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Optimal Design of System of Cross-Beams

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Abstract. The aim of this paper is to create a method for reduction of constructive weights of system of cross-beams (grillages) optimizing its elements. The study has practical application and aimed primarily to companies who design civil and industrial buildings, different types of mechanical constructions, vehicle industry, etc.

Introduction

Consideration and solution of first problem of optimal design projection was initally introduced by Galileo Galiley in his 1638's work devoted to cross-beams.

In consequtive almost 350 year period there were a lot of reasearches upon optimal design of construction. Most of these works, like this one, in order to enlighten the task, deal with static tasks and are the basis for optimizing soltuions in the field of concrete practical applications mentioning character of dynamic impacts.

Let us consider the possible frame construction of trailer Fig. 1. It consists of longitudional and lateral beams, one borders of which are fixated on hinged movable and unmovable supports.

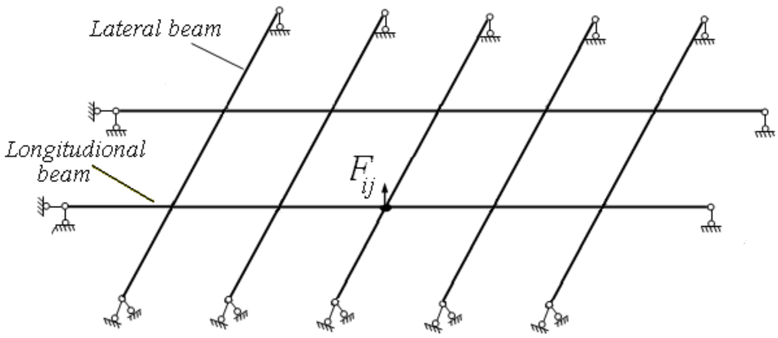

Fig. 1. System of cross-beams (grillage)

The main idea of proposed solution was published in 1970, when B.Lisitsyn presented his doctoral dissertation on the topic: "Method of defining conditions and its usage for solution of basic classes of elasticity theory" [1]. It this paper we describe the general principles of method of elastic's elements optimization. In early 90's, developing the ideas of Lisitsyn, G.Aryassov created the base of general algorithm for optimization of the elastic beam elements [2]. The aim of article is development and application of Aryassov's ideas for reduction of trailer weight by optimizing its elements.

Optimal design of the elastic elements of ladder frames

We replace observing with i-th beam for simplification of solution, where F_{ij} (i–number of longitudional, j-number of lateral beam) denote all reactions of beams. After that we observe i-th beam loaded with distributed load Fig. 2.

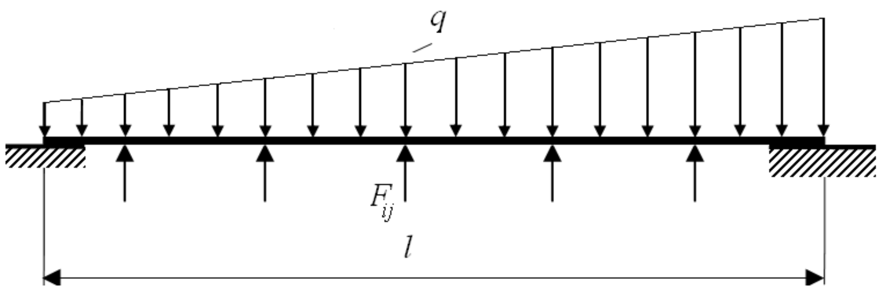

Fig. 2. Longitudinal beams of grillage

We use the analytic form of the finite element method [3]. Looking for the approximate bending function $w = W_n$ by the method of defining conditions we obtain

$$w = W_n = \sum_{i=0}^{n} \alpha_i \varphi_i \tag{1}$$

and the approximate function of angle of rotation

$$\varphi = \sum_{i=0}^{n} \alpha_i \frac{d\varphi_i}{dx} \tag{1*}$$

where n = 5, α_i - unknown coefficients and the functions φ_i may not satisfy anything, so we take

$$\varphi_i = (x/l)^i \tag{1**}$$

That in Eq. 1 includes 6 unknown coefficients. To find it we take defining six conditions (k = 1,2, ..., 6), which consistently substitute in the integral relation of the theorem of reciprocity of work. In general, this relation can be written as

