpha \circ \mu$; since the matrices defining ∂ and μ are fixed, these equations are described by a system of linear equations in the variables describing α. Thus, the fibre $\operatorname{Stab}(M)$ is smooth.

Now, each closed fibre of $\varrho(F)$ is isomorphic to $\operatorname{Stab}(M)$ by translation, so it is smooth over F. Hilbert's Nullstellensatz implies that $\varrho(F)$ maps closed points to closed points, so it follows from [26, Théorème (17.5.1)] that $\varrho(F)$ is smooth at every closed point of $\underline{\operatorname{GL}}(W)_0(F)$. Since smoothness is an open condition on the source by [25, Corollaire (6.8.7)], it follows that $\varrho(F)$ is smooth at every point (closed or not) of $\underline{\operatorname{GL}}(W)_0(F)$. The fact that $\varrho(F)$ is smooth implies that ϱ is smooth, by [14, I.4.4.1].

Finally, because $\varrho(F)$ is faithfully flat, it is surjective. We know that $\operatorname{GL}(W)_0$ is smooth over F, and $\varrho(F)$ is smooth, so $\operatorname{GL}(W)_0 \cdot M$ is also smooth over F by [25, Proposition (6.8.3)(ii)]. It follows from [14, I.4.4.1] that $\underline{\operatorname{GL}}(W)_0 \cdot M$ is smooth. □

Lemma 4.8. *We work in the setting of Notations 4.1 and 4.5. Given an element $M = (\partial, \mu) \in \operatorname{Mod}^U(W)$, the tangent space $\mathsf{T}_M^{\underline{\operatorname{Mod}}^U(W)}$ is the set of all ordered pairs $(\overline{\partial}, \overline{\mu}) \in \underline{\operatorname{Mod}}^U(W)(F[\epsilon])$ that give rise to M modulo ϵ. Equivalently, $\mathsf{T}_M^{\underline{\operatorname{Mod}}^U(W)}$ is the set of all ordered pairs $(\overline{\partial}, \overline{\mu}) = (\{\overline{\partial}_i\}, \{\overline{\mu}_i\})$ satisfying the following conditions:*

1. *For each i, we have $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & \gamma_i \\ 0 & \partial_i \end{smallmatrix}\right]$ where $\gamma_i \colon W_i \to W_{i-1}$ is an F-linear transformation such that $\partial_i \gamma_{i+1} + \gamma_i \partial_{i+1} = 0$.*
2. *There is a degree-0 graded homomorphism $\theta \colon U \otimes_F W \to W$ of F-vector spaces such that the map $\overline{\mu} \colon U[\epsilon] \otimes_{F[\epsilon]} W[\epsilon] \to W[\epsilon]$ is given by the formula*
$$\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \theta_{i+j}(a \otimes w) + \mu_{i+j}(a \otimes w') + \mu_{i+j}(a' \otimes w) \\ \mu_{i+j}(a \otimes w) \end{smallmatrix}\right]$$
for all $\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \in U_i \oplus U_i$ and $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in W_j \oplus W_j$, and $\overline{\mu}$ is a degree-0 graded homomorphism of $F[\epsilon]$-modules.
3. *For each $a \in U_i$ and $w \in W_j$, we have*
$$\begin{aligned}\gamma_{i+j}(\mu_{i+j}(a \otimes w)) &+ \partial_{i+j}(\theta_{i+j}(a \otimes w)) \\ &= \theta_{i-1+j}(\partial_i^U(a) \otimes w) + (-1)^i \theta_{i+j-1}(a \otimes \partial_j(w)) + (-1)^i \mu_{i-1+j}(a \otimes \gamma_j(w)).\end{aligned}$$
4. *For each $a \in U_i$, $b \in U_p$ and $w \in W_j$, we have*
$$\theta_{i+p+j}((ab) \otimes w) = \theta_{i+p+j}(a \otimes \mu_{p+j}(b \otimes w)) + \mu_{i+p+j}(a \otimes \theta_{p+j}(b \otimes w)).$$

Proof. The natural map $F[\epsilon] \to F$ induces a morphism
$$\operatorname{Mod}^{U[\epsilon]}(W[\epsilon]) = \underline{\operatorname{Mod}}^U(W)(F[\epsilon]) \to \underline{\operatorname{Mod}}^U(W)(F) = \operatorname{Mod}^U(W)$$
and the tangent space $\mathsf{T}_M^{\underline{\operatorname{Mod}}^U(W)}$ is the fibre of this morphism over M. Thus, an element of $\mathsf{T}_M^{\underline{\operatorname{Mod}}^U(W)}$ is precisely a DG $U[\epsilon]$-module structure on $W[\epsilon]$ that gives rise to M modulo ϵ.

Let $N = (\overline{\partial}, \overline{\mu}) \in \mathsf{T}_M^{\underline{\operatorname{Mod}}^U(W)}$; we show that conditions (1)–(4) are satisfied. The fact that $\overline{\partial}$ is $F[\epsilon]$-linear and gives rise to ∂ modulo ϵ, implies that $\overline{\partial}$ has the form $\overline{\partial}_i = \left[\begin{smallmatrix} \delta_i & \gamma_i \\ \beta_i & \partial_i \end{smallmatrix}\right]$ where $\beta_i, \gamma_i, \delta_i \colon W_i \to W_{i-1}$. Since the ordered pair $(\overline{\partial}, \overline{\mu})$ endows $W[\epsilon]$

with a DG $U[\epsilon]$-module structure, the Leibniz rule must be satisfied. In particular, for all $w \in W_j$, we have

$$\begin{aligned}
\overline{\partial}_j\left(\overline{\mu}_j\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\otimes\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]\right)\right) &= \overline{\mu}_{j-1}\left(\partial_j^{U[\epsilon]}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right)\otimes\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]\right)+\overline{\mu}_{j-1}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\otimes\overline{\partial}_j\left(\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]\right)\right)\\
\left[\begin{smallmatrix}\delta_j & \gamma_j\\ \beta_j & \partial_j\end{smallmatrix}\right]\left[\begin{smallmatrix}w\\0\end{smallmatrix}\right] &= 0+\overline{\mu}_{j-1}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\otimes\left(\left[\begin{smallmatrix}\delta_j & \gamma_j\\ \beta_j & \partial_j\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]\right)\right)\\
\left[\begin{smallmatrix}\delta_j(w)\\ \beta_j(w)\end{smallmatrix}\right] &= \overline{\mu}_{j-1}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\otimes\left[\begin{smallmatrix}\gamma_j(w)\\ \partial_j(w)\end{smallmatrix}\right]\right)\\
\left[\begin{smallmatrix}\delta_j(w)\\ \beta_j(w)\end{smallmatrix}\right] &= \left[\begin{smallmatrix}\partial_j(w)\\ 0\end{smallmatrix}\right].
\end{aligned}$$

It follows that $\beta_j = 0$ and $\partial_j = \delta_j$, so we have $\overline{\partial}_i = \left[\begin{smallmatrix}\partial_i & \gamma_i\\ 0 & \partial_i\end{smallmatrix}\right]$. Also for each i the condition $\overline{\partial}_i\overline{\partial}_{i+1} = 0$ implies that $\partial_i\gamma_{i+1}+\gamma_i\partial_{i+1} = 0$ for all i. This establishes (1).

