Higher order processes in electromagnetic production of electron positron pairs in relativistic heavy ion collisions

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Abstract

We study higher-order effects in the electromagnetic production of electron-positron pairs in relativistic heavy ion collisions. Treating the field of the heavy ions as an external field and neglecting the interaction among electrons and positrons, we show that the N-pair creation amplitude is the antisymmetrised product of N one-pair creation amplitudes and the vacuum amplitude. Neglecting contributions coming from exchange terms, we show that the total probability for N pairs is approximately a Poisson distribution. We investigate further the structure of the reduced one-pair amplitude, concentrating especially on multiple-particle corrections. We calculate the first of these corrections in second order Magnus theory based on our previous result in second-order Born approximation for impact parameter b zero. Explicit calculations show that the total probability is increased up to 10 % by this correction for realistic collider parameters. The calculations can also be used to confirm the use of the Poisson distribution for the total probability.

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I. INTRODUCTION

The electromagnetic production of electron-positron pairs in relativistic heavy ion collisions has gained some interest recently due to the observation that the total probability violates unitarity if calculated in lowest order even for realistic energies and impact parameters as large as the Compton wavelength [1]. It can be expected therefore that higher-order effects — especially the multiple-pair production — are of importance. Several models have been used in order to cure the unitarity violation and predict multiple-pair probabilities, based on the summation of some subclasses of diagrams [2–5]. All authors find that the pair multiplicity can be approximated by a Poisson distribution and that the lowest-order results should be interpreted as the pair multiplicity. Unitarity is no longer violated due to the inclusion of the vacuum amplitude, which reduces all N-pair probabilities by the same factor. Recently Ionescu [6] has proposed a model, which differs from the previous ones, but which is not applicable for multiple-pair production.

We are going to extend these existing models into two directions: First we address the question how the N-pair creation amplitude is exactly reducible to one-pair creation amplitudes and what approximations are necessary in order to get the Poisson distribution. We then compare with earlier results. Secondly we study higher-order processes contributing to the lowest-order one-pair creation amplitude. We concentrate on multiple-particle effects, which have not been included in previous models.

This paper is arranged as follows: In Sec. II we show that the N-pair creation amplitude can be written as an antisymmetrised product of reduced one-pair creation amplitudes and the vacuum amplitude. For this we use Feynman boundary conditions for the fermion field. Neglecting contributions from exchange terms in the calculation of the total probability we get a Poisson distribution for the total N-pair probability quite easily. In Sec. III the same result is derived using the perturbation expansion of the S-matrix, which gives us some insight into the processes contributing to the reduced pair creation amplitude and the vacuum amplitude. Finally we compare in Sec. IV our results with other existing ones.

In Sec. V we derive the general form of the S-operator in second-order Magnus theory and use it in Sec. VI in order to extract from this the lowest-order multiple-particle correction to the one-pair creation probability. This correction is then calculated in Sec. VII based on our analytic form of the differential pair creation amplitude in second-order Born approximation for impact parameter b zero [7]. The same calculation is used also to study the deviation of the total two-pair creation probability from the Poisson distribution, which is found to be rather small. Results of these calculations together with conclusions are then summarized in Sec. VIII.

In the Appendix, we show a way to calculate the coefficients appearing in the Magnus theory expansion, which can therefore be calculated to arbitrary order.

Throughout this paper, we treat the fields of the heavy ions as external fields. We neglect the electromagnetic interaction between electrons and positrons. This seems to be justified for almost all fermions, apart from electrons and positrons, which are produced with very low relative velocitiy, and even in a bound e^+–e^- state, due to the smallness of the coupling constant α between the fermions compared to the effective coupling constant $Z\alpha$ for the interaction with the external field. Due to these assumptions, our many particle system is essentially a system of independent fermions in an external field. For such a system the S

operator is known to be of the form of a time-ordered exponential

$$\begin{aligned} S &= \mathcal{T} \exp\left\{-i \int_{-\infty}^{+\infty} H_I(t) dt\right\} \\ &= \mathcal{T} \exp\left\{\int d^4x : \bar{\Psi}(x)[-ie \not{A}(x)]\Psi(x) :\right\}, \end{aligned} \tag{1}$$

where H_I is the Hamiltonian in the interaction picture. This form of the S operator is the starting point for our calculations.

The general theory of a fermion interacting with an external electromagnetic field is rather old and has been developed already from the beginning of QED. It has been developed especially by Feynman [8] and Schwinger [9]. An overview can also be found in [10–12].

Although we discuss only the electromagnetic pair production by heavy ion collisions, the general theory can also be applied to other pair production processes in external fields, for example, pair creation due to bremsstrahlung [13].

II. REDUCTION USING FEYNMAN BOUNDARY CONDITIONS

The amplitude for the N-pair creation is given with the help of the S operator above as

$$S_{fi} = \langle f | \, S \, | i \rangle \, . \tag{2}$$

As we want to calculate pair creation, the initial state is the vacuum state $|0\rangle$ and the final state an N-pair state given by

$$|f\rangle = b_{k_1}^+ d_{l_1}^+ \cdots b_{k_N}^+ d_{l_N}^+ \, |0\rangle \, , \tag{3}$$

where k_i denotes the electron quantum numbers, for example, momentum and spin, and l_i the same for the positrons. The S matrix is therefore

$$S_N := S(k_1, l_1 \cdots, k_N, l_N) = \langle 0 | \, d_{l_N} b_{k_N} \cdots d_{l_1} b_{k_1} S \, |0\rangle \, . \tag{4}$$

If we neglect the interaction of the electrons and positrons among each other, we are essentially dealing with a system of independent particles. Therefore the field operator at any given time is linearly connected to its value at the boundary. As we can decompose the field operator into creation and annihilation operators, the same is also true for them.

Using retarded boundary conditions this means that the creation and annihilation operators for electrons and positrons at $t \to +\infty$ are connected through a unitary matrix with those at $t \to -\infty$. In the interaction picture this transformation is expressed as

$$b_p S = \sum_{i>0} a_{pi} \; S b_i + \sum_{j<0} a_{pj} \; S d_j^+, \tag{5a}$$

$$d_q S = \sum_{i>0} a_{qi}^* \; S b_i^+ + \sum_{j<0} a_{qj}^* \; S d_j. \tag{5b}$$

The main idea is now not to use these boundary conditions but Feynman boundary conditions instead. These are of a mixed retarded and advanced type. With these the

electron creation operators for $t \to +\infty$ are connected with the electron creation operators for $t \to -\infty$ and the positron annihilation operators for $t \to +\infty$. This corresponds to the Stückelberg-Feynman interpretation of electrons moving forward, positrons moving backwards in time. Again in the interaction picture this is written as [11,14,15]

$$b_p S = \sum_{q>0} s^{++}_{pq} S b_q + \sum_{q'<0} s^{+-}_{pq'} d^+_{q'} S. \tag{6}$$

It has been shown already by Feynman that these boundary conditions are completely equivalent to the retarded ones, but that they are better suited for QED [8]. For this relation to be applicable, we assume also that the external field vanishes asymptotically.

Similar relations exist for all other creation and annihilation operators also; Eq. (6) is the only relation we will need in the following. Applying it to b_{k_1} we get for S_N:

$$\begin{aligned} S_N = &\langle 0| \, d_{l_N} b_{k_N} \cdots b_{k_2} d_{l_1} \\ &\times \left[\sum_{q_1>0} s^{++}_{k_1 q_1} S b_{q_1} + \sum_{q'_1<0} s^{+-}_{k_1 q'_1} d^+_{q'_1} S \right] |0\rangle \, . \end{aligned} \tag{7}$$

The b_{q_1} operator annihilates the vacuum state, therefore we drop this term:

$$\begin{aligned} S_N &= \langle 0| \, d_{l_N} b_{k_N} \cdots b_{k_2} d_{l_1} \sum_{q'_1<0} s^{+-}_{k_1 q'_1} d^+_{q'_1} S \, |0\rangle \\ &= \sum_{q'_1<0} s^{+-}_{k_1 q'_1} \langle 0| \, d_{l_N} b_{k_N} \cdots b_{k_2} d_{l_1} d^+_{q'_1} S \, |0\rangle \, . \end{aligned} \tag{8}$$

Using the fact that b_{k_2} anticommutes with d_{l_1} and $d^+_{q'_1}$ we get

$$S_N = \sum_{q'_1<0} s^{+-}_{k_1 q'_1} \langle 0| \, d_{l_N} b_{k_N} \cdots d_{l_1} d^+_{q'_1} b_{k_2} S \, |0\rangle \, . \tag{9}$$

Now one replaces b_{k_2} using again Eq. (6) and then commutes the next b_{k_i} to the most right. Doing this for all b_{k_i} we finally get:

$$\begin{aligned} S_N = &\sum_{q'_1 \cdots q'_N<0} s^{+-}_{k_1 q'_1} \cdots s^{+-}_{k_N q'_N} \\ &\times \langle 0| \, d_{l_N} \cdots d_{l_1} d^+_{q'_1} \cdots d^+_{q'_N} S \, |0\rangle \, . \end{aligned} \tag{10}$$

The vacuum expectation value can be calculated easily using the anticommutation relations

$$\left\{ d_{l_i}, d^+_{q'_j} \right\} = \delta_{l_i, q'_j} \tag{11}$$

in order to move $d^+_{q'_1}$ to the left, until it reaches the left side, where it annihilates the vacuum. Doing this for all creation operators we see that we get a nonvanishing result only, if every q'_i is equal to some l_j, and the overall sign of the vacuum expectation value is equal to the signum of the permutation to arrange $l_1 \cdots l_N$ in the same sequence as the corresponding $q'_1 \cdots q'_N$. We get

$$\langle 0| d_{l_N} \cdots d_{l_1} d^+_{q'_1} \cdots d^+_{q'_N} S |0\rangle = \langle 0| S |0\rangle \sum_\sigma \mathrm{sgn}(\sigma) \delta(l_{\sigma(1)}, q'_1) \cdots \delta(l_{\sigma(N)}, q'_N) \tag{12}$$

and therefore for S_N:

$$S_N(k_1, l_1, \cdots, k_N, l_N) = \langle 0| S |0\rangle \sum_\sigma \mathrm{sgn}(\sigma) s^{+-}_{k_1 l_{\sigma(1)}} \cdots s^{+-}_{k_N l_{\sigma(N)}}, \tag{13}$$

where the sum consists of $N!$ terms. This can also be written formally as

$$S_N = \langle 0| S |0\rangle \det\left[s^{+-}_{k_i l_j}\right]. \tag{14}$$

This is the main result, which we will use. The N-pair creation amplitude can be written as an antisymmetrised product of the s^{+-}_{kl} times the vacuum amplitude. It is easy to see that the s^{+-}_{kl} are just the one-pair creation amplitudes divided by the vacuum amplitude. In the following we call this amplitude the "reduced" amplitude.

Normally the vacuum amplitude does not show up in calculations, as it is often of magnitude one, for example, for time independent systems. In our case the external fields are explicitly time dependent, therefore the vacuum amplitude can not be neglected. This can be understood easily, if one notices that the vacuum amplitude is the amplitude for the "no pair creation process", that is, the probability amplitude for the vacuum to remain the vacuum. As pair creation does occur, this probability must be smaller than one and therefore the amplitude has to be of magnitude less than one. Also we see here that the vacuum amplitude appears as a factor in all N-pair amplitudes and therefore reduces all these amplitudes by the same factor.