$$\int_0^l q'_k w dx + \sum_{x=0}^{x=l} M'_k \varphi + \sum_{x=0}^{x=l} Q'_k w = \int_0^l q W'_k dx + \sum_{x=0}^{x=l} M \varphi'_k + \sum_{x=0}^{x=l} Q W'_k \tag{2}$$

where q - distributed load, M - bending moment, Q - shear force, W - bending function existing only in k-conditions. The components of the k-defining condition in Eq. 2 are with apostrophe.
Equation of the bending beam can be written as

$$D(x)\frac{d^4 w}{dx^4} + 2\frac{dD(x)}{dx} \cdot \frac{d^3 w}{dx^3} + \frac{d^2 D(x)}{dx^2} \cdot \frac{d^2 w}{dx^2} = q(x) \tag{3}$$

where $D(x)$ – bending stiffness of the beam, $q(x)$ – distributed load and with the border conditions

x = 0: w = 0, M = 0;

x = l: w = 0, M = 0. (4)

Elastic reactions F_k Fig. 1 are included in the distributed load $q(x)$. Taken into account the border conditions Eq. 4, we obtain in Eq. 2, that

$$\sum_{x=0}^{x=l} Q'_k w = 0, \quad \sum_{x=0}^{x=l} M \varphi'_k = 0 \tag{4*}$$

then Eq. 2 becomes

$$\int_0^l q'_k w dx + \sum_{x=0}^{x=l} M'_k \varphi - \sum_{x=0}^{x=l} Q W'_k = \int_0^l q W'_k dx \tag{5}$$

Bending function of determining the conditions $W_k^{'}$ also may not satisfy neither one of the conditions of the problem, but to simplify the solution we agree to select $W_k^{'}$ so that to satisfy the conditions $W_k^{'}$ at $x = 0$ and $x = l$. Then becomes

$$\sum_{x=0}^{x=l} QW_k^{'} = 0 \tag{5*}$$

and finally from Eq. 5 we get

$$\int_0^l q_k^{'} w dx + \sum_{x=0}^{x=l} M_k^{'} \varphi = \int_0^l q W_k^{'} dx \tag{6}$$

In Eq. 6 are the components (functions) of w, φ and q of the desired state.
From Eqs. (1-1**) we get the approximate functions w and φ

$$w = \alpha_0 + \alpha_1 (x/l) + \alpha_2 (x/l)^2 + ... + \alpha_5 (x/l)^5$$

$$\varphi = \frac{dw}{dx} = \frac{1}{l}\alpha_1 + \frac{2}{l}\alpha_2 (x/l) + ... + \frac{5}{l}\alpha_5 (x/l)^4 \tag{7}$$

Then

at $x = 0 : w = \alpha_0$; at $x = l : w = \alpha_0 + \alpha_1 + ... + \alpha_5$;

at $x = 0 : \varphi = \frac{1}{l}\alpha_1$; at $x = l$; $x = l : \varphi = \frac{1}{l}\alpha_1 + \frac{2}{l}\alpha_2 + ... + \frac{5}{l}\alpha_5$ (7*)

Determining state will take the form

$$W_k{}' = (x/l)^k - (x/l)^{k+1}, \quad k = 1, 2, ..., 6 \tag{8}$$

the remaining components of the defining condition from Eq. 8

$$x = 0: \quad M_k^{'} = -D\frac{d^2 W_k^{'}}{dx^2} = D\left[k(k-1)\left(\frac{x}{l}\right)_{x=0}^{k-2} - (k+1)k\left(\frac{x}{l}\right)_{x=0}^{k-1}\right]\frac{1}{l^2}$$

$$x = l: \quad M_k^{'} = -D\frac{d^2 W_k^{'}}{dx^2} = D\left[k(k-1)\left(\frac{x}{l}\right)_{x=l}^{k-2} - (k+1)k\left(\frac{x}{l}\right)_{x=l}^{k-1}\right]\frac{1}{l^2} \tag{9}$$

In accordance with Eq. 3

$$q_k^{'} = D\frac{d^4 W_k^{'}}{dx^4} + 2\frac{dD(x)}{dx}\cdot\frac{d^3 w}{dx^3} + \frac{d^2 D(x)}{dx^2}\cdot\frac{d^2 w}{dx^2} \tag{10}$$

for example with $k = 3$, we have

$$q_3^{'} = D\left(-\frac{24}{l^4}\right) + 2\frac{dD}{dx}\cdot\frac{1}{l^3}\left(6 - 24\frac{x}{l}\right) + \frac{d^2 D}{dx^2}\cdot\frac{1}{l^2}\left[6\frac{x}{l} - 12\left(\frac{x}{l}\right)^2\right] \tag{11}$$

