Fourier Transform for Option Pricing under Affine Jump-Diffusions: An Overview

Artur Sepp
Institute of Mathematical Statistics,
Faculty of Mathematics and Computer Science
University of Tartu, J. Liivi 2, 50409 Tartu, Estonia,
Mail: artursepp@hotmail.com, Web: www.hot.ee/seppar

7 September, 2003

Abstract

This paper surveys the developments in the finance literature with respect to applying the Fourier transform for option pricing under affine jump-diffusions. We provide a broad description of the issues and a detailed summary of the main points and features of the models proposed.

First, we consider a wide class of affine jump-diffusions proposed for the asset price dynamics: jump-diffusions, diffusions with stochastic volatility, jump-diffusions with stochastic volatility, and jump-diffusions with stochastic volatility and jump intensity. Next we apply the Fourier transform for solving the problem of European option pricing under these price processes. We present two solution methods: the characteristic formula and the Black-Scholes-style formula. Finally, we discuss numerical implementation of pricing formulas and apply the considered processes for modeling the DAX options volatility surface.

Key words: stochastic volatility, jump-diffusion processes, volatility smile, option pricing, characteristic function, Fourier transform, DAX volatility surface.

1 Introduction

We will concern with calculating fair prices for financial contracts derived from the underlying assets. Financial derivative securities are used by individuals and companies who wish to buy or sell an asset or commodity in advance to insure themselves against adverse market movements or to make speculative bets. By arranging now to buy an asset in the future, an investor can ensure that the price at which he can trade is fixed. This is a forward, the simplest example of a financial derivative.

An option is a derivative which gives to its holder the right to buy or sell a security in the future for a given price. A call gives the right to buy while a put gives the right to sell. Standard options are European if they can only be exercised at option maturity. American options can be exercised at any time up to the maturity. In this paper, we consider the problem of pricing European put and call options, which are usually called vanilla options.

Like all financial contracts, an option has its value and its market price. The value is usually obtained by means of a mathematical model, which we usually call the pricing model. This model makes an assumption about the distribution of the underlying assets and proposes a hedging strategy to replicate the option payoff. A good pricing model leads to a perfect replication of the option payoff at the expiry date. The market price of an option is an amount of money which is needed to acquire this option. A good pricing model implies the price of option which is very close to the market price of this option. The primary goal of financial practitioners is to develop such pricing models that satisfy both criteria.

The first methodology for pricing options was developed by Black and Scholes in 1973. The Black-Scholes model assumes that under the martingale measure $\mathbb{Q}$ the price of an underlying asset follows a geometric Brownian motion with constant drift and diffusion parameters

$$dS(t) = (r-d)S(t)dt + \sigma S(t)dW(t) \tag{1.1}$$

where $S(t)$ is the asset (underlying) price, r is a risk-free (domestic) interest rate, d is a dividend (foreign interest) rate, σ is volatility, $W(t)$ is a standard Wiener process.

Assuming that the option payoff can be replicated by risk-neutral delta-hedging strategy, Black-Scholes (1973) showed that the value $F(S,t)$ of European option satisfies the following partial differential equation (PDE)

$$\begin{aligned} &F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} + (r-d)SF_S - rF = 0, \\ &F(S,T) = \max\{\varphi[S-K], 0\} \end{aligned} \tag{1.2}$$

where the contract parameters are strike K, maturity T and type φ, with $\varphi = +1$ for a call option and $\varphi = -1$ for a put option; subscripts denote partial derivatives.

The solution to PDE (1.2) is given by the celebrated Black-Scholes (1973) formula

$$F(S,t) = \varphi[Se^{-d(T-t)}N(\varphi d_+) - Ke^{-r(T-t)}N(\varphi d_-)] \tag{1.3}$$

where

$$
\begin{aligned}
d_{+,-} &= \frac{\ln(S/K) + (r - d \pm \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \\
n(y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, \ N(x) = \int_{-\infty}^{x} n(y)dy.
\end{aligned} \tag{1.4}
$$

In the above formula, $n(y)$ is the standard normal probability density function (PDF) and $N(x)$ is the corresponding cumulative distribution function (CDF).

One of the key concepts in option pricing is volatility, which is an indicator of uncertainty associated with the asset price. Volatility has several meanings. On one hand, it can be calculated empirically as the standard derivation of price values over some time interval, on the other hand, volatility can be calculated from option prices quoted in the market. By pricing options, we are mostly interested in the last quantity which we call implied volatility $\sigma_{imp}(K, T)$ and which is the solution to equation

$$
F_{BS}(S, t, \sigma_{imp}(K, T); K, T) = F_{market}(S, t; K, T) \tag{1.5}
$$

where $F_{BS}(\sigma_{imp}(K, T); K, T)$ is the Black-Scholes price with $\sigma = \sigma_{imp}$ and $F_{market}(K, T)$ is the market price (or the price obtained by using an alternative pricing model) of a vanilla option with strike K and expiry K.

Equation (1.5) can quickly be solved via Newton-Raphson method. The implied volatility is often referred to as the Black-Scholes implied volatility.

The next key concept is the volatility smile, which is a plot with market implied volatilities versus strikes. The term structure of implied volatility is a plot with market implied volatilities for at-the-money (for $K \approx S(t)e^{(r-d)(T-t)}$) options versus maturities. Combined together, these two features produce the volatility surface where market implied volatilities are plotted across all possible strikes and maturities.

To understand relevance of the volatility smile, let us recall that in the Black-Scholes model the diffusion parameter, which is called volatility, is constant. Accordingly, the volatility surface should be flat across all stikes and maturities. However, in practice all option markets exhibit the volatility smile phenomenon which means that options with different maturities and strikes have different implied Black-Scholes volatilities. The assumption of constant volatility is now out-dated. The Black-Scholes model is now used as a reference. Almost all European-style calls and puts, which are usually called vanilla options, are quoted in terms of the Black-Scholes volatility.

The implied volatility surface of the DAX (The Deutscher Aktienindex) vanilla options is shown in Figure 1 in Appendix D. This is a typical pattern for equity and foreign exchange options. As opposed to the constant volatility in the Black-Scholes framework, volatility depends on both strikes and maturities. As a result, accurate pricing and hedging of options is hard to achieve within the standard Black-Scholes model.

The problem of pricing vanilla options consistently with the volatility smile has been attracting much attention in the finance literature. A number of approaches have been proposed to deal with the smile. Lipton (2001) provides a good review of these approaches and their implications for option pricing and hedging. Generally, we can distinguish two main approaches.

The first one was developed by Dupire (1994), and it assumes that the volatility is a deterministic function of time and the asset price. This approach is often referred to as as the local volatility approach. In short, this approach requires computing partial derivatives from the discrete market prices, which makes it very sensitive to the choice of interpolation method. In addition, Hagan *et al* (2002) stated that local volatility models predict the wrong dynamics of volatility which results in inaccurate hedges. They concluded that local volatility models perform even worse than the naive Black-Scholes model.

The second approach consists in modifying the asset price dynamics by means of introducing price-jumps, stochastic volatility, their combination. This approach has been studied by a number of researches [Merton (1976), Heston (1993), Bates (1996), Duffie-Pan-Singleton (2000), and others] resulting in a number of different models. Let us note that there are no accepted models and every market has its own favorites. One of the well-known models are affine jump-diffusions. These processes have a few appealing features: a good consistency with empirical data, analytical tractability, a rich variety of specifications. In subsequent section, we will consider some of the best-known affine jump-diffusions.

2 Modeling the Asset Price Dynamics with Affine Jump-Diffusions

Here we consider a series of increasingly complex affine jump-diffusion processes.

2.1 Jump-Diffusion Processes

To obtain more realistic models of stock prices, researches have added jumps and crashes to the standard Black-Scholes diffusion of stock prices. This approach was inspired by Merton (1976) and studied by a number of authors afterwards. Price jumps are important because they do exist in the market (for example, those that happened in October 1987 and 1997), so taking them into account makes the pricing model more realistic. Jump-diffusion models can also explain the smile in a stationary way. Jumps are rather important for explaining shorter maturity smiles.

A serious drawback of all jump-diffusion models is that they always produce increasing at-the-money-forward (for $K \approx S(t)e^{(r-d)(T-t)}$) term structure. This is because jump models always increase the overall level of volatility. As a result, jump-diffusions cannot match a steeply decreasing term structure of the implied volatility. Another disadvantage of jump-diffusions is that these models are incomplete from the point of view of replication - we cannot hedge jumps with the stock alone or with an additional option.

Now we consider the market where the governing SDE for $S(t)$ under the objective measure $\mathbb{P}$ is given by

$$dS(t) = \mu(S,t)dt + \sigma(S,t)dW(t) + \gamma(S,t)JdN(t) \tag{2.1}$$

where $N(t)$ is a Poisson processes with constant intensity λ. When jump occurs, its magnitude J is a random variable with the PDF $\varpi(J)$.

The risk-neutral dynamics for $S(t)$ are built in such a way that $e^{-rt}S(t)$ remains a martingale. To calculate the expected value of the increment, $dN(t)$, we use the property of the Poisson process that the arrival of a jump is independent of the arrival of previous jumps, and that the probability of two simultaneous jumps is zero. Let us assume that j jumps have occurred. Then, over the next small time interval, dt, one jump will arrive with probability λdt, and no jumps will arrive with probability $1 - \lambda dt$. Therefore

$$\mathbb{E}[dN(t)] = 1 \cdot \lambda dt + 0 \cdot (1 - \lambda dt) = \lambda dt.$$

We define the process $dM(t)$ as

$$dM(t) = dN(t) - \lambda dt.$$

The expectation of $dM(t)$, $\mathbb{E}[dM(t)]$, is zero. Thus, the process $M(t)$ is a martingale. The process $M(t)$ is often referred to as the compensated counting process.

Jumps are usually specified for the logarithmic stock price. As a result, the risk-neutral dynamics for the price of an underlying asset S under the martingale measure $\mathbb{Q}$ is given by

$$dS(t) = (r - d - \lambda m)S(t)dt + \sigma S(t)dW(t) + (e^J - 1)S(t)dN(t) \tag{2.2}$$

where m is the average jump amplitude: $m = \int [e^J - 1]\varpi(J)dJ$.

Next we will consider possible PDFs for the jump sizes.