From this we can get easily an approximate Poisson distribution for the total N-pair probability, which has also been found by earlier calculations. For this we need the absolute value squared of S_N. As S_N consists of $N!$ terms, we get a total of $(N!)^2$ terms. In most of these terms we have the product coming from two different permutations. Normally the produced electrons and positrons are correlated to some extent with each other. If we therefore assume that terms, where two different permutations appear, are much smaller than those with the same permutation and can therefore be neglected, we get only $N!$ terms:

$$P(N) \approx |\langle 0| S |0\rangle|^2 \sum_\sigma \left|s^{+-}_{k_1 l_{\sigma(1)}}\right|^2 \cdots \left|s^{+-}_{k_N l_{\sigma(N)}}\right|^2. \tag{15}$$

Summing or integrating now over all states we get for the total probability

$$P_{\mathrm{total}}(N) \approx \frac{1}{(N!)^2} |\langle 0| S |0\rangle|^2 \times \sum_{k_1, l_1, \cdots, k_N, l_N} \sum_\sigma \left|s^{+-}_{k_1 l_{\sigma(1)}}\right|^2 \cdots \left|s^{+-}_{k_N l_{\sigma(N)}}\right|^2, \tag{16}$$

where the factor $1/(N!)^2$ has been introduced in order to correct for the multiple summation over electrons and positrons. As every term in the sum over σ gives the same result, we get

$$
\begin{aligned}
P_{\text{total}}(N) &\approx \frac{1}{N!} \left|\langle 0| S |0\rangle\right|^2 \sum_{k_1,l_1} \left|s^{+-}_{k_1 l_1}\right|^2 \cdots \sum_{k_N,l_N} \left|s^{+-}_{k_N l_N}\right|^2 \\
&= \frac{1}{N!} \left|\langle 0| S |0\rangle\right|^2 \left[\sum_{k,l} \left|s^{+-}_{kl}\right|^2\right]^N \\
&= P(0) \frac{\left[P^R_{\text{total}}\right]^N}{N!},
\end{aligned} \tag{17}
$$

with $P(0)$ the "no pair" probability and $P^R_{\text{total}} = \sum_{k,l} \left|s^{+-}_{kl}\right|^2$ the total reduced one-pair probability. This is just a Poisson distribution, and $P(0)$ can be calculated using the fact that the sum over all N has to be one:

$$
P(0) \sum_N \frac{\left[P^R_{\text{total}}\right]^N}{N!} = P(0) \exp\left[P^R_{\text{total}}\right] = 1. \tag{18}
$$

From this we finally get

$$
P_{\text{total}}(N) = \exp\left[-P^R_{\text{total}}\right] \frac{\left[P^R_{\text{total}}\right]^N}{N!}. \tag{19}
$$

The only approximation that was necessary in order to get this result was the neglection of all "exchange terms" in the calculation of the probability. We show later that at least for the two-pair creation this neglection seems to be justified.

Therefore for the calculation of the N-pair creation probability it suffices to calculate the reduced one-pair creation probability. All other probabilities are given then by the Poisson distribution. If one wants to go beyond the Poisson distribution, it suffices to calculate the reduced one-pair creation amplitude and the vacuum amplitude.

This derivation is rather simple, but it is not useful for explicit calculations. Especially the nature of the processes contributing to the reduced pair creation amplitude as well as the vacuum amplitude remains unexplained. Therefore we derive the same result in the next section using the perturbation-theory expansion of the S operator together with the Wick theorem.

III. REDUCTION USING PERTURBATION THEORY

In this section we use the field operator $\Psi(x)$ directly instead of the creation and annihilation operators as in the previous section, that is, we are going to calculate

$$
S_N(y_1, \cdots, y_N, y'_1, \cdots, y'_N) = \langle f| S |i\rangle , \tag{20}
$$

where the initial state is again the vacuum $|0\rangle$, and the final state is

$$
|f\rangle = \bar{\Psi}^{(+)}(y_N)\Psi^{(-)}(y'_N) \cdots \bar{\Psi}^{(+)}(y_1)\Psi^{(-)}(y'_1) |0\rangle . \tag{21}
$$

Here we have made use of the frequency parts of the field operators for electrons and positrons. The electron coordinates are y_i, the positron coordinates y'_i and the time coordinate is the same for all coordinates and assumed to go to ∞. S_N gives the probability

amplitude in occupation number space to get electrons finally at y_i and positrons at y'_i. We rewrite the final state a little bit by introducing normal ordering of the field operators:

$$\langle f| = \langle 0| : \bar{\Psi}(y'_1)\Psi(y_1)\cdots\bar{\Psi}(y'_N)\Psi(y_N) : , \tag{22}$$

where we can use the whole field operators instead of the frequency parts, as the normal ordering together with the vacuum state guarantees that only the appropriate part of each field operator contributes. Throughout the calculation in this section the Dirac indices will be suppressed. In the final form we arrange the operators in the appropriate form, which is not always possible at intermediate steps, but generally Dirac indices can be looked at, as if they are incorporated into the coordinates.

Using the form of the S matrix from Eq. (1) we get

$$\begin{aligned} S(y_1,\cdots,y_N,y'_1,\cdots,y'_N) = \\ \langle 0| : \bar{\Psi}(y'_1)\Psi(y_1)\cdots\bar{\Psi}(y'_N)\Psi(y_N) : \\ \times\mathcal{T}\exp\left\{\int d^4x \ : \bar{\Psi}(x)[-ie\,\not{A}(x)]\Psi(x) :\right\}|0\rangle . \end{aligned} \tag{23}$$

As the time, when we detect electrons and positrons, is ∞, the time ordering can be extended to the whole expression.

We use now the Wick theorem in the following form [16], where O_i is an arbitrary fermionic operator:

$$\begin{aligned} \langle 0|\,\mathcal{T}O_1\cdots O_M\,|0\rangle = \sum_{j,(j\neq i)} \mathrm{sgn}(\pi(j,i))\,\langle 0|\,\mathcal{T}O_jO_i\,|0\rangle \\ \times\langle 0|\,\mathcal{T}O_1\cdots\widehat{O_j}\cdots\widehat{O_i}\cdots O_M\,|0\rangle . \end{aligned} \tag{24}$$

Here i is a fixed index, $\pi(j,i)$ is the permutation to put O_j and O_i to the left of all other operators, and the hat in the vacuum expectation value means that these operators have been removed from the expression. In our case the O_i are either $\bar{\Psi}$ or Ψ and we can use the fact that

$$\langle 0|\,\mathcal{T}\Psi(x)\Psi(y)\,|0\rangle = \langle 0|\,\mathcal{T}\bar{\Psi}(x)\bar{\Psi}(y)\,|0\rangle = 0 \tag{25}$$

and

$$\langle 0|\,\mathcal{T}\Psi(x)\bar{\Psi}(y)\,|0\rangle = iS_F(x-y), \tag{26}$$

where S_F is the usual Feynman propagator. As we have also normal-ordered products, the sum over j in this case also excludes those operators, which are in the same normal-ordered product as the O_i.

In a first step we apply Eq. (24) to $\Psi(y_N)$ in the expansion of S_N:

$$\begin{aligned} I_M := &\int dx_1\cdots dx_M\,\langle 0|\,\mathcal{T} : \bar{\Psi}(y'_1)\Psi(y_1)\cdots\bar{\Psi}(y'_N)\Psi(y_N) : \\ &\times : \bar{\Psi}(x_1)\,[-ie\,\not{A}(x_1)]\,\Psi(x_1) : \cdots : \bar{\Psi}(x_M)\,[-ie\,\not{A}(x_M)]\,\Psi(x_M) : |0\rangle \\ = &\int dx_1\cdots dx_M\ \sum_{i=1}^{M}\langle 0|\,\mathcal{T}\Psi(y_N)\bar{\Psi}(x_i)\,|0\rangle\,[-ie\,\not{A}(x_i)] \\ &\times\langle 0|\,\mathcal{T} : \bar{\Psi}(y'_1)\Psi(y_1)\cdots\bar{\Psi}(y'_N) : \Psi(x_i) : (1) : \cdots : \widehat{(i)} : \cdots : (M) : |0\rangle \\ = &\ M\int dz_1\ dx_1\cdots dx_{M-1}\ S_F(y_N-z_1)e\,\not{A}(z_1) \\ &\times\langle 0|\,\mathcal{T} : \bar{\Psi}(y'_1)\Psi(y_1)\cdots\bar{\Psi}(y'_N) : \Psi(z_1) : (1) : \cdots : (M-1) : |0\rangle , \end{aligned} \tag{27}$$

where we have used the fact that each term in the sum is identical after an even permutation and a renumbering of the variables x_i. This integral is just one term in the expansion of the exponential in S and applying this reduction to every term we get

$$\begin{aligned} S_N = & \int dz_1 \; S_F(y_N - z_1) e \not{A}(z_1) \, \langle 0 | \, \mathcal{T} : \bar{\Psi}(y_1') \Psi(y_1) \cdots \bar{\Psi}(y_N') : \Psi(z_1) \\ & \times \exp\left\{ \int dx \;\; : \bar{\Psi}(x) \left[-ie \not{A}(x) \right] \Psi(x) : \right\} |0\rangle \, . \end{aligned} \tag{28}$$

In a next step we apply the Wick theorem to $\Psi(z_1)$ and get similar as in the previous case for the integral

$$\begin{aligned} I_M' := & \int dx_1 \cdots dx_M \, \langle 0 | \, \mathcal{T} : \bar{\Psi}(y_1') \Psi(y_1) \cdots \bar{\Psi}(y_N') : \Psi(z_1) : (1) : \cdots : (M) : |0\rangle \\ = & \sum_{i=1}^{N} -i S_F(z_1 - y_i') \int dx_1 \cdots dx_M \\ & \times \langle 0 | \, \mathcal{T} : \bar{\Psi}(y_1') \Psi(y_1) \cdots \widehat{\bar{\Psi}(y_i')} \Psi(y_i) \cdots \bar{\Psi}(y_N') : : (1) : \cdots : (M) : |0\rangle \\ & + M \int dz_2 \; dx_1 \cdots dx_{M-1} \; S_F(z_1 - z_2) e \not{A}(z_2) \\ & \times \langle 0 | \, \mathcal{T} : \bar{\Psi}(y_1') \Psi(y_1) \cdots \bar{\Psi}(y_N') : \Psi(z_2) : (1) : \cdots : (M-1) : |0\rangle \, , \end{aligned} \tag{29}$$

which is again a term in the expansion of S_N, so that our amplitude is

$$\begin{aligned} S_N = & \sum_{i=1}^{N} -i \int dz_1 \; S_F(y_N - z_1) e \not{A}(z_1) S_F(z_1 - y_i') \\ & \times \langle 0 | \, \mathcal{T} : \bar{\Psi}(y_1') \Psi(y_1) \cdots \widehat{\bar{\Psi}(y_i')} \Psi(y_i) \cdots \bar{\Psi}(y_N') : \\ & \times \exp\left\{ \int dx \;\; : \bar{\Psi}(x) \left[-ie \not{A}(x) \right] \Psi(x) : \right\} |0\rangle \\ & + \int dz_1 dz_2 \; S_F(y_N - z_1) e \not{A}(z_1) S_F(z_1 - z_2) e \not{A}(z_2) \\ & \times \langle 0 | \, \mathcal{T} : \bar{\Psi}(y_1') \Psi(y_1) \cdots \bar{\Psi}(y_N') : \Psi(z_2) \exp\left\{ \int dx \;\; : \bar{\Psi}(x) \left[-ie \not{A}(x) \right] \Psi(x) : \right\} |0\rangle \, . \end{aligned} \tag{30}$$

The vacuum expectation value in the first term is just the $(N-1)$-pair creation amplitude, that is,

$$\begin{aligned} S_N = & \sum_{i=1}^{N} (-i) \int dz_1 \; S_F(y_N - z_1) e \not{A}(z_1) S_F(z_1 - y_i') \\ & \times \mathrm{sgn}(\sigma) S_{N-1}(y_1, \cdots, y_{N-1}, y_1', \cdots, \widehat{y_i'}, \cdots, y_N') \\ & + \int dz_1 dz_2 \; S_F(y_N - z_1) e \not{A}(z_1) S_F(z_1 - z_2) e \not{A}(z_2) \\ & \times \langle 0 | \, \mathcal{T} : \bar{\Psi}(y_1') \Psi(y_1) \cdots \bar{\Psi}(y_N') : \Psi(z_2) \exp\left\{ \int dx \;\; : \bar{\Psi}(x) \left[-ie \not{A}(x) \right] \Psi(x) : \right\} |0\rangle \, , \end{aligned} \tag{31}$$

where σ is the permutation to rearrange the $\bar{\Psi}(y_j')$ and $\Psi(y_j)$ to the standard form. We can use Eq. (29) again for the vacuum expectation value in the second term. Doing this recursively and assuming that the series converges we finally get

$$
\begin{aligned}
S_N = \sum_{i=1}^{N} \mathrm{sgn}(\sigma) S_{N-1}(y_1, \cdots, y_{N-1}, y_1', \cdots, \widehat{y_i'}, \cdots, y_N') \\
\times (-i) \Big[\int dz_1 \; S_F(y_N - z_1) e \not{A}(z_1) S_F(z_1 - y_i') \\
+ \int dz_1 dz_2 \; S_F(y_N - z_1) e \not{A}(z_1) S_F(z_1 - z_2) e \not{A}(z_2) S_F(z_2 - y_i') \\
+ \cdots \Big] .
\end{aligned} \tag{32}
$$

The infinite series is just the normal perturbation theory expansion for the pair creation without any disconnected parts. We identify it with the reduced one-pair creation amplitude $S^R(y_N, y_i')$. Therefore we have found a way to reduce the N-pair amplitude to a $(N-1)$-pair amplitude. Recursively we can write for S_N:

$$
\begin{aligned}
S_N(y_1, \cdots, y_N, y_1', \cdots, y_N') = \\
S_0 \sum_{\sigma} \mathrm{sgn}(\sigma) S^R(y_1, y'_{\sigma(1)}) \cdots S^R(y_N, y'_{\sigma(N)})
\end{aligned} \tag{33}
$$

with the vacuum amplitude

$$
\begin{aligned}
S_0 &= \langle 0 | \, \mathcal{T} \exp \left\{ \int dx \; : \bar{\Psi}(x) [-ie \not{A}(x)] \Psi(x) : \right\} | 0 \rangle \\
&= \langle 0 | \, S \, | 0 \rangle \, .
\end{aligned} \tag{34}
$$

This is identical to the result of the last section. Therefore we have found an expression for S^R, that is, for s^{+-}. It is just the perturbation theory expansion of a fermion interacting an arbitrary number of times with an external field without any disconnected parts.