Substituting the components of the desired condition in Eq. 6, we obtain

$$\int_0^l q_k^{'} \sum_{i=0}^{5} \alpha_i \left(\frac{x}{l}\right)^i dx + M_k^{'}\left(\frac{1}{l}\alpha_1\right)^{(x=0)} + M_k^{'}\left(\frac{1}{l}\alpha_1 + \frac{2}{l}\alpha_2 + ... + \frac{5}{l}\alpha_5\right)^{(x=l)} = \int_0^l q W_k^{'} dx \tag{12}$$

From Eq. 12 we get the allowing equation in general form

$$\sum_{i=0}^{5} C_{ki}\alpha_i = d_k, \quad k = 1,2,...,6 \tag{13}$$

where

$$C_{ki} = \int_0^l q_k^{'}\left(\frac{x}{l}\right)^i dx + M_k^{'}\left(\frac{1}{l}\right)^{(x=0)} + M_k^{'}\left(\frac{1}{l}\right)^{(x=l)}; \qquad d_k = \int_0^l qW_k^{'}dx \tag{14}$$

We substitute in Eq. 12 the components of defining states for k = 1,2, ..., 6 continuously and obtain a system of 6 linear algebraic equations to calculate the coefficients of the approximate bending function $w = W_n$ Eq. 1.

So it would be if the bending stiffness D is given, but the stiffness is unknown. We can write it in a form of a polynomial with undetermined coefficients

$$D_m = \sum_{j=1}^{m} \beta_j \psi_j \tag{15}$$

If we substitute Eq. 15 into Eq. 9 and Eq. 10, whereas the expression of the stiffness Eq. 15 enters in Eq. 12 and the system Eq. 13 becomes a system of nonlinear equations with unknown α_i and β_i. This system has many solutions, therefore it is necessary to include additional conditions.

Suppose formula of bending - functions W_n Eq. 1 is defined for a given load $q(x)$ and limit conditions Eq. 4 by the defining states. However, the function $D_m(x)$ is unknown. This function is in the form Eq.15 depends on the parameters β_i. If we substitute it into the left side of the equation Eq. 3, that in this case $q(x)$ becomes a function $P(\beta_j)$

$$P(\beta_j) = D_m \frac{d^4W_n}{dx^4} + 2\frac{dD_m}{dx}\cdot\frac{d^3W_n}{dx^3} + \frac{d^2D_m}{dx^2}\cdot\frac{d^2W_n}{dx^2} \tag{16}$$

Ideally, if the number of β_j were such that

$$P(\beta_j) = q(x) \tag{17}$$

then the exercise would be solved.
However, the Eq. 17 fails and the error is expressed at every point [4]

$$\delta(\beta_j) = q(x) - P(\beta_j). \tag{18}$$

Then the total square error is

$$\Delta^2 = \int_0^l \delta^2(\beta_j)dx = \int_0^l \left[q(x) - P(\beta_j)\right]^2 dx \tag{19}$$

The unknown coefficients β_j need to pick up so that Δ^2 would be minimal

$$\int_0^l \left[q(x) - P(\beta_j)\right]^2 dx = \min \tag{20}$$

After substituting Eq. 16 in Eq. 20, we obtain

$$\int_0^l \left\{ q(x) - \left[\sum_{j=1}^m \beta_j \psi_j \sum_{i=1}^n \frac{d^4\varphi_i}{dx^4} + 2\sum_{j=1}^m \beta_j \frac{d\psi_j}{dx} \sum_{i=1}^n \alpha_i \frac{d^3\varphi_i}{dx^3} + \sum_{j=1}^m \beta_j \frac{d^2\psi_j}{dx^2} \sum_{i=1}^n \alpha_i \frac{d^2\varphi_i}{dx^2} \right] \right\}^2 dx = \min$$

$$j = 1,2,...,m \tag{21}$$

Differentiating the Eq. 21 respect to β_j we get m equations for the calculation of m unknown β_j

