Bits Through Relay Cascades with Half-Duplex Constraint

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Abstract

Consider a relay cascade, i.e. a network where the source node, the sink node and a certain number of intermediate relay nodes are arranged in a line. We assume that adjacent node pairs are connected by error-free $(q+1)$-ary pipes. The following communication scenario is treated. The source and a subset of the relays wish to communicate independent information to a common sink under the condition that each relay in the cascade is half-duplex constrained. We introduce a simple channel model for half-duplex constrained links and provide a coding scheme which transfers information by an information-dependent, non-deterministic allocation of the transmission and reception slots of the relays. The coding scheme requires synchronization on the symbol level through a shared clock. In the case of a relay cascade with a single source, the coding strategy is capacity achieving. Numerical values for the capacity of cascades of various lengths are provided, and it turns out that the capacities are significantly higher than the rates which are achievable with a deterministic time-sharing approach. If the cascade includes a source and a certain number of relays with their own information, the strategy achieves the cut-set bound when the rates of the relay sources fall below individual thresholds. Hence, a partial characterization of the boundary of the capacity region follows. For cascades composed of an infinite number of half-duplex constrained relays and a single source, we derive an explicit capacity expression. Remarkably, the capacity for $q = 1$ is equal to the logarithm of the golden ratio. We finally show that the proposed coding strategy is superior to network coding in the case of the wireless, half-duplex constrained butterfly network.

Index Terms

Half-duplex constraint, relay networks, network coding, timing, constrained coding, capacity, capacity region, method of types, golden ratio.

I. Introduction

A relay cascade is a network where the source node, the sink node and a certain number of intermediate relay nodes are arranged in a line. In this paper we consider the problem that a source node and an arbitrary but fixed number of relay nodes from the cascade wish to communicate independent messages to a common sink under the condition that each relay is half-duplex constrained, i.e. is not able to transmit and receive simultaneously. Throughout the paper, we assume that adjacent node pairs are connected by error-free $(q+1)$-ary pipes. This approach allows us to gain a better understanding of half-duplex constrained transmission without having to distinguish which effects are due to channel noise and which result from the half-duplex constraint. Moreover, the problem becomes more feasible since combinatorial arguments can be used instead of statistical arguments.

How could we construct an efficient coding scheme which takes the half-duplex constraint into account? A first approach would probably be to define a protocol such that the time-division schedule is determined a priori. Under this assumption, the capacity or rate region of various half-duplex constrained relay channels [2], [3] and networks [4] has been determined. We will show that time-sharing falls considerably short of the theoretical optimum or, conversely, higher rates are possible by an information-dependent, non-deterministic allocation of the transmission and reception slots of the relays.

The meaning of information-dependent allocation scheme is illustrated in following example. Let $\mathcal{W}_0 = \{0, \ldots, 7\}$ be a message set. In each block $i = 1, 2, \ldots$ of length 4, the source wishes to communicate a randomly chosen message $w_0(i) \in \mathcal{W}_0$ to the destination via a single half-duplex constrained relay node. The alphabet of both source and relays equals $\{0, 1, \mathrm{N}\}$ where "N" indicates a channel use without transmission and $\{0, 1\}$ is a $q = 2$-ary transmission alphabet. Let $\mathbf{x}_0(i)$ be the codeword chosen by the source encoder to represent $w_0(i)$ in block i and let $\mathbf{x}_1(i)$ indicate the codeword chosen by the relay encoder for representing $w_0(i-1)$ in block i. The coding scheme is illustrated in Table I. The source encoder maps each message $w_0(i)$ to $\mathbf{x}_0(i)$ by allocating the corresponding binary representation of $w_0(i)$, i.e. three bits, to four time slots. The precise allocation of the three bits to four slots is determined by the first two binary digits of codeword $\mathbf{x}_0(i-1)$. Based on the first two binary digits of the noiselessly received codeword $\mathbf{x}_0(i)$, the relay encoder determines which time slot to use for transmission in $\mathbf{x}_1(i+1)$. The binary value to be transmitted in $\mathbf{x}_1(i+1)$ is equal to the third bit in $\mathbf{x}_0(i)$. Hence, the relay encodes a part of its information in the timing of the transmission symbols. Since the source encoder knows the scheme used to determine the relay's transmission slot, it can allocate its three new bits in $\mathbf{x}_0(i+1)$ to those slots in which the relay is able to listen. The sink then determines the message from the received relay codeword using both the position of the transmission

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symbol and its value. In this example, a rate of 0.75 bit/use is achievable. By allowing arbitrarily long codewords, we will show that the strategy approaches 1.1389 bits per use which is also the capacity of the single relay cascade with half-duplex constraint when the transmission alphabet is binary.

$R = 0.75$ bit per use				
block i	$w_0(i)$	$\mathbf{x}_0(i)$	$\mathbf{x}_1(i)$	$\hat{w}_0(i)$
$i = 1$	1 (001)	001N	NNNN	-
$i = 2$	2 (010)	N010	1NNN	1
$i = 3$	4 (100)	1N00	N0NN	2
$i = 4$	7 (111)	11N1	NN0N	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE I

THE RELAY ENCODES A PART OF THE INFORMATION BY THE POSITION OF THE TRANSMISSION SYMBOLS.

The example suggests that information encoding by means of timing is beneficial in the context of half-duplex constrained transmission. A similar example for $q = 1$ was shown in [5]. In [6], Kramer applied the achievable decode and forward rates of the relay channel due to Cover and El Gamal [7] to a half-duplex constrained relay channel and noticed that higher rates are possible when the transmission and reception time slots of the relay are random. The randomness results from the fact that one can send information through the timing of operating modes. Timing is not a new idea in the information theoretical literature and has already been used in conjunction with queuing channels. Anantharam and Verdú showed [8] that encoding information into the distances of arrival to the queue achieves the capacity of the single server queue with exponential service distribution. The analog in discrete-time was analyzed in [9].

