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An ultimate bound on the trajectories of the Lorenz system and its applications

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Abstract
In this paper, a new bound on the trajectories of the Lorenz system is derived. This result is useful to show that the transverse stability of the origin in two Lorenz systems coupled in a drive-response manner is a necessary and sufficient condition for global asymptotic synchrony of the two systems, and to simplify the derivation of the upper bound to the Hausdorff dimension of the Lorenz attractor.

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1. Introduction

It is well known that all trajectories of the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= -bz + xy\end{aligned} \tag{1}$$

are ultimately bounded for arbitrary positive σ, r, b, that is, there exists a trapping region $R(\sigma, r, b) \subset \mathbb{R}^3$ which ultimately bounds all trajectories. Many expressions on the boundedness of its trajectories exist (see, e.g., [4, 8] which we will refer to throughout this paper). In this paper, we derive a new ultimate bound for the trajectories of the Lorenz system, which finds two straightforward applications. We first prove that, for two systems of type (1) coupled in a drive-response manner, the threshold of global asymptotic stability of the synchronous state equals the threshold of transverse stability of the zero fixed point, for a certain range of parameters σ, r, b. We then apply our result to derive an upper bound for the Hausdorff dimension of the Lorenz attractor, as the local Lyapunov dimension of the origin.

2. Bounds for solutions of the Lorenz system

It is convenient to call a set R a *trapping region* for a system if each trajectory either enters the set R and thereafter never leaves it or tends to its boundary. Clearly, the intersections of some trapping regions is a trapping region as well. In the following, whenever we write that, for example, the inequality $f(x, y, z) \geqslant c$ holds in some trapping region, we mean $\limsup_{t\to\infty}(f(x(t), y(t), z(t)) - c) \geqslant 0$. The main result of this section is the following lemma.

Lemma 1. *Suppose $b < 2\sigma$ and $2 \leqslant b \leqslant 4$, and let*

$$\alpha = -2\sigma r + 2\sigma - b\sigma + 2 - 3b + b^2. \tag{2}$$

Then for arbitrary positive μ the following inequality is satisfied

$$z^2 + \frac{\mu - \alpha}{\sigma} z \geqslant \left(1 - \frac{(2-b)^2}{2\mu}\right) y^2 \tag{3}$$

in some trapping region.

The proof of this lemma makes use of the following known result (see, e.g., [4, 8] for a proof).

Lemma 2. *Suppose $2\sigma > b$, then the following estimate*

$$z \geqslant \frac{x^2}{2\sigma}$$

holds in some trapping region.

The proof of lemma 1 follows.

Proof. Consider the following scalar function

$$V = \frac{(\alpha+\delta)x^2}{2\sigma} - \frac{x^4}{4\sigma} + (2-b)xy + \sigma y^2 + (x^2 - \delta)z$$

where δ is to be determined later. Its time derivative satisfies

$$\dot{V} = x^4 + ((2-b)r - \alpha - \delta)\, x^2 - b\sigma y^2 + \delta b z - 2(\sigma+1)x^2 z + b(b-2)xy.$$

For simplicity, we substituted expression (2) for α only in the last term. It follows then from lemma 2 that in some trapping region

$$(b - 2(\sigma+1))x^2 z \leqslant \frac{(b - 2(\sigma+1))}{2\sigma} x^4$$

and if one chooses δ such that

$$\alpha + \delta = -\frac{2r\sigma(b-2)}{2\sigma - b}$$

it follows that

$$\dot{V} + bV \leqslant \left(\frac{b - 2(\sigma+1)}{2\sigma} + 1 - \frac{b}{4\sigma}\right) x^4 = \frac{b-4}{4\sigma} x^4 \leqslant 0.$$