Finally we want to find also a perturbation theory expression for S_0. Expanding the exponential we get

$$
S_0 = \sum_{M=0}^{\infty} \frac{1}{M!} \tilde{I}_M , \tag{35}
$$

with $\tilde{I}_M$ defined as

$$
\begin{aligned}
\tilde{I}_M := \int dx_1 \cdots dx_M \\
\times \langle 0 | \, \mathcal{T} : \bar{\Psi}(x_1) \left[-ie \not{A}(x_1) \right] \Psi(x_1) : \cdots : (M) : | 0 \rangle \, .
\end{aligned} \tag{36}
$$

We are now using the Wick theorem again in order to reduce this expression. Applying it to $\Psi(x_1)$ we get

$$
\begin{aligned}
\tilde{I}_M = (-i)(M-1) \int dx_1 dx_2 \cdots dx_M \\
\times e \not{A}(x_1) S_F(x_1 - x_2) e \not{A}(x_2) \\
\times \langle 0 | \, \mathcal{T} \bar{\Psi}(x_1) \Psi(x_2) : (3) : \cdots : (M) : | 0 \rangle \, .
\end{aligned} \tag{37}
$$

Similar we get for the following expression

$$\begin{aligned}\tilde{J}_M(x_1,x_2) &:= \int dx_3\cdots dx_M\,\langle 0|\,\mathcal{T}\bar{\Psi}(x_1)\Psi(x_2):\bar{\Psi}(x_3)\,[-ie\,\not{A}(x_3)]\,\Psi(x_3):\cdots:(M):|0\rangle \\ &= -iS_F(x_2-x_1)\int dx_3\cdots dx_M\,\langle 0|\,\mathcal{T}:(3):\cdots:(M):|0\rangle \\ &\quad +(M-2)\int dx_3dx_4\cdots dx_M\;iS_F(x_2-x_3)\,[-ie\,\not{A}(x_3)] \\ &\quad \times\langle 0|\,\mathcal{T}\bar{\Psi}(x_1)\Psi(x_3):(4):\cdots:(M):|0\rangle\,. \end{aligned} \tag{38}$$

That is,

$$\tilde{I}_M = -i(M-1)\int dx_1dx_2\;e\,\not{A}(x_1)S_F(x_1-x_2)e\,\not{A}(x_2)\tilde{J}_M(x_1,x_2) \tag{39}$$

and

$$\tilde{J}_M(x_1,x_2) = -iS_F(x_2-x_1)\tilde{I}_{M-2} + (M-2)\int dx_3\;S_F(x_2-x_3)e\,\not{A}(x_3)\tilde{J}_{M-1}(x_1,x_3). \tag{40}$$

Using both relations in order to reduce $\tilde{I}_M$ in terms of $\tilde{I}_k$ we get

$$\begin{aligned}\tilde{I}_M = &(M-1)\;C_2\tilde{I}_{M-2} + (M-1)(M-2)\;C_3\tilde{I}_{M-3} \\ &+\cdots+(M-1)\cdots 2\;C_{M-1}\tilde{I}_1 \\ &+(M-1)\cdots 2\;1\;C_M\tilde{I}_0,\end{aligned} \tag{41}$$

with C_N defined as

$$\begin{aligned}C_N := (-1)\int dx_1\cdots dx_N \\ \mathrm{Tr}\Big[e\,\not{A}(x_1)S_F(x_1-x_2)e\,\not{A}(x_2)\cdots \\ \times e\,\not{A}(x_N)S_F(x_N-x_1)\Big],\end{aligned} \tag{42}$$

writing explicitly the Dirac indices to get the trace. For ease of writing we define also $C_0 = C_1 = 0$ and use that $\tilde{I}_1 = 0$ and $\tilde{I}_0 = 1$. With these we can write more compactly

$$\begin{aligned}\tilde{I}_M &= \sum_{k=1}^{M} C_k\tilde{I}_{M-k}\frac{(M-1)!}{(M-k)!} \\ &= \frac{M!}{M}\sum_{k=1}^{M} C_k\frac{\tilde{I}_{M-k}}{(M-k)!}.\end{aligned} \tag{43}$$

This reduction formula can now be used in order to find an expression for S_0. For this we write $\tilde{I}_M$ in a series ordered by the number of C_i's (that is, as we will see later, the number of loops) that appear in each term after a complete reduction. We see that the formula above just gives us a recipe, how the next order can be derived from the lower one. Only one term with no C_i in it exists:

$$\tilde{I}_0^{(0)} = 1, \tag{44}$$

giving therefore in zeroth order for S_0:

$$S_0^{(0)} = 1. \tag{45}$$

Using this only nonzero $\tilde{I}_M^{(0)}$ in Eq. (43) we get in first order only a contribution if $k = M$

$$\tilde{I}_M^{(1)} = \frac{M!}{M} C_M \tag{46}$$

and for S_0

$$S_0^{(1)} = \sum_{M=0}^{\infty} \frac{1}{M!} \tilde{I}_M^{(1)} = \sum_{M=1}^{\infty} \frac{C_M}{M}. \tag{47}$$

In second order we get

$$\begin{aligned}
\tilde{I}_M^{(2)} &= \frac{M!}{M} \sum_{k=1}^{M} C_k \frac{\tilde{I}_{M-k}^{(1)}}{(M-k)!} = \frac{M!}{M} \sum_{k=1}^{M} k \frac{C_k}{k} \frac{C_{M-k}}{M-k} \\
&= \frac{M!}{M} \frac{1}{2} \sum_{k=1}^{M} \left[k \frac{C_k}{k} \frac{C_{M-k}}{M-k} + (M-k) \frac{C_{M-k}}{M-k} \frac{C_k}{k} \right] \\
&= \frac{M!}{2} \sum_{k=1}^{M} \frac{C_k}{k} \frac{C_{M-k}}{M-k},
\end{aligned} \tag{48}$$

where we have changed the summation index from k to $M-k$ in one step. For S_0 we get

$$\begin{aligned}
S_0^{(2)} &= \sum_{M=0}^{\infty} \frac{1}{M!} \tilde{I}_M^{(2)} = \frac{1}{2} \sum_{M=0}^{\infty} \sum_{k=1}^{M} \frac{C_k}{k} \frac{C_{M-k}}{M-k} \\
&= \frac{1}{2} \left[\sum_{k=1}^{\infty} \frac{C_k}{k} \right] \left[\sum_{l=1}^{\infty} \frac{C_l}{l} \right] = \frac{1}{2} \left[S_0^{(1)} \right]^2.
\end{aligned} \tag{49}$$

Using the same transformations we can express also all higher terms of S_0, for which we get

$$S_0^{(n)} = \frac{1}{n!} \left[S_0^{(1)} \right]^n, \tag{50}$$

so that we finally get the vacuum amplitude as

$$\begin{aligned}
S_0 = \exp\left[S_0^{(1)} \right] = \exp\Big\{ &- \sum_{M=2}^{\infty} \frac{1}{M} \int dx_1 \cdots dx_M \\
&\times \mathrm{Tr}\left[e \not{A}(x_1) S_F(x_1 - x_2) \cdots e \not{A}(x_M) S_F(x_M - x_1) \right] \Big\}.
\end{aligned} \tag{51}$$

This is a well known result [9–11,14]. Its interpretation is straightforward. The sum in the exponential is just the sum over all single loops, whereas the exponential accounts for the fact that we can have also two or more loops.

With this we have found now perturbation-theory expressions for all terms occurring in the reduction formula Eq. (33), that is, for the reduced one-pair amplitude and the vacuum amplitude.

A final remark has to be made about the vacuum amplitude S_0. Already Schwinger found out that the expression for S_0 is not finite due to infinities in the imaginary part of $S_0^{(1)}$ [9](see also [12]). But he also mentioned that in the total probabilities we need only the absolute value squared of S_0, where only the real part of $S_0^{(1)}$ contributes, which is finite.

IV. COMPARISON WITH EARLIER RESULTS

The results of the previous two sections suggest the following picture of the pair production: In the Feynman picture the N-pair production can be described by two forms of processes (Fig. 1). N positron lines enter the interaction region coming from the future. They interact with the external field an arbitrary number of times, where they may change also their direction in time. Finally they leave the interaction region as electron lines moving into the normal time direction. Besides these processes, which are characterized each by a continuous line coming from and leaving to the future, there are also processes, which consists of closed loops. As they remain entirely inside the interaction region, they are not visible as physical processes. These closed loops form the vacuum amplitude. From this picture it is also clear that the vacuum amplitude is the same for all N-pair amplitudes and therefore a common factor in all of them. The fact that we are dealing with multiple particles only shows up in the calculation through the antisymmetrisation with respect to all electrons (or equivalent to all positrons).

This means that for a complete calculation of the N-pair creation probabilities we need to know only the reduced one-pair creation amplitude together with the vacuum amplitude. Both can be calculated in principle using perturbation theory, where the reduced one-pair amplitude is identical to the usual perturbation series result and setting the vacuum amplitude to one, that is, neglecting all diagrams with disconnected parts.

If one neglects the exchange terms in the calculation of the probability, it suffices to know the reduced one-pair creation probability, as this is the only result needed for the Poisson distribution of the multiple-pair probabilities.

Note that according to the rules of the Feynman diagrams we have neglected the antisymmetrisation and therefore the Pauli principle for all intermediate states. There is a deeper reason for doing so, which has to do with the connection of the higher-order processes and the occurrence of the vacuum processes through antisymmetrisation. This is, for example, discussed in detail in [11,14]. Let us look, for example, at a typical higher-order diagram in perturbation theory, where we would expect corrections because of the Pauli principle. In Fig. 2 we would expect a deviation as we have two electrons, which are not allowed to be in the same state. (The same is also true for the two positrons.) But it can be shown that the contributions from this process are just canceled by those of the vacuum correction shown in Fig. 3, where the electron in the loop and the produced electron are not allowed to be in the same state also. Both processes are connected with each other, as we get one from the other, if we exchange the two electron lines or the two positron lines. This result is of a very general nature, so that the antisymmetrisation of all intermediate particles can be dropped, if we also include the vacuum processes (again without antisymmetrisation).

This shows that the higher-order multiple-particle corrections to the one-pair creation are connected with the vacuum process. Therefore we think that these processes should be studied in more detail. Especially as we know that the vacuum processes are not negligible, as they decrease the reduced probabilities larger than one to values less than one.

The inclusion of these higher-order processes is straightforward in our model. As the Poisson distribution needs only the neglection of the exchange terms, we can include the higher-order terms into the calculation of the reduced probability and use this result then in the Poisson distribution for the multiple-pair probabilities. This procedure is consistent, as

it includes automatically the higher-order corrections to the vacuum amplitude. Therefore the calculation changes only the reduced probability used in the Poisson distribution but not the explicit form.

Let us compare this result with the other models [2,4,5]: All three models get a Poisson distribution for the multiple-pair creation as well but only based on the summation of a restricted class of diagrams. Studying further these approximations one finds that all of them are essentially "quasi boson" approximations. This means that they have as fundamental processes a pair creation and a pair annihilation process (neglecting at the moment the Coulomb scattering). Combining these processes all previous models assume that only pairs, which have been produced together in a creation process, can be annihilated. Therefore the electron and the positron are seen as an unbreakable pair, which behaves more or less like a boson. And for bosons in an external field the Poisson distribution comes out exactly [10]. Even the inclusion of Coulomb scattering in some of these models does not change the "quasi bosonic" nature of them. Our calculation indicates that there are higher-order processes in which electrons and positrons do not behave as "quasi bosons". These processes are multiple-particle processes in the sense that more than one electron or positron are essentially needed in an intermediate step.

The fact that these are multiple-particle effects also shows the advantage of using the Feynman boundary conditions instead of the retarded boundary conditions used in the Dirac sea picture. In the Feynman picture the production of a pair is described by a continuous fermion line, where the inclusion of multiple-particle effects presents no difficulties. The advantage of the Dirac sea picture is mainly that its particle hole interpretation allows to treat the creation of one electron-positron pair in a single particle formalism. As the higher-order processes need the existence of more than one electron and one hole, this advantage of the Dirac sea picture is no longer existent then.