In Section II we introduce a channel model which captures the half-duplex constraint in a simple way. A capacity achieving coding strategy based on allocating the transmission and reception time slots of a node relying on the node's previously received data is introduced in Section III. The proposed strategy requires synchronization on the symbol level through a shared clock. In Section IV, the performance of the coding strategy is analyzed yielding several capacity results. In the case of a relay cascade with a single source, it will be shown that the coding strategy is capacity achieving, i. e. approaches a rate equal to

$$C_{m-1}(q) = \max_{p_{X_0 \dots X_m}} \min_{1 \le i \le m} H(Y_i | X_i) \tag{I.1}$$

where $m - 1$ indicates the number of relays in the cascade and X_i and Y_i are the sent and received symbol of relay i. If the cascade includes a source and a certain number of relays with their own information, the strategy achieves the cut-set bound given that the rates of the relay sources fall below individual thresholds. Hence, a partial characterization of the boundary of the capacity region follows. For cascades composed of an infinite number of half-duplex constrained relays, we show that the capacity is given by

$$C_\infty(q) = \log_2 \left(\frac{1 + \sqrt{4q+1}}{2} \right) \text{ bits per use.} \tag{I.2}$$

Remarkably, $C_\infty(1)$ is equal to the logarithm of the golden ratio. In Section V the capacity results are applied to various special cases. In particular, we transform (I.1) into a convex optimization program with linear objective and provide numerical solutions for $C_{m-1}(q)$ for different values of m and q. In the case of a single relay channel with a source and a relay source, an explicit expression of the cut-set bound and of the achievable segment on the cut-set bound will be stated. We finally show that the proposed coding strategy is superior to network coding in the case of the wireless, half-duplex constrained butterfly network.

II. NETWORK MODEL

We consider a discrete memoryless relay cascade as depicted in Fig. 1. The underlying topology corresponds to a directed path graph in which each node is labeled by a distinct number from $\mathcal{V} = \{0, \dots, m\}$ with $m > 0$. The integers 0 and m belong to source and sink, respectively, while all remaining integers 1 to $m-1$ represent half-duplex constrained relays, i. e. relays which cannot transmit and receive at the same time. The connectivity within the network is described by the set of edges $\mathcal{E} = \{(i, i+1) : 0 \le i \le m-1\}$, i.e. the ordered pair $(i, i+1)$ represents the communications link from node i to node $i+1$. The output of the ith node, which is the input to channel $(i, i+1)$ is denoted as X_i and takes values on the alphabet $\mathcal{X} = \{0, \dots, q-1\} \cup \{\text{N}\}$ where $\mathcal{Q} = \{0, \dots, q-1\}$ denotes the q-ary transmission alphabet while "N" is meant to signify a channel use in which node i is not transmitting. The input of the ith node, which is the output of channel $(i-1, i)$ is denoted as Y_i. Each message w_0, sent via multiple hops from node 0 to m at a transmission rate R_0, is uniformly drawn from the index set $\mathcal{W}_0 = \left\{1, 2, \dots, 2^{nR_0}\right\}$ where n is the block length of the coding scheme. Besides forwarding previously received information, an arbitrary but fixed number of relay nodes also act as sources, i. e. each relay $v \in \mathcal{V}_s$ intends to transmit its own messages at rate $R_{\alpha(v)}$ from $\mathcal{W}_{\alpha(v)} = \left\{1, 2, \dots, 2^{nR_{\alpha(v)}}\right\}$ to the destination, where $\mathcal{V}_s$ summarizes all relays with their

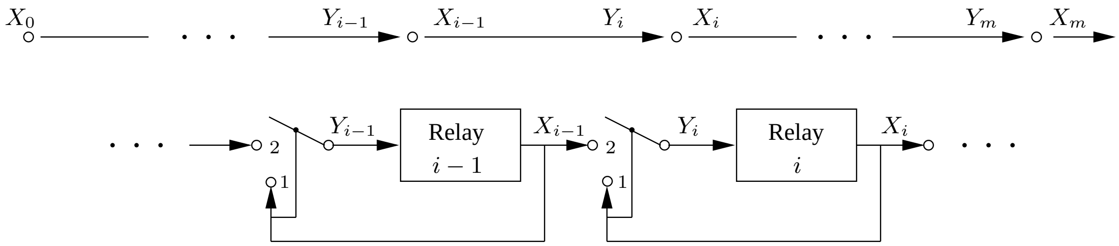


Fig. 1. A noiseless relay cascade (top) and the link model illustrated by means of feedback. If relay i is transmitting, the switch is in position 1 otherwise in position 2.

own messages and source node 0. Note that the bijection $\alpha : \mathcal{V}_s \rightarrow \{0, \ldots, |\mathcal{V}_s| - 1\}$ numbers the elements of $\mathcal{V}_s$ according to the order in which they appear in the cascade. Again, the transmission involves a multi-hop scheme since the information flow associated with a particular message $w_{\alpha(v)} \in \mathcal{W}_{\alpha(v)}$ has to pass all nodes between the corresponding (relay) source v and the destination.

The output symbol of channel $(i-1, i)$ is given by

$$Y_i = \begin{cases} X_{i-1}, & \text{if } X_i = \text{N} \\ X_i, & \text{if } X_i \in \mathcal{Q} \end{cases} \tag{II.1}$$

where $1 \leq i \leq m$. Our channel model (II.1) captures the half-duplex constraint as follows. Assume relay i is in transmission mode, i.e. $x_i \in \mathcal{Q}$. Then relay i hears itself ($Y_i = X_i$) but cannot listen to relay $i-1$ or, equivalently, relay i and relay $i-1$ are disconnected. However, if relay i is not transmitting, i. e. $x_i = \text{N}$, it is able to listen to relay $i-1$ via a noise-free $(q+1)$-ary pipe ($Y_i = X_{i-1}$). The sink listens all the time, i. e. x_m is always equal to N, and therefore its input is given by $Y_m = X_{m-1}$. Another interpretation of the channel model is that the output of relay i controls the position of a switch which is placed at its input. If relay i is transmitting, the switch is in position 1 otherwise it is in position 2 (see Fig. 1). Since a pair of nodes is either perfectly connected or disconnect, we obtain a deterministic network with $p(y_1, \ldots, y_m | x_0, \ldots, x_m) \in \{0, 1\}$.