The map $\overline{\mu}$ is a chain map over $F[\epsilon]$ from $U[\epsilon]\otimes_{F[\epsilon]}(W[\epsilon],\overline{\partial})$ to $(W[\epsilon],\overline{\partial})$. The fact that $\overline{\mu}$ is $F[\epsilon]$-linear and gives rise to μ modulo ϵ, implies that $\overline{\mu}$ satisfies the following conditions:

$$\begin{aligned}
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix}0\\a\end{smallmatrix}\right]\otimes\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]\right) &= \left[\begin{smallmatrix}\theta_{i+j}(a\otimes w)\\ \mu_{i+j}(a\otimes w)\end{smallmatrix}\right]\\
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix}0\\a\end{smallmatrix}\right]\otimes\left[\begin{smallmatrix}w\\0\end{smallmatrix}\right]\right) &= \left[\begin{smallmatrix}\mu_{i+j}(a\otimes w)\\ 0\end{smallmatrix}\right]\\
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix}a\\0\end{smallmatrix}\right]\otimes\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]\right) &= \left[\begin{smallmatrix}\mu_{i+j}(a\otimes w)\\ 0\end{smallmatrix}\right]\\
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix}a\\0\end{smallmatrix}\right]\otimes\left[\begin{smallmatrix}w\\0\end{smallmatrix}\right]\right) &= \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right].
\end{aligned}$$

Here we have $a \in U_i$ and $w \in W_j$, and $\theta\colon U\otimes_F W \to W$ is a degree-0 graded homomorphism of F-vector spaces. Condition (2) follows by linearity. Condition (3) follows from the Leibniz rule for elements of the form $\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right] \in W[\epsilon]_j$ and $\left[\begin{smallmatrix}0\\a\end{smallmatrix}\right] \in U[\epsilon]_i$, and condition (4) follows from the associativity of the scalar multiplication $\overline{\mu}$.

Similar reasoning shows that any ordered pair $(\overline{\partial},\overline{\mu}) \in \underline{\mathrm{Mod}}^U(W)(F[\epsilon])$ satisfying conditions (1)–(4) is a DG $U[\epsilon]$-module structure on $W[\epsilon]$ that gives rise to M modulo ϵ, that is, an element of $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$. Note that condition (4) implies that $\theta_j(1\otimes w) = 0$ for all $w \in W_j$ for all j, which is used in this implication. □

Lemma 4.9. *We work in the setting of Notations 4.1 and 4.5. Given an element $M = (\partial,\mu) \in \mathrm{Mod}^U(W)$, the tangent space $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ is an F-vector space under the following operations: Let $N^{(1)}, N^{(2)} \in \mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ where $N^{(n)} = (\overline{\partial}^{(n)},\overline{\mu}^{(n)})$ such that $\overline{\partial}_i^{(n)} = \left[\begin{smallmatrix}\partial_i & \gamma_i^{(n)}\\ 0 & \partial_i\end{smallmatrix}\right]$ and*

$$\overline{\mu}_{i+j}^{(n)}\left(\left[\begin{smallmatrix}a'\\a\end{smallmatrix}\right]\otimes\left[\begin{smallmatrix}w'\\w\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}\theta_{i+j}^{(n)}(a\otimes w)+\mu_{i+j}(a\otimes w')+\mu_{i+j}(a'\otimes w)\\ \mu_{i+j}(a\otimes w)\end{smallmatrix}\right]$$

for $n = 1, 2$ as in Lemma 4.8. For $\alpha_1, \alpha_2 \in F$ the element $\alpha_1 N^{(1)} + \alpha_2 N^{(2)}$ in $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ is given using the functions $\alpha_1\gamma^{(1)}+\alpha_2\gamma^{(2)}$ and $\alpha_1\theta^{(1)}+\alpha_2\theta^{(2)}$, that is, we have $\alpha_1N^{(1)}+\alpha_2N^{(2)} = (\overline{\partial},\overline{\mu})$ where $\overline{\partial}_i = \left[\begin{smallmatrix}\partial_i & \alpha_1\gamma_i^{(1)}+\alpha_2\gamma_i^{(2)}\\ 0 & \partial_i\end{smallmatrix}\right]$ and

$$\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix}a'\\a\end{smallmatrix}\right]\otimes\left[\begin{smallmatrix}w'\\w\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}\alpha_1\theta_{i+j}^{(1)}(a\otimes w)+\alpha_2\theta_{i+j}^{(2)}(a\otimes w)+\mu_{i+j}(a\otimes w')+\mu_{i+j}(a'\otimes w)\\ \mu_{i+j}(a\otimes w)\end{smallmatrix}\right].$$

Proof. It is straightforward to show that the ordered pair $(\overline{\partial},\overline{\mu})$ satisfies conditions (1)–(4) from Lemma 4.8. That is, the tangent space $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ is closed under linear combinations. The other vector space axioms follow readily. □

Lemma 4.10. *We work in the setting of Notations 4.1 and 4.5. The tangent space* $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W}$ *is the set of all elements of* $\underline{\mathrm{GL}}(W)_0(F[\epsilon])$ *that give rise to* id_W *modulo* ϵ*. Equivalently,* $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W}$ *is the set of all matrices of the form* $\xi = \left[\begin{smallmatrix} \mathrm{id}_W & D \\ 0 & \mathrm{id}_W \end{smallmatrix}\right]$*, where* $D \in \mathrm{End}_F(W)_0$*.*

Proof. Arguing as in the proof of Lemma 4.8, one checks readily that $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W}$ is the set of all elements of $\underline{\mathrm{GL}}(W)_0(F[\epsilon])$ that give rise to id_W modulo ϵ. To describe the elements of $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W}$ explicitly, recall from Notation 4.1 that we write $W[\epsilon]$ as $W \oplus W$. Thus, the elements of $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W} \subseteq \underline{\mathrm{GL}}(W)_0(F[\epsilon])$ have the form $\xi = \left[\begin{smallmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{smallmatrix}\right]$ where each $\xi_{ij} \colon W \to W$. Since ξ gives rise to id_W modulo ϵ, we must have $\xi_{22} = \mathrm{id}_W$. Also, the condition $\epsilon\xi(\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]) = \xi(\left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right])$ for all $w \in W$ implies that $\xi_{21} = 0$ and $\xi_{11} = \mathrm{id}_W$, and hence ξ has the desired form. One checks similarly that every matrix of the form $\left[\begin{smallmatrix} \mathrm{id}_W & D \\ 0 & \mathrm{id}_W \end{smallmatrix}\right]$, where $D \in \mathrm{End}_F(W)_0$ is an element of $\underline{\mathrm{GL}}(W)_0(F[\epsilon])$ that gives rise to id_W modulo ϵ, that is, it is in $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W}$. □

4.11 (Proof of Theorem B)**.** Using the notation of Lemma 4.8, let $N = (\overline{\partial}, \overline{\mu})$ be an element of $\mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_M$. Since N is a DG $U[\epsilon]$-module, restriction of scalars along the natural inclusion $U \to U[\epsilon]$ makes N a DG U-module with scalar multiplication given by the following formula

$$a\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] := \overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \mu_{i+j}(a\otimes w')+\theta_{i+j}(a\otimes w) \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right] \tag{4.11.1}$$

for all $a \in U_i$ and $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in N_j = W_j \oplus W_j$.