2.1.1 Log-Normal Jump-Diffusions

Merton (1976) proposed jump-diffusions where the logarithm of jump size is normally distributed:

$$\varpi(J) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(J-\nu)^2}{2\delta^2}}.$$

As a result, variable e^J is log-normally distributed. Merton showed that for this jump-diffusion it is possible to represent the price of a vanilla call ot put as a weighted average of the standard Black-Scholes prices:

$$F(S, \sigma, \lambda, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda' \tau}(\lambda' \tau)^n}{n!} F_{BS}(S, \sigma_n, r_n, \tau) \tag{2.3}$$

where $\lambda' = \lambda(1 + m)$, $\sigma_n^2 = \sigma^2 + \frac{n\delta^2}{\tau}$, $r_n = r - \lambda m + \frac{n \ln(1+m)}{\tau}$, $m = e^{\nu + 1/2\delta^2} - 1$, $\tau = T - t$.

Although the formula includes an infinite number of terms, we use a local stopping criterion to cut-off the summation when a specified level of accuracy is achieved.

2.1.2 Double-Exponential Jump-Diffusions

This jump-diffusion was studied by Kou (2002). The jump size has an asymmetric double exponential distribution:

$$\varpi(J) = p\frac{1}{\eta_u}e^{-\frac{1}{\eta_u}J}\mathbf{1}_{\{J\geq 0\}} + q\frac{1}{\eta_d}e^{\frac{1}{\eta_d}J}\mathbf{1}_{\{J<0\}} \tag{2.4}$$

where $1 > \eta_u > 0$, $\eta_d > 0$ are means of positive and negative jumps, respectively; $p,\ q \geq 0,\ p+q=1$. p and q represent the probabilities of positive and negative jumps. Requirement that $\eta_u < 1$ is needed to ensure that $\mathbb{E}[e^J] < \infty$ and $\mathbb{E}[S] < \infty$.

The double-exponential jump-diffusion has one more parameter than the log-normal jump-diffusion, so it is capable of producing more flexible smile shapes. Papapantoleon (2002) provided some details on this account.

2.1.3 Jump-Diffusion with a Mixture of Independent Jumps

The Fourier transform allows obtaining convenient pricing formulas for the jump-diffusion with a mixture of independent price-jumps with jump size PDF defined by

$$\varpi(J) = \sum_{j=1}^{n}\Omega_j\varpi_j(J) \tag{2.5}$$

where Ω_j is a weighting function, $\sum_{j=1}^{n}\Omega_j = 1$, and $\varpi_j(J)$ is a PDF corresponding to an individual jump size.

2.2 Stochastic Volatility Models

These models assume that volatility itself is volatile and fluctuating towards a long-term mean. These models were introduced to reflect empirical evidence that the volatility of asset returns is not constant. A number of models were proposed for the volatility dynamics [Hull-White (1987), Stein-Stein (1991), Schobel-Zhu (1999)], but the most popular model, which is also known as the square root model, was developed by Heston (1993):

$$\begin{cases} dS(t) &= (r-d)S(t)dt + \sqrt{V(t)}S(t)dW^s(t),\ S(0)=S; \\ dV(t) &= \kappa(\theta - V(t))dt + \varepsilon\sqrt{V(t)}dW^v(t),\ V(0)=V. \end{cases} \tag{2.6}$$

where κ is a mean-reverting rate, θ is a long-term variance, ε is a volatility of volatility, $W^s(t)$ and $W^v(t)$ are correlated Wiener processes with constant correlation ρ.

Stochastic volatility models have a few advantages: models are in good agreement with implied volatility surfaces, especially with long-term parts of the surface. Models' implied smiles are quite stable and unchanging over time.

However, these models have a serious disadvantage: we have to hedge stochastic volatility to replicate and price the option. Unlike a stock or a currency, volatility is not a traded variable with an observable price. As a matter of fact, we must hedge one option with another, which is harder both in theory and in practice.

2.3 Jump-Diffusions with Stochastic Volatility

There is strong evidence that stochastic volatility models cannot handle short-term smiles properly. The addition of jumps [Bates (1996)] makes these models more realistic:

$$\begin{cases} dS(t) &= (r-d-\lambda m)S(t)dt + \sqrt{V(t)}S(t)dW^s(t) + (e^J-1)S(t)dN(t), \\ dV(t) &= \kappa(\theta - V(t))dt + \varepsilon\sqrt{V(t)}dW^v(t). \end{cases} \tag{2.7}$$

Jumps can be drawn from either normal or double-exponential distribution.

Perhaps these models propose the most realistic dynamics for the smile. Models combine advantages and disadvantages of both jumps and stochastic volatility. They are also supported by numerous empirical studies [for example, Bates (1996), Fang (2000), Duffie-Pan-Singleton (2000)].

2.4 Jump-Diffusions with Stochastic Volatility and Jump Intensity

Fang (2000) studied the Bates model with a stochastic jump intensity rate:

$$\begin{cases} dS(t) &= (r-d-\lambda(t)m)S(t)dt + \sqrt{V}S(t)dW^s(t) + (e^J-1)S(t)dN(t),\ S(0)=S; \\ dV(t) &= \kappa(\theta - V(t))dt + \varepsilon\sqrt{V(t)}dW^v(t),\ V(0)=V; \\ d\lambda(t) &= \kappa_\lambda(\theta_\lambda - \lambda(t))dt + \varepsilon_\lambda\sqrt{\lambda(t)}dW^\lambda(t),\ \lambda(0)=\lambda. \end{cases} \tag{2.8}$$

where κ_λ is a mean-reverting rate, θ_λ is a long-term intensity, ε_λ is volatility of jump intensity, a Wiener process $W^\lambda(t)$ is independent of $W^s(t)$ and $W^v(t)$.

This is a very ambitious model, and it should be avoided in practice. We applied this model for modeling the DAX volatility surface and found that there is no reason to assume that the jump intensity is stochastic. However, jump-diffusions with time-dependent intensity perform slightly better that those with constant intensity.

2.5 Jump-Diffusions with Deterministic Volatility and Jump Intensity

Jump-diffusions with stochastic volatility result in two-dimensional pricing problem and quite complicated hedging strategy. We can avoid this by introducing time-dependent volatility and jump intensity:

$$\begin{cases} dS(t) &= (r-d-\lambda(t)m)S(t)dt + \sqrt{V}S(t)dW^s(t) + (e^J-1)S(t)dN(t),\ S(0)=S; \\ dV(t) &= \kappa(\theta - V(t))dt,\ V(0)=V; \\ d\lambda(t) &= \kappa_\lambda(\theta_\lambda - \lambda(t))dt,\ \lambda(0)=\lambda. \end{cases} \tag{2.9}$$

Empirical results of calibrating this model to the DAX surface imply that the model produces the fit which is comparable to those produced by jump-diffusion with stochastic volatility. A similar model, but in a discrete GARCH specification, was considered in the empirical study by Maheu&McCurdy (2003). They found that the model provides a good fit to the data, and it is capable of producing a rich variety of implied distributions.

2.6 Jump-Diffusion with Price and Volatility Jumps

Duffie-Pan-Singleton (2000) proposed a jump diffusion with both price and volatility jumps (SVJ):

$$\begin{cases} dS(t) &= (r - d - \lambda m)S(t)dt + \sqrt{V}S(t)dW^s(t) + (e^{J^s} - 1)S(t)dN^s(t),\ S(0) = S; \\ dV(t) &= \kappa(\theta - V(t))dt + \varepsilon\sqrt{V(t)}dW^v(t) + J^v dN^v(t),\ V(0) = V. \end{cases} \tag{2.10}$$

The general SVJ model nests four types of jumps: 1)jumps in the asset price; 2) jumps in the variance with exponentially distributed jump size; 3) double jumps model with jumps in the asset price and independent jumps in the variance with exponentially distributed jump size; and 4) simultaneous jumps model (SVSJ) with simultaneous correlated jumps in price and variance. The marginal distribution of the jump size in variance is exponential with mean η. Conditional distribution on a realization J^v of a jump size in variance, a jump in logarithmic asset price is normally distributed with mean $\nu + \rho_J J^v$ and variance δ^2:

$$J^v \sim \exp(\frac{1}{\eta}),\ J^s|J^v \sim N(\nu + \rho_J J^v, \delta^2). \tag{2.11}$$

This model is supported by a number of studies [for example, Duffie-Pan-Singleton (2000), Eraker-Johannes-Polson (2003)], and it provides a remarkable fit to observed volatility surfaces.

3 Solution of the Pricing Problem

Now we solve the problem of pricing European-style options on the considered price processes.

3.1 Preliminaries

We consider the following process which is a generalization of processes considered previously:

$$\begin{cases} dS(t)/S(t) &= (r - d - \lambda(t)m)dt + \sqrt{V(t)}dW^s(t) + (e^J - 1)dN(t),\ S(0) = S; \\ dV(t) &= \kappa(\theta - V(t))dt + \varepsilon\sqrt{V(t)}dW^v(t) + J^v dN^v(t),\ V(0) = V; \\ d\lambda(t) &= \kappa_\lambda(\theta_\lambda - \lambda(t))dt + \varepsilon_\lambda\sqrt{\lambda(t)}dW^\lambda(t),\ \lambda(0) = \lambda. \end{cases} \tag{3.1}$$

We will solve the pricing problem for European-style claims under general dynamics (3.1). Therefore, solutions for particular models can be obtained by simplifications of a general solution.

Using the martingale pricing, we can represent option value as an integral of a discounted probability density times the payoff function and apply the Feynman-Kac theorem to derive the PIDE satisfied by the value of an option. We make a change of variable from S to $x = \ln S$ and from t to $\tau = T - t$. Applying Feynman-Kac theorem [for example, Duffie-Pan-Singleton (2000)] for the price dynamics (3.1), we obtain that the value of a European-style claim denoted by $f(x, V, \lambda, \tau)$ satisfies the following partial integro-differential equation (PIDE)

$$\begin{aligned}
&-f_\tau + (r - d - \frac{1}{2}V - \lambda m)f_x + \frac{1}{2}V f_{xx} + \kappa(\theta - V)f_V + \frac{1}{2}\varepsilon^2 V f_{VV} + \rho\varepsilon V f_{xV} \\
&\quad + \kappa_\lambda(\theta_\lambda - \lambda)f_\lambda + \frac{1}{2}\varepsilon_\lambda^2 \lambda f_{\lambda\lambda} + \lambda \int_{-\infty}^{\infty} [f(x+J) - f(x)]\varpi(J)dJ \\
&\quad + \lambda^v \int_0^{\infty} [f(V+J^v) - f(V)]\overline{\psi}(J^v)dJ^v = rf, \\
&f(x, V, \lambda, 0) = g(e^x, K).
\end{aligned} \tag{3.2}$$

where g is the payoff function. We assume that the payoff function is independent of the variance. The payoff for an European call or put option is given by

$$g(e^x, K) = \max\{\varphi[e^x - K], 0\}. \tag{3.3}$$

We will apply the Fourier transform to solve PIDE (3.2). The forward Fourier transform of $f(x)$, which is denoted by $\hat{f}(x)$, or $\mathcal{F}[f(x)]$, is defined by

$$\hat{f}(z) = \mathcal{F}[f(x)] = v.p. \int_{-\infty}^{\infty} e^{izx} f(x)dx \tag{3.4}$$

and the inverse Fourier transform is defined by

$$f(x) = \mathcal{F}^{-1}[\hat{f}(z)] = \frac{1}{2\pi}\ v.p. \int_{iz_i-\infty}^{iz_i+\infty} e^{-izx}\hat{f}(z)dz \tag{3.5}$$

where $i = \sqrt{-1}$ and $z \in \mathbb{C}$, $z = z_r + z_i i$, $z_r = \Re z \in \mathbb{R}$, $z_i = \Im z \in \mathbb{R}$, is the transform variable.