III. A Timing Code for Line Networks with Multiple Sources

A. General Idea and Codebook Sizes

A coding strategy is introduced which relies on the observation that information can be represented not only by the value of code symbols but also by timing the transmission and reception slots of the relay nodes. The strategy requires synchronization on the symbol level through a shared clock.

The encoding technique applied at the source and the relays is as follows. The source uses a $(q+1)$-ary alphabet $\mathcal{X}$ for encoding without transmitting information in the timing of the symbols while relay i represents information by taking n_i transmission symbols per block from the q-ary transmission alphabet $\mathcal{Q}$ combined with allocating the n_i symbols to the transmission block. Then, at most $q^{n_{m-1}}\binom{n}{n_{m-1}}$ different sequences $\mathbf{x}_{m-1}$ of length n can be generated by relay $m-1$ where $q^{n_{m-1}}$ denotes the number of distinct sequences when the q-ary symbols are located at fixed slots while $\binom{n}{n_{m-1}}$ denotes the number of possible slot allocations. Due to the half-duplex constraint, the effective codeword length of relay $m-2$ reduces to $n - n_{m-1}$. This results from the fact that relay $m-1$ cannot pay attention to relay $m-2$ when relay $m-1$ transmits and, therefore, the number of length n sequences producible by relay $m-2$ is at most $q^{n_{m-2}}\binom{n-n_{m-1}}{n_{m-2}}$. The same argument holds for each relay in the cascade. In general, relay $i \in \{1, \ldots, m-1\}$ is able to generate $q^{n_i}\binom{n-n_{i+1}}{n_i}$ distinct sequences where $n_m = 0$ since the sink listens all the time. Finally, the effective length of the source codeword is $n - n_1$ which enables the source to generate $(q+1)^{n-n_1}$ different sequences $\mathbf{x}_0$.

Next, the maximum size of $\mathcal{W}_0$, $\mathcal{W}_1, \ldots,$ $\mathcal{W}_{|\mathcal{V}_s|-1}$ is given. Since the node with the smallest number of available sequences is obviously a bottleneck in the cascade from source to destination, we immediately obtain an upper bound on $|\mathcal{W}_0|$ which is

$$|\mathcal{W}_0| \leq \min\left\{(q+1)^{n-n_1}, \min_{1 \leq i \leq m-1} q^{n_i}\binom{n-n_{i+1}}{n_i}\right\}. \tag{III.1}$$

Both the source and the relay sources choose their messages uniformly and independent of each other. Hence, relay source v is required to have $\prod_{i=0}^{\alpha(v)} |\mathcal{W}_i|$ sequences available in order to represent an arbitrary set of arriving messages together with its own message. Consequently, for all $v \in \mathcal{V}_s \setminus \{0\}$ we obtain

$$|\mathcal{W}_{\alpha(v)}| \leq \left\lfloor \left(\prod_{i=0}^{\alpha(v)-1} |\mathcal{W}_i|\right)^{-1} \min_{v \leq i \leq m-1} q^{n_i}\binom{n-n_{i+1}}{n_i} \right\rfloor \tag{III.2}$$

where the minimization in (III.2) yields the bottleneck, i. e. the least number of available sequences, between relay source v and the destination.

The transmission rate $R_{\alpha(v)}$ of (relay) source $v \in \mathcal{V}_s$ is defined (in the standard way) as $R_{\alpha(v)} = \log |\mathcal{W}_{\alpha(v)}|/n$. Assume the cascade contains $\mathcal{V}_s \geq 2$ nodes with their own information. Further, assume that $\sum_{i=0}^{\alpha(v)-1} R_i > 0$ where $\mathcal{V}_s \ni v > 0$, i.e. relay source v has to forward external information. Then the external information, which flows from node $v-1$ to node v in block $b-1$, has to determine the slot allocation used by node v in block b, i.e. the transmission pattern selected by node v is not allowed to depend on $w_{\alpha(v)}(b)$. Otherwise, node $v-1$ would not know when transmission in block b is possible without collision where collision means that at least one transmission of two adjacent nodes occurs in the same time slot. Thus, the message sets have to satisfy for all $v \in \mathcal{V}_s \setminus \{0\}$

$$|\mathcal{W}_{\alpha(v)}| \leq \begin{cases} q^{n_v}, & \text{if} \sum_{i=0}^{\alpha(v)-1} R_i > 0 \\ q^{n_v}\binom{n-n_{v+1}}{n_v}, & \text{else} \end{cases} . \tag{III.3}$$

Note that the bottom constraint is already contained in (III.2).

B. Example

We now illustrate the ideas introduced in the previous section by constructing a code for a relay cascade with $\mathcal{V} = \{0, \dots, 3\}$ and $\mathcal{V}_s = \{0, 2\}$. The transmission alphabet is binary, i. e. $q = 2$, and the code parameters are $n = 4$, $n_1 = 1$, $n_2 = 2$. According to (III.1) to (III.3), the maximum size of the message sets is $|\mathcal{W}_0| = |\mathcal{W}_1| = 4$, which corresponds to a sum rate of 1 bit per use. Table II depicts possible codebooks $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ for nodes 0, 1 and 2, respectively. The codewords in the first row are used for representing source index 0, the codewords in the second row for representing source index 1 and so forth. The last row emphasizes that a codeword $\mathbf{x}_k(i) \in \mathcal{C}_k$, which is sent in block i by node k, represents a message $w_0(i-k)$ injected by the source encoder in block $i-k$ with $k \in \{0, 1, 2\}$.