Define $\rho \colon M \to N$ and $\pi \colon N \to M$ by the formulas $\rho(w) := \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]$ and $\pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) := w$. Using the equation $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & \gamma_i \\ 0 & \partial_i \end{smallmatrix}\right]$ from Lemma 4.8, it is straightforward to show that ρ and π are chain maps. From equation (4.11.1), we conclude that ρ and π are U-linear. In other words, we have an exact sequence

$$0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$$

of DG U-module morphisms. So, we obtain a map $\tau\colon\ \mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_M \to \mathrm{YExt}^1_U(M,M)$ where $\tau(N)$ is the equivalence class of the displayed sequence in $\mathrm{YExt}^1_U(M,M)$. We show that τ is a surjective abelian group homomorphism with $\mathrm{Ker}(\tau) = \mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_M$.

To show that τ is additive, let $N^{(1)}, N^{(2)} \in \mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_M$ where $N^{(n)} = (\overline{\partial}^{(n)}, \overline{\mu}^{(n)})$ such that $\overline{\partial}^{(n)}_i = \left[\begin{smallmatrix} \partial_i & \gamma^{(n)}_i \\ 0 & \partial_i \end{smallmatrix}\right]$ and

$$\overline{\mu}^{(n)}_{i+j}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \theta^{(n)}_{i+j}(a\otimes w)+\mu_{i+j}(a\otimes w')+\mu_{i+j}(a'\otimes w) \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right]$$

for $n = 1, 2$ as in Lemma 4.8. Then $\tau(N^{(n)})$ is represented by the exact sequence

$$0 \to M \xrightarrow{\rho} N^{(n)} \xrightarrow{\pi} M \to 0$$

for $n = 1, 2$.[5] The Baer sum $\tau(N^{(1)}) + \tau(N^{(2)})$ is represented by the exact sequence

$$0 \to M \xrightarrow{\rho'} T \xrightarrow{\pi'} M \to 0 \tag{4.11.2}$$

which is constructed in the following four steps:

[5]We abuse notation slightly here: for instance, the maps $M \to N^{(1)}$ and $M \to N^{(2)}$ have different domains, so they should not both be called ρ. However, the maps of underlying vector spaces are the same, so there should be no confusion.

(1) Let L denote the pull-back of $\pi\colon N^{(1)} \to M$ and $\pi\colon N^{(2)} \to M$, which is a DG U-module with[6]

$$\begin{aligned}
L_i &= \left\{\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ v \end{smallmatrix}\right]\right) \in N_i^{(1)} \oplus N_i^{(2)} \mid \pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \pi\left(\left[\begin{smallmatrix} v' \\ v \end{smallmatrix}\right]\right)\right\} \\
&= \left\{\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ v \end{smallmatrix}\right]\right) \in N_i^{(1)} \oplus N_i^{(2)} \mid w = v\right\} \\
&= \left\{\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right) \in N_i^{(1)} \oplus N_i^{(2)}\right\} \\
\partial_i^L\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right) &= \left(\overline{\partial}_i^{(1)}\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right), \overline{\partial}_i^{(2)}\left(\left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right)\right) \\
&= \left(\left[\begin{smallmatrix} \partial_i(w') + \gamma_i^{(1)}(w) \\ \partial_i(w) \end{smallmatrix}\right], \left[\begin{smallmatrix} \partial_i(v') + \gamma_i^{(2)}(w) \\ \partial_i(w) \end{smallmatrix}\right]\right) \\
a\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right) &= \left(a\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], a\left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right) \\
&= \left(\left[\begin{smallmatrix} \theta_{i+j}^{(1)}(a\otimes w) + \mu_{i+j}(a\otimes w') \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right], \left[\begin{smallmatrix} \theta_{i+j}^{(2)}(a\otimes w) + \mu_{i+j}(a\otimes w') \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right]\right)
\end{aligned}$$

for all $a \in U$.

(2) The map $\sigma\colon M \to N^{(1)} \oplus N^{(2)}$ given by $\sigma(m) = \left(\left[\begin{smallmatrix} -m \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right]\right)$ is a well-defined DG U-module morphism such that $\operatorname{Im}(\sigma) \subseteq L$. Let $\overline{\sigma}\colon M \to L$ denote the induced DG U-module morphism.

(3) Set $T = \operatorname{Coker}(\overline{\sigma})$. Let the element of T represented by the ordered pair $\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right)$ be denoted $\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right]$. Then T is a DG U-module with differential and scalar multiplication induced from L:

$$\begin{aligned}
\partial_i^T\left(\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right]\right) &= \left[\left[\begin{smallmatrix} \partial_i(w') + \gamma_i^{(1)}(w) \\ \partial_i(w) \end{smallmatrix}\right], \left[\begin{smallmatrix} \partial_i(v') + \gamma_i^{(2)}(w) \\ \partial_i(w) \end{smallmatrix}\right]\right] \\
a\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right] &= \left[\left[\begin{smallmatrix} \theta_{i+j}^{(1)}(a\otimes w) + \mu_{i+j}(a\otimes w') \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right], \left[\begin{smallmatrix} \theta_{i+j}^{(2)}(a\otimes w) + \mu_{i+j}(a\otimes w') \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right]\right]
\end{aligned}$$

(4) Let $\rho'\colon M \to T$ be given by $\rho'(m) = \left[\left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right] = \left[\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right]\right]$. Let $\pi'\colon T \to M$ be given by $\pi'\left(\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right]\right) = w$. It is straightforward to show that ρ' and τ' are well-defined DG U-module morphisms making (4.11.2) exact.

On the other hand, Lemma 4.9 implies that the element $N = N^{(1)} + N^{(2)}$ in $\mathsf{T}_{M}^{\underline{\mathrm{Mod}}^U}(W)$ is given using the functions $\gamma^{(1)} + \gamma^{(2)}$ and $\theta^{(1)} + \theta^{(2)}$, that is, we have $N = (\overline{\partial}, \overline{\mu})$ where $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & \gamma_i^{(1)} + \gamma_i^{(2)} \\ 0 & \partial_i \end{smallmatrix}\right]$ and

$$\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \theta_{i+j}^{(1)}(a\otimes w) + \theta_{i+j}^{(2)}(a\otimes w) + \mu_{i+j}(a\otimes w') + \mu_{i+j}(a'\otimes w) \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right].$$

The element $\tau(N)$ is represented by the exact sequence

$$0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$$

where ρ and π are the canonical inclusion and surjection. To prove that $\tau(N) = \tau(N^{(1)}) + \tau(N^{(2)})$, we need to construct a DG U-module morphism $\phi\colon T \to N$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\rho'} & T & \xrightarrow{\pi'} & M & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow{\phi} & & \Big\downarrow{=} & & \\
0 & \longrightarrow & M & \xrightarrow{\rho} & N & \xrightarrow{\pi} & M & \longrightarrow & 0.
\end{array}$$

[6] We conveniently abuse our notational protocol for direct sums here.

Define $\phi_i\left(\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right]\right) = \left[\begin{smallmatrix} w'+v' \\ w \end{smallmatrix}\right]$ for each i. It is straightforward to show that ϕ is a well-defined DG U-module morphism making the above diagram commute. Thus, the map τ is additive.