For typical option payoffs, integral 3.4 exists only if $\Im z$ is restricted to a strip $\alpha < \Im z < \beta$, where $\Im$ means the imaginary part of z. We will refer to this strip as the strip of regularity.

To solve the pricing problem, we need 1) to find an explicit representation of Fourier integral (3.5); and 2) to invert the transform with the z-plain integration keeping $\Im z$ in an appropriate strip of regularity. Integral (3.5) can be computed using the standard methods of numerical integration.

The advantage of employing the Fourier transform is obvious. Because a typical option payoff depends only on the underlying (so that we need to apply only one-dimensional Fourier transform), we can solve a multi-dimensional PIDE corresponding to some tractable market models by means of a one-dimensional integration!

It makes sense to derive a general formula for pricing calls and puts. We consider variable $x(t) = \ln S(t)$. Under the martingale measure $\mathbb{Q}$, $x(t)$ satisfies

$$\mathbb{E}^{\mathbb{Q}}[x(T)] = x(t) + (r-d)(T-t). \tag{3.6}$$

It is more convenient to consider the option with the bounded payoff function $g(x) = \min\{e^x, K\}$. Let $f(x, V, \lambda, t)$ denote the value function of this option.

Using martingale pricing, we can represent the value of a call or put $F(x,t)$ as

$$\begin{aligned} F(x(t),t) &= \mathbb{E}^{\mathbb{Q}}[e^{-(T-t)r} \max\{\varphi[e^{x(T)} - K], 0\}] \\ &= e^{-(T-t)r} \left[\frac{1+\varphi}{2} \mathbb{E}^{\mathbb{Q}}[e^{x(T)}] + \frac{1-\varphi}{2} \mathbb{E}^{\mathbb{Q}}[K] - \mathbb{E}^{\mathbb{Q}}[\min\{e^{x(T)}, K\}] \right] \\ &= \frac{1+\varphi}{2} e^{x(t)-(T-t)d} + \frac{1-\varphi}{2} e^{-(T-t)r} K - f(x, V, \lambda, t). \end{aligned} \tag{3.7}$$

Feynman-Kac theorem implies that the option value function $f(x, V, \lambda, \tau)$, where $\tau = T - t$, satisfies PIDE (3.2) with the following initial condition

$$g(e^x, K) = \min\{e^x, K\}. \tag{3.8}$$

As a result, we need to solve PIDE (3.2) for the function $f(x, V, \lambda, \tau)$ and price calls and puts using formula (3.7).

For our subsequent analysis, we need to find the transformed initial condition $\widetilde{g}(z) = \mathcal{F}[\min\{e^x, K\}]$. A relatively simple calculation yields

$$\begin{aligned} \widetilde{g}(z) &= \int_{-\infty}^{\infty} e^{izx} \min\{e^x, K\} dx = \lim_{U\to\infty} \int_{-U}^{\ln K} e^{izx} e^x dx + K \lim_{U\to\infty} \int_{\ln K}^{U} e^{izx} dx \\ &= \lim_{U\to\infty} \left. \frac{e^{(iz+1)x}}{iz+1} \right|_{x=-U}^{x=\ln K} + K \lim_{U\to\infty} \left. \frac{e^{izx}}{iz} \right|_{x=\ln K}^{x=U} \\ &= \frac{e^{(iz+1)\ln K}}{iz+1} - 0 + K \left(0 - \frac{e^{iz \ln K}}{iz} \right) = \frac{K^{iz+1}}{iz+1} - \frac{K^{iz+1}}{iz} = \frac{K^{iz+1}}{z^2 - iz}. \end{aligned} \tag{3.9}$$

The lower limit exists only if $\Im z < 1$. The upper limit exists only if $\Im z > 0$. Thus, the transform is defined for $0 < \Im z < 1$. We will denote this strip of regularity as the payoff strip S_f.

3.2 The Characteristic Formula

Now we consider a powerful approach for solving a general pricing problem. We assume that the characteristic function corresponding to the price dynamics is given in closed-form. The characteristic function of $x(T) = \ln S(T)$ is defined by

$$\phi_T(z) = \mathbb{E}^Q[e^{izx}] = \int_{-\infty}^{\infty} e^{izx} \varpi_T(x) dx \tag{3.10}$$

where $\varpi_T(x)$ is the risk-neutral density of the logarithmic price $x(T)$.

Now we state a modified version of Theorem 3.2 in Lewis (2001), which is very important for option pricing under general stochastic processes. Lewis proposed this formula for option pricing under Lévy processes; we apply his result for general stochastic processes.

Theorem 3.1 (The Characteristic Formula)**.** *We assume that $x(T)$ has the analytic characteristic function $\phi_T(z)$ with the strip of regularity $S_z = \{z : \alpha < \Im z < \beta\}$. Next we assume that $e^{-z_i x} f(x) \in L^1(\mathbb{R})$ where z_i is located in the payoff strip S_f with transform $\hat{f}(z)$, $\Im z \in S_f$.*

Then, if $S_F = S_f \cap S_z$ is not empty, the option value is given by

$$f(x(t)) = \frac{e^{-r(T-t)}}{2\pi} \int_{iz_i-\infty}^{iz_i+\infty} \phi_T(-z)\hat{f}(z)dz \tag{3.11}$$

where $z_i = \Im z$, $z \in S_F = S_f \cap S_z$.

Proof. Using risk-neutral pricing, we have

$$\begin{aligned} f(x(t)) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} f(x(T))\right] \overset{(3.5)}{=} e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} e^{-izx(T)}\hat{f}(z)dz\right] \\ &= \frac{e^{-r(T-t)}}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} \mathbb{E}^{\mathbb{Q}}[e^{-izx(T)}]\hat{f}(z)dz \overset{(3.10)}{=} \frac{e^{-r(T-t)}}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} \phi_T(-z)\hat{f}(z)dz. \end{aligned}$$

This exchange of integration order is allowed by the Fubini theorem. By assumption $\phi_T(-z)$ exists if $z \in S_z$ and z is already restricted to $z \in S_z$. Accordingly, the whole integrand exists if $z \in S_F = S_f \cap S_z$. □

3.3 The Black-Scholes-style Formula

Now we present another pricing formula which we call the Black-Scholes-style formula. This formula is a generalization of previous work by Heston (1993), Bates(1996) and others. A similar result was first considered by Lewis (2001) for pricing under Lévy processes. Here, we consider it in more details and apply for pricing under general processes.

Theorem 3.2 (The Black-Scholes-style Formula)**.** *We assume that the characteristic function $\phi_T(z) = \mathbb{E}^{\mathbb{Q}}[e^{izx(T)}]$ corresponding to the market model is analytic and bounded in the strip $0 \le \Im z \le 1$. Two characteristics, $\phi_j(k)$ $(j = 1, 2)$, $k \in \mathbb{R}$, are given by $\phi_1(k) = e^{-\ln S(t)-(r-d)(T-t)}\phi_T(k-i)$ and $\phi_2(k) = \phi_T(k)$. The CDF-s, Π_j, in the variable $y = \ln K$ of the log-spot price $x(T) = \ln S(T)$ are given by*

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{\phi_j(k)e^{-iky}}{ik}\right] dk \tag{3.12}$$

and variables $P_j(\varphi)$ are defined by

$$P_j(\varphi) = \frac{1-\varphi}{2} + \varphi\Pi_j. \tag{3.13}$$

Then the current value of a European-style contingent claim, $F(S,t)$, that pays off $\max\{\varphi[S_T - K], 0\}$, *where the binary variable $\varphi = +1$ for a call and $\varphi = -1$ for a put, at time of the expiration date T has the form*

$$F(S,t) = \varphi\left[e^{-d(T-t)} S P_1(\varphi) - e^{-r(T-t)} K P_2(\varphi)\right]. \qquad (3.14)$$

To derive the Black-Scholes-style formula (3.14), we will employ characteristic formula 3.11 and the Cauchy's theorem. The proof is given in Appendix A.

3.4 Analysis of Pricing Formulas

Thus, the pricing problem for European calls and puts is reduced to the evaluation of Fourier integrals, which has to be done numerically. This method is quite easy to implement and it is sufficiently fast for calibration.

We consider the proposed formulas in more details. The Black-Scholes formula (3.14) includes integrals which are already expressed in terms of the real-valued transform parameter k.

Now, we consider the characteristic approach. Given an explicit expression for the moment generating function (MGF) $G(\Phi, x, V, \lambda, \tau)$, the complex-valued characteristic function is given by $\phi_T(z) = G(iz)$ and, similarly, $\phi_T(-z) = G(-iz)$.