Let us first consider $\mathcal{C}_2$. Four out of six possible transmission patterns are shown where a binary transmission slot is marked with B $\in \{0, 1\}$. Node 2 uses each transmission pattern for representing a particular source message $w_0 \in \mathcal{W}_0$. Node 2's message $w_1 \in \mathcal{W}_1$ is encoded by the transmission symbols B. Each transmission pattern is identified with a unique color $r \in \{a, b, c, d\}$.

w_0	$\mathcal{C}_0$			$\mathcal{C}_1$				$\mathcal{C}_2$
0	N0NN e	0NNN f	0NNN g	0NNN (a,e)	N0NN (b,f)	0NNN (c,e)	N0NN (d,f)	NBNB a
1	N1NN e	1NNN f	1NNN g	1NNN (a,e)	N1NN (b,f)	1NNN (c,e)	N1NN (d,f)	BNBN b
2	NN0N e	NN0N f	N0NN g	NN0N (a,g)	NNN0 (b,g)	NNN0 (c,g)	NN0N (d,g)	NBBN c
3	NN1N e	NN1N f	N1NN g	NN1N (a,g)	NNN1 (b,g)	NNN1 (c,g)	NN1N (d,g)	BNNB d
	$w_0(i) \mapsto \mathbf{x}_0(i)$			$w_0(i-1) \mapsto \mathbf{x}_1(i)$				$w_0(i-2) \mapsto \mathbf{x}_2(i)$

TABLE II
EXAMPLE CODEBOOKS FOR SOURCE, RELAY AND RELAY SOURCE.

Next, $\mathcal{C}_1$ is considered. Since node 1 knows the message w_0 to be forwarded by node 2 as well as codebook $\mathcal{C}_2$, it can always figure out both time slots t_1 and t_2 in which node 2 listens. Let $x_{1t_1}, x_{1t_2} \in \{0, 1, \mathrm{N}\}$ denote the symbols used by node 1 in t_1 and t_2 for encoding a particular source message. The following mapping $w_0 \mapsto (x_{1t_1}, x_{1t_2})$ is chosen: $0 \mapsto (0, \mathrm{N})$, $1 \mapsto (1, \mathrm{N})$, $2 \mapsto (\mathrm{N}, 0)$, $3 \mapsto (\mathrm{N}, 1)$. By allocating each of the four values of (x_{1t_1}, x_{1t_2}) to the listen slots of pattern $r \in \{a, b, c, d\}$ and, further, by requiring that node 1 is quiet when node 2 sends a binary symbol B, we obtain the codewords in $\mathcal{C}_1$ which are colored by (r, s). Color $s \in \{e, f, g\}$ labels the resulting transmission patterns in $\mathcal{C}_1$. Color r helps node 1 to pick the new codeword from the correct column, i.e. when node 2 uses a pattern with color r node 1 uses in the same transmission block a codeword whose first color is equivalent to r.

Finally, we consider $\mathcal{C}_0$. In each transmission block, source node 0 can use three time slots t_1, t_2 and t_3 for encoding since node 1 sends once per block. Let $x_{0t_1}, x_{0t_2}, x_{0t_3} \in \{0, 1, \mathrm{N}\}$ denote the symbols used by node 0 for encoding a particular message $w_0 \in \mathcal{W}_0$. We use a similar mapping as before, i. e. $w_0 \mapsto (x_{0t_1}, x_{0t_2}, x_{0t_3})$ is given by $0 \mapsto (0, \mathrm{N}, \mathrm{N})$, $1 \mapsto (1, \mathrm{N}, \mathrm{N})$, $2 \mapsto (\mathrm{N}, 0, \mathrm{N})$, $3 \mapsto (\mathrm{N}, 1, \mathrm{N})$. Now, by allocating all possible values of $(x_{0t_1}, x_{0t_2}, x_{0t_3})$ to the listen slots of codewords in $\mathcal{C}_1$ whose second color is s and, further, by requiring that node 0 is quiet when node 1 transmits, we obtain all codewords in $\mathcal{C}_0$ which are colored with s. It should be noted that merely four from 27 possible sequences are used in the mapping $w_0 \mapsto (x_{0t_1}, x_{0t_2}, x_{0t_3})$. Hence, $\mathcal{C}_0$ could be designed such that node 0 is able to send $\lfloor 27/4 \rfloor$ additional messages to a sink at node 1 at a rate of 0.6462 bit per use. In summary, the source encoder applies the following strategy. Based on message $w_0(i-2)$, the first color r of codeword $\mathbf{x}_1(i)$ is determined. Subsequently, based on this information, the source determines the second color s of $\mathbf{x}_1(i)$ by means of $w_0(i-1)$. This color tells node 0 from which column the new codeword has to be picked, namely from a column whose codewords are colored with s. The precise choice within the picked column depends on the new source message $w_0(i)$.

C. Rate Region

We now turn towards gaining an achievable rate region $\mathcal{R}$ from the expressions derived in section III-A. The following abbreviations are used for the portion of time in which relay i listens or transmits: $p_i = n^{-1}(n - n_i), 0 \leq n_i < n$, and $\bar{p}_i = 1 - p_i$ where $p_i = p_{X_i}(\mathrm{N})$, $1 \leq i \leq m$.

The method of types [10] provides important tools for relating combinatorial expressions to information theoretic expressions. An example very useful for the problem considered here is [11, Th. 1.4.5]

$$n^{-1} \log \binom{n}{n_i} = H(p_i) + o(1) \quad \text{for } n \to \infty \tag{III.4}$$

where $H(p_i)$ denotes the binary entropy function evaluated at $p_i = n^{-1}(n - n_i)$. By (III.4) and $R_i = \log |\mathcal{W}_i|/n$, we obtain from (III.1) to (III.3) for $n \to \infty$

$$R_0 \leq \min \left\{ p_1 \log(q+1), \min_{1 \leq i \leq m-1} \left(\bar{p}_i \log q + p_{i+1} H\left(\bar{p}_i p_{i+1}^{-1}\right) + o(1) \right) \right\} \tag{III.5}$$

$$\sum_{i=0}^{\alpha(v)} R_i \leq \min_{v \leq i \leq m-1} \left\{ \bar{p}_i \log q + p_{i+1} H\left(\bar{p}_i p_{i+1}^{-1}\right) + o(1) \right\}, \quad \forall v \in \mathcal{V}_s \setminus \{0\} \tag{III.6}$$

$$R_{\alpha(v)} \leq \bar{p}_v \log q, \qquad \text{if } \sum_{i=0}^{\alpha(v)-1} R_i > 0, \quad \forall v \in \mathcal{V}_s \setminus \{0\}. \tag{III.7}$$

Further simplifications are possible by taking into account the optimal structure of the marginal distributions $p_{X_0X_1}, \ldots, p_{X_{m-1}X_m}$ as shown in Tables III and IV. The zero probabilities in Table III and IV result from following consideration.