The other advantage of the Feynman boundary conditions is that the different processes can be separated from each other. Especially the vacuum processes are independent of the reduced pair creation processes. This is not the case in the Dirac sea picture, where vacuum processes and pair creation processes can not be separated in the intermediate steps but only in the final state.

In the following we show that the processes of the type of Fig. 2 are the lowest-order corrections to the one-pair creation processes if we neglect Coulomb scattering terms. For this we restrict ourself to second-order Magnus theory.

V. THE S OPERATOR IN MAGNUS THEORY

The Magnus theory can be seen as an expansion in the interaction time. For large values of γ the interaction time of the two heavy-ion fields is very short, therefore the use of this approximation seems to be justified. Up to second order the S operator is given by [17,18]

$$\begin{aligned} S &= \mathcal{T} \exp\left[-i \int_{-\infty}^{+\infty} H_I(t)\, dt\right] \\ &\approx \exp\left[-i \int_{-\infty}^{+\infty} H_I(t) dt + \frac{1}{2}(-i)^2 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{t_2} dt_1 \left[H_I(t_2), H_I(t_1)\right] + \cdots\right]. \end{aligned} \tag{52}$$

The commutator of the H_I's for $t_1 < t_2$ is the difference of the time-ordered and the anti-time-ordered product. We rewrite S as

$$\begin{aligned} S &= \exp\Big\{-i\int_{-\infty}^{+\infty} H_I(t)dt + \frac{1}{4}(-i)^2\int_{-\infty}^{+\infty} dt_2\int_{-\infty}^{+\infty} dt_1 \\ &\quad \times\mathcal{T}\left[H_I(t_2)H_I(t_1)\right] - \mathcal{A}\left[H_I(t_2)H_I(t_1)\right] + \cdots\Big\} \qquad (53) \\ &= \exp\Big\{\int d^4x\; e : \bar\Psi(x)[-ie\,\not{A}(x)]\Psi(x) : \\ &\quad +\frac{1}{4}\int d^4x_1\; d^4x_2\; (\mathcal{T}-\mathcal{A}) : \bar\Psi(x_2)[-ie\,\not{A}(x_2)]\Psi(x_2) : : \bar\Psi(x_1)[-ie\,\not{A}(x_1)]\Psi(x_1) :\Big\}. \qquad (54) \end{aligned}$$

Now we use again the Wick theorem in order to put the field operators into normal-ordered form. We use it for the time-ordered and anti-time-ordered products in the form

$$\mathcal{T}AB = :AB: + \langle 0|\mathcal{T}AB|0\rangle = :AB: + \langle \mathcal{T}AB\rangle\,, \qquad (55a)$$

$$\mathcal{A}AB = :AB: + \langle 0|\mathcal{A}AB|0\rangle = :AB: + \langle \mathcal{A}AB\rangle\,. \qquad (55b)$$

As the first term in S is in normal-ordered form already, only the second one has to be rearranged. For the time-ordered product we get

$$\begin{aligned} \mathcal{T} : \bar\Psi_2\Psi_2 : : \bar\Psi_1\Psi_1 : \; &= : \bar\Psi_2\Psi_2\bar\Psi_1\Psi_1 : + : \Psi_2\bar\Psi_1 : \left\langle \mathcal{T}\bar\Psi_2\Psi_1\right\rangle \\ &\quad + : \bar\Psi_2\Psi_1 : \left\langle \mathcal{T}\Psi_2\bar\Psi_1\right\rangle + \left\langle \mathcal{T}\bar\Psi_2\Psi_1\right\rangle\left\langle \mathcal{T}\Psi_2\bar\Psi_1\right\rangle, \qquad (56) \end{aligned}$$

and the same for the anti-time-ordered product by replacing $\mathcal{T}$ with $\mathcal{A}$. We get for S

$$\begin{aligned} S = \exp\Big\{&\int dx\; [-ie\,\not{A}(x)_{\alpha\beta}(x)] : \bar\Psi_\alpha\Psi_\beta : + \frac{1}{4}\int dx_1dx_2\; [-ie\,\not{A}(x_2)_{\alpha\beta}][-ie\,\not{A}(x_1)_{\gamma\delta}] \\ &\times[: \Psi_\beta(x_2)\bar\Psi_\gamma(x_1) : \left\langle \mathcal{D}\bar\Psi_\alpha(x_2)\Psi_\delta(x_1)\right\rangle + : \bar\Psi_\alpha(x_2)\Psi_\delta(x_1) : \left\langle \mathcal{D}\Psi_\beta(x_2)\bar\Psi_\gamma(x_1)\right\rangle \\ &+ \left\langle \mathcal{T}\bar\Psi_\alpha(x_2)\Psi_\delta(x_1)\right\rangle\left\langle \mathcal{T}\Psi_\beta(x_2)\bar\Psi_\gamma(x_1)\right\rangle - \left\langle \mathcal{A}\bar\Psi_\alpha(x_2)\Psi_\delta(x_1)\right\rangle\left\langle \mathcal{A}\Psi_\beta(x_2)\bar\Psi_\gamma(x_1)\right\rangle]\Big\}, \end{aligned} \qquad (57)$$

where we have defined $\mathcal{D}$ as $\mathcal{T}-\mathcal{A}$. The last two terms can be rewritten to give

$$\left\langle \mathcal{D}\bar\Psi_\alpha(x_2)\Psi_\delta(x_1)\right\rangle\left\langle \mathcal{T}\Psi_\beta(x_2)\bar\Psi_\gamma(x_1)\right\rangle + \left\langle \mathcal{A}\bar\Psi_\alpha(x_2)\Psi_\delta(x_1)\right\rangle\left\langle \mathcal{D}\Psi_\beta(x_2)\bar\Psi_\gamma(x_1)\right\rangle. \qquad (58)$$

Exchanging the integration of dx_1 and dx_2 we see that the second and the third term of Eq. (57) can be combined and also the last two terms, so that we get

$$\begin{aligned} S = \exp\Big\{&\int dx\; [-ie\,\not{A}(x)_{\alpha\beta}(x)] : \bar\Psi_\alpha\Psi_\beta : + \frac{1}{4}\int dx_1dx_2\; [-ie\,\not{A}(x_2)_{\alpha\beta}][-ie\,\not{A}(x_1)_{\gamma\delta}] \\ &\times\Big[2 : \Psi_\beta(x_2)\bar\Psi_\gamma(x_1) : \left\langle \mathcal{D}\bar\Psi_\alpha(x_2)\Psi_\delta(x_1)\right\rangle \\ &+ \left\langle (\mathcal{T}+\mathcal{A})\Psi_\beta(x_2)\bar\Psi_\gamma(x_1)\right\rangle\left\langle \mathcal{D}\bar\Psi_\alpha(x_2)\Psi_\delta(x_1)\right\rangle\Big]\Big\}. \end{aligned} \qquad (59)$$

Now we transform S into momentum space. For this we need the Fourier transform of the field operator as well as of the vacuum expectation values.

The field operator and its conjugate can be written as

$$\Psi(x) = \sum_s \int d\tilde{p} \Big[b(p,s) u(p,s) \exp(-ipx) + d^+(p,s) v(p,s) \exp(ipx) \Big], \tag{60a}$$

$$\bar{\Psi}(x) = \sum_s \int d\tilde{p} \Big[b^+(p,s) \bar{u}(p,s) \exp(ipx) + d(p,s) \bar{v}(p,s) \exp(-ipx) \Big]. \tag{60b}$$

where we have introduced $d\tilde{p}$, which is defined as

$$d\tilde{p} := \frac{d^3p}{(2\pi)^{3/2}(2p_0)^{1/2}}. \tag{61}$$

In the following we need also the factor of $d\tilde{p}$ alone together with the Lorentz invariant phase space, therefore we define also

$$\gamma(p) := \frac{1}{(2\pi)^{3/2}(2p_0)^{1/2}}, \tag{62}$$

and

$$d\Gamma(p) := \frac{d^3p}{(2\pi)^3 2p_0}. \tag{63}$$

With the usual relations between vacuum expectation values, propagators, and singular function, one gets for the vacuum expectation values [19–21]

$$\begin{aligned} \langle 0| \mathcal{T} \Psi(x) \bar{\Psi}(x') |0\rangle &= iS_F(x-x') \\ &= i \int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \exp(-ip(x-x')), \end{aligned} \tag{64a}$$

$$\begin{aligned} \langle 0| \mathcal{A} \Psi(x) \bar{\Psi}(x') |0\rangle &= -iS_A(x-x') \\ &= -i \int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 - i\epsilon} \exp(-ip(x-x')). \end{aligned} \tag{64b}$$

Combining both we get for $\mathcal{D}$ and $\mathcal{T} + \mathcal{A}$:

$$\begin{aligned} \langle 0| \mathcal{D} \Psi(x) \bar{\Psi}(x') |0\rangle &= i \left[S_F(x-x') + S_A(x-x') \right] \\ &= i \int \frac{d^4p}{(2\pi)^4} (\not{p} + m) \left(\frac{1}{p^2 - m^2 + i\epsilon} + \frac{1}{p^2 - m^2 - i\epsilon} \right) \\ &\times \exp(-ip(x-x')) \\ &= 2i \int \frac{d^4p}{(2\pi)^4} (\not{p} + m) \frac{\text{P.P.}}{p^2 - m^2} \exp(-ip(x-x')), \end{aligned} \tag{65}$$

where P.P. denotes the principal part of the integral, and

$$\langle 0|\,(\mathcal{T}+\mathcal{A})\Psi(x)\bar{\Psi}(x')\,|0\rangle = i\,[S_F(x-x') - S_A(x-x')]$$
$$= \int d\Gamma(p)\ (\not{p}+m)\exp(-ip(x-x'))$$
$$-\int d\Gamma(p)\ (\not{p}-m)\exp(ip(x-x')), \tag{66}$$

which describes the propagation of on-shell electrons and positrons.

These forms of the vacuum expectation values are put into S together with the decomposition of the field operators $\Psi(x)$ and $\bar{\Psi}(x)$ and using also the Fourier transform of the external field

$$A(x) = \int \frac{d^4q}{(2\pi)^4} A(q)\exp(-iqx). \tag{67}$$

The integration over the coordinate space can be done, giving deltafunctions for the momenta. Reducing the momentum integrals with the help of these, we finally get

$$\begin{aligned}
S = \exp\Bigg\{ & -ie\int d\tilde{p}_1\ d\tilde{p}_2 \\
& \times\Big[:b^+(p_1)b(p_2):\ \bar{u}(p_1)\,\not{A}(p_1-p_2)u(p_2) \\
& + :b^+(p_1)d^+(p_2):\ \bar{u}(p_1)\,\not{A}(p_1+p_2)v(p_2) \\
& + :d(p_1)b(p_2):\ \bar{v}(p_1)\,\not{A}(-p_1-p_2)u(p_2) \\
& + :d(p_1)d^+(p_2):\ \bar{v}(p_1)\,\not{A}(-p_1+p_2)v(p_2)\Big] \\
& -ie^2\int d\tilde{p}_1\ d\tilde{p}_2\ \frac{dp}{(2\pi)^4} \\
& \times\Big[:b^+(p_1)b(p_2):\ \bar{u}(p_1)\,\not{A}(p_1-p)\,(\not{p}+m)\,\frac{\text{P.P.}}{p^2-m^2}\,\not{A}(p-p_2)u(p_2) \\
& + :b^+(p_1)d^+(p_2):\ \bar{u}(p_1)\,\not{A}(p_1-p)\,(\not{p}+m)\,\frac{\text{P.P.}}{p^2-m^2}\,\not{A}(p+p_2)v(p_2) \\
& + :d(p_1)b(p_2):\ \bar{v}(p_1)\,\not{A}(-p_1-p)\,(\not{p}+m)\,\frac{\text{P.P.}}{p^2-m^2}\,\not{A}(p-p_2)u(p_2) \\
& + :d(p_1)d^+(p_2):\ \bar{v}(p_1)\,\not{A}(-p_1-p)\,(\not{p}+m)\,\frac{\text{P.P.}}{p^2-m^2}\,\not{A}(p+p_2)v(p_2)\Big] \\
& +i\frac{e^2}{2}\int\frac{dp}{(2\pi)^4}d\Gamma(p')\ \text{Tr}\left[(\not{p}'+m)\,\not{A}(p'-p)(\not{p}+m)\frac{\text{P.P.}}{p^2-m^2}\,\not{A}(p-p')\right] \\
& -i\frac{e^2}{2}\int\frac{dp}{(2\pi)^4}d\Gamma(p')\ \text{Tr}\left[(\not{p}'-m)\,\not{A}(-p'-p)(\not{p}+m)\frac{\text{P.P.}}{p^2-m^2}\,\not{A}(p+p')\right]\Bigg\}.
\end{aligned} \tag{68}$$

Again, we have not written explicitly the spins of the leptons, which we look at as included in the momenta.