For an option with payoff function $g(x) = \min\{e^x, K\}$, the integral (3.11) can be represented as

$$f(x, V, \lambda, \tau) = \frac{Ke^{-\tau r}}{2\pi} \int_{iz_i - \infty}^{iz_i + \infty} e^{iz \ln K} \frac{G(-iz, x, V, \lambda, \tau)}{z^2 - iz} dz. \qquad (3.15)$$

It is more convenient to evaluate integral (3.15) along a straight line $v = 1/2$ in the complex z-plain parallel to the real axis. Substituting $z = k + i/2$, $k \in \mathbb{R}$, into (3.15), we obtain that

$$f(x, V, \lambda, \tau) = \frac{Ke^{-\tau r}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-(-ik+1/2)\ln K} G(-ik + \frac{1}{2}, x, V, \lambda, \tau)}{k^2 + 1/4} dk. \qquad (3.16)$$

For brevity, we introduce

$$Q(k, x, V, \lambda, \tau) = e^{-\tau r - (-ik+1/2)\ln K} G(-ik + \frac{1}{2}, x, V, \lambda, \tau). \qquad (3.17)$$

It can be shown that integrand in (3.16) is a symmetric function, i.e. $Q(-k) = \overline{Q(k)}$, so that for option pricing we need to evaluate the following integral

$$f(x, V, \lambda, \tau) = \frac{Ke^{-\tau r}}{\pi} \int_0^{\infty} \Re\left[\frac{Q(k, x, V, \lambda, \tau)}{k^2 + 1/4}\right] dk. \qquad (3.18)$$

In Appendix C, we report explicit expressions of characteristic functions and $Q(k, x, V, \lambda, \tau)$ for particular market dynamics.

Given the value of $f(x, V, \lambda, \tau)$, values of European calls and puts are calculated using formula (3.7). Thus, the pricing problem is reduced to a one-dimensional integration along the real axis. It can be shown that integrals (3.16) and (3.18) are uniformly convergent so that the partial derivatives of the option value can be computed by differentiating the integrand and inverting the Fourier transform. The same is true for the Black-Scholes-style formula.

As a result, we have obtained two distinct formulas (3.14) and (3.18) for solving the pricing problem. We have extensively studied convergence properties of the integrals arising from both approaches. We found that using the Black-Scholes-style formula one can compute the option price about three times faster than using the transform-based approach. This property does not depend on a specific market model. The advantage of characteristic integrals is that they can be computed by virtue of the Fast Fourier Transform (FFT), which can only be implemented for non-singular integrands. The FFT allows for simultaneous calculation of option values for a given level of strikes. This can be very useful for calibrating a model to the implied volatility surface. Implementation of the FFT for option pricing is presented by Carr and Madan (1999).

3.5 Solution to the Moment Generating Function

For implementing the pricing formulas, we need to find explicit expression for the characteristic function. Here we illustrate our approach using the square root processes given by the dynamics (3.1). Other stochastic processes can be analyzed in a similar manner. Let us recall the following relationship between the characteristic function $\phi(k)$ and MGF $G(\Phi)$: $\phi(k) = G(ik)$. We apply the standard considerations employed by Heston (1993) and consider the MGF $G(\Phi, x, V, \lambda, \tau)$ associated with the log of the terminal asset price $x(\tau) = \ln S(\tau)$ under the measure $\mathbb{Q}$:

$$G(\Phi, x, V, \lambda, \tau) = \mathbb{E}^{\mathbb{Q}}[e^{\Phi x(\tau)}] = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[e^{r\tau}e^{\Phi x(\tau)}]. \tag{3.19}$$

Accordingly, MGF $G(\Phi, x, V, \lambda, \tau)$ can be interpreted as a contingent claim that pays off $e^{r\tau+\Phi x}$ at time τ.

For the jump-diffusion with stochastic volatility and intensity rate specified by SDE(3.1), Feynman-Kac theorem implies that $G(\Phi, x, V, \lambda, \tau)$ solves

$$\begin{aligned}
&- G_\tau + (r - d - \frac{1}{2}V - \lambda m)G_x + \frac{1}{2}VG_{xx} + \kappa(\theta - V)G_V + \frac{1}{2}\varepsilon^2 VG_{VV} \\
&+ \rho\varepsilon VG_{xV} + \kappa_\lambda(\theta_\lambda - \lambda)G_\lambda + \frac{1}{2}\varepsilon_\lambda^2\lambda G_{\lambda\lambda} + \lambda\int_{-\infty}^{\infty}[G(x + J) - G]\varpi(J)dJ = 0, \\
&G(\Phi, x, V, \lambda, 0) = e^{\Phi x}.
\end{aligned} \tag{3.20}$$

We can solve PIDE (3.20) in closed-form by the method of indetermined coefficients using a guess $G = e^{A(\tau)+B(\tau)V+C(\tau)\lambda}$. Thus, the solution has an affine-form and, accordingly, these models are called affine (jump-)diffusions. The full solution is specified by

Proposition 3.1. *The solution to PIDE (3.20) is given by*

$$G(\Phi, x, V, \lambda, \tau) = e^{x\Phi+(r-d)\tau\Phi+A(\Phi,\tau)+B(\Phi,\tau)V+C(\Phi,\tau)+D(\Phi,\tau)\lambda} \tag{3.21}$$

where

$$A(\Phi, \tau) = -\frac{\kappa\theta}{\varepsilon^2}\left[\psi_+\tau + 2\ln\left(\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right)\right],$$

$$B(\Phi, \tau) = -(\Phi - \Phi^2)\frac{1 - e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}},$$

$$A_d(\Phi, \tau) = -(\kappa\tau + e^{-\kappa\tau} - 1)\frac{\theta(\Phi - \Phi^2)}{2\kappa},\ B_d(\Phi, \tau) = -(1 - e^{-\kappa\tau})\frac{(\Phi - \Phi^2)}{2\kappa},$$

$$C(\Phi, \tau) = -\frac{\kappa_\lambda\theta_\lambda}{\varepsilon_\lambda^2}\left[\chi_+\tau + 2\ln\left(\frac{\chi_- + \chi_+ e^{-\xi\tau}}{2\xi}\right)\right],$$

$$D(\Phi, \tau) = 2\Lambda(\Phi)\frac{1 - e^{-\xi\tau}}{\chi_- + \chi_+ e^{-\xi\tau}},$$

$$C_d(\Phi, \tau) = (\kappa_\lambda\tau + e^{-\kappa_\lambda\tau} - 1)\frac{\theta_\lambda\Lambda(\Phi)}{\kappa_\lambda},\ D_d(\Phi, \tau) = (1 - e^{-\kappa_\lambda\tau})\frac{\Lambda(\Phi)}{\kappa_\lambda},$$

$$\psi_\pm = \mp(\kappa - \rho\varepsilon\Phi) + \zeta,\ \zeta = \sqrt{(\kappa - \rho\varepsilon\Phi)^2 + \varepsilon^2(\Phi - \Phi^2)},$$

$$\chi_\pm = \mp\kappa_\lambda + \xi,\ \xi = \sqrt{\kappa_\lambda^2 - 2\varepsilon_\lambda^2\Lambda(\Phi)},$$

$$\Lambda(\Phi) = \int_{-\infty}^{\infty} e^{J\Phi}\varpi(J)dJ - 1 - m\Phi,\ m = \int_{-\infty}^{\infty} e^{J}\varpi(J)dJ - 1,$$

where $A = A_d$, $B = B_d$ and $C = C_d$, $D = D_d$ in case of deterministic volatility and jump intensity, respectively.

Proof is given in Appendix B.

In the above formula, $\Lambda(\Phi)$ is the so-called jump transform. Simple calculations yield that for log-normal price-jumps we have

$$\Lambda(\Phi) = e^{\nu\Phi+\delta^2\Phi^2/2} - 1 - \Phi(e^{\nu+\delta^2/2} - 1), \tag{3.22}$$

and for double-exponential jumps we have

$$\Lambda(\Phi) = \frac{p}{1 - \Phi\eta_u} + \frac{q}{1 + \Phi\eta_d} - 1 - \Phi(\frac{p}{1 - \eta_u} + \frac{q}{1 + \eta_d} - 1). \tag{3.23}$$

provided that $-1/\eta_d < \Im z < 1/\eta_u$.

For a mixture of independent jumps defined by (2.5), we obtain that

$$\Lambda(\Phi) = \sum_{j=1}^{n} \Omega_j\Lambda_j(\Phi) \tag{3.24}$$

where $\Lambda_j(\Phi)$ is the transform of a corresponding jump size PDF.

Once the explicit expression for MGF has been obtained, we can use it to implement the pricing formulas. In Appendix C, we report explicit expressions for the characteristic formula and the Black-Scholes-style formula.

The expression for the MGF corresponding to a SVSJ process is derived in a similar manner. We omit details to obtain

Proposition 3.2. *The MGF corresponding to the SVSJ process is given by*

$$G(\Phi, x, V, \lambda, \tau) = e^{\Phi x+(r-d)\tau\Phi+A(\Phi,\tau)+B(\Phi,\tau)V+\Delta(\Phi,\tau)} \tag{3.25}$$

where

$$\begin{aligned}\Delta(\Phi,\tau) = \lambda\left(-\Phi\left(\frac{e^{\nu+\frac{1}{2}\delta^2}}{1-\rho_J\eta}-1\right)-1\right)\tau + \lambda e^{\nu\Phi+\frac{1}{2}\delta^2\Phi^2}\left(\frac{\psi_-}{\psi_- L+\eta U}\tau\right.\\ \left.-\frac{2\eta U}{(\zeta L)^2-(ML+\eta U)^2}\ln\left[1-\frac{\psi_+ L-\eta U}{2\zeta L}(1-e^{-\zeta\tau})\right]\right),\end{aligned} \tag{3.26}$$

$U = \Phi - \Phi^2$, $M = \kappa - \rho\varepsilon\Phi$, $L = 1 - \rho_J\eta\Phi$, and $A(\Phi, \tau)$, $B(\Phi, \tau)$, $\psi_\pm$, ζ are defined in (3.21).

An outline of the proof is given in Appendix B.

4 Computing Fourier Integral

The Fourier integral can be computed by means of standard procedures of numerical integration. An extensive list of corresponding formulas can be found in Numerical Recepies in C [Press *et al* (1992)]. We use the ten-point Gaussian quadrature routine. To approximate the Fourier integral, we re-write it as

$$I(k) = \int_0^\infty f(k)dk = \lim_{N\to\infty}\sum_{j=1}^{N}\int_{(j-1)\cdot h}^{j\cdot h} f(k)dk. \tag{4.1}$$

Sub-integrals in (4.1) are computed by means of Gaussian quadrature formula. N is not fixed but determined by a local stopping criterion: if the contribution of the last strip, $f(k_j)$, becomes smaller than $10e-12$ the integration is stopped.

By using formula (4.1) we also have to specify a suitable step size h. We have two opportunities. The first one is to employ Runge method and let h be fixed locally according to some convergence criterion. The second approach is to determine the suitable step size h by trials or establish a function $h(X)$ where $X = \ln S/K+(r-d)\tau$. We analyzed the second approach with different choices of h. We found that choice $h = 1$ for the transform approach and choice $h = 10$ for the Black-Scholes-style formula lead to very satisfactory results. For these choices of h, price differences between two approaches do not exceed 10e-4 for most realistic values of model and option parameters. This is the case for all models we have presented.