Now we show that τ is onto. Fix an arbitrary element $\zeta \in \operatorname{YExt}^1_U(M, M)$, represented by the sequence $0 \to M \xrightarrow{f} Z \xrightarrow{g} M \to 0$. In particular, this is an exact sequence of F-complexes, so it is degree-wise split. This implies that we have a commutative diagram of graded vector spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & Z & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \vartheta} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{\rho} & W[\epsilon] & \xrightarrow{\pi} & M & \longrightarrow & 0 \end{array}$$

where $\rho(w) = \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]$, $\pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = w$, and ϑ is an isomorphism of graded F-vector spaces. The map ϑ allows us to transfer a DG U-module structure to $W[\epsilon]$ as follows: let the differential on $W[\epsilon]$ be given by the formula $\overline{\partial}_i = \vartheta_{i-1}\partial^Z_i\vartheta_i^{-1}$, and define scalar multiplication μ' over U on $W[\epsilon]$ by the formula $\mu'_{i+j}\left(a \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \vartheta_{i+j}\left(\mu_{i+j}\left(a \otimes \vartheta_j^{-1}\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right)\right)\right)$ for all $a \in U_i$ and $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in W[\epsilon]_j$. These definitions provide an exact sequence

$$0 \to M \xrightarrow{\rho} \left(W[\epsilon], \overline{\partial}, \mu'\right) \xrightarrow{\pi} M \to 0 \tag{4.11.3}$$

of DG U-modules equivalent to the original sequence $0 \to M \xrightarrow{f} Z \xrightarrow{g} M \to 0$. Next, define scalar multiplication $\overline{\mu}$ over $U[\epsilon]$ on $W[\epsilon]$ by the formulas

$$\begin{aligned} \overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) &= \mu'_{i+j}\left(a \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \vartheta_{i+j}\left(\mu_{i+j}\left(a \otimes \vartheta_j^{-1}\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right)\right)\right) \\ \overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) &= \left[\begin{smallmatrix} \mu_{i+j}(a \otimes w) \\ 0 \end{smallmatrix}\right] \end{aligned}$$

for all $a \in U_i$ and $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in W[\epsilon]_j$. These definitions endow $W[\epsilon]$ with a DG $U[\epsilon]$-module structure $(\overline{\partial}, \overline{\mu})$ that gives rise to M modulo ϵ, so $N = (\overline{\partial}, \overline{\mu}) \in \mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_M$. Furthermore, since the sequence (4.11.3) is equivalent to the sequence representing ζ, we have $\tau(N) = \zeta$, so τ is surjective.

By Lemma 4.7, the map $\varrho : \underline{\mathrm{GL}}(W)_0 \to \underline{\mathrm{GL}}(W)_0 \cdot M$ defined by $g \mapsto g \cdot M$ is smooth. Thus, the induced map on tangent spaces $\mathsf{T}^{\underline{\mathrm{GL}}(W)}_{\mathrm{id}_W} \xrightarrow{\varpi} \mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_M$ is surjective; see [14, I.4.4.15].

To describe $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_M$ as a subset of $\mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_M$, consider the next commutative diagram of morphisms of schemes where $\underline{\Psi}_M$ is (induced by) multiplication by M

$$\begin{array}{ccc} \underline{\mathrm{GL}}(W)_0 & \xrightarrow{\quad \underline{\Psi}_M \quad} & \underline{\mathrm{Mod}}^U(W) \\ & {\scriptstyle \cdot M}\searrow \qquad \nearrow & \\ & \underline{\mathrm{GL}}(W)_0 \cdot M & \end{array}$$

and the unspecified arrow is the natural inclusion. Note that for each F-algebra S, the map $\underline{\Psi}_M(S)\colon \underline{\mathrm{GL}}(W)_0(S) \to \underline{\mathrm{Mod}}^U(W)(S)$ is given by multiplication on $S \otimes_F M$. This diagram induces the following commutative diagram of maps of tangent spaces:

$$\begin{array}{ccc} \mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W} & \xrightarrow{\quad\psi_M\quad} & \mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_{M} \\ & \searrow^{\varpi} \qquad \nearrow_{\upsilon} & \\ & \mathsf{T}^{\underline{\mathrm{GL}}(W)_0\cdot M}_{M} & \end{array}.$$

Since the orbit $\mathrm{GL}(W)_0 \cdot M$ is a locally closed subset of $\mathrm{Mod}^U(W)$, the map υ is injective. Since ϖ is surjective, it follows that $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0\cdot M}_{M}$ is isomorphic to the image of ψ_M. Thus, we identify $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0\cdot M}_{M}$ with $\mathrm{Im}(\psi_M) \subseteq \mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_{M}$.

To continue our description of $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0\cdot M}_{M}$, we describe ψ_M explicitly. Again, $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W}$ is the fibre over id_W in the map $\underline{\mathrm{GL}}(W)_0(F[\epsilon]) \to \underline{\mathrm{GL}}(W)_0(F) = \mathrm{GL}(W)_0$ induced by the natural ring epimorphism $F[\epsilon] \to F$. And $\mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_{M}$ is the fibre over M in the induced map $\underline{\mathrm{Mod}}^U(W)(F[\epsilon]) \to \underline{\mathrm{Mod}}^U(W)(F) = \mathrm{Mod}^U(W)$. Thus, the map ψ_M is induced by $\underline{\Psi}_M(F[\epsilon])\colon \underline{\mathrm{GL}}(W)_0(F[\epsilon]) \to \underline{\mathrm{Mod}}^U(W)(F[\epsilon])$, so it is given by multiplication on $F[\epsilon]\otimes_F M$. In a variation of the notation of Lemma 4.8, we have $F[\epsilon]\otimes_F M = (\partial', \mu')$ where $\partial'_i = \left[\begin{smallmatrix} \partial_i & 0 \\ 0 & \partial_i \end{smallmatrix}\right]$ and

$$\mu'_{i+j}(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]) = \left[\begin{smallmatrix} \mu_{i+j}(a\otimes w') + \mu_{i+j}(a'\otimes w) \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right]$$

for all $\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \in U[\epsilon]_i$ and all $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in W[\epsilon]_j$.

By definition, the map $\underline{\Psi}_M(F[\epsilon])\colon \underline{\mathrm{GL}}(W)(F[\epsilon]) \to \underline{\mathrm{Mod}}^U(F[\epsilon])$ is given by $\underline{\Psi}_M(F[\epsilon])(\xi) = (\partial'', \mu'')$ where $\partial'' = \xi\circ\partial'\circ\xi^{-1}$ and $\mu'' = \xi\circ\mu'\circ(U[\epsilon]\otimes\xi^{-1})$. Then ψ_M is the restriction of $\underline{\Psi}_M(F[\epsilon])$ to $\mathsf{T}^{\underline{\mathrm{GL}}(W)}_{\mathrm{id}_W}$.

Lemma 4.10 implies that each element $\xi \in \mathsf{T}^{\underline{\mathrm{GL}}(W)}_{\mathrm{id}_W}$ is of the form

$$\xi = \left[\begin{smallmatrix} \mathrm{id}_W & D \\ 0 & \mathrm{id}_W \end{smallmatrix}\right] \tag{4.11.4}$$

where $D \in \mathrm{End}_F(W)_0$. Note that $\xi^{-1} = \left[\begin{smallmatrix} \mathrm{id}_W & -D \\ 0 & \mathrm{id}_W \end{smallmatrix}\right]$. It follows that

$$\begin{aligned} \mu''_{i+j}(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]) &= \xi_{i+j}\left(\mu'_{i+j}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \xi_j^{-1}\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right)\right)\right) \\ &= \left[\begin{smallmatrix} \mu_{i+j}(a'\otimes w)+\mu_{i+j}(a\otimes w')-\mu_{i+j}(a\otimes D_j(w))+D_{i+j}(\mu_{i+j}(a\otimes w)) \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right] \end{aligned}$$

and similarly $\partial''_i = \left[\begin{smallmatrix} \partial_i & D_{i-1}\partial_i - \partial_i D_i \\ 0 & \partial_i \end{smallmatrix}\right]$. So, we have $\psi_M(\xi) = (\partial'', \mu'')$ where ∂'' and μ'' are given by the above formulas.