$X_0 \backslash X_1$	0	$\cdots$	$q-1$	N
0	0	$\cdots$	0	$p_1/(q+1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$q-1$	0	$\cdots$	0	
N	$\bar{p}_1/q$	$\cdots$	$\bar{p}_1/q$	$p_1/(q+1)$

TABLE III
OPTIMAL $p_{X_0X_1}$.

$X_{i-1} \backslash X_i$	0	$\cdots$	$q-1$	N
0	0	$\cdots$	0	$\bar{p}_{i-1}/q$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$q-1$	0	$\cdots$	0	$\bar{p}_{i-1}/q$
N	$\bar{p}_i/q$	$\cdots$	$\bar{p}_i/q$	$p_i - \bar{p}_{i-1}$

TABLE IV
OPTIMAL $p_{X_{i-1}X_i}$ FOR $2 \leq i \leq m$. NOTE THAT $p_m = 1$.

Assume relay i is transmitting, i. e. $x_i \in \mathcal{Q}$. According to the underlying channel model, relay i is not able to detect the input of node $i-1$ and, consequently, node $i-1$ should not transmit when node i transmits. Or, to be more precise, a channel input pair (x_{i-1}, x_i) is negligible if it produces the same channel output pair (y_i, y_{i+1}) as another channel input pair and this with the same probabilities. Hence only one non-zero entry remains in each of the first q columns of Table III and IV whereas the assignment of the non-zero entry within a column is not unique from an information theoretic viewpoint. However, from an engineering point of view, the assignment as depicted in both tables is reasonable since node $i-1$ should not transmit and, therefore, waste transmit power when its input cannot be detected by node i. As a simple consequence of the zero probability assignment, we have the relation $p_{X_{i-1}X_i}(\mathrm{N}, k) = p_{X_iX_{i+1}}(k, \mathrm{N})$ for all $k \in \mathcal{Q}$ and $1 \leq i \leq m-1$.

Let us now address the remaining values in Table III and IV. First, consider the time slots in which the first relay listens. During this fraction of time, the source should make optimum use of the channel by encoding with uniformly distributed input symbols. Hence, $p_{X_0X_1}(k, \mathrm{N}) = p_{X_0X_1}(l, \mathrm{N})$ for all $k, l \in \mathcal{X}$. By taking the relative frequency of the transmission symbols into account, we have $p_{X_0X_1}(k, \mathrm{N}) = p_1/(q+1)$. Moreover, in order to achieve the maximum information flow from relay $i-1$ to node i or, likewise, from the fact that a permutation of the transmission symbols $x_{i-1} \in \mathcal{Q}$ obviously yields the same information flow, we can choose $p_{X_{i-1}X_i}(k, \mathrm{N}) = p_{X_{i-1}X_i}(l, \mathrm{N})$ for all $k, l \in \mathcal{Q}$. Due to the relative frequency of transmission symbols within a block, we have $p_{X_{i-1}X_i}(k, \mathrm{N}) = \bar{p}_{i-1}/q$ for all $k \in \mathcal{Q}$ where $2 \leq i \leq m$.

It is now fairly easy to check that the following equalities hold

$$\begin{aligned} H(Y_1|X_1) &= p_1 \log(q+1) && \text{(III.8)} \\ H(Y_{i+1}|X_{i+1}) &= \bar{p}_i \log q + p_{i+1} H\left(\bar{p}_i p_{i+1}^{-1}\right), \quad 1 \le i \le m-1. && \text{(III.9)} \end{aligned}$$

From (III.5) to (III.7) together with (III.8) and (III.9), we obtain

$$\mathcal{R} = Co\left(\bigcup_{p:=p_{X_0\ldots X_m}} \mathcal{R}_p\right) \tag{III.10}$$

where $\mathcal{R}_p$ for $n \to \infty$ is given by[1]

$$\begin{aligned} \mathcal{R}_p = \quad & \left\{\mathbf{R} : \sum_{k=0}^{\alpha(v)} R_k \le \min_{v+1 \le i \le m} H(Y_i|X_i), \quad \forall v \in \mathcal{V}_s\right\} \\ & \bigcap \left\{\mathbf{R} : R_{\alpha(v)} \le \bar{p}_v \log q \vee \sum_{i=0}^{\alpha(v)-1} R_i = 0, \quad \forall v \in \mathcal{V}_s \setminus \{0\}\right\}. \end{aligned} \tag{III.11}$$

and $Co(-)$ denotes the convex hull, i.e. takes timesharing into account. Note that $\mathbf{R}$ is a $\mathcal{V}_s$-dimensional rate vector with R_i as its ith entry.

IV. Capacity Results

In this section we shall investigate the optimality of the coding strategy.

Theorem 1*:* A part $\mathcal{C}'$ of the capacity region $\mathcal{C}$ of a noise-free relay cascade with $|\mathcal{V}_s|$ sources and $m-1$ half-duplex constrained relays is given by

$$\mathcal{C}' = \bigcup_{p_{X_0\ldots X_m}} \left\{\mathbf{R} : \sum_{k=0}^{\alpha(v)} R_k \le \min_{v+1 \le i \le m} H(Y_i|X_i), \quad \forall v \in \mathcal{V}_s\right\} \tag{IV.1}$$

i.e. if the elements of $\mathbf{R}$ satisfy

$$\left\{\mathbf{R} : R_{\alpha(v)} \le \bar{p}_v \log q \vee \sum_{i=0}^{\alpha(v)-1} R_i = 0, \quad \forall v \in \mathcal{V}_s \setminus \{0\}\right\} \tag{IV.2}$$

then (IV.1) yields the corresponding boundary points of $\mathcal{C}$ for some joint distribution $p_{X_0\ldots X_m}$.