It is very important to carefully handle the complex logarithm arising in our formulas because complex logarithm $\ln(z)$ is a multivalued function. Therefore, we should implement our formula carefully keeping track of the complex logarithm along the integration path. This gives a smooth function and leads to correct values of $f(k)$.

We use the following algorithm adopted from Theorem 4.17 in Priestley (1990):
1) at $k = 0$ an integer b_0 (branch) is initialized to 0 and the principal branch of the logarithm is initialized to $L_0 = \ln|z| + i \cdot arg(z)$;
2) given b_{k-1}, the b_k is determined as follows:
if $\Im(L_k - L_{k-1}) < -\pi$ then $b_k := b_{k-1} + 1$
if $\Im(L_k - L_{k-1}) > \pi$ then $b_k := b_{k-1} - 1$
otherwise, $b_k := b_{k-1}$.
Then the correct logarithm to use at the point k is $L_k + i \cdot 2\pi \cdot b_k$.

We use the principal square root to evaluate $\sqrt{z}$ in our pricing formulas.

5 Empirical Study

In this section, we report results of applying jump-diffusions presented here to the DAX options data.

5.1 Model Calibration

In previous sections, we derived formulas to calculate prices of European-style options for given parameters characterizing jump and volatility components. For practical purposes, we have to specify these parameters. We have two possibilities: either to use historical time series in order to estimate parameters of the stock process or to use the market prices of liquid options in order to imply these parameters from option prices. By pricing derivative securities, market makers are primarily concerned with parameters implied by the market, because they need to model their option portfolios consistently with the market. A hedge fund might be interested in both historical and implied parameters in order to make some arbitrage bets.

Parameter estimation using historical time series has been treated in several studies: Bates (1996), Duffie-Pan-Singleton (2000), Eraker-Johannes-Polson (2003). In our study, we consider parameter estimation using the implied volatility matrix which encapsulates information of liquid options across all possible strikes and maturities. A typical volatility matrix is shown in Table 2, which contains the DAX (The Deutscher Aktienindex) option implied volatilities of July 5, 2002. To be sure that our choice of parameters is compatible with the given volatility matrix, we need to solve the calibration problem. Because we are interested in the model's fit into the implied volatility surface, we use the following "goodness of fit" measure

$$\min_{\Theta} \sum_{j=1}^{N} \left(\sigma_j^{market}(K,T) - \sigma_j^{model}(K,T;\Theta)\right)^2 \tag{5.1}$$

where $\sigma_j^{market}(K,T)$ is the market implied volatility for strike K and maturity T, and $\sigma_j^{model}(K,T;\Theta)$ is the Black-Scholes volatility implied from the model price of a call option with strike K and maturity T.

To solve the optimization problem, we use Excel Solver 5.0, which is sufficiently efficient to deal with such problems. Our formulas are implemented in C++, which guarantees high computational speed.

5.2 Results

In the first part of Table 1, we report estimated parameters for jump-diffusions with log-normal (LN) jumps. BS denotes the Black-Scholes model, which is used as a benchmark. SV denotes stochastic volatility component (the Heston model), DV - deterministic volatility component, SI - stochastic jump intensity component, DI - deterministic jump intensity component, SVSJ - jump-diffusion with simultaneous price and volatility jumps. In the second part of Table 1, we report estimated parameters for jump-diffusion with log-double-exponential (LE) jumps.

The sum of squared differences between market and model implied volatility (SSE) can be viewed as a measure of in-sample fit. We also report the ratio of SSE produced by the model to that of the Black-Scholes model. We can assume that the lower the ratio, the better the model is compatible with the vanilla market.

The implied volatility surface and differences between market and model implied volatilities for representative models are shown in Appendix D.

5.3 Comments

Jump-diffusions reduce squared errors by about 72 percent. The jump-diffusions hold an advantage in fitting shorter maturity smile.

Diffusion with stochastic volatility cuts errors by more than 96 percent and thus explains a significant part of the smile. As expected, the stochastic volatility model implies a strongly negative correlation, ρ, between the asset price and volatility, which is common for equity markets. Somehow unusual is the large volatility of volatility ε and mean reversion κ. This effect may be caused by a large number of shorter maturity out-of-the-money and in-the-money options in our data. It also indicates that we need to incorporate jumps into the Heston model in order to account for short maturities.

Jump-diffusions with stochastic volatility. Combing stochastic volatility and jump models does a better job than either alone. It is clear that the pure stochastic volatility model cannot handle very short maturities properly. By introducing jumps it is possible to improve the overall quality of calibration. Jumps also take off some pressure from volatility of volatility ε and mean reversion κ in the Heston model.

Jump-diffusions with stochastic intensity rate. An interesting fact is that spot variance becomes zero. Thus, the asset price follows a pure jump process and the main source of uncertainty is the stochastic jump intensity, which has a quite

	BS	LN	LN-SI	LN-SV	LN-SV-DI	LN-DV-DI	SVSJ
V_0	0.0942	0.0433	0.0013	0.1372	0.1170	0.1228	0.1254
κ				9.5110	3.8988	10.8850	9.6952
θ				0.0285	0.0701	0.0001	0.0113
ε				0.8010	1.1536		0.3846
ρ				-0.5483	-0.5646		-1.0000
λ		1.1098	22.2782	0.3635	1.6872	1.0995	1.1587
κ_λ			15.1160		43.6666	10.2252	
θ_λ			7.5368		0.0001	1.4383	
ε_λ			15.0947				
ν		-0.1285	-0.0550	-0.2459	-0.1538	-0.1614	-0.0982
δ		0.1702	0.0772	0.2547	0.1856	0.1591	0.1801
η_J							0.0696
ρ_J							-0.0584
SSE	5701.9	1588.1	381.7	36.6	28.4	60.9	19.7
Ratio	100.0%	27.85%	6.69%	0.64%	0.50%	1.07%	0.35%
Rank		12	9	5	3	7	2

	SV	LE	LE-SI	LE-SV	LE-SV-DI	LE-DV-DI
V_0	0.1979	0.0340	0.0001	0.1287	0.1006	0.1209
κ	16.2144			9.2908	3.3114	11.0413
θ	0.0741			0.0204	0.0720	0.0002
ε	3.3904			0.6389	1.0130	
ρ	-0.5113			-0.7400	-0.6092	
λ		4.4044	55.1657	2.2445	11.1592	2.0977
κ_λ			13.4850		23.7478	9.4700
θ_λ			19.1176		0.0001	2.5076
ε_λ			0.0768			
ν_{up}		0.0428	0.0231	0.0441	0.0325	0.1234
ν_{down}		0.1205	0.0643	0.2078	0.1387	0.1231
p		0.5957	0.6553	0.7365	0.8131	0.0193
SSE	177.2	1584.6	307.8	29.9	15.7	50.6
Ratio	3.11%	27.79%	5.40%	0.52%	0.28%	0.89%
Rank	10	11	8	4	1	6

Table 1: Estimated parameters and in-sample fit

V_0 is spot variance; κ is mean-reversion speed; θ is long-term variance; ε is volatility of volatility; ρ is correlation between price and variance;
λ, ν, δ are jump intensity, mean and volatility for the Merton jump-diffusion, respectively;
λ, η_{up}, η_{down}, p are intensity, mean of positive jumps, mean of negative jumps, probability of positive jumps for the double-exponential jump-diffusion, respectively;
κ_λ, θ_λ, ε_λ are mean-reversion speed, long-term rate, volatility of the jump intensity, respectively;
λ_J and η_J are volatility jump mean and correlation for SVSJ model, respectively;

large initial value and long-term level.

Jump-diffusions with stochastic volatility and deterministic intensity. The quality of fitting for jump-diffusions with stochastic volatility and intensity is equivalent to that for jump-diffusions with deterministic intensity. This justifies our motivation of analyzing jump diffusions with deterministic jump intensity. From the in-sample fit results, we can conclude that the jump-diffusions with deterministic jump intensity are superior to the jump-diffusions with constant intensity.

Jump-diffusions with deterministic volatility and intensity provide a close fit to data, thus they are worth studying. Their fit is comparable to that implied by jump-diffusions with stochastic volatility, but these models are more simpler.

Jump-diffusion with price and volatility jumps fits the surface very well. An interesting fact is that the correlation between the underlying and volatility ρ is -1. As expected, the jump correlation ρ_J is negative, which means that a jump in volatility affects the asset jump size rather negatively.

A good observation is that all parameters are important, that is increasing the number of model parameters improves the quality of calibration. The best in-sample fit is given by the double-exponential jump-diffusion with stochastic volatility and deterministic jump intensity. The SVSJ process is very close to it.

6 Conclusions

In this research, we examined a number of affine jump-diffusions for the asset price dynamics. We considered pricing problem of European-style options and derived pricing formulas for European call and put option values using two approaches: the characteristic formula and the Black-Scholes-style formula. We found that the Black-Scholes-style formula yields the option value considerably faster than the characteristic formula. However, the characteristic formula can be computed using the Fast Fourier Transform.

Our empirical results with DAX options data show that, to achieve a close fit to the vanilla market, one needs to incorporate both components into pricing models - the stochastic volatility and jumps. Another good alternative is a jump-diffusion with deterministic volatility and jump intensity.

References

[1] Bates D, (1996), "Jumps and stochastic volatility: exchange rate processes implicit in Deutsche mark options," *Review of Financial Studies 9*, 69-107.

[2] Black F and M Scholes, (1973), "The pricing of options and corporate liabilities," *Journal of Political Economy 81*, 637-659.

[3] Carr P and D Madan (1999), " Option Valuation using the Fast Fourier Transform," *Journal of Computational Finance, 2, No.4, Summer*, 61-73.

[4] Duffie D, Pan J, and K Singleton, (2000), "Transform analysis and asset pricing for affine jump-diffusion," *Econometrica, Vol.68, No.6*, 1343-1376.

[5] Dupire B, (1994), "Pricing with a smile," *Risk January*, 18-20.

[6] Eraker B, Johannes M, and N Polson (2003), "The Impact of Jumps in Volatility and Returns," forthcoming in *Journal of Finance.*

[7] Fang H, (2000), "Option Pricing Implications of a Stochastic Jump Rate," *University of Virginia, Working Paper.*

[8] Press W, Teukolsky S, W Vetterling, and Flannery B (1992), "Numerical recipes in C: The Art of Scientific Computing," *Cambridge University Press*, available at www.nr.com.