We now show that $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0\cdot M}_{M} \subseteq \mathrm{Ker}(\tau)$. Let $N \in \mathsf{T}^{\underline{\mathrm{GL}}(W)_0\cdot M}_{M}$, and write $N = \psi_M(\xi)$ where ξ is as in (4.11.4). Define $h\colon M \to N$ by the formula $h_j(w) = \left[\begin{smallmatrix} D_j(w) \\ w \end{smallmatrix}\right]$. By definition, the DG U-module structure on N gives

$$a\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mu_{i+j}(a\otimes w')-\mu_{i+j}(a\otimes D_j(w))+D_{i+j}(\mu_{i+j}(a\otimes w)) \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right].$$

From this (using the explicit description of $\psi_M(\xi)$) it is straightforward to show that h is a morphism of DG U-modules such that $\pi\circ h = \mathrm{id}_M$. Therefore the exact sequence $0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$ representing $\tau(N)$ splits. This means that $N \in \mathrm{Ker}(\tau)$, as desired.

We conclude the proof by showing that $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0\cdot M}_{M} \supseteq \mathrm{Ker}(\tau)$. Given an element $N = (\overline{\partial}, \overline{\mu}) \in \mathrm{Ker}(\tau)$, the short exact sequence $0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$ representing

$\tau(N)$ splits over U. Therefore there exists a morphism of DG U-modules $h\colon M \to N$ such that $\pi \circ h = \mathrm{id}_M$. The condition $\pi \circ h = \mathrm{id}_M$ implies that $h_j(w) = \left[\begin{smallmatrix} D_j(w) \\ w \end{smallmatrix}\right]$ for some $D \in \mathrm{End}_F(W)_0$. The fact that h is a chain map implies that $\gamma_i = D_{i-1}\partial_i - \partial_i D_i$, in the notation of Lemma 4.8, so we have $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & D_{i-1}\partial_i - \partial_i D_i \\ 0 & \partial_i \end{smallmatrix}\right]$. By checking the condition of h being a DG U-homomorphism, we get

$$\theta_{i,j}(a \otimes w) = D_{i+j}(\mu_{i+j}(a \otimes w)) - \mu_{i+j}(a \otimes D_j(w))$$

for $a \in U_i$ and $w \in W_j$. Thus

$$\overline{\mu}_{i+j}(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]) = \left[\begin{smallmatrix} \mu_{i+j}(a'\otimes w)+\mu_{i+j}(a\otimes w')-\mu_{i+j}(a\otimes D_j(w))+D_{i+j}(\mu_{i+j}(a\otimes w)) \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right].$$

This means that $N = \psi_M(\xi) \in \mathrm{Im}(\psi_M) = \mathsf{T}_M^{\underline{\mathrm{GL}}(W)_0 \cdot M}$, where ξ is as in (4.11.4). $\square$

The next result follows the ideas of Gabriel [21, 1.2 Corollary].

Corollary 4.12. *We work in the setting of Notations 4.1 and 4.5. Let C be a degree-wise finite graded-projective (e.g., semi-free) semidualizing DG U-module, and let $s \geqslant \sup(C)$. Set $M = \tau(C)_{(\leqslant s)}$ and $W = M^\natural$. Then the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$ is open in $\underline{\mathrm{Mod}}^U(W)$.*

Proof. Proposition 3.12 implies that $\mathrm{YExt}^1_U(M, M) = 0$, so by Theorem B we have $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)} = \mathsf{T}_M^{\underline{\mathrm{GL}}(W)_0 \cdot M}$. Lemma 4.8 implies that the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$ is smooth. This explains the first step in the next sequence

$$\begin{aligned}
\dim(\mathcal{O}_{\underline{\mathrm{GL}}(W)_0 \cdot M, M}) &= \mathrm{rank}_F(\mathsf{T}_M^{\underline{\mathrm{GL}}(W)_0 \cdot M}) \\
&= \mathrm{rank}_F(\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}) \\
&\geqslant \dim(\mathcal{O}_{\underline{\mathrm{Mod}}^U(W), M}) \\
&\geqslant \dim(\mathcal{O}_{\underline{\mathrm{GL}}(W)_0 \cdot M, M}).
\end{aligned}$$

The second step follows from the equality $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)} = \mathsf{T}_M^{\underline{\mathrm{GL}}(W)_0 \cdot M}$. The third step is standard, and the last step follows from the fact that $\underline{\mathrm{GL}}(W)_0 \cdot M$ is a locally closed subscheme of $\underline{\mathrm{Mod}}^U(W)$. It follows that $\underline{\mathrm{Mod}}^U(W)$ is smooth at M such that $\dim(\mathcal{O}_{\underline{\mathrm{Mod}}^U(W), M}) = \dim(\mathcal{O}_{\underline{\mathrm{GL}}(W)_0 \cdot M, M})$. Since $\underline{\mathrm{GL}}(W)_0 \cdot M$ is a locally closed subscheme of $\underline{\mathrm{Mod}}^U(W)$, the ring $\mathcal{O}_{\underline{\mathrm{GL}}(W)_0 \cdot M, M}$ is a localization of a quotient of $\mathcal{O}_{\underline{\mathrm{Mod}}^U(W), M}$. However, since $\mathcal{O}_{\underline{\mathrm{Mod}}^U(W), M}$ is a regular local ring, any proper quotient or localization has strictly smaller Krull dimension. It follows that $\mathcal{O}_{\underline{\mathrm{GL}}(W)_0 \cdot M, M} = \mathcal{O}_{\underline{\mathrm{Mod}}^U(W), M}$, so $\mathrm{GL}(W)_0 \cdot M$ and $\mathrm{Mod}^U(W)$ are equal in an open neighborhood V of M in $\mathrm{Mod}^U(W)$.

Every closed point $M' \in \mathrm{GL}(W)_0 \cdot M$ is of the form $M' = \sigma \cdot M$ for some element $\sigma \in \mathrm{GL}(W)_0$. Translating by σ, we see that $\mathrm{GL}(W)_0 \cdot M$ and $\mathrm{Mod}^U(W)$ are equal in an open neighborhood $\sigma \cdot V$ of $\sigma \cdot M = M'$ in $\mathrm{Mod}^U(W)$. Since this is true for every closed point of $\mathrm{GL}(W)_0 \cdot M$, it is true for every point of $\mathrm{GL}(W)_0 \cdot M$. This uses the fact that $\mathrm{Mod}^U(W)$ is of finite type over a field. We conclude that $\mathrm{GL}(W)_0 \cdot M$ is open in $\mathrm{Mod}^U(W)$. Hence, $\underline{\mathrm{GL}}(W)_0 \cdot M$ is open in $\underline{\mathrm{Mod}}^U(W)$. $\square$

5. Answering Vasconcelos' Question

The final steps of our proof of Theorem A begin with the next result which is motivated by Happel [27].

Lemma 5.1. *We work in the setting of Notations 4.1 and 4.5. Let $\mathfrak{S}_W(U)$ denote the set of quasiisomorphism classes of degree-wise finite semi-free semidualizing DG U-modules C such that $s \geqslant \sup(C)$, $C_i = 0$ for all $i < 0$, and $(\tau(C)_{(\leqslant s)})^{\natural} \cong W$. Then $\mathfrak{S}_W(U)$ is a finite set.*

Proof. Fix a representative C for each quasiisomorphism class in $\mathfrak{S}_W(U)$, and write $[C] \in \mathfrak{S}_W(U)$ and $M_C = \tau(C)_{(\leqslant s)}$.