Therefore, $V \leqslant 0$ in the trapping region. This inequality can be rewritten as follows

$$\begin{aligned} \delta z &\geqslant \frac{\alpha+\delta}{2\sigma} x^2 - \frac{x^4}{4\sigma} + (2-b)xy + \sigma y^2 + x^2 z \\ &\geqslant \frac{\alpha+\delta}{2\sigma} x^2 - \frac{x^4}{4\sigma} + (2-b)xy + \sigma y^2 \end{aligned} \tag{4}$$

where the last inequality follows from lemma 2. Recall that by assumption $2 \leqslant b < 2\sigma$, for which $\alpha + \delta \leqslant 0$ and therefore, applying the previous lemma we have

$$\frac{\alpha+\delta}{2\sigma}x^2 \geqslant (\alpha+\delta)z, \qquad -\frac{x^4}{4\sigma} \geqslant -\sigma z^2.$$

Substituting these inequalities in (4) gives

$$-\alpha z \geqslant -\sigma z^2 + (2-b)xy + \sigma y^2. \tag{5}$$

Using the following inequality

$$(2-b)xy \geqslant -\frac{\mu x^2}{2\sigma} - \frac{(2-b)^2}{2\mu}\sigma y^2$$

which is valid for arbitrary positive μ, we derive from (5) that

$$-\alpha z \geqslant -\sigma z^2 - \frac{\mu x^2}{2\sigma} - \frac{(2-b)^2}{2\mu}\sigma y^2 + \sigma y^2. \tag{6}$$

Finally, using the inequality

$$-\frac{\mu x^2}{2\sigma} \geqslant -\mu z$$

which follows from lemma 2, we derive (3) from (6). ■

Parameter values $\sigma = 10, r = 28, b = \frac{8}{3}$, which may give rise to a chaotic evolution, are a standard choice, in the sense that they are most often found in scientific literature, and they satisfy the restrictions imposed on the parameters in lemma 1. Hence, the following result holds for Lorenz system (1).

Proposition 1. *Let $\sigma = 10, r = 28, b = \frac{8}{3}$. Then the set defined by the inequality*

$$\frac{y^2}{b} + (z-2r)^2 \leqslant 4r^2 \tag{7}$$

contains a trapping region.

The proof of this result is based on lemma 1. One can take, for example, $\mu = 40$ and make use of the following well-known estimate (see, e.g., [4, 8])

$$y^2 + (z-r)^2 \leqslant \frac{b^2}{4(b-1)}r^2. \tag{8}$$

The intersection of region (3) defined in lemma 1 and region (8) lies inside ellipsoid (7). This situation is schematically reproduced in figure 1. It is clear that for large z the estimate (7) is very conservative, but sufficient for the applications considered in this paper. Though we have only proved estimate (7) for the standard parameters, one can use lemma 1 combined with some optimization procedure for μ to find a set of parameters for which estimate (7) is satisfied.

This new result on the ultimate boundedness of the trajectories of Lorenz system (1) can find several applications. In the subsequent sections, we apply proposition 1 to derive two lower bounds: the first one (section 3) on the gain for global asymptotic stability for synchronization of two copies of system (1), and then (section 4) on the Hausdorff dimension of the Lorenz attractor.