[9] Hagan P, Kumar D, Lesniewski A, and D Woodward (2002), "Managing Smile Risk," *Wilmott magazine September*, 84-108.

[10] Heston S, (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *Review of Financial Studies 6*, 327-343.

[11] Hull J and A White, (1987), "The pricing of options with stochastic volatilities," *Journal of Finance 42*, 281-300.

[12] Kou S, (2002), "A jump diffusion model for option pricing," *Management Science Vol. 48*, 1086-1101.

[13] Lewis A, (2001), "A simple option formula for general jump-diffusion and other exponential Lévy processes," *www.optioncity.net, Working Paper.*

[14] Lipton A, (2002), "The vol smile problem," *Risk, February*, 81-85.

[15] Maheu J and T McCurdy (2003), "News Arrival, Jump Dynamics, and Volatility Components for Individual Stock Returns," forthcoming in *Journal of Finance.*

[16] Merton R, (1976), "Option pricing when underlying stock returns are discontinuous," *Journal of Financial Economics 3*, 125-144.

[17] Papapantoleon A, (2002), "Option Pricing in a Jump Diffusion model with Double Exponential Jumps," *Commerzbank FX, Quantitative Research*, available at www.institute.mathfinance.de.

[18] Priestley H, (1990), "Introduction to Complex Analysis," *Oxford University Press.*

[19] Schobel R and J Zhu, (1999), "Stochastic Volatility With an OrnsteinUhlenbeck Process: An Extension," *European Finance Review 3*, 23-46.

[20] Stein E and J Stein, (1991), "Stock price distributions with stochastic volatility: an analytic approach," *Review of Financial Studies 4*, 727-752.

A Proof of Black-Scholes-style Formula

We assume that $\phi_T(-z)$ has the strip of regularity $0 \leq \Im z \leq 1$. First, we re-write the transform-based integral (3.11) as

$$\begin{aligned} f_{\min(S,K)}(S(t)) &= \frac{e^{-r(T-t)}}{2\pi} \int_{iz_i-\infty}^{iz_i+\infty} \phi_T(-z) \frac{K^{1+iz}}{z^2 - iz} dz \\ &= \frac{e^{-r(T-t)}K}{2\pi} \left(\int_{iz_i-\infty}^{iz_i+\infty} \phi_T(-z) \frac{iK^{iz}}{z} dz - \int_{iz_i-\infty}^{iz_i+\infty} \phi_T(-z) \frac{iK^{iz}}{z-i} dz \right) \\ &= \frac{e^{-r(T-t)}K}{2\pi} \left(\Re(I_1) + \Re(I_2) \right). \end{aligned} \tag{A.1}$$

As it is customary in complex analysis, to evaluate I_1, we employ a contour integral over the contour given by 6 parametric curves: $\Gamma_1 : z = k,\ k \in (q, R)$ with $q, R > 0$; $\Gamma_2 : z = R + ib,\ b \in (0, z_i)$; $\Gamma_3 : z = k + iz_i,\ k \in (R, -R)$; $\Gamma_4 : z = -R + ib,\ b \in (z_i, 0)$; $\Gamma_5 : z = k,\ k \in (-R, -q)$; $\Gamma_6 : z = qe^{i\theta},\ \theta \in (0, \pi)$. As the integrand is analytic on this contour, Cauchy's theorem implies that

$$\sum_{j=1}^{6} \int_{\Gamma_j} F(z_j) dz_j = 0.$$

Evaluating the corresponding integrals, taking limits $q \to 0$ and $R \to \infty$, and changing variable $k \to -k$, we obtain that

$$\Re(I_1) = -\pi - \int_{-\infty}^{\infty} \Re \left[\phi_T(k) \frac{iK^{-ik}}{k} \right] dk. \tag{A.2}$$

Again, to evaluate I_2 we employ a contour integral over the contour given by 6 parametric curves: $\Gamma_1 : z = k + i,\ k \in (q, R)$ with $q, R > 0$; $\Gamma_2 : z = R + ib,\ b \in (1, 1 + z_i)$; $\Gamma_3 : z = k + i(1 + z_i),\ k \in (R, -R)$; $\Gamma_4 : z = -R + ib,\ b \in (z_i, 1)$; $\Gamma_5 : z = k + i,\ k \in (-R, -q)$; $\Gamma_6 : z = i + qe^{i\theta},\ \theta \in (0, \pi)$. Evaluating the corresponding integrals, taking limits $q \to 0$ and $R \to \infty$, changing variable $k \to k - i$, and noting that $\phi_T(-i) = Se^{(r-d)(T-t)}$ we obtain that

$$\Re(I_2) = \frac{Se^{(r-d)(T-t)}}{K} \left(-\pi - \int_{-\infty}^{\infty} \Re \left[\phi_T(k - i) \frac{iK^{-ik}}{k} \right] dk \right). \tag{A.3}$$

Substituting (A.2) and (A.3) into (A.1), using relationship (3.7) and introducing terms Π_j, we obtain the Black-Scholes-style formula (3.14).

B Solution to the Moment Generating Function

Here, we derive the solution to the MGF corresponding to the jump-diffusion (3.1). We guess that the solution for $G(\Phi, x, V, \lambda, \tau)$ in (3.20) has the form

$$G(\Phi, x, V, \lambda, \tau) = e^{\Phi x + (r-d)\tau\Phi + O(\Phi,\tau) + B(\Phi,\tau)V + D(\Phi,\tau)\lambda}$$

with initial conditions $O(\Phi, 0) = 0$, $B(\Phi, 0) = 0$ and $D(\Phi, 0) = 0$.

Plugging these into (3.20), yields

$$\begin{aligned} O_\tau + B_\tau V + D_\tau \lambda = & -1/2(\Phi - \Phi^2)V + \kappa(\theta - V)B + 1/2\varepsilon^2 V B^2 + \\ & \rho\varepsilon V \Phi B + \kappa_\lambda(\theta_\lambda - \lambda)D + 1/2\varepsilon_\lambda^2 \lambda D^2 + \lambda\Lambda(\Phi) \end{aligned}$$

where $\Lambda(\Phi) = \int_{-\infty}^{\infty} e^{J\Phi}\varpi(J)dJ - 1 - m\Phi$.

This leads to a system of three ODEs

$$\begin{cases} O_\tau &= \kappa\theta B + \kappa_\lambda\theta_\lambda D \\ B_\tau &= -\frac{1}{2}(\Phi - \Phi^2) - (\kappa - \rho\varepsilon\Phi)B + \frac{1}{2}\varepsilon^2 B^2 \\ D_\tau &= -\kappa_\lambda D + \frac{1}{2}\varepsilon_\lambda^2 D^2 + \Lambda(\Phi). \end{cases}$$

This system can be solved explicitely. Finally, we can represent $O(\Phi, \tau)$ as

$$O(\Phi, \tau) \equiv A(\Phi, \tau) + C(\Phi, \tau)$$

where $A_\tau(\Phi, \tau) = \kappa\theta B$ and $C_\tau(\Phi, \tau) = \kappa_\lambda\theta_\lambda D$.

First, we solve the second ODE, which is the Riccati differential equation. Making the substitution

$$B = -\frac{E'(\tau)}{\frac{\varepsilon^2}{2}E(\tau)},$$

we obtain the second order differential equation

$$E''(\tau) + (\kappa - \Phi\rho\varepsilon)E'(\tau) + \frac{1}{4}\varepsilon^2(\Phi - \Phi^2)E(\tau) = 0.$$

A general solution has the form

$$E(\tau) = C_1 e^{-\frac{1}{2}\psi_- \tau} + C_2 e^{\frac{1}{2}\psi_+ \tau}$$

where

$$\psi_\pm = \mp(\kappa - \Phi\rho\varepsilon) + \zeta,\ \zeta = \sqrt{(\kappa - \Phi\rho\varepsilon)^2 - \varepsilon^2(\Phi - \Phi^2)}.$$

We note that $\psi_+ + \psi_- = 2\zeta$ and $\psi_+ \cdot \psi_- = -\varepsilon^2(\Phi - \Phi^2)$.

The boundary conditions

$$\begin{cases} E(0) &= C_1 + C_2 \\ E'(0) &= -\frac{1}{2}\psi_- C_1 + \frac{1}{2}\psi_+ C_2 = 0 \end{cases}$$

yield

$$\begin{cases} C_1 &= \frac{\psi_+ E(0)}{2\zeta} \\ C_2 &= \frac{\psi_- E(0)}{2\zeta} \end{cases}.$$

Thus,

$$B(z,\tau) = -\frac{E'(\tau)}{\frac{\varepsilon^2}{2}E(\tau)} = -\frac{-\frac{1}{2}\psi_- \frac{\psi_+ E(0)}{2\zeta} e^{-\frac{1}{2}\psi_-\tau} + \frac{1}{2}\psi_+ \frac{\psi_- E(0)}{2\zeta} e^{\frac{1}{2}\psi_+\tau}}{\frac{\varepsilon^2}{2}\left(\frac{\psi_+ E(0)}{2\zeta} e^{-\frac{1}{2}\psi_-\tau} + \frac{\psi_- E(0)}{2\zeta} e^{\frac{1}{2}\psi_+\tau}\right)}$$
$$= -(\Phi - \Phi^2)\frac{-e^{-\frac{1}{2}\psi_-\tau} + e^{-\frac{1}{2}\psi_+\tau}}{\psi_+ e^{-\frac{1}{2}\psi_-\tau} + \psi_- e^{\frac{1}{2}\psi_+\tau}} = -(\Phi - \Phi^2)\frac{1 - e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}}.$$

Finally, the expression for $A(z,\tau)$ is obtained by integrating the first ODE.

$$A(z,\tau) = \kappa\theta \int_0^\tau B(s)ds = -\frac{2\kappa\theta}{\varepsilon^2}\int_0^\tau \frac{E'(s)}{E(s)}ds = -\frac{2\kappa\theta}{\varepsilon^2}\left(\ln E(s)|_{s=0}^{s=\tau}\right)$$
$$= -\frac{2\kappa\theta}{\varepsilon^2}\ln\left[\frac{E(\tau)}{E(0)}\right] = -\frac{2\kappa\theta}{\varepsilon^2}\ln\left[\frac{\psi_+}{2\zeta}e^{-\frac{1}{2}\psi_-\tau} + \frac{\psi_-}{2\zeta}e^{\frac{1}{2}\psi_+\tau}\right]$$
$$= -\frac{2\kappa\theta}{\varepsilon^2}\ln\left[\frac{\psi_+ e^{-\frac{1}{2}\psi_-\tau} + \psi_- e^{\frac{1}{2}\psi_+\tau}}{2\zeta}\right] = -\frac{\kappa\theta}{\varepsilon^2}\left(\psi_+\tau + 2\ln\left[\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right]\right).$$

The ODE for the stochastic jump intensity is solved by analogy. Jump-diffusions with deterministic volatility and jump intensity are treated in a similar manner.