Let $[C], [C'] \in \mathfrak{S}_W(U)$. If $\underline{\mathrm{GL}}(W)_0 \cdot M_C = \underline{\mathrm{GL}}(W)_0 \cdot M_{C'}$, then $[C] = [C']$: indeed, Remark 4.4 explains the second step in the next display

$$C \simeq M_C \cong M_{C'} \simeq C'$$

and the remaining steps follow from the assumptions $s \geqslant \sup(C)$ and $s \geqslant \sup(C')$.

Now, each orbit $\underline{\mathrm{GL}}(W)_0 \cdot M_C$ is open in $\underline{\mathrm{Mod}}^U(W)$ by Corollary 4.12. Since $\underline{\mathrm{Mod}}^U(W)$ is quasi-compact, it can only have finitely many open orbits. By the previous paragraph, this implies that there are only finitely many distinct elements $[C] \in \mathfrak{S}_W(U)$. □

Theorem A is a corollary of the following result whose proof uses techniques we learned from Avramov and Iyengar. Recall the notation $\mathfrak{S}(R)$ from Definition 2.20.

Theorem 5.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then the set $\mathfrak{S}(R)$ is finite.*

Proof. A result of Grothendieck [24, Proposition (0.10.3.1)] provides a flat local ring homomorphism $R \to (R', \mathfrak{m}', k')$ such that k' is algebraically closed. Composing with the natural map from R' to its $\mathfrak{m}'$-adic completion, we assume that R' is complete. By [19, Theorem II(c)], the induced map $\mathfrak{S}(R) \to \mathfrak{S}(R')$ is a monomorphism. Thus it suffices to prove the result for R', so we assume that R is complete with algebraically closed residue field.

Let $\underline{t} = t_1, \cdots, t_n$ be a minimal generating sequence for $\mathfrak{m}$, and set $K = K^R(\underline{t})$, the Koszul complex. The map $\mathfrak{S}(R) \to \mathfrak{S}(K)$ induced by $C \mapsto K \otimes_R C$ is bijective by [31, Corollary 3.10]. Thus, it suffices to show that $\mathfrak{S}(K)$ is finite. Note that for each semidualizing R-complex C, we have $\mathrm{amp}(C) \leqslant \dim(R) - \mathrm{depth}(R)$ by [11, (3.4) Corollary]. A standard result about K (see, e.g., [18, 1.3]) implies that

$$\mathrm{amp}(K \otimes_R C) \leqslant \mathrm{amp}(C) + n \leqslant \dim(R) - \mathrm{depth}(R) + n. \tag{5.2.1}$$

Set $s = \dim(R) - \mathrm{depth}(R) + n$.

Since R is complete, the Cohen Structure Theorem provides a complete regular local ring $(A, \mathfrak{M}, k)$ and an epimorphism $A \to R$ such that $\mathfrak{M}$ is generated by a sequence $\underline{a} = a_1, \ldots, a_n \in \mathfrak{M}$ where a_i is a lifting of t_i to A. The Koszul complex $K^A(\underline{a})$ is a minimal A-free resolution of k. The lifting assumption for $\underline{a}$ implies that $K \cong K^A(\underline{a}) \otimes_A R$. The fact that A is regular and local implies that $q := \mathrm{pd}_A(R) < \infty$.

From [6, Proposition 2.2.8] we know that there is a "DG algebra resolution" B of R over A such that $B_i = 0$ for all $i > q$. This means that B is a DG A-algebra with a quasiisomorphism of DG A-algebras $B \xrightarrow{\simeq} R$ such that each B_i is finitely generated and free over A and $B_i = 0$ for all $i > q$. Since $K^A(\underline{a})$ and B are DG A-algebras that are bounded below and consist of flat A-modules, we have the following (quasi)isomorphisms of DG A-algebras:

$$K \cong K^A(\underline{a}) \otimes_A R \xleftarrow{\simeq} K^A(\underline{a}) \otimes_A B \xrightarrow{\simeq} k \otimes_A B =: U. \tag{5.2.2}$$

Note that the assumptions on B imply that U is a finite dimensional DG k-algebra, as in Notation 4.1 with $F = k$. With the second paragraph of this proof, Lemma 2.22(c) says that base change yields bijections

$$\mathfrak{S}(R) \xrightarrow{\cong} \mathfrak{S}(K) \cong \mathfrak{S}(K^A(\underline{a}) \otimes_A R) \xleftarrow{\cong} \mathfrak{S}(K^A(\underline{a}) \otimes_A B) \xrightarrow{\cong} \mathfrak{S}(U).$$

Thus, it suffices to show that $\mathfrak{S}(U)$ is finite. Note that each algebra in (5.2.2) is a local DG A-algebra, as is R.

Let C' be a semidualizing DG U-module, and let C be a semidualizing R-complex corresponding to C' under the bijections given above. Assume without loss of generality that C is not shift-isomorphic in $\mathcal{D}(R)$ to R. (Removing this from consideration only removes a single semidualizing DG U-module, so does not affect the discussion of the finiteness of $\mathfrak{S}(U)$.) Since R is local, it follows from [11, (8.1) Theorem] that $\operatorname{pd}_R(C) = \infty$.

From Lemma 2.22(b) and the display (5.2.1), we have

$$\operatorname{amp}(C') = \operatorname{amp}(K \otimes_R C) \leqslant s.$$

By applying an appropriate shift we assume without loss of generality that $\inf(C) = 0 = \inf(C')$, so we have $\sup(C') \leqslant s$. Let $L \xrightarrow{\simeq} C'$ be a minimal semi-free resolution of C' over U. The conditions $\sup(L) = \sup(C') \leqslant s$ imply that L (and hence C') is quasiisomorphic to the truncation $\widetilde{L} := \tau(L)_{\leqslant s}$. We set $W := \widetilde{L}^\natural$ and work in the setting of Notations 4.1 and 4.5.

We claim that $\beta_p^R(C) \leqslant \mu_R^{p+\operatorname{depth} R}(R)$ for all $p \geqslant 0$. To see this, first note that the isomorphism $\mathbf{R}\operatorname{Hom}_R(C,C) \simeq R$ implies the following equality of power series $I_R^R(t) = P_R^C(t) I_C^R(t)$. See [7, (1.5.3)]. We conclude that for each m we have

$$\mu_R^m(R) = \sum_{t=0}^{m} \beta_t^R(C) \mu_R^{m-t}(C).$$

In particular, for $m < \operatorname{depth}(R)$, we have

$$0 = \mu_R^m(R) = \sum_{t=0}^{m} \beta_t^R(C) \mu_R^{m-t}(C) \geqslant \beta_0^R(C) \mu_R^m(C).$$

The equality $\inf(C) = 0$ implies that $\beta_0^R(C) \neq 0$ by [11, (1.7.1)], so it follows that $\mu_R^m(C) = 0$. For $m = \operatorname{depth}(R)$, we conclude from this that

$$0 \neq \mu_R^{\operatorname{depth}(R)}(R) = \sum_{t=0}^{\operatorname{depth}(R)} \beta_t^R(C) \mu_R^{\operatorname{depth}(R)-t}(C) = \beta_0^R(C) \mu_R^{\operatorname{depth}(R)}(C)$$

and hence $\mu_R^{\operatorname{depth}(R)}(C) \neq 0$. Similarly, for $m = p + \operatorname{depth}(R)$, we have

$$\begin{aligned} \mu_R^{p+\operatorname{depth}(R)}(R) &= \sum_{t=0}^{p+\operatorname{depth}(R)} \beta_t^R(C) \mu_R^{p+\operatorname{depth}(R)-t}(C) \\ &\geqslant \beta_p^R(C) \mu_R^{\operatorname{depth}(R)}(C) \\ &\geqslant \beta_p^R(C) \end{aligned}$$

as claimed.