$$\int_0^l \left\{ \left[q(x) - \sum_{j=1}^m \beta_j \psi_j \sum_{i=1}^n \frac{d^4\varphi_i}{dx^4} + 2\sum_{j=1}^m \beta_j \frac{d\psi_j}{dx} \sum_{i=1}^n \alpha_i \frac{d^3\varphi_i}{dx^3} + \sum_{j=1}^m \beta_j \frac{d^2\psi_j}{dx^2} \sum_{i=1}^n \alpha_i \frac{d^2\varphi_i}{dx^2} \right] \cdot \right.$$
$$\left. \cdot \left[-\psi_j \sum_{i=1}^n \alpha_i \frac{d^4\varphi_i}{dx^4} - 2\frac{d\psi_j}{dx} \sum_{i=1}^n \alpha_i \frac{d^3\varphi_i}{dx^3} - \frac{d^2\psi_j}{dx^2} \sum_{i=1}^n \alpha_i \frac{d^2\varphi_i}{dx^2} \right] \right\} dx = 0$$

$$j = 1,2,...,m \tag{22}$$

The system of equations Eq. 22 is a non-linear system of the second order.

Substituting Eq. 15 and the system Eq. 21 in Eq. 12, we obtain a system $m + n$ equations with two groups of unknowns (It is marked as system of equations Eq. 23, and we shall give reference to it in the future but due to its big size we shall not citate it).

To solve the system Eq. 23 we accept the initial values of coefficients α_i which are entered in Eq. 13 and obtain a linear system with β_j as unknowns. Solving this system we find initial values of coefficients β_j. Substituting determined β_j from the Eq. 13 in the system Eq. 23 we obtain a linear system, from which we find α_i. Substituting the finding coefficients α_i in Eq. 21 we find required β_j of the bending stiffness D. If the values of coefficients β_j satisfy the condition of the optimal design Eq. 20 this the corresponding bending stiffness D Eq. 15 is optimal with the minimal usage of material

$$\int_0^l A(x)dx = \min \tag{23}$$

where $A(x) = b\,h(x)$ is an area of the rectangular cross-sections.

During the process of integration it is necessary to check the obtained stiffness of the strength condition. As an initial approximation we take the coefficients α_i corresponding to the exact solution for a given load and a constant bending stiffness D. Doing some calculations, we find for that kind of load $q = 2 - \frac{x}{l}$ the function of bending stiffness $D(x)$ for which the height $h(x)$of the rectangular cross-section will be as follows Fig. 3. According to calculations, beam is not symmetrical. If the lenght of a beam is divided on 15 conditional parts, the widest cross-section is situated on the level $8l/15$. The spared material along length is highlighted with white.

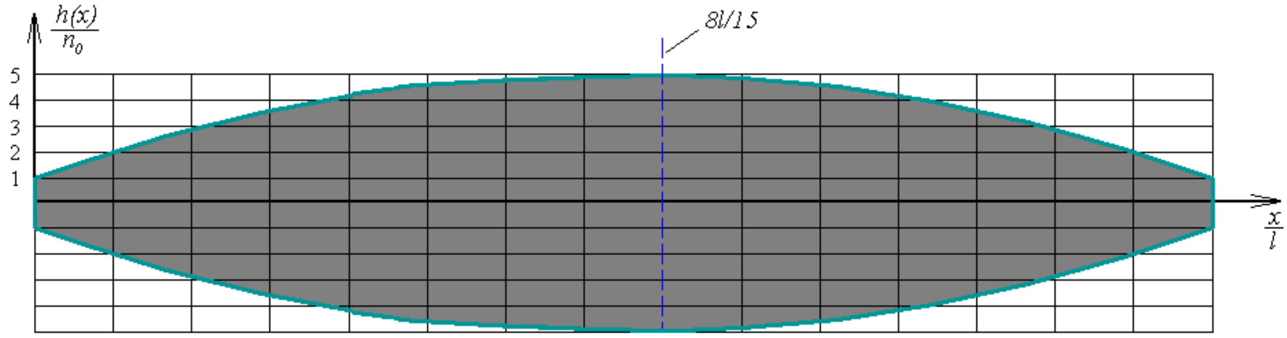

Fig. 3. Theoretical reduction of material usage

Conclusion

In this work is submitted the algorithm to optimize beam-type elements. One of the ways of logical development of this work could be determination of dynamical conditions, making the task more concrete. Some of researches are going to be based on articles [1-4]. The algorithm described in this work takes in attention the most important mathematical and physical parameters, so it can be considered as universal for every beam-type element of trailer construction. It can be widely applied for optimization of trailers and other dynamic constructions in future.

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