Proof: We first show which subset of the network cuts is sufficient for the considered line network. Recall that an upper bound on the sum rate $\sum_{k=0}^{\alpha(v)} R_k$ is given by [12, chap. 14.10]

$$\sum_{k=0}^{\alpha(v)} R_k \le \max_{p_{X_v\ldots X_m}} \min_{S \in \mathcal{M}} I(X_v, X_{S^c}; Y_S, Y_m|X_S), \tag{IV.3}$$

where $\mathcal{M} = \mathcal{P}\left(\{v+1, \ldots, m-1\}\right)$ and S^c is the complement of S in $\{v+1, \ldots, m-1\}$. Since our network model is deterministic, (IV.3) simplifies to

$$\sum_{k=0}^{\alpha(v)} R_k \le \max_{p_{X_v\ldots X_m}} \min_{S \in \mathcal{M}} H(Y_S, Y_m|X_S). \tag{IV.4}$$

Now assume that S is nonempty and let $i \in \{v+1, \ldots, m-1\}$ denote the smallest integer in S. By the chain rule for entropy, we can expand $H(Y_S, Y_m|X_S)$ as

$$\begin{aligned} H(Y_S, Y_m|X_S) &= H(Y_i|X_S) + H(Y_{S\setminus\{i\}}|X_S, Y_i) + H(Y_m|X_S, Y_S) \\ &\ge H(Y_i|X_S). \end{aligned} \tag{IV.5}$$

For each cut S with smallest entry i, a cut here called S_i can be found such that $H(Y_{S_i}, Y_m|X_{S_i})$ is less than or equal to $H(Y_S, Y_m|X_S)$. Simply choose $S_i := \{i, \ldots, m-1\}$. This eliminates the second and third term on the rhs of (IV.5) due to the underlying channel model. Further, since $S \subseteq S_i$ we have $H(Y_i|X_S) \ge H(Y_i|X_{S_i})$ and, thus, each non-empty cut S with

[1]Note that the Landau symbols are neglected for the sake of simple notation.

smallest element i is dominated by S_i in terms of delivering a smaller entropy value. Finally, $S=\emptyset$ has to be considered in (IV.4); $S=\emptyset$ yields[2] $H(Y_m)$. To sum up, $\sum_{k=0}^{\alpha(v)} R_k$ is upper bounded by

$$\sum_{k=0}^{\alpha(v)} R_k \leq \max_{p_{X_v \dots X_m}} \min_{v+1 \leq i \leq m} H(Y_i|X_{S_i}) \tag{IV.6}$$

$$\leq \max_{p_{X_v \dots X_m}} \min_{v+1 \leq i \leq m} H(Y_i|X_i) \tag{IV.7}$$

where the last inequality follows from the fact that conditioning reduces entropy. Therefore, the cut-set bound $\hat{\mathcal{C}}$ is given by

$$\hat{\mathcal{C}} = \bigcup_{p_{X_0 \dots X_m}} \left\{ \mathbf{R} : \sum_{k=0}^{\alpha(v)} R_k \leq \min_{v+1 \leq i \leq m} H(Y_i|X_i), \quad \forall v \in \mathcal{V}_s \right\}. \tag{IV.8}$$

Let $\mathcal{C}$ denote the capacity region, i.e. $\mathcal{C} \subseteq \hat{\mathcal{C}}$. If we focus on rate vectors $\mathbf{R}$ whose elements satisfy (IV.2), then $\mathcal{R}'$ defined as

$$\mathcal{R}' = \bigcup_{p := p_{X_0 \dots X_m}} \mathcal{R}_p \tag{IV.9}$$

(i.e. $\mathcal{R}$ from (III.10) and (III.11) without the timesharing points) equals $\hat{\mathcal{C}}$. Thus, $\mathcal{R}' = \mathcal{C}$ under constraint (IV.2). ■

Corollary 1*:* The capacity of a noise-free relay cascade with a single source-destination pair and $m-1$ half-duplex constrained relays is given by

$$C_{m-1}(q) = \max_{p_{X_0 \dots X_m}} \min_{1 \leq i \leq m} H(Y_i|X_i) \tag{IV.10}$$

where q equals the number of transmission symbols.

The capacity of a single source line network with an infinite number of half-duplex constrained relays is stated in Theorem 2.

Theorem 2*:* For $m \to \infty$, i. e. for an unbounded number of relays, and q transmission symbols, the capacity of the noise-free and half-duplex constrained relay cascade with a single source-destination pair is given by

$$C_\infty(q) = \log_2 \left(\frac{1 + \sqrt{4q+1}}{2} \right) \text{ bits per use.} \tag{IV.11}$$

Proof: Theorem 2 is proved in the Appendix. ■

Remarks:

i) In order to achieve $C_\infty(q)$ it follows from equation (A.6) that each relay has to transmit τ percentage of the time where

$$\tau = 50 \left(1 - \frac{1}{\sqrt{4q+1}} \right). \tag{IV.12}$$

ii) $C_\infty(1) = 0.6942$ bit per use is equal to the logarithm of the *golden ratio*. Also remarkable, $C_\infty(2)$ is exactly 1 bit per use.

iii) The maximum achievable rates with time-sharing and, thus, no timing, are given by $R_{ts}(q) = \log_2 \sqrt{q+1}$ bits per use. For $q = 1, 2$ we have 0.5 and 0.7925 b/u, respectively. Since $C_\infty(q)$ is obviously a lower bound on the capacity of each finite length cascade, a comparison of the time-sharing rates with $C_\infty(1)$ and $C_\infty(2)$ shows that time-sharing falls considerably short of the theoretical achievability for small transmission alphabets. For very large transmission alphabets the gap between the rates due to time-sharing and timing becomes negligible, i.e. $\lim_{q\to\infty} (C_\infty(q) - R_{ts}(q)) = 0$.