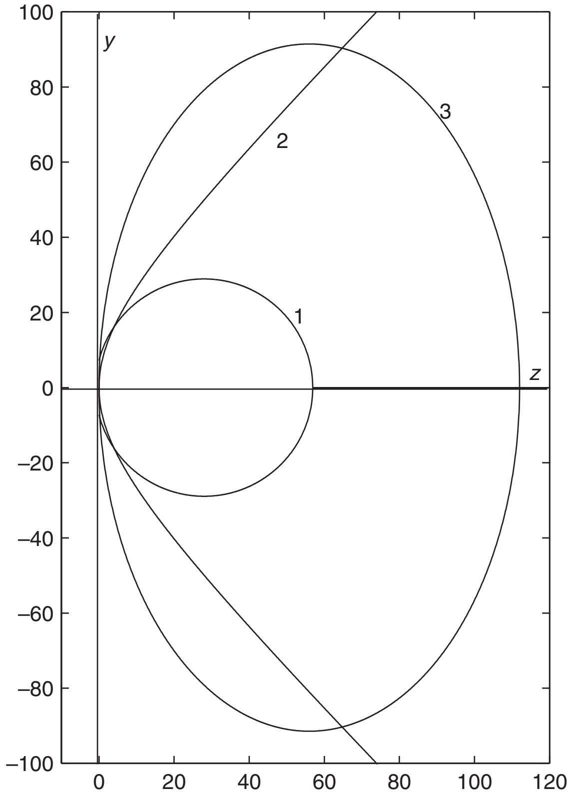

Figure 1. 1: the circle $y^2 + (z - r)^2 = b^2r^2/4(b - 1)$, equation (8), 2: the hyperbola $z^2 + (\mu - \alpha)z/\sigma = (1 - (2 - b)^2/2\mu)y^2$, equation (3), and 3: the ellipse $y^2/b + (z - 2r)^2 = 4r^2$, equation (7). Parameter values are $\sigma = 10, r = 28, b = \frac{8}{3}, \mu = 40$.

3. Synchronization of two Lorenz systems

Suppose that the signal $x(t)$ from (1) drives an identical copy of the Lorenz system as follows:

$$\begin{aligned}\dot{\hat{x}} &= \sigma(\hat{y} - \hat{x}) - \gamma(\hat{x} - x)\\ \dot{\hat{y}} &= r\hat{x} - \hat{y} - \hat{x}\hat{z}\\ \dot{\hat{z}} &= -b\hat{z} + \hat{x}\hat{y}.\end{aligned} \tag{9}$$

The purpose of this section is to find the smallest synchronization threshold $\bar{\gamma}$ for which the set

$$\mathcal{D} := \{(x, y, z, \hat{x}, \hat{y}, \hat{z}) : x = \hat{x}, y = \hat{y}, z = \hat{z}\}$$

contains a globally asymptotically stable compact subset Ω.

Numerical simulations on systems (1) and (9) show that randomly initialized trajectories converge on $\mathcal{D}$ when γ is greater than about $\gamma = 10$, but not all trajectories initialized on unstable periodic points of (1) converge on $\mathcal{D}$. The stability in $\mathcal{D}$ of any such trajectory is a necessary condition for the global asymptotic stability in $\mathcal{D}$ of Ω. The simplest unstable periodic point of (1) is the zero fixed point, which gains stability in $\mathcal{D}$ for

$$\gamma \geqslant \sigma(r - 1). \tag{10}$$

To prove that the inequality (10) is a necessary bound for the synchronization threshold we assume that the subset Ω contains the origin (this is true for the values of the parameters previously considered, see [9]). In this case, a necessary condition for the asymptotic stability of the error dynamics is that all unstable manifolds of the zero equilibrium of systems (1) and (9) belong to the 'diagonal' set $\mathcal{D}$. By linearization around the origin one can observe that the origin of the coupled system is transversely stable if the linear system

$$\begin{aligned} \dot{\xi}_1 &= \sigma(\xi_2 - \xi_1) - \gamma\xi_1 \\ \dot{\xi}_2 &= r\xi_1 - \xi_2 \end{aligned} \tag{11}$$

is stable, which is true when condition (10) is satisfied.

We now want to prove that the condition

$$\gamma > \sigma(r-1) \tag{12}$$

is a sufficient condition for which the set $\mathcal{D}$ contains a globally asymptotically stable compact subset Ω. To this end, consider the following Lyapunov function:

$$V = \frac{1}{2}\left(\frac{r}{\sigma}(x-\hat{x})^2 + (y-\hat{y})^2 + (z-\hat{z})^2\right).$$

Its time derivative satisfies

$$\dot{V} = -\frac{r(\sigma+\gamma)}{\sigma}e_1^2 + (2r-z)e_1e_2 + ye_1e_3 - e_2^2 - be_3^2$$

where we denoted $e_1 = x - \hat{x}$, $e_2 = y - \hat{y}$, $e_3 = z - \hat{z}$. This expression is negative definite with respect to e_1, e_2, e_3 if the following inequality is satisfied:

$$r\left(1+\frac{\gamma}{\sigma}\right) > \frac{1}{4b}(y^2(t) + b(2r - z(t))^2). \tag{13}$$

Using proposition 1 one can see that this inequality is satisfied in the trapping region for the standard parameters of the Lorenz system as soon as (12) is satisfied, which proves sufficiency of (12). Hence, necessary (10) and sufficient (12) conditions for global asymptotic synchrony define the exact threshold $\bar{\gamma} = \sigma(r-1)$.