Next, we claim that there is an integer $\lambda \geqslant 0$, depending only on R and U, such that $\sum_{i=0}^{s} r_i \leqslant \lambda$. (Recall that r_i and other quantities are fixed in Notation 4.1.) To see this, first note that for $i = 1, \ldots, s$ we have $L_i = \bigoplus_{j=0}^{i} U_j^{\beta_{i-j}^U(C')}$; see Fact 2.17. From [1, p. 44, Proposition] and the previous claim, we conclude that

$$\beta_j^U(C') = \beta_j^R(C) \leqslant \mu_R^{j+\operatorname{depth}(R)}(R)$$

for all j. It follows that

$$r_i \leqslant \operatorname{rank}_F(L_i) = \sum_{j=0}^{i} n_{i-j}\beta_j^U(C') = \sum_{j=0}^{i} n_{i-j}\beta_j^R(C) \leqslant \sum_{j=0}^{i} n_{i-j}\mu_R^{j+\operatorname{depth}(R)}(R).$$

And we conclude that

$$\sum_{i=0}^{s} r_i \leqslant \sum_{i=0}^{s}\sum_{j=0}^{i} n_{i-j}\mu_R^{j+\operatorname{depth}(R)}(R).$$

Since the numbers in the right hand side of this inequality only depend on R and U, we have found the desired value for λ.

Because there are only finitely many $(r_0, \ldots, r_s) \in \mathbb{N}^{s+1}$ with $\sum_{i=0}^{s} r_i \leqslant \lambda$, there are only finitely many W that occur from this construction, say $W^{(1)}, \ldots, W^{(b)}$. Lemma 5.1 implies that $\mathfrak{S}(U) = \mathfrak{S}_{W^{(1)}}(U) \cup \cdots \cup \mathfrak{S}_{W^{(b)}}(U) \cup \{[U]\}$ is finite. □

5.3 (Proof of Theorem A)**.** The set $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$ is finite by Theorem 5.2. □

Now, we prove versions of Theorems A and 5.2 for semilocal rings. We note that, over a non-local ring, the set $\mathfrak{S}(R)$ may not be finite. For instance, the *Picard group* $\operatorname{Pic}(R)$, consisting of finitely generated rank-1 projective R-modules, is contained in $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$, so $\mathfrak{S}(R)$ can even be infinite when R is a Dedekind domain. We use some notions from [20] to deal with this.

Definition 5.4. A *tilting R-complex* is a semidualizing R-complex of finite projective dimension. The *derived Picard group of R* is the set $\operatorname{DPic}(R)$ of isomoprhism classes in $\mathcal{D}(R)$ of tilting R-complexes. The isomorphism class of a tilting R-complex L is denoted $[L] \in \operatorname{DPic}(R)$.

Remark 5.5. A homologically finite R-complex L is tilting if and only if $L_\mathfrak{m} \sim R_\mathfrak{m}$ for all maximal (equivalently, for all prime) ideals $\mathfrak{m} \subset R$, by [20, Proposition 4.4 and Remark 4.7]. In [9] tilting complexes are called "invertible".

The derived Picard group $\operatorname{DPic}(R)$ is an abelian group under the operation $[L][L'] := [L \otimes_R^{\mathbf{L}} L']$. The identity in $\operatorname{DPic}(R)$ is $[R]$, and $[L]^{-1} = [\mathbf{R}\operatorname{Hom}_R(L, R)]$. The classical Picard group $\operatorname{Pic}(R)$ is naturally a subgroup of $\operatorname{DPic}(R)$. The group $\operatorname{DPic}(R)$ acts on $\mathfrak{S}(R)$ in a natural way: $[L][C] := [L \otimes_R^{\mathbf{L}} C]$. See [20, Properties 4.3 and Remark 4.9]. This action restricts to an action of $\operatorname{Pic}(R)$ on $\mathfrak{S}_0(R)$ given by $[L][C] := [L \otimes_R C]$.

Notation 5.6. The set of orbits in $\mathfrak{S}(R)$ under the action of $\operatorname{DPic}(R)$ is denoted $\overline{\mathfrak{S}}(R)$.[7] Given $[C] \in \mathfrak{S}(R)$, the orbit in $\overline{\mathfrak{S}}(R)$ is denoted $\langle C\rangle$. The set of orbits in $\mathfrak{S}_0(R)$ under the action of $\operatorname{Pic}(R)$ is denoted $\overline{\mathfrak{S}_0}(R)$, and the orbit in $\overline{\mathfrak{S}_0}(R)$ of a given semidualizing R-module C is denoted $\langle C\rangle$.

Fact 5.7. Given semidualizing R-complexes A and B, the following conditions are equivalent by [20, Proposition 5.1]:

(i) there is an element $[P] \in \operatorname{DPic}(R)$ such that $B \simeq P \otimes_R^{\mathbf{L}} A$; and

[7]Observe that the notations $\mathfrak{S}(R)$ and $\overline{\mathfrak{S}}(R)$ represent different sets in [20].

(ii) $A_{\mathfrak{m}} \sim B_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subset R$ and $\operatorname{Ext}^i_R(A, M) = 0$ for $i \gg 0$.

It is straightforward to show that the natural inclusion $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$ gives an inclusion $\overline{\mathfrak{S}_0}(R) \subseteq \overline{\mathfrak{S}}(R)$.

Lemma 5.8. *Assume that R is Cohen-Macaulay (not necessarily local), and let C be a semidualizing R-complex. There is an element $[L] \in \operatorname{DPic}(R)$ such that $L \otimes_R C$ is isomorphic in $\mathcal{D}(R)$ to a module.*

Proof. For each $\mathfrak{p} \in \operatorname{Spec}(R)$, the fact that R is Cohen-Macaulay implies that $\operatorname{amp}(C_{\mathfrak{p}}) = 0$ by [11, (3.4) Corollary], that is, $C_{\mathfrak{p}} \sim \operatorname{H}_i(C)_{\mathfrak{p}} \neq 0$ for some i. As $\operatorname{amp}(C) < \infty$, this implies that $\operatorname{Spec}(R)$ is the disjoint union

$$\operatorname{Spec}(R) = \bigcup_{i=\inf(C)}^{\sup(C)} \operatorname{Supp}_R(\operatorname{H}_i(C)).$$

It follows that each set $\operatorname{Supp}_R(\operatorname{H}_i(C))$ is both open and closed. So, if $\operatorname{Supp}_R(\operatorname{H}_i(C))$ is non-empty, then it is a union of connected components of $\operatorname{Spec}(R)$.

Let $e_1, \ldots, e_p$ be a "complete set of orthogonal primitive idempotents of R" as in [9, 4.8]. Then $R \cong R_{e_1} \times \cdots \times R_{e_p}$ and each $\operatorname{Spec}(R_{e_i})$ is naturally homeomorphic to a connected component of $\operatorname{Spec}(R)$. From the previous paragraph, for $i = 1, \ldots, p$ we have $C_{e_i} \simeq \Sigma^{u_i} \operatorname{H}_{u_i}(C_{e_i})$, and $\operatorname{H}_{u_i}(C_{e_i})$ is a semidualizing R_{e_i}-module. Each R-module M has a natural decomposition $M \cong \oplus_{i=1}^p M_{e_i}$ that is compatible with the product decomposition of R, and it follows that $C \simeq \oplus_{i=1}^p \Sigma^{u_i} \operatorname{H}_{u_i}(C_{e_i})$.