The interpretation of the individual terms is straightforward (see Fig. 4). Terms of the form $:b^+b:$ are electron Coulomb scattering terms, $:dd^+:$ the corresponding positron scattering terms. $:b^+d^+:$ corresponds to pair creation and $:db:$ to pair annihilation. The last two terms have no operators in them. They correspond to the lowest-order vacuum corrections. As they are only numbers, they commute with all other terms.

VI. APPLICATION TO ELECTROMAGNETIC PAIR CREATION

We apply this result, which is true for an arbitrary external field, to the special case of pair production in heavy-ion collisions. In this case there are some kinematical restrictions, so that some of the terms in S can be dropped. Especially there is no pair creation or pair annihilation in first-order Magnus theory and we can replace the principal value integrals in the pair creation and annihilation term in the second order by ordinary integrals, as the intermediate state is not allowed to be on-shell in this case.

As we want to concentrate on the multiple-particle effects we will neglect also all Coulomb scattering terms. It is generally assumed that these terms do change the differential probabilities, but are only of minor importance for the total probabilities. Also their influence seems to become smaller for higher energies. Therefore we will neglect all terms of the form $: b^+b :$ and $: dd^+ :$. Also the vacuum terms will be dropped, as we are only interested in the calculation of the reduced amplitude, where vacuum corrections do not appear. The S operator with these approximations is

$$\begin{aligned} S = \exp\Big\{ & -ie^2 \int d\tilde{p}_1 \; d\tilde{p}_2 \frac{dp}{(2\pi)^4} \\ & \times \Big[b^+(p_1) d^+(p_2) \\ & \times \bar{u}(p_1) \, \not{A}(p_1 - p) \frac{\not{p} + m}{p^2 - m^2} \, \not{A}(p + p_2) v(p_2) \\ & + d(p_1) b(p_2) \\ & \times \bar{v}(p_1) \, \not{A}(-p_1 - p) \frac{\not{p} + m}{p^2 - m^2} \, \not{A}(p - p_2) u(p_2) \Big] \Big\}. \end{aligned} \tag{69}$$

The integrals over p in this expression are just the pair creation and pair annihilation amplitude in second-order Born approximation. We define the pair creation and annihilation potential V and V^* as

$$\begin{aligned} V(p_1, p_2) = e^2 & \int \frac{dp}{(2\pi)^4} \\ & \times \bar{u}(p_1) \, \not{A}(p_1 - p) \frac{\not{p} + m}{p^2 - m^2} \, \not{A}(p + p_2) v(p_2) \end{aligned} \tag{70a}$$

$$\begin{aligned} V^*(p_2, p_1) = e^2 & \int \frac{dp}{(2\pi)^4} \\ & \times \bar{v}(p_1) \, \not{A}(-p_1 - p) \frac{\not{p} + m}{p^2 - m^2} \, \not{A}(p - p_2) u(p_2) \end{aligned} \tag{70b}$$

(where V^* is just the complex conjugate of V) in order to write S as:

$$\begin{aligned} S = \exp\Big\{ & -i \int d\tilde{p} \; d\tilde{q} \\ & \times \Big[b^+(p) d^+(q) V(p, q) + d(q) b(p) V^*(p, q) \Big] \Big\}. \end{aligned} \tag{71}$$

The matrix element of the pair creation in second order is connected with the potential V through

$$M(p,q) = -iV(p,q). \tag{72}$$

For the reduced one-pair creation amplitude we have to calculate

$$\langle f|\, S\,|i\rangle = \langle 0|\, d(q_f)b(p_f)S\,|0\rangle\,. \tag{73}$$

We are now expanding the exponential in S in order to get the different contributions to the one-pair creation. In lowest order we expect to get back the second-order Born result. We get

$$\begin{aligned}\langle f|\, S^{(1)}\,|i\rangle &= -i\int d\tilde{p}\; d\tilde{q}\; V(p,q)\,\langle 0|\, d(q_f)b(p_f)b^+(p)d^+(q)\,|0\rangle \\ &= -i\gamma(p_f)\gamma(q_f)V(p_f,q_f) \\ &= M(p_f,q_f)\gamma(p_f)\gamma(q_f),\end{aligned} \tag{74}$$

The differential probability in first order therefore is

$$P(p_f,q_f) = |M(p_f,q_f)|^2\,\gamma^2(p_f)\gamma^2(q_f) \tag{75}$$

and the total probability is given by

$$P_{\text{total}} = \int |M(p_f,q_f)|^2\, d\Gamma(p_f)d\Gamma(q_f), \tag{76}$$

which is indeed the Born result.

As the vacuum expectation values, we get from higher orders, vanish, if the number of creation and annihilation operators is not equal for each kind of particles, we see easily that only odd orders in the expansion of S contribute. The next order is therefore the third one, where we get

$$\begin{aligned}\langle f|\, S^{(3)}\,|i\rangle = \frac{i}{3!}\int d\tilde{p}_1\; d\tilde{p}_2\; d\tilde{p}_3\; d\tilde{q}_1\; d\tilde{q}_2\; d\tilde{q}_3 \\ \times\Big[V(p_1,q_1)V^*(p_2,q_2)V(p_3,q_3)\,\langle 0|\, d(q_f)b(p_f)b^+(p_1)d^+(q_1)d(q_2)b(p_2)b^+(p_3)d^+(q_3)\,|0\rangle \\ +V^*(p_1,q_1)V(p_2,q_2)V(p_3,q_3)\,\langle 0|\, d(q_f)b(p_f)d(q_1)b(p_1)b^+(p_2)d^+(q_2)b^+(p_3)d^+(q_3)\,|0\rangle\Big].\end{aligned} \tag{77}$$

Here we have used the fact that only two of the eight possible combinations do not vanish trivially. Again the vacuum expectation value can be calculated using the anticommutation relations of the particle operators. The first expectation value is

$$\begin{aligned}\langle 0|\, d(q_f)b(p_f)b^+(p_1)d^+(q_1)d(q_2)b(p_2)b^+(p_3)d^+(q_3)\,|0\rangle = \\ \delta(p_f-p_1)\delta(p_2-p_3)\delta(q_f-q_1)\delta(q_2-q_3)\end{aligned} \tag{78}$$

and the second one

$$\begin{aligned}\langle 0|\, d(q_f)b(p_f)d(q_1)b(p_1)b^+(p_2)d^+(q_2)b^+(p_3)d^+(q_3)\,|0\rangle = \\ \delta(p_1-p_2)\delta(q_1-q_2)\delta(p_f-p_3)\delta(q_f-q_3) \\ -\delta(p_1-p_2)\delta(p_f-p_3)\delta(q_f-q_2)\delta(q_1-q_3) \\ -\delta(p_f-p_2)\delta(p_1-p_3)\delta(q_1-q_2)\delta(q_f-q_3) \\ +\delta(p_f-p_2)\delta(p_1-p_3)\delta(q_f-q_2)\delta(q_1-q_3). \end{aligned} \tag{79}$$

Terms of the form $\delta(p_2-p_3)\delta(q_2-q_3)$ describe closed fermion loops, and we drop them in the calculation of the reduced amplitude. Therefore only two terms remain. In third order we get

$$\langle f|\, S^{(3)}\,|i\rangle = -i\frac{2}{3!}\gamma(p_f)\gamma(q_f)\int d\Gamma(p_1)d\Gamma(q_1)V(p_f,q_1)V^*(p_1,q_1)V(p_1,q_f). \tag{80}$$

Combining both results we get up to third order

$$\langle f|\, S^{(1+3)}\,|i\rangle = \gamma(p_f)\gamma(q_f)\left[M(p_f,q_f)+\frac{1}{3}\int d\Gamma(p_1)d\Gamma(q_1)M(p_f,q_1)M^*(p_1,q_1)M(p_1,q_f)\right]. \tag{81}$$

The interpretation of this higher-order process is straightforward. If we keep in mind that $M(p,q)$ is the amplitude for the production of a pair and $M^*(p,q)$ the amplitude for the annihilation of a pair, we get the Feynman diagram as shown in Fig. 5. These are just those higher-order processes, we already discussed before in Sec. IV (see Fig. 2). Two pairs are created and then one of the electrons annihilates with the other positron, so that finally only one electron and one positron remain. The Magnus theory restricts these intermediate particles to the mass shell, so that we only need the on-shell M, whereas they may be off-shell in the general case.

Using the same technique higher orders can also be calculated. The explicit calculation of the fifth and seventh order gives

$$\begin{aligned}\langle f|\, S^{(5)}\,|i\rangle &= \frac{16}{5!}\gamma(p_f)\gamma(q_f)\int d\Gamma(p_1)d\Gamma(q_1)d\Gamma(p_2)d\Gamma(q_2) \\ &\quad\times M(p_f,q_1)M^*(p_1,q_1)M(p_1,q_2)M^*(p_2,q_2)M(p_2,q_f) \qquad (82a) \\ \langle f|\, S^{(7)}\,|i\rangle &= \frac{272}{7!}\gamma(p_f)\gamma(q_f)\int d\Gamma(p_1)d\Gamma(q_1)d\Gamma(p_2)d\Gamma(q_2)d\Gamma(p_3)d\Gamma(q_3) \\ &\quad\times M(p_f,q_1)M^*(p_1,q_1)M(p_1,q_2)M^*(p_2,q_2)M(p_2,q_3)M^*(p_3,q_3)M(p_3,q_f), \qquad (82b)\end{aligned}$$

where in the calculation of the vacuum expectation values also more complicated fermion loops occur, which have to be neglected. The following properties are remarkable, as they are true for all higher orders: First all contributions coming from the same order are of the same type and can therefore be summed. Second all amplitudes have the same sign, that is, they are all added coherently. Therefore they increase all the reduced total probability, no cancellation or reduction occurs. This same sign can also be understood from the connection of the higher-order processes with the vacuum corrections. Vacuum loops contribute with a negative sign to the amplitude (see Sec. III and [14]). As our diagrams are connected with a vacuum diagram by the permutation of some lines, it is clear that it has to have the same sign as the lowest-order diagram. The general form of the nth order therefore is

$$\langle f|\, S^{(n)}\,|i\rangle = c_n\gamma(p_f)\gamma(q_f)\int d\Gamma(p_1)d\Gamma(q_1)\cdots d\Gamma(p_{(n-1)/2})d\Gamma(q_{(n-1)/2}) \times M(p_f,q_1)M^*(p_1,q_1)M(p_1,q_2)\cdots M^*(p_{(n-1)/2)},q_{(n-1)/2})M(p_{(n-1)/2},q_f), \quad (83)$$

For higher orders the number of terms occurring in the reduction of the vacuum expectation values gets rather large and the number of diagrams increases rapidly. However we show in the appendix how these total number of processes and therefore c_n can be calculated using combinatorial arguments and a recursive formula.

Finally we calculate the lowest-order correction to the reduced total probability. Squaring the pair production amplitude up to third order (Eq. (81)) we get

$$\begin{aligned}
P^{(1+3)}(p_f,q_f) &= \left|\langle f|\, S^{(1+3)}\,|i\rangle\right|^2 \\
&= \left[M^*(p_f,q_f) + \frac{1}{3}\int d\Gamma(p)d\Gamma(q)M^*(p_f,q)M(p,q)M^*(p,q_f)\right] \\
&\quad \times\left[M(p_f,q_f) + \frac{1}{3}\int d\Gamma(p')d\Gamma(q')M(p_f,q')M^*(p',q')M(p',q_f)\right] \\
&\quad \times\gamma^2(p_f)\gamma^2(q_f) \\
&= \Big\{|M(p_f,q_f)|^2 + \frac{1}{3}\Big[\int d\Gamma(p)d\Gamma(q)M(p_f,q_f)M^*(p_f,q)M(p,q)M^*(p,q_f) \\
&\quad + M^*(p_f,q_f)M(p_f,q)M^*(p,q)M(p,q_f)\Big] + \cdots\Big\}\gamma^2(p_f)\gamma^2(q_f) \\
&= \Big\{|M(p_f,q_f)|^2 + \frac{2}{3}\int d\Gamma(p)d\Gamma(q) \\
&\quad \times\mathrm{Re}\left[M(p_f,q_f)M^*(p_f,q)M(p,q)M^*(p,q_f)\right] + \cdots\Big\}\gamma^2(p_f)\gamma^2(q_f). \qquad (84)
\end{aligned}$$

Integrating over p_f and q_f we get the total probability (Fig. 6)

$$\begin{aligned}
P_{\text{total}} &= \int d\Gamma(p_f)d\Gamma(q_f)|M(p_f,q_f)|^2 + \frac{2}{3}\int d\Gamma(p_f)d\Gamma(q_f)d\Gamma(p)d\Gamma(q) \\
&\quad \times\mathrm{Re}\left[M(p_f,q_f)M^*(p_f,q)M(p,q)M^*(p,q_f)\right] \qquad (85) \\
&=: P^{(B)} + P^{(M)}. \qquad (86)
\end{aligned}$$

This result is easily generalized. The general series is of the form

$$\begin{aligned}
P_{\text{total}} &= \int d\Gamma(p_1)d\Gamma(q_1)|M(p_1,q_1)|^2 \\
&\quad + d_3\int d\Gamma(p_1)d\Gamma(q_1)d\Gamma(p_2)d\Gamma(q_2)\mathrm{Re}\left[M(p_1,q_1)M^*(p_1,q_2)M(p_2,q_2)M^*(p_2,q_1)\right] \\
&\quad + \cdots \\
&\quad + d_{2l-1}\int d\Gamma(p_1)\cdots d\Gamma(q_l)\mathrm{Re}\underbrace{\left[M(p_1,q_1)M^*(p_1,q_2)\cdots M(p_l,q_l)M^*(p_l,q_1)\right]}_{l\times} \\
&\quad + \cdots. \qquad (87)
\end{aligned}$$

The coefficients d_n are again calculated in the appendix. They can be derived easily from the coefficients c_n of the amplitudes.