Therefore, for the Lorenz system (1) coupled as in (9), the transverse stability of the origin gives the exact threshold for global asymptotic synchrony. Computer simulations shows that the synchronization phenomenon between two coupled Lorenz systems can be observed for gains significantly smaller than $\sigma(r-1) = 270$. However, in this case the diagonal set can contain an invariant set which is an attractor only in some weaker sense, e.g. in the Milnor sense.

In practical implementation of the synchronization between two Lorenz systems the difference between asymptotic stability and attractivity in the sense of Milnor is crucial even though a set of zero measure can be thought of as negligible in a physical experiment. To illustrate a phenomenon peculiar to this situation, let us include an additive noise $\eta(t)$ which affects both master and slave system. Assume also that transverse Lyapunov exponents for the Sinai–Ruelle–Bowen (SRB) measure are negative (the existence of the SRB measure for the Lorenz attractor is proven in [9]), but $\gamma < \sigma(r-1)$. Since the origin of (1) belongs to the chaotic attractor, there are trajectories for which this point is an ω-limit point. Since for $\gamma < \sigma(r-1)$ the origin is unstable, an arbitrarily small noise $\eta(t)$ can drive the trajectory of the system (1) and (9) along the unstable manifolds of the zero equilibrium, i.e. far from the set $\mathcal{D}$.

However, negativity of transverse Lyapunov exponents for the SRB measure ensures that $\mathcal{D}$ contains a Milnor (essential) locally riddled attractor (see [1], proposition 2.21). So, a typical 'bursting' phenomenon occurs, that is, for an arbitrarily small level of the noise signal η, the time evolution of the error signal demonstrates some peaks between which the level of the error signal is very small (see figure 2). Contrarily, in the case where (12) holds, the level

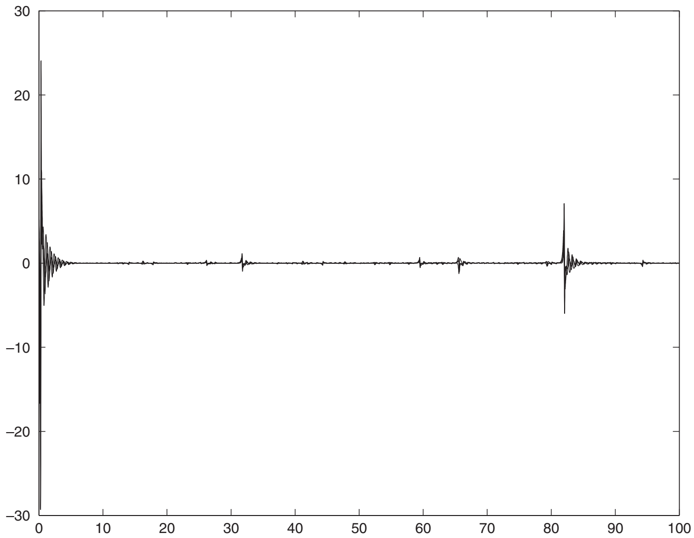

Figure 2. A typical bursting phenomenon.

of the error signal depends continuously on the level of noise, as it follows from the existence of a smooth Lyapunov function, which proves stability of the set Ω, and hence in this case the bursting is not observed.