Let $L = \oplus_{i=1}^p \Sigma^{-u_i} R_{e_i}$. Then L is a tilting R-complex by Remark 5.5, and

$$\begin{aligned} L \otimes_R C &\simeq (\oplus_{i=1}^p \Sigma^{-u_i} R_{e_i}) \otimes_R (\oplus_{i=1}^p \Sigma^{u_i} \operatorname{H}_{u_i}(C_{e_i})) \\ &\simeq \oplus_{i=1}^p (\Sigma^{-u_i} R_{e_i}) \otimes_{R_{e_i}} (\Sigma^{u_i} \operatorname{H}_{u_i}(C_{e_i})) \\ &\simeq \oplus_{i=1}^p \operatorname{H}_{u_i}(C_{e_i}). \end{aligned}$$

Since $\oplus_{i=1}^p \operatorname{H}_{u_i}(C_{e_i})$ is an R-module, this establishes the lemma. □

Definition 5.9. The *non-Gorenstein locus* of R is

$$\operatorname{nGor}(R) := \{\text{maximal ideals } \mathfrak{m} \subset R \mid R_{\mathfrak{m}} \text{ is not Gorenstein}\} \subseteq \text{m-Spec}(R)$$

where m-Spec(R) is the set of maximal ideals of R.

Remark 5.10. For "nice" rings, e.g. rings with a dualizing complex [33], the set $\operatorname{nGor}(R)$ is closed in m-Spec(R), so it is small in some sense.

Theorem 5.11. *Assume that R satisfies one of the following conditions:*

(1) *R is semilocal, or*

(2) *R is Cohen-Macaulay and* $\operatorname{nGor}(R)$ *is finite.*

Then the sets $\overline{\mathfrak{S}_0}(R)$ and $\overline{\mathfrak{S}}(R)$ are finite.

Proof. Because of the containment $\overline{\mathfrak{S}_0}(R) \subseteq \overline{\mathfrak{S}}(R)$, it suffices to show that $\overline{\mathfrak{S}}(R)$ is finite. Let $X = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \subseteq \text{m-Spec}(R)$, and let $f \colon \overline{\mathfrak{S}}(R) \to \prod_{i=1}^n \mathfrak{S}(R_{\mathfrak{m}_i})$ be given by the formula $f(\langle C \rangle) := ([C_{\mathfrak{m}_1}], \ldots, [C_{\mathfrak{m}_n}])$. This is well-defined because if $\langle B \rangle = \langle C \rangle$, then there is an element $[P] \in \operatorname{DPic}(R)$ such that $C \simeq P \otimes^{\mathbf{L}}_R B$, and the fact that $P_{\mathfrak{m}} \sim R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset R$ implies that $B_{\mathfrak{m}_i} \sim C_{\mathfrak{m}_i}$ for $i = 1, \ldots, n$.

In each case (1)–(2) we show that there is a finite set X such that f is 1-1. Then Theorem 5.2 implies that the set $\prod_{i=1}^n \mathfrak{S}(R_{\mathfrak{m}_i})$ is finite, so $\overline{\mathfrak{S}}(R)$ is also finite.

(1) Assume that R is semilocal, and set $X := \text{m-Spec}(R)$. To show that f is 1-1, let $\langle B\rangle, \langle C\rangle \in \overline{\mathfrak{S}}(R)$ such that $B_{\mathfrak{m}_i} \sim C_{\mathfrak{m}_i}$ for $i = 1, \ldots, n$. We need to show that there is an element $[P] \in \mathrm{DPic}(R)$ such that $C \simeq P \otimes^{\mathbf{L}}_R B$. According to Fact 5.7, it suffices to show that $\mathrm{Ext}^j_R(B, C) = 0$ for $j \gg 0$. Since we know that $\mathrm{Ext}^j_{R_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}, B_{\mathfrak{m}_i}) = 0$ for all $j \geqslant 1$, we conclude that there are integers $j_1, \ldots, j_n$ such that for $i = 1, \ldots, n$ we have $\mathrm{Ext}^j_{R_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}, C_{\mathfrak{m}_i}) = 0$ for all $j \geqslant j_i$. Since B is homologically finite, we have

$$0 = \mathrm{Ext}^j_{R_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}, C_{\mathfrak{m}_i}) \cong \mathrm{Ext}^j_R(B, C)_{\mathfrak{m}_i}$$

for all $j \geqslant \max_i j_i$. Since vanishing is a local property, it follows that $\mathrm{Ext}^j_R(B, C) = 0$ for $j \gg 0$, as desired.

(2) Now, assume that R is Cohen-Macaulay and $\mathrm{nGor}(R)$ is finite, and set $X := \mathrm{nGor}(R)$. To show that f is 1-1, let $\langle B\rangle, \langle C\rangle \in \overline{\mathfrak{S}}(R)$ such that $B_{\mathfrak{m}_i} \sim C_{\mathfrak{m}_i}$ for $i = 1, \ldots, n$. Lemma 5.8 provides tilting R-complexes L and M such that $L \otimes_R B$ and $M \otimes_R C$ are isomorphic in $\mathcal{D}(R)$ to modules B' and C', respectively. Thus, we have $\langle B\rangle = \langle L \otimes_R B\rangle = \langle B'\rangle$ and $\langle C\rangle = \langle M \otimes_R C\rangle = \langle C'\rangle$, so we may replace B and C with B' and C' to assume that B and C are modules.

We claim that $B_{\mathfrak{m}} \sim C_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subset R$ and $\mathrm{Ext}^i_R(B, C) = 0$ for all $i \geqslant 1$. (Then the desired conclusion follows from Fact 5.7.) Since B is a finitely generated R-module, it suffices to show that $B_{\mathfrak{m}} \sim C_{\mathfrak{m}}$ and $\mathrm{Ext}^i_{R_{\mathfrak{m}}}(B_{\mathfrak{m}}, C_{\mathfrak{m}}) = 0$ for all $i \geqslant 1$ and for all maximal ideals $\mathfrak{m} \subset R$.

Case 1: $\mathfrak{m} \in \mathrm{nGor}(R)$. In this case, we have $B_{\mathfrak{m}} \sim C_{\mathfrak{m}}$, by assumption. Since B and C are both modules, this implies that $B_{\mathfrak{m}} \cong C_{\mathfrak{m}}$, so the fact that $B_{\mathfrak{m}}$ is semidualizing over $R_{\mathfrak{m}}$ implies that

$$\mathrm{Ext}^i_{R_{\mathfrak{m}}}(B_{\mathfrak{m}}, C_{\mathfrak{m}}) \cong \mathrm{Ext}^i_{R_{\mathfrak{m}}}(B_{\mathfrak{m}}, B_{\mathfrak{m}}) = 0.$$

Case 2: $\mathfrak{m} \notin \mathrm{nGor}(R)$. In this case, the ring $R_{\mathfrak{m}}$ is Gorenstein, so we have $B_{\mathfrak{m}} \cong R_{\mathfrak{m}} \cong C_{\mathfrak{m}}$ by [11, (8.6) Corollary], and the desired vanishing follows. □

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