V. Numerical Examples

In this section we shall provide numerical capacity results for various scenarios by means of Theorem 1 and Corollary 1. In particular, we show how to obtain the capacity of a half-duplex constrained relay cascade with one source-destination pair for an arbitrary number of relays. Further, in case of a single relay cascade with source and relay source, an explicit expression of the region due to Theorem 1 is derived. Throughout the section, the base of the logarithm is assumed to be two.

[2]Note that $H(Y_m) = H(Y_m|X_m)$.

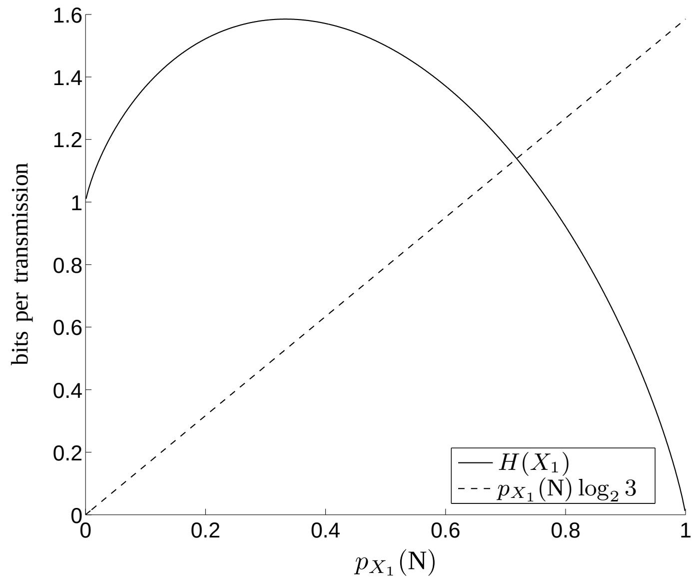


Fig. 2. Graphical solution of optimization problem (V.1).

A. One Source

Let us first consider a relay cascade with $\mathcal{V} = \{0, 1, 2\}$, $\mathcal{V}_s = \{0\}$ and $q = 2$, i. e. source node 0 intends to communicate with sink node 2 via the half-duplex constrained relay 1. By Corollary 1 and the optimum input pmf stated in Table III, we have

$$C_1(2) = \max_{p_{X_0X_1X_2}} \min \{p_{X_1}(\mathrm{N}) \log_2 3, H(X_1)\}\,. \tag{V.1}$$

Problem (V.1) exhibits a single degree of freedom. Since the maximum does not occur in the maximum of one of the two (concave) functions, the problem is readily solved by finding a $p_{X_1}(\mathrm{N})$ which satisfies $p_{X_1}(\mathrm{N}) \log_2 3 = H(X_1)$ (see Fig. 2). The optimum value for $p_{X_1}(\mathrm{N})$ equals 0.7185 which yields

$$C_1(2) = 1.1389 \text{ bits per use.} \tag{V.2}$$

Remarks:

i) Assume the relay does not have the capability to decide whether the source has transmitted or not, i. e. $p_{X_0X_1}(\mathrm{N}, \mathrm{N}) = 0$. In this case an identical approach shows that the capacity equals 0.8295 bit per use, which is still greater than the time-sharing rate of $\log_2 \sqrt{3}$ bit per use.

ii) For $q = 1$, the outlined procedure yields $C_1(1) = 0.7729$ bit per use achieved by $p_{X_1}(\mathrm{N}) = 0.7729$. The capacity value of this specific case has also been obtained in [13]. Therein, the focus was not on half-duplex constrained transmission but on finding the capacity of certain classes of deterministic relay channels. In [5], the same channel model was considered and the author noticed that the capacity is greater than 0.5 bit per use though a half-duplex constrained relay is modeled. A simple coding scheme was outlined which approaches $2/3$ bit per use.

In order to compute $C_{m-1}(q)$ for $m > 2$, we transform (IV.10) into a convex program with linear cost function $H(Y_1|X_1)$ and convex equality constraints $H(Y_1|X_1) - H(Y_{i+1}|X_{i+1}) = 0$ for all $i \in \{1, \dots, m-1\}$[3]. The resulting program reads as

$$\begin{aligned} \text{maximize} \quad & p_1 \log(q+1) \\ \text{subject to} \quad & p_1 \log(q+1) + \bar{p}_i \log \frac{\bar{p}_i}{q p_{i+1}} + (p_{i+1} - \bar{p}_i) \log \frac{p_{i+1} - \bar{p}_i}{p_{i+1}} = 0 \\ & 1 - \sum_{j=i}^{i+1} p_j \leq 0 \\ & p_i \in [0, 1] \end{aligned} \tag{V.3}$$

By adopting a standard algorithm for constrained optimization problems, the capacity $C_{m-1}(q)$ was computed for various values of m. A brief summary is given in Table V.

[3]See proof in the Appendix why the constraints are satisfied with equality.

m	$C_{m-1}(1)$	$C_{m-1}(2)$
2	0.7729 b/u	1.1389 b/u
3	0.7324 b/u	1.0665 b/u
4	0.7173 b/u	1.0400 b/u
5	0.7099 b/u	1.0271 b/u
11	0.6981 b/u	1.0066 b/u
21	0.6954 b/u	1.0020 b/u
41	0.6946 b/u	1.0006 b/u
101	0.6943 b/u	1.0001 b/u
∞	0.6942 b/u	1 b/u
TS	0.5 b/u	0.7925 b/u

TABLE V
CAPACITY RESULTS FOR VARIOUS CASCADES COMPOSED OF $m-1$ HALF-DUPLEX CONSTRAINED RELAYS. THE TRANSMISSION ALPHABET IS EITHER UNARY OR BINARY. ROW "TS" SHOWS THE CORRESPONDING TIME-SHARING RATES.