VII. CALCULATION FOR IMPACT PARAMETER ZERO

For the calculation of the lowest-order multiple-particle correction to the one-pair creation probability we make use of the analytic form of the matrixelement for impact parameter b zero. Details of this calculation can be found in a previous publication [7], therefore we will only review rather briefly the main properties. The Feynman diagrams contributing to the lowest order Born calculation are shown in Fig. 7, where (1) and (2) denotes the interaction with the external field of ion 1 and 2 , respectively. The matrix element is

$$\begin{aligned} M(p,q) &= -ie^2\bar{u}(p) \\ &\times\Big[\int\frac{d^4k}{(2\pi)^4}\not{A}_1(p-k)\frac{\not{k}+m}{k^2-m^2}\not{A}_2(q+k) \\ &+\int\frac{d^4k}{(2\pi)^4}\not{A}_2(p-k)\frac{\not{k}+m}{k^2-m^2}\not{A}_1(q+k)\Big]\,v(q). \end{aligned} \tag{88}$$

The external fields are given by

$$A_\mu^{(1,2)}(q) = -2\pi Z_{(1,2)}eu_\mu^{(1,2)}\delta(qu^{(1,2)})\frac{F(q^2)}{q^2}. \tag{89}$$

$u^{(1,2)}$ are the four velocities of the ions and $F(q^2)$ is their form factor. All calculations are done for collisions of the same type of ions and in the center of mass system. For the form factor a simple dipole form factor of the form

$$F_{\text{dipole}}(q^2) = \frac{\Lambda^2}{\Lambda^2-q^2}, \tag{90}$$

with $\Lambda = 83$MeV, has been used together with a second form factor, which is a linear combination of two dipole form factors. In the previous publication the integral has been solved in order to calculate the lowest-order Born result. Here we use this analytic result for M in Eq. (86). The usual way to calculate expressions of this kind is to rewrite the spin summation as a trace over gamma matrices. But as each of the M consists of two diagrams, we finally get a total of sixteen different traces, which all give large expressions, which are not manageable by an algebraic calculation program and therefore can not be used for numerical calculations. Therefore we prefer to calculate M directly and to do the spin summation numerically.

Several methods have been proposed in the literature how to calculate amplitudes directly instead of their squares [22–24]. We use the method described by Fearing and Silbar [25]. It consists of multiplying and dividing M with $\bar{v}(q)u(p)$ in order to get

$$\begin{aligned} M &= \bar{u}(p)\hat{M}v(q) \\ &= \frac{1}{\bar{v}(q)u(p)}\mathrm{Tr}\Big\{v(q)\bar{v}(q)u(p)\bar{u}(p)\hat{M}\Big\} \\ &= \frac{1}{\bar{v}(q)u(p)}\mathrm{Tr}\Big\{(\not{q}-m)\,\frac{1}{2}\,(1+\gamma_5\lambda_q\,\not{s}_q) \\ &\times(\not{p}+m)\,\frac{1}{2}\,(1+\gamma_5\lambda_p\,\not{s}_p)\,\hat{M}\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4S(p,q)} \mathrm{Tr}\Big\{ (\not{q} - m) \\
&\times (1 + \gamma_5 \lambda_q \not{s}_q + \gamma_5 \lambda_p \not{s}_p - \lambda_p \lambda_q \not{s}_q \not{s}_p) \\
&\times (\not{p} + m) \hat{M} \Big\},
\end{aligned} \tag{91}
$$

where s_p, s_q are the spinvectors of the electron and the positron, and λ_p, λ_q are the eigenvalues of the spinors with respect to the spinvector, that is, they have values of ± 1. $S(p,q)$ is a complex number, which has been calculated using an explicit form for the spinors for the standard form of the gamma matrices. In order to make the numerical calculations easier we have used polarisation vectors s_p and s_q, which are longitudinal vectors:

$$
s_p = \frac{1}{\sqrt{p_0^2 - p_z^2}} (p_z, 0, 0, p_0) \tag{92}
$$

and similar for s_q. The calculation of the trace has been done with the help of the algebraical calculation program FORM [26]. All sixteen different spin combinations have been calculated and summed. The final expression has to be real, which has been used to test the program (As the final expression can be expressed as a trace, which does not contain any γ_5 matrices, it must be equivalent to an expression containing only real numbers and real scalar products, therefore it has to be real). The integration over four particles gives a 12-dimensional integral, of which one angular integration is trivial. It was calculated using a Monte Carlo (MC) integration routine (VEGAS [27,28]). Together with the total probability we have calculated also single differential probabilities by sorting the points into appropriate bins. The error of the total probability is always less then 1%. An explicit error estimate is given for the differential probabilities. As our calculation is symmetric with respect to all four momenta and all four electron-positron combinations, this error has been calculated from the standard deviation of the four results.

Finally we want to point out that the expression in Eq. (86) can be interpreted also in a different way: If we calculate the reduced total two-pair creation probability exactly in lowest order according to the previous section, we get

$$
\begin{aligned}
P^{(2)}_{\mathrm{total}} &= \frac{1}{(2!)^2} \int d\Gamma(p_1) d\Gamma(p_2) d\Gamma(q_1) d\Gamma(q_2) \\
&\times [M^*(p_1,q_1) M^*(p_2,q_2) - M^*(p_1,q_2) M^*(p_2,q_1)] \\
&\times [M(p_1,q_1) M(p_2,q_2) - M(p_1,q_2) M(p_2,q_1)] \\
&= \frac{1}{2} \int d\Gamma(p_1) d\Gamma(p_2) d\Gamma(q_1) d\Gamma(q_2) \\
&\times \Big\{ |M(p_1,q_1)|^2 \, |M(p_1,q_1)|^2 \\
&- \mathrm{Re}\left[M^*(p_1,q_1) M(p_2,q_1) M^*(p_2,q_2) M(p_1,q_2) \right] \Big\} \\
&= P^{(D)}_{\mathrm{total}} - P^{(X)}_{\mathrm{total}}.
\end{aligned} \tag{93}
$$

The first term is the direct part, which is used in order to get the Poisson distribution, whereas the second term is the exchange part, which we neglect in the Poisson distribution

(see also Fig. 8). Comparing the second expression with that in Eq. (86) we see that they are identical. Comparing Fig. 8 with Fig. 6 we see that we only have to change the interpretation of the diagram by cutting it at a different point. Therefore the result of the calculation for the total probability can also be used to test, whether the neglection of the exchange term in order to get the Poisson distribution is justified.

VIII. RESULTS AND CONCLUSIONS

Figure 9 shows the results of our calculations for the reduced probability as a function of γ. In all our diagrams we set $Z\alpha = 1$, as this is a common factor in all results. Also shown is the result for the corresponding probability of the lowest-order calculation (see [7]). In order to test the dependence on the form factor the calculations have been done for the dipole form factor and the double dipole form factor. Both agree within the error interval, therefore we show here only the results for the double dipole form factor. We see that the correction is similar in size as the lowest-order result for large values of γ. Using realistic values for $Z\alpha$ its importance is reduced, as the higher-order correction has to be multiplied by $(Z\alpha)^8$, the lowest-order result by $(Z\alpha)^4$. In Table I we give predictions for the contribution of the higher-order correction to the reduced total probability. For γ and Z we have used typical values for relativistic heavy-ion colliders. The higher-order correction increases the reduced probability for Pb and U up to about 5–10%. Therefore they should be observable in principle. Based on this calculation we expect that higher-order corrections (5th order and more) are again only of about 5–10% of the third-order results and are therefore corrections of less than 1% to the total probability.

Also shown are the results for the differential probabilities as a function of the energy E (Fig. 10) and the angle with the beam axis θ (Fig. 11). In these as in all other single differential probabilities as well the correction follows more or less the lowest-order result.

In Fig. 12 we compare both terms contributing to the total two-pair production probability. Here the first and the second diagrams of Fig. 8 are multiplied by the same factor $(Z\alpha)^8$, therefore the ratio of both curves gives directly the contribution of the exchange term to the first term, which is used in the Poisson distribution alone. We see that the exchange diagram contributes only with about 1% to the reduced total probability. Also its importance gets smaller for larger values of γ, therefore the use of the Poisson distribution seems to be justified. This could be different, of course, if one looks at differential probabilities in some region of the phase space.

Let us summarize our results: We have shown that the N-pair creation amplitude can be reduced exactly to the reduced one-pair creation amplitudes and the vacuum amplitude. Therefore for the calculation of all N-pair probabilities, it suffices to calculate those two amplitudes. The use of the external field approximation and the neglection of the interaction among electrons and positrons is essential in order to get this result. We have shown that we get a Poisson distribution for the total probabilities very generally, if we neglect all exchange terms. A calculation of the reduced one-pair creation probability then suffices alone, as all higher-order processes are given by the Poisson distribution.

We have found perturbation theory expressions for the reduced amplitude as well as for the vacuum amplitude, so that both can in principal be calculated using perturbation theory. None of them is of a principal nonpertubative character.

We have compared this result with earlier calculation, where we have pointed out that they have mainly made use of a quasi boson approximation of the electron-positron pairs in order to get the Poisson distribution. On the other hand our form of the Poisson distribution needs only the neglection of the exchange terms, therefore higher-order processes beyond the quasi-boson model can easily be incorporated.

Based on the general form of the S operator in second-order Magnus theory we have calculated the lowest-order contribution of such higher-order multiple-particle effects to the total probabilities for collisions with impact parameter b zero. The contribution were found to be on the $1-10\%$ level for high Z, and should therefore be incorporated into the calculations.

Finally we compared the deviation of the two-pair creation probability from the Poisson result. This deviation was found to be small, therefore the use of the Poisson distribution seems to be justified.

APPENDIX A: COEFFICIENTS IN MAGNUS THEORY

As shown in Sec. V all higher-order processes in the Magnus theory without Coulomb rescattering terms are of the same type. The number of terms in each order is given by the reduction of the vacuum expectation values. The explicit reduction is not useful because the complexity of the expressions increases quickly. Therefore we show here how these can be calculated more easily.

This is a combinatorial problem, which can be formulated in the following way: In Nth order we have N interaction points, each of which either corresponds to a pair creation or pair annihilation process. In terms of the fermion lines this means that a line entering one point from the left also leaves it to the left and the same from the right. We need the total number of paths through these N points using the above condition, where the line initially enters from the right and finally leaves to the right. As we look only at reduced amplitudes, no closed loops are allowed.

Let us find a recursive formula for the number of paths: We define $A(n,i)$ as the number of distinct paths for i fermion lines (coming from and leaving to the right) going through n interaction points. If we add now another interaction point to the right, there are two possibilities: If it is a pair creation process, the number of lines is increased by one. If we have a pair annihilation process, we have to connect it with one outgoing and one ingoing line, therefore reducing their number by one. For this we have to choose one of the i outgoing lines and one of the ingoing lines, where we have to be careful not to make a closed loop; therefore only $i-1$ of them are allowed. This means that there are $i(i-1)$ possible combinations. This gives us the recursion relation

$$A(n+1,i) = A(n,i-1) + i(i+1)A(n,i+1), \tag{A1}$$

together with the boundary conditions

$$A(1,1) = 1 \tag{A2a}$$

$$A(1,i) = 0 \qquad \text{for } i > 1 \tag{A2b}$$

$$A(n,0) = 0 \qquad \text{for all } n. \tag{A2c}$$

With this recursion relation we can calculate $A(n, 1)$ for all n. This is just the number of diagrams that we were looking for. Dividing it by $n!$ gives the coefficient c_n in the expansion of the amplitude. These coefficients are given in Table II. For the coefficients in the reduced total probability we have to multiply c_l and c_{n+1-l} and sum over all $l \in 0 \ldots n$. These are the coefficients d_n of Eq. (87). They are also given in Table II.