Finally, we consider jump-diffusion wit price and volatility jumps. By analogy, Feynman-Kac theorem implies that in this case $G(\Phi, x, V, \lambda, \tau)$ satisfies

$$\begin{aligned}
&-G_\tau + (r - d - \frac{1}{2}V - \lambda m)G_x + \frac{1}{2}VG_{xx} + \kappa(\theta - V)G_V + \frac{1}{2}\varepsilon^2 V G_{VV} \\
&+ \rho\varepsilon V G_{xV} + \lambda\mathbb{E}\left[G(x + J^s, V + J^v) - G\right] = 0, \\
&G(\Phi, x, V, \lambda, 0) = e^{\Phi x}
\end{aligned} \tag{B.1}$$

where $J^v \sim \exp(\frac{1}{\eta})$ and $J^s|J^v \sim N(\nu + \rho_J J^v, \delta^2)$.

PIDE (B.1) can be solved by analogy with the PIDE (3.20). The solution is given by formula (3.25).

C Summary of Pricing Formulas

Here, we report explicit expressions corresponding to jump-diffusion (3.1) for the characteristic formula (3.16) and the Black-Scholes-style formula (3.14).

C.1 The Characteristic Formula

The complex-valued inverse Fourier transform corresponding to the market model (3.1) for the payoff function $f(x) = \min\{e^x, K\}$ is given by formula (3.16) where the general form of $Q(k, x, V, \lambda, \tau)$ is determined using the expression for the MGF given by (3.21).

A general form of $Q(k, x, V, \lambda, \tau)$ is given by

$$Q(k, x, V, \lambda, \tau) = e^{(-ik+1/2)X + A(k,\tau) + B(k,\tau)V + C(k,\tau) + D(k,\tau)\lambda} \tag{C.1}$$

where $X = \ln(S/K) + (r-d)\tau$ and other coefficients are specified with accordance to

1) Volatility

- Constant
  $A(k,\tau) \equiv 0, \;\; B(k,\tau) = -(k^2+1/4)\tau/2.$

- Deterministic
  $A(k,\tau) = -\frac{\theta}{2\kappa}(\kappa\tau - 1 + e^{-\kappa\tau})(k^2+\frac{1}{4}), \;\; B(k,\tau) = -\frac{1}{2\kappa}(1-e^{-\kappa\tau})(k^2+\frac{1}{4}).$

- Stochastic

$$A(k,\tau) = -\frac{\kappa\theta}{\varepsilon^2}\left[\psi_+\tau + 2\ln\left(\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right)\right], \; B(k,\tau) = -(k^2+1/4)\frac{1-e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}}.$$

$$\psi_\pm = \mp(u + ik\rho\varepsilon) + \zeta, \;\; \zeta = \sqrt{(k^2\varepsilon^2(1-\rho^2) + 2ik\rho\varepsilon u + u^2 + \varepsilon^2/4}, \;\; u = \kappa - \rho\varepsilon/2.$$

2) Jump Rate Intensity

- Constant
  $C(k,\tau) \equiv 0, \; D(k,\tau) = \tau\Lambda(k).$

- Deterministic

$$C(k,\tau) = \left(\kappa_\lambda\tau - 1 + e^{-\kappa_\lambda\tau}\right)\frac{\theta_\lambda\Lambda(k)}{\kappa_\lambda}, \;\; D(k,\tau) = \left(1 - e^{-\kappa_\lambda\tau}\right)\frac{\Lambda(k)}{\kappa_\lambda}.$$

- Stochastic

$$C(k,\tau) = -\frac{\kappa_\lambda\theta_\lambda}{\varepsilon_\lambda^2}\left[\chi_+\tau + 2\ln\left(\frac{\chi_- + \chi_+ e^{-\xi\tau}}{2\xi}\right)\right], \;\; D(k,\tau) = 2\Lambda(z)\frac{1-e^{-\xi\tau}}{\chi_- + \chi_+ e^{-\xi\tau}}$$

$$\chi_\pm = \mp\kappa_\lambda + \xi, \;\; \xi = \sqrt{\kappa_\lambda^2 - 2\varepsilon_\lambda^2\Lambda(k)}.$$

3) Jump Size Distribution

- Log-Normal

$$\Lambda(k) = e^{-ik(\nu+\delta^2/2)-(k^2-1/4)\delta^2/2+\nu/2} - 1 - (-ik+1/2)(e^{\nu+\delta^2/2} - 1).$$

- Double-Exponential

$$\Lambda(k) = \frac{p}{1-(-ik+1/2)\eta_u} + \frac{q}{1+(-ik+1/2)\eta_d} - 1 - (-ik+1/2)(\frac{p}{1-\eta_u} + \frac{q}{1+\eta_d} - 1).$$

C.2 The Black-Scholes-style Formula

Here, we report the expressions for characteristic functions for option pricing using the Black-Scholes-style formula (3.14). The characteristics $\phi_j(k)$ $(j = 1, 2)$ are determined using the expression for the MGF given by (3.21). A general form of $\phi_j(k)$ $(j = 1, 2)$ is given by

$$\phi_j(k) = e^{ikX+A(k,\tau)+B(k,\tau)V+C(k,\tau)+D(k,\tau)\lambda}. \tag{C.2}$$

Variables u, I, b are defined as follows

$$\text{if } j = 1: \ u = +1, \ I = 1, \ b = \kappa - \rho\varepsilon;$$
$$\text{if } j = 2: \ u = -1, \ I = 0, \ b = \kappa.$$

The coefficients in (C.2) are specified with accordance to
1) Volatility

- Constant

$$A(k,\tau) \equiv 0, \ B(k,\tau) = -(k^2 - uik)\tau/2.$$

- Deterministic
$A(k,\tau) = -\frac{\theta}{2\kappa}(\kappa\tau - 1 + e^{-\kappa\tau})(k^2 - uik), \ B(k,\tau) = -\frac{1}{2\kappa}(1 - e^{-\kappa\tau})(k^2 - uik).$

- Stochastic

$$A(k,\tau) = -\frac{\kappa\theta}{\varepsilon^2}\left[\psi_+\tau + 2\ln\left(\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right)\right], \ B(k,\tau) = -(k^2 - uik)\frac{1 - e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}},$$
$$\psi_\pm = \mp(b - \rho\varepsilon ik) + \zeta, \ \zeta = \sqrt{(\kappa - \rho\varepsilon ik)^2 + \varepsilon^2(k^2 - uik)}.$$

2) Jump Rate Intensity

- Constant

$$C(k,\tau) \equiv 0, \ D(k,\tau) = \tau\Lambda(k).$$

- Deterministic

$$C(k,\tau) = \left(\kappa_\lambda\tau - 1 + e^{-\kappa_\lambda\tau}\right)\frac{\theta_\lambda\Lambda(k)}{\kappa_\lambda}, \ D(k,\tau) = \left(1 - e^{-\kappa_\lambda\tau}\right)\frac{\Lambda(k)}{\kappa_\lambda}.$$

- Stochastic

$$C(k,\tau) = -\frac{\kappa_\lambda\theta_\lambda}{\varepsilon_\lambda^2}\left[\chi_+\tau + 2\ln\left(\frac{\chi_- + \chi_+ e^{-\xi\tau}}{2\xi}\right)\right], \ D(k,\tau) = 2\Lambda(k)\frac{1 - e^{-\xi\tau}}{\chi_- + \chi_+ e^{-\xi\tau}},$$
$$\chi_\pm = \mp\kappa_\lambda + \xi, \ \xi = \sqrt{\kappa_\lambda^2 - 2\varepsilon_\lambda^2\Lambda(k)}.$$

3) Jump Size Distribution

- Log-Normal

$$\Lambda(k) = e^{(\nu + I\delta^2/2)ik - \delta^2 k^2/2 + I(\nu+\delta^2/2)} - 1 - (ik + I)(e^{\nu+\delta^2/2} - 1).$$

- Double-Exponential

$$\Lambda(k) = \frac{p}{1-(ik+I)\eta_u} + \frac{q}{1+(ik+I)\eta_d} - 1 - (ik+I)(\frac{p}{1-\eta_u} + \frac{q}{1+\eta_d} - 1).$$

D Quality of Calibration for Selected Models

In Table 2, we report the implied volatility matrix of DAX (The Deutscher Aktienindex) options of July 5, 2002 and discount rates, which we used in model calibration. The spot price is $S = 4468.17$. DAX options are European type options and no dividend yield applies. Figures below illustrate DAX volatility surface and the quality of calibration for selected models.

Expiry	Jul-02	Aug-02	Sep-02	Dec-02	Mar-03	Jun-03	Dec-03	Jun-04
Time	2 weeks	1 month	3m	6m	9m	12m	18m	24m
Date	19-07-02	16-08-02	20-09-02	20-12-02	21-03-03	20-06-03	19-12-03	18-06-04
T	0.0389	0.1139	0.2083	0.4583	0.7111	0.9583	1.4556	1.9528
r	0.0357	0.0349	0.0341	0.0355	0.0359	0.0368	0.0386	0.0401
strike								
3400	0.6625	0.4875	0.4204	0.3667	0.3431	0.3267	0.3121	
3600	0.6007	0.4543	0.3967	0.3511	0.3279	0.3154	0.2984	0.2921
3800	0.5084	0.4221	0.3718	0.3327	0.3155	0.3027	0.2919	0.2880
4000	0.4541	0.3869	0.3492	0.3149	0.2963	0.2926	0.2819	0.2800
4200	0.4060	0.3607	0.3330	0.2999	0.2887	0.2811	0.2751	0.2775
4400	0.3726	0.3396	0.3108	0.2871	0.2788	0.2722	0.2661	0.2686
4500	0.3550	0.3277	0.3012	0.2781	0.2718	0.2661		
4600	0.3428	0.3209	0.2958	0.2740	0.2688	0.2627	0.2580	0.2620
4800	0.3302	0.3062	0.2799	0.2631	0.2573	0.2533	0.2504	0.2544
5000	0.3343	0.2959	0.2705	0.2540	0.2504	0.2464	0.2448	0.2462
5200	0.3460	0.2845	0.2624	0.2463	0.2425	0.2385	0.2373	0.2422
5400	0.3857	0.2860	0.2578	0.2399	0.2357	0.2327	0.2312	0.2351
5600	0.3976	0.2860	0.2607	0.2356	0.2297	0.2268	0.2241	0.2320

Table 2: DAX implied volatility matrix of July 5, 2002.