The situation in which the set $\mathcal{D}$ contains a Milnor attractor can have some advantages too. Suppose that the response system is described by the following equations

$$\begin{aligned} \dot{\hat{x}} &= \sigma(\hat{y} - \hat{x}) - \gamma[\hat{x} - (x + \eta(t))] \\ \dot{\hat{y}} &= r\hat{x} - \hat{y} - \hat{x}\hat{z} \\ \dot{\hat{z}} &= -b\hat{z} + \hat{x}\hat{y} \end{aligned} \tag{14}$$

where $\eta(t)$ is bounded noise. This situation is possible when the master and slave systems are coupled through some channel that adds some distortions to the signal from the master system. Then due to noise amplification by high gain, performance of the synchronization scheme for relatively small γ can be better than the performance of the synchronization for $\gamma > \sigma(r - 1)$.

We carried out computer simulation for two different observer gains: $\gamma_1 = 80$, for which the diagonal set is unstable and $\gamma_2 = 276$, for which the diagonal set is asymptotically stable. We assumed that the output signal in both cases is measured with additive noise $\eta(t) = 0.1 \sin 160t$. The results of computer simulation are presented in figure 3. From the simulation it follows that after some transient the error for small gain is smaller than that for the high gain.

4. Hausdorff dimension of the Lorenz attractor

The Hausdorff dimension of the invariant set lying in (7) is bounded above by the Lyapunov dimension of the origin. This statement is known as the Eden conjecture [2] which has been proved by Ljashko [5]. The method proposed by Ljashko does not depend on the estimates of the ultimate bounds for the system trajectories but leads to very cumbersome calculations. Recently, Leonov extended Ljashko's method and found certain restrictions on the parameters of the Lorenz system for which the Lyapunov dimension of the Lorenz attractor equals the

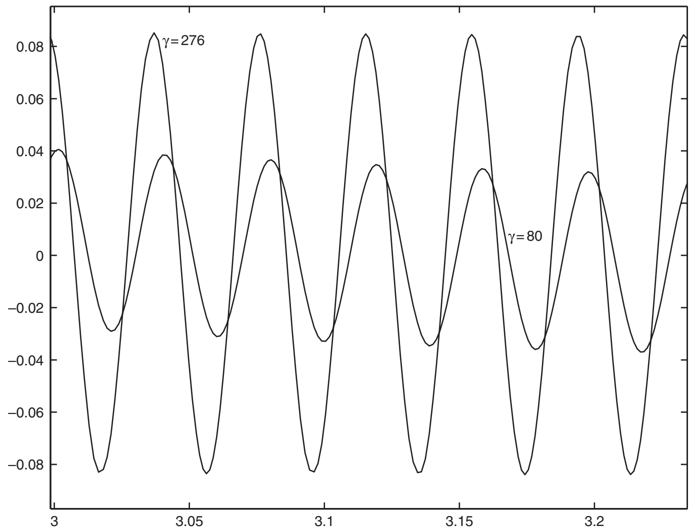

Figure 3. The error $x_1(t) - \hat{x}_1(t)$ versus time for different observer gains.

local Lyapunov dimension of the origin [6]. In this section, we obtain the same result as in [5, 6], but with a much simpler derivation.

According to [6, 7] we have to find solutions $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$, functions of (x, y, z), to the following generalized eigenvalue problem

$$\det[PJ + J^{\mathrm{T}}P - \lambda P] = 0$$

for some positive definite matrix P, where J stands for Jacobian of the Lorenz system. Let us recall theorem 7 from [7], where a proof is found. We consider the local version of the result proved in [7] and the case in which P is independent of the state vector.

Theorem 1. *Suppose that for some positive definite matrix P there exists a number $s \in [0, 1)$ such that*

$$\lambda_1(x, y, z) + \lambda_2(x, y, z) + s\lambda_3(x, y, z) < 0 \tag{15}$$

for all x, y, z in some trapping region, then the Hausdorff dimension $\dim_H K$ of the Lorenz attractor is bounded by above by $2 + s$.