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FIGURES

FIG. 1. Graphical illustration of the general form of the N-pair production process. The creation of a pair is described by a connected fermion line entering from and leaving to the future. The vacuum processes are described by all sorts of closed fermion loops. The interaction with the external field is shown as a cross.

FIG. 2. One possible higher-order pair creation process, where the Pauli principle is neglected in the intermediate state. Interaction with the external field is shown as a cross.

FIG. 3. Vacuum correction to the one-pair creation connected by an exchange of two lines with the process shown in Fig. 2.

FIG. 4. Graphical illustration of the processes occurring in first (a) and second-order Magnus theory (b and c). Dotted lines denote the propagator, where only the principal part is taken, double lines denote electrons or positrons, which are on-shell. Interaction with the external field is shown as a cross.

FIG. 5. Third order correction to the one-pair creation process in second-order Magnus theory; compare with Fig. 2. Double lines mean that the particles are on shell.

FIG. 6. Lowest order (a) and the first two correction terms (b and c) to the reduced one-pair creation probability.

FIG. 7. The two Feynman diagrams contributing to the one-pair creation in lowest order. The interaction with the external field of ion 1 and 2 is denoted by (1) and (2), respectively.

FIG. 8. The two Feynman diagrams for the lowest-order two-pair creation process; direct term (a) and exchange term (b).

FIG. 9. Comparison of the correction to the reduced one-pair creation probability with the lowest-order result as a function of γ. $Z\alpha$ is set to 1. The points are the results of the calculation of the multiple-particle corrections, the solid line a fitted $\ln\gamma$ dependence. The dotted line is the result of the lowest-order Born approximation.

FIG. 10. Comparison of the differential probability $P(E)$ for $\gamma = 3400$. Data points are the results of the calculation of the multiple-particle correction, the solid line results of the lowest-order Born calculation. $Z\alpha$ is set to 1.

FIG. 11. Comparison of the differential probability $P(\theta)$ for $\gamma = 3400$. Definitions are the same as in Fig. 10.

FIG. 12. Comparison of the contributions of the direct process $P^{(D)}$ (dotted line) and the exchange process $P^{(X)}$ (solid line and data points) to the reduced two-pair creation process. $Z\alpha$ is set to 1.

TABLES

	γ	Ion	$P^{(B)}$	$P^{(M)}$	
SPS	10	Pb	0.63	0.013	2.1 %
RHIC	100	Au	1.6	0.059	3.7 %
LHC	3400	Pb	3.9	0.21	5.3 %
LHC	3400	U	6.1	0.49	8.1 %

TABLE I. Comparison of the contribution of the multiple-particle correction $P^{(M)}$ to the reduced probability with the lowest-order Born result $P^{(B)}$ (see Eq. (86)).

n	# Diags.	c_n	d_n
1	1.0000×10^0	1.0000000000	1.0000000000
3	2.0000×10^0	0.3333333333	0.6666666667
5	1.6000×10^1	0.1333333333	0.3777777778
7	2.7200×10^2	0.0539682540	0.1968253968
9	7.9360×10^3	0.0218694885	0.0974955908
11	3.5379×10^5	0.0088632355	0.0466976645
13	2.2368×10^7	0.0035921280	0.0218375158
15	1.9038×10^9	0.0014558344	0.0100304665

TABLE II. The number of diagrams and the coefficients c_n and d_n appearing in Magnus theory for different orders n.

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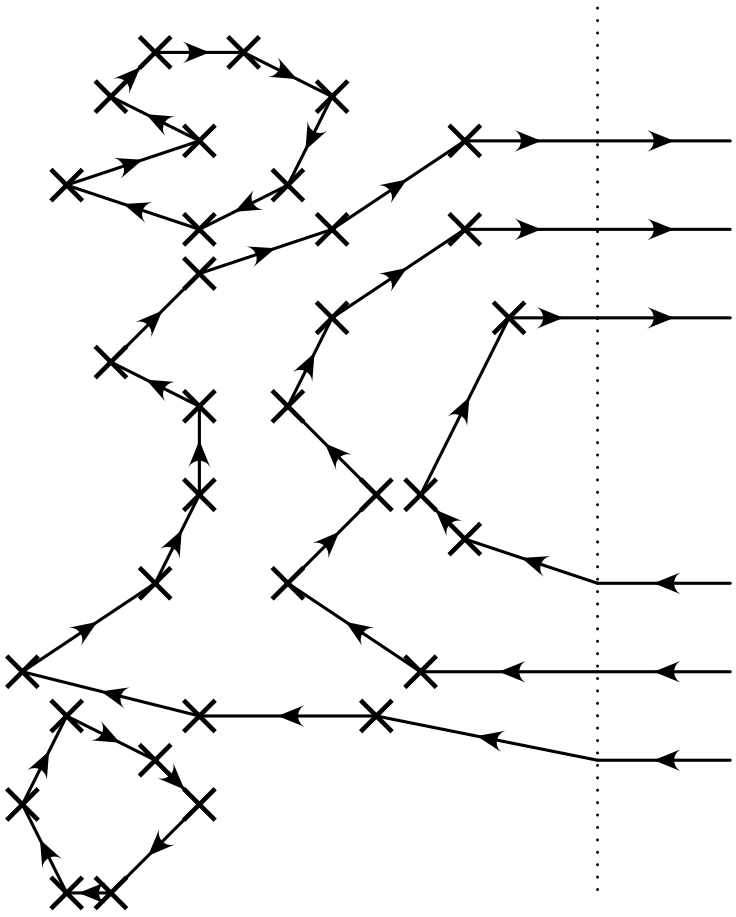
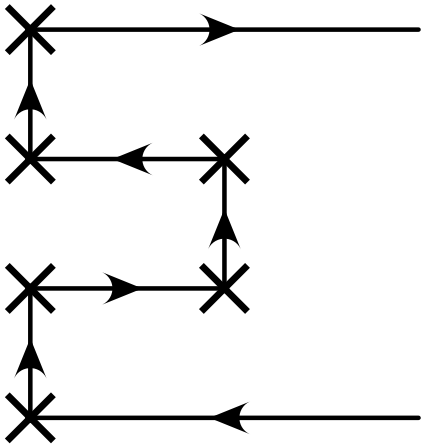


Figure 1



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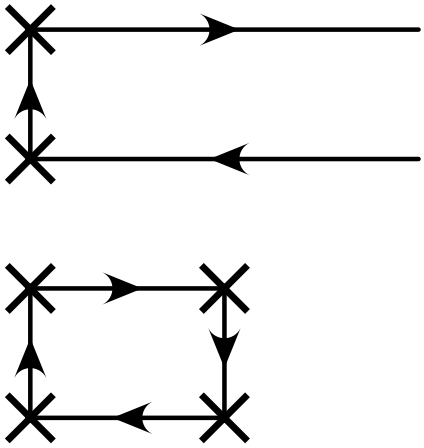

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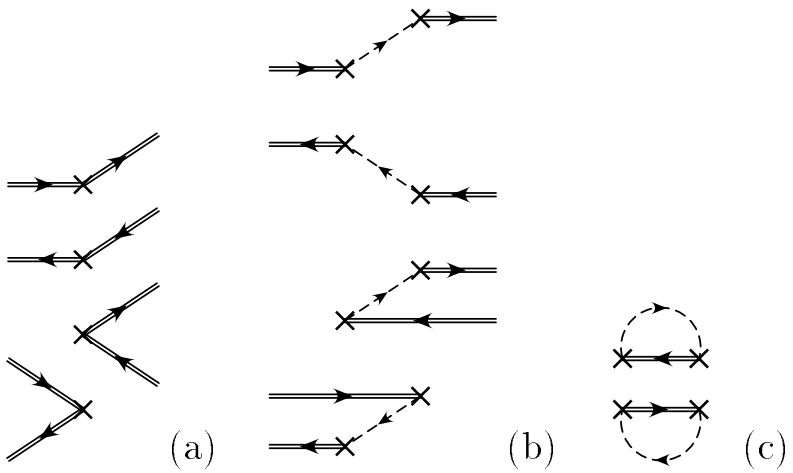

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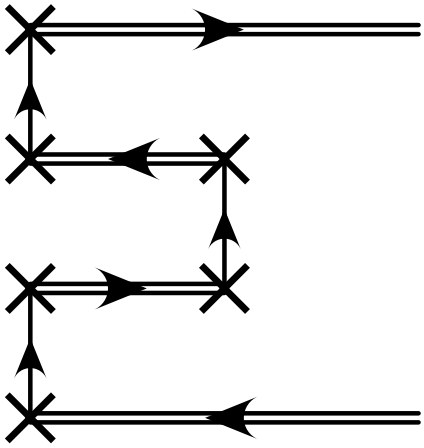

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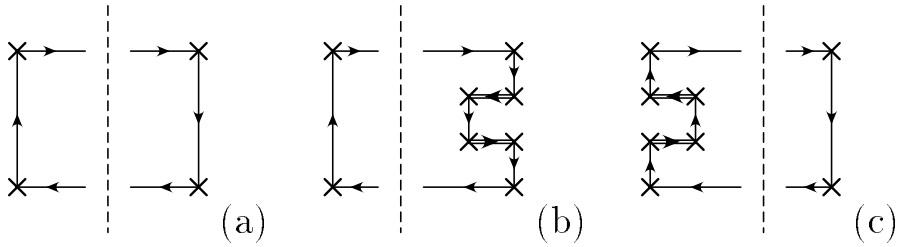

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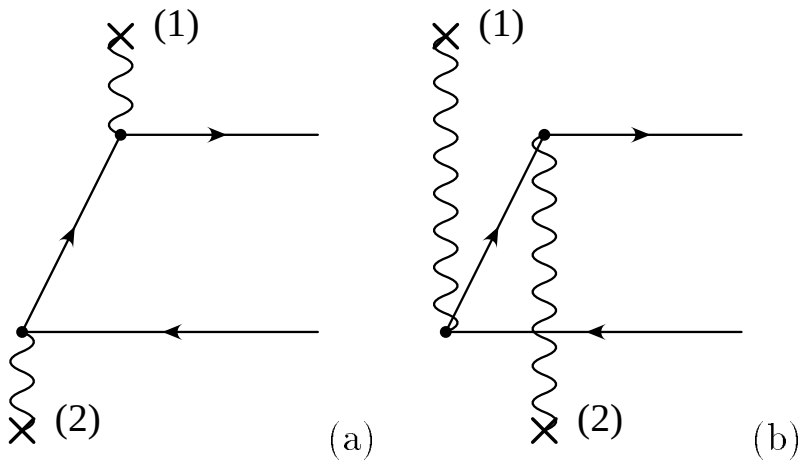