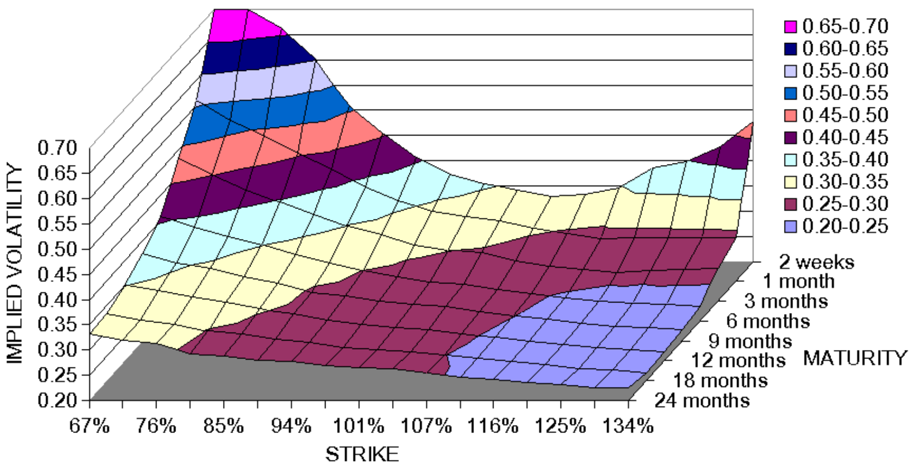

Figure 1: DAX implied volatility surface of July 5, 2002.

Jump-Diffusion with Stochastic Volatility

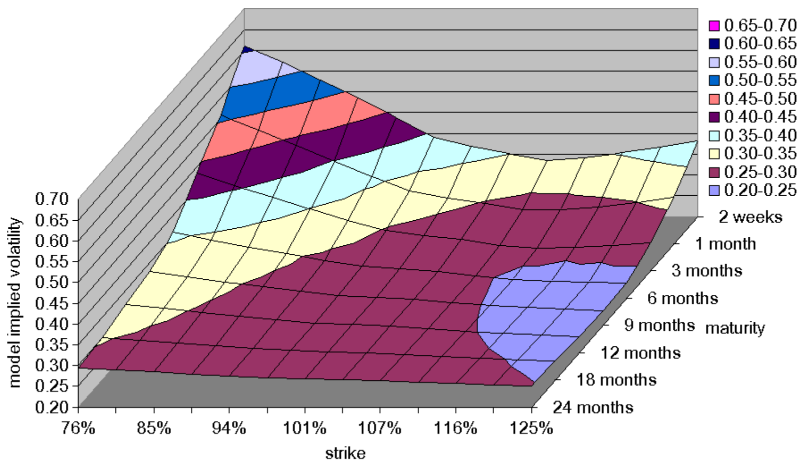

Figure 2: Model implied volatility.

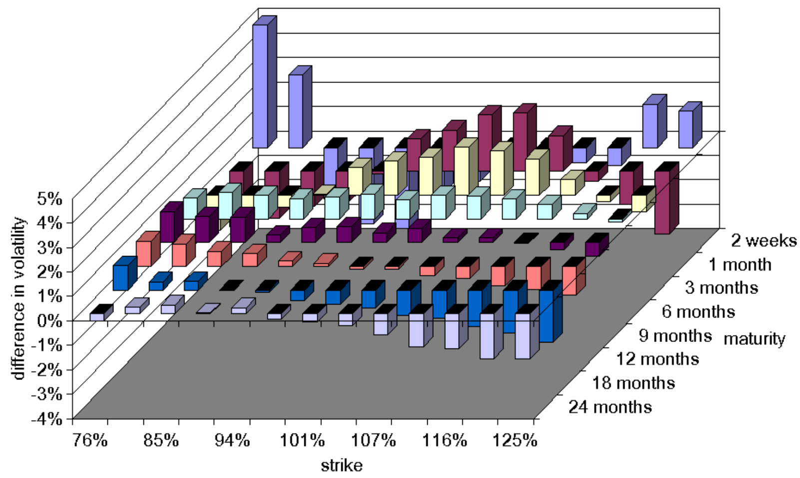

Figure 3: Model differences in implied volatility.

The Heston model is capable of matching middle-term and longer-term maturities. The model cannot fit shorter maturities in a proper fashion.

Log-Normal Jump-Diffusion with Deterministic Volatility and Intensity

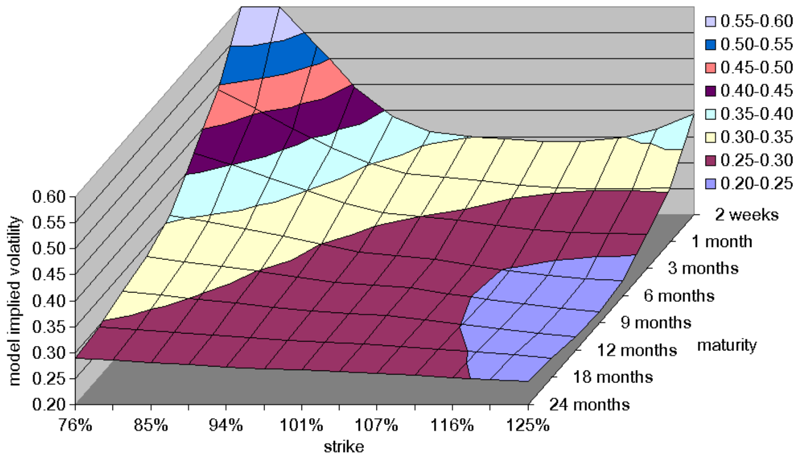

Figure 4: Model implied volatility.

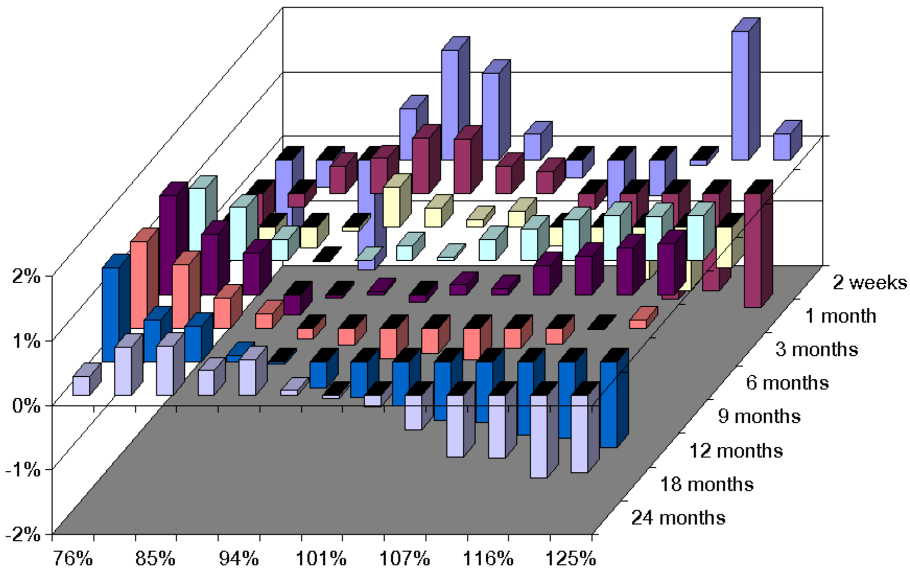

Figure 5: Model differences in implied volatility.

The Merton jump-diffusion with deterministic volatility and intensity is capable of fitting shorter-term as well as longer-term maturities.

Log-Normal Jump-Diffusion with Stochastic Volatility

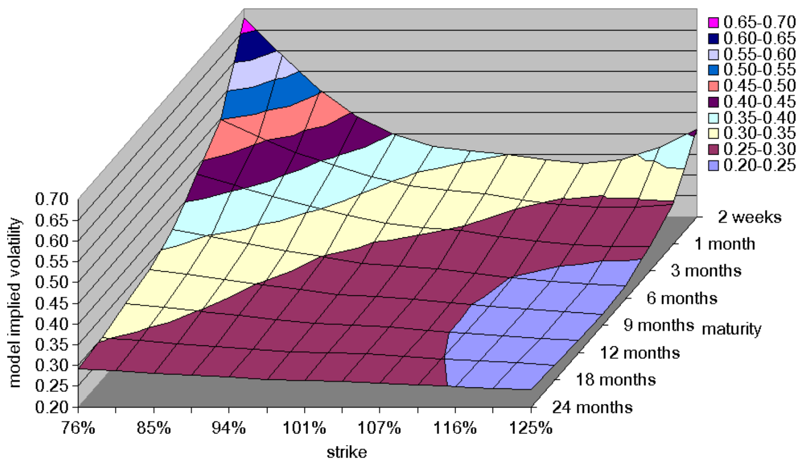

Figure 6: Model implied volatility.

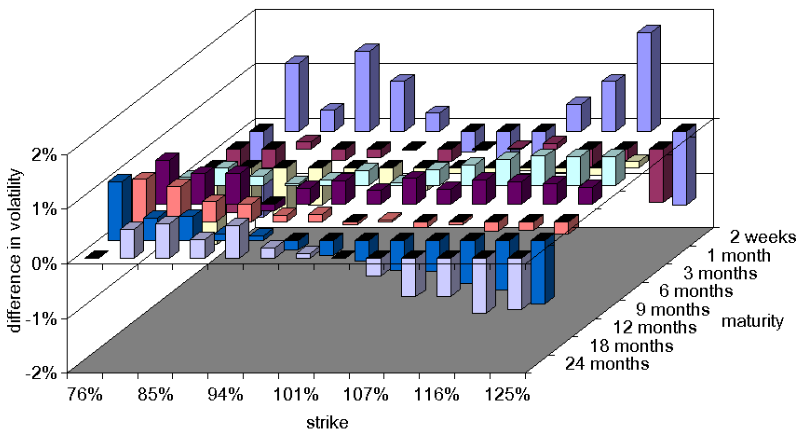

Figure 7: Model differences in implied volatility.

The Bates model leads to a perfect fit to all maturities.