The Jacobian of the Lorenz system is

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

and let P be

$$P = \begin{pmatrix} \frac{r}{\sigma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that

$$PJ + J^{\mathrm{T}}P = \begin{pmatrix} -2r & 2r - z & y \\ 2r - z & -2 & 0 \\ y & 0 & -2b \end{pmatrix}.$$

We know that $\lambda_1 + \lambda_2 + \lambda_3 = -2(\sigma + 1 + b)$, and from $-2(\sigma + 1 + b) < 0$ it follows that $\lambda_3 < 0$ for all x, y, z. Hence, by the theorem mentioned above, to find an upper estimate of s it is sufficient to find a lower estimate of $\bar{\lambda}_3$. Given this estimate, we then evaluate the upper bound on the Hausdorff dimension as

$$\dim_H K \leqslant 3 - \frac{2(\sigma + b + 1)}{|\bar{\lambda}_3|}.$$

The smallest negative eigenvalue of some symmetric matrix Q is such a number $\bar{\lambda}$ that the matrix $Q - \lambda I$ is positive definite as long as $\lambda < \bar{\lambda}$. For all nonpositive λ the following matrix inequality

$$PJ + J^{\mathrm{T}}P - \lambda P \geqslant PJ + J^{\mathrm{T}}P - \lambda P_1$$

is satisfied, where $P_1 = \mathrm{diag}\{r/\sigma, 1, 0\}$. Therefore, to find a lower estimate for λ_3 it is sufficient to find a lower estimate of the smallest nonpositive solution of the following equation

$$\det(PJ + J^{\mathrm{T}}P - \lambda P_1) = 0$$

or,

$$r\left(2 + \frac{\lambda}{\sigma}\right)(2 + \lambda) = \frac{y^2}{b} + (2r - z)^2 + \lambda\frac{y^2}{2b}.$$

Using proposition 1 and neglecting the term $\lambda y^2/2b$ (since we are looking for a negative lower estimate of the smallest solution for λ) it is then straightforward to derive the following result.

Proposition 2. *Let* $\sigma = 10, r = 28, b = \frac{8}{3}$. *Then the Hausdorff dimension of the invariant compact set of the Lorenz system* $\dim_H K$ *satisfies*

$$\dim_H K \leqslant 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4r\sigma}} \approx 2.4013 \tag{16}$$

which is the same bound as derived in [5].

5. Conclusion

In this paper, we have derived a new ultimate bound on the trajectories of the Lorenz system (1). This new expression has proved to be advantageous in the two applications we have considered. In the first one, we have derived, not just a new upper bound on the threshold for global asymptotic stability of two linearly unidirectionally coupled Lorenz systems, but that the transverse stability of the zero fixed point triggers the global asymptotic stability of the invariant set. Then, we used this result to obtain an upper bound on the Hausdorff dimension of system (1).

The first result validates a conjecture proposed by Hunt and Ott [3], who deal with the following problem: in the realm of chaotic systems, specifically, which unstable periodic orbits on the attractor yields the largest (optimal) value of the long-time average of some function of the state vector? Hunt and Ott [3] presented numerical evidence and analysis in support of the conjecture that the optimal value is achieved by the periodic orbits with the lowest period. Apart from the case of some coupled discrete, low-dimensional maps, there are no results on coupled continuous systems which show that the ‘optimal’ value of the threshold of asymptotic stability of synchronous motion is achieved at the onset of transverse stability of the lowest possible unstable periodic orbit: a fixed point.

For the second application we discussed, it is remarkable that the same bound is obtained by Ljashko in [5] and Leonov in [6] with a different and more cumbersome analytic derivation.

However, our proof is presented for the standard parameters of the Lorenz system while in [5, 6] the set of the system parameters for which the estimate (16) is true is found. The largest set of such parameters is found in [6].

To conclude, proofs of stability using Lyapunov functions are rigorous, but also known to be difficult to produce, which may discourage researchers from embarking on them. Even when a suitable Lyapunov function is found, then the estimate of the threshold is often too conservative an approximation of the real one, which is then much lower than the derived one. We present here a case in which an estimate through the Lyapunov function of the upper bound of a threshold is precisely the threshold itself.

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