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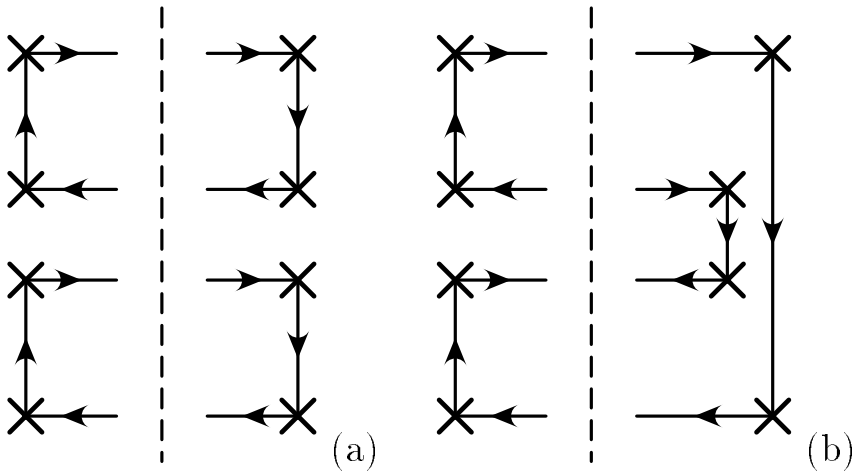
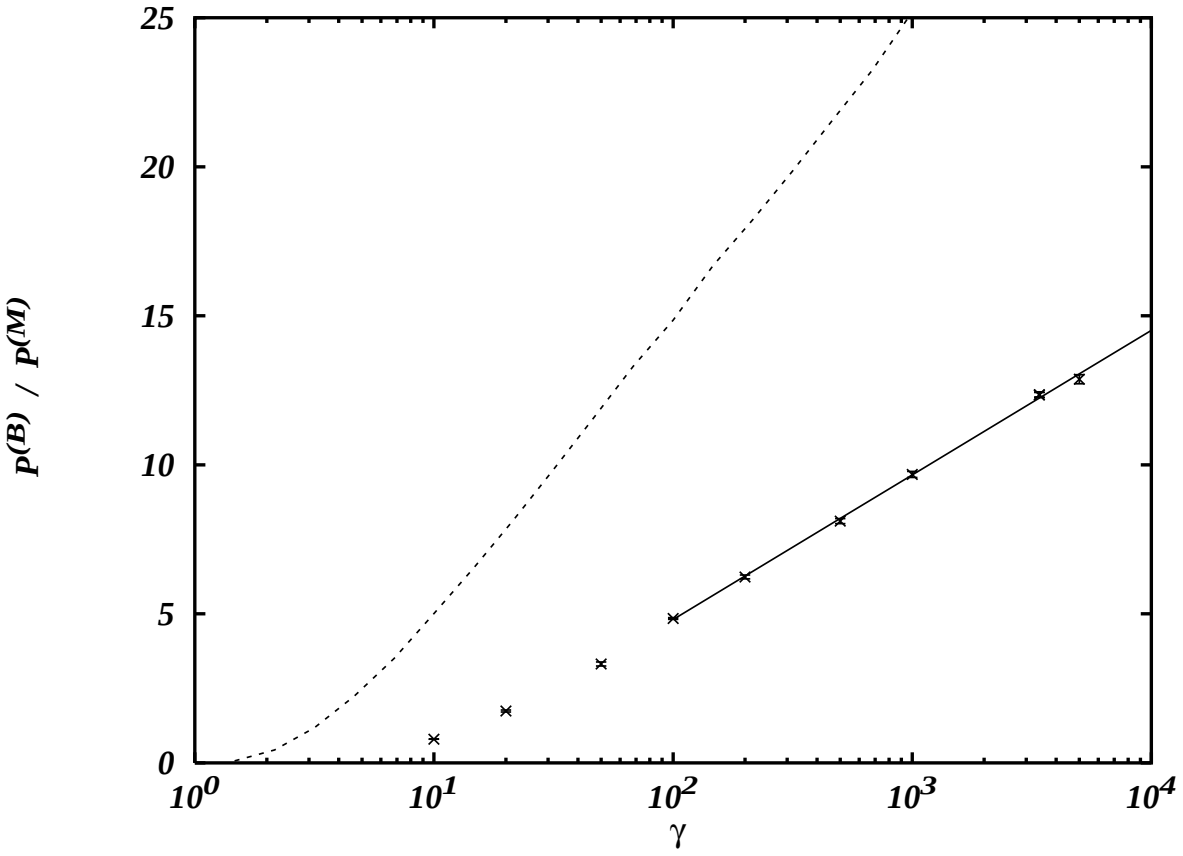
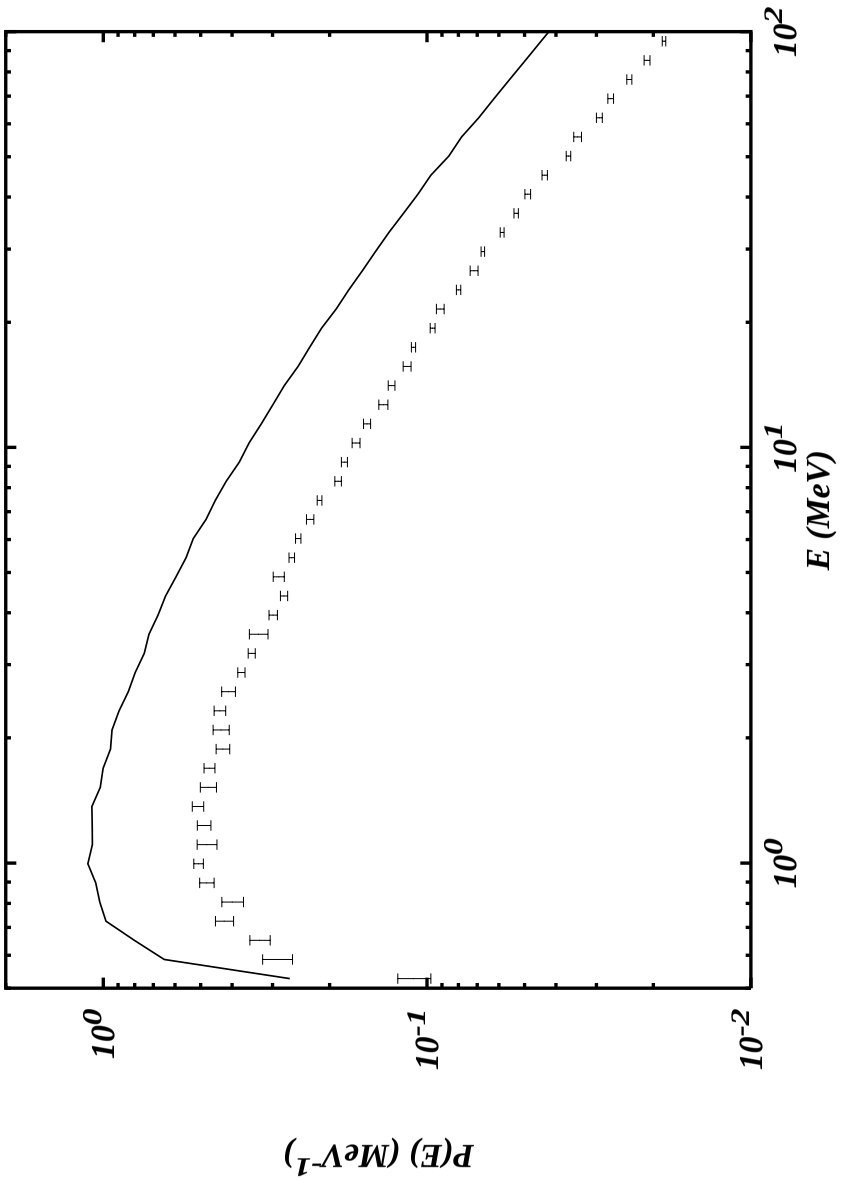


Figure 8

Figure 9





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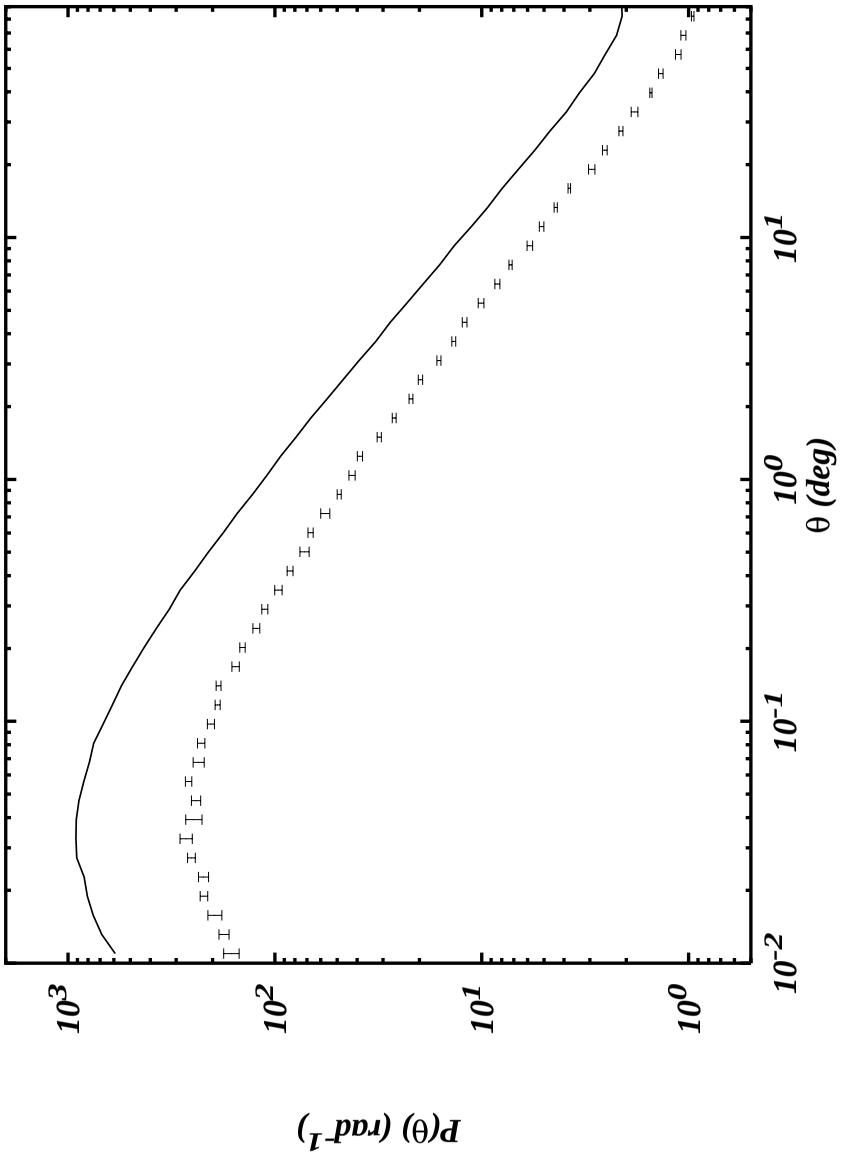

Figure 11

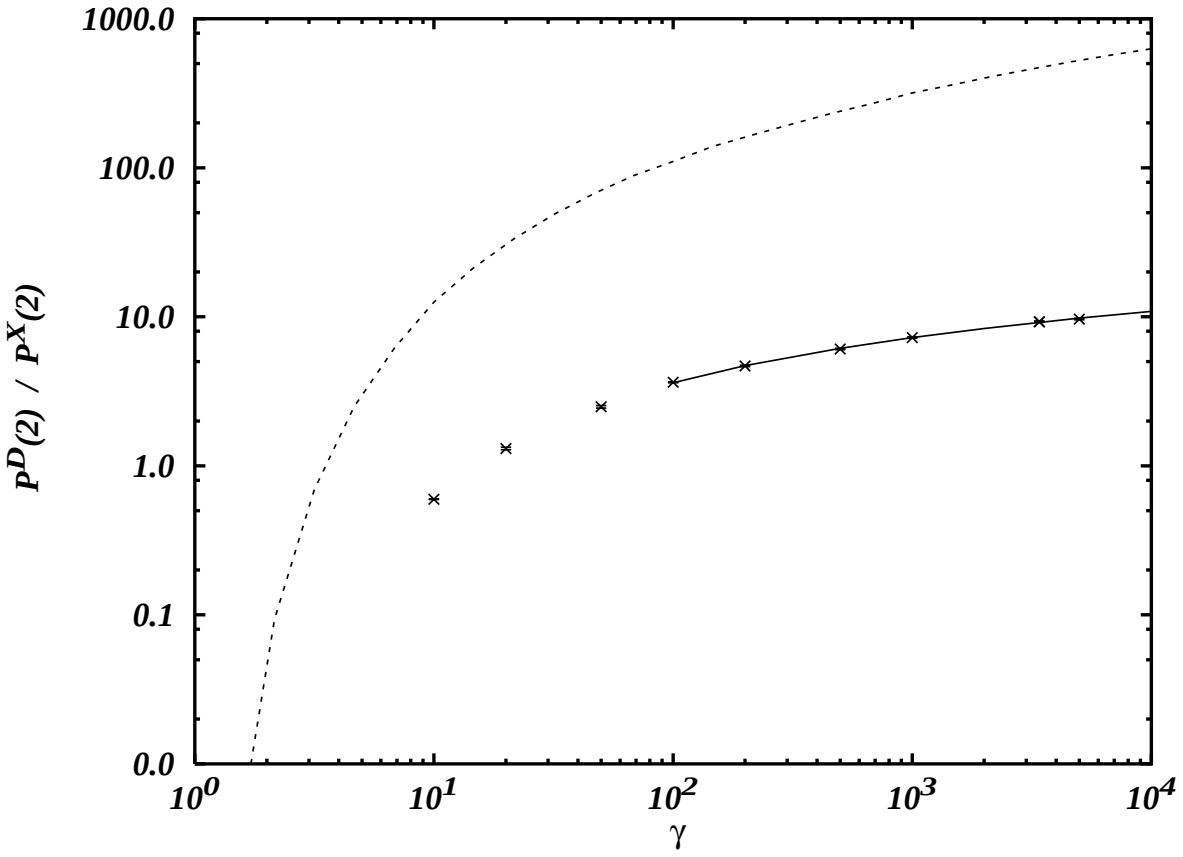

Figure 12