B. Two Sources

The considered relay network is characterized by $\mathcal{V} = \{0, 1, 2\}$, $\mathcal{V}_s = \{0, 1\}$ and $q = 2$. By Theorem 1, a part $\mathcal{C}'$ of the capacity region is given by[4]

$$\begin{aligned} R_0 &\leq H(X_0|X_1) &\text{(V.4)}\\ R_0 + R_1 &\leq H(X_1) &\text{(V.5)}\\ R_1 &\leq \bar{p}_1, \quad \text{if } R_0 > 0 &\text{(V.6)} \end{aligned}$$

for some $p_{X_0X_1}$. We will first derive an explicit expression for the boundary of $\hat{\mathcal{C}}$ which is characterized by (V.4) and (V.5). Subsequently, $\mathcal{C}'$ and $\mathcal{R}$ are given.

Two cases have to be considered depending on whether an optimum input pmf for the source or the relay source is used. An optimum input pmf for the relay source is shown in Table VI. It yields a maximum sum rate $H(X_1)$ of $\log_2 3$ bits per use for all valid y (i.e. $y \in [0, 1/6]$). When y varies from 0 to $1/6$ all points on $R_1 = \log_2 3 - R_0$ for $0 \leq R_0 \leq \frac{1}{3}\log_2 3$ result.

X_0 \ X_1	0	1	N
0	0	0	y
1	0	0	y
N	$1/3$	$1/3$	$1/3 - 2y$

TABLE VI
OPTIMAL RELAY $p_{X_0X_1}$ WHICH YIELDS A SUM RATE OF $\log_2 3$ BITS PER USE.

It remains to focus on the interval $\frac{1}{3}\log_2 3 < R_0 \leq 1.1389$ bits per use. Under consideration of the optimum input pmf for source node 0 (Table III) and (III.8), we can express $R_1 = H(X_1) - R_0$ as shown in the second line of (V.7). Hence, the boundary of $\hat{\mathcal{C}}$ is given by

$$R_1 = \begin{cases} \log_2 3 - R_0, & 0 \leq R_0 \leq \frac{1}{3}\log_2 3 \\ H\left(\frac{R_0}{\log_2 3}\right) + \left(1 - \frac{R_0}{\log_2 3}\right) - R_0, & \frac{1}{3}\log_2 3 < R_0 \leq 1.1389 \end{cases}. \tag{V.7}$$

In order to determine $\mathcal{C}'$, (V.6) has to be taken into account. For $R_0 > 0$ (V.6) yields an upper bound on R_1 or, equivalently, a lower bound on R_0. This lower bound is given by the right hand side of

$$H(X_0|X_1) \geq H(X_1) - \bar{p}_1. \tag{V.8}$$

Since $H(X_0|X_1)$ is linear in p_1 while $H(X_1) - \bar{p}_1$ is concave in p_1, the smallest value for the lower bound follows by finding a p_1 which achieves equality in (V.8). We obtain $p_1 = 0.6091$ what gives $R_0 \geq 0.9654$ and $R_1 \leq 0.3909$ bit per use. If $R_0 = 0$, (V.6) is not valid anymore and we have $R_1 \leq \log_2 3$. Thus, the boundary of $\mathcal{C}'$ is given by (V.7) for $0.9654 \leq R_0 \leq 1.1389$ bits per use together with the rate vector $\mathbf{R} = (0, \log_2 3)$ bits per use. $\mathcal{R}$ follows from $\mathcal{C}'$ by taking the convex hull. In particular, all points on the connecting line between $(0, \log_2 3)$ and $(0.9654, 0.3909)$ bits per use are added. The three regions are depicted in Fig. 3.

The derivation reveals the following interesting fact. Even when the source transmits at a rate beyond the time-sharing rate of $\log_2 \sqrt{3}$ bit per use, the relay is still able to send its own information at a non-zero rate.

[4] $H(X_1)$ is not considered on the right hand side of (V.4) since it already appears in (V.5)

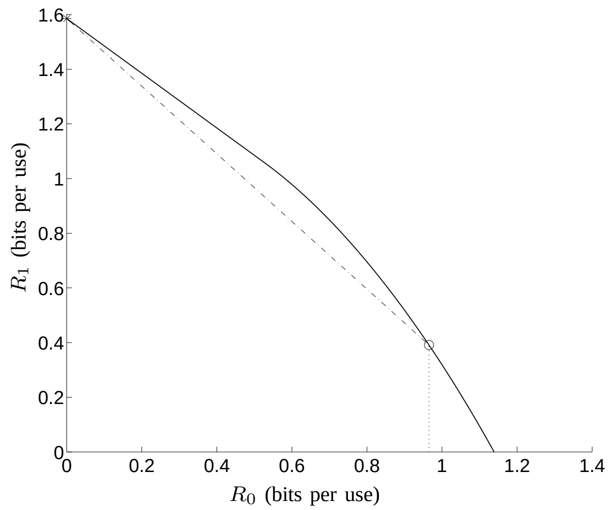


Fig. 3. $\hat{\mathcal{C}}$, $\mathcal{C}'$ and $\mathcal{R}$ are depicted. The points marked with a star and a circle correspond to $(0, \log_2 3)$ and $(0.9654, 0.3909)$ bits per use. $\hat{\mathcal{C}}$ equals the region bounded by the solid curve, $\mathcal{C}'$ is the region bounded by the dotted line and the fraction of the solid curve to the right of $(0.9654, 0.3909)$ together with the isolated point $(0, \log_2 3)$. $\mathcal{R}$ is the union of $\mathcal{C}'$ and the region bounded by the dash dotted line.

VI. Extension to other Networks

Relay cascades are fundamental building blocks in communication networks. Therefore, the results derived in the previous sections may be instrumental in order to determine the capacity of half-duplex constrained networks with more elaborate topologies.

A. Wireless Trees

Consider, for instance, the tree structured network depicted in Fig. 4. The root (node 1) wants to multicast information to all leaves (nodes 2 to 8) via four half-duplex constrained relays. We assume noise-free bit pipes (i. e. $q = 1$) and broadcast behavior at nodes with more than one outgoing arrow. The multicast capacity is limited by the capacity of the longest path in the tree which goes from node 1 to node 7 or 8. Hence, the multicast capacity in the considered example is equal to the capacity of a cascade containing two intermediate relay nodes, i. e. $C_2(1) = 0.7324$ bit per use (see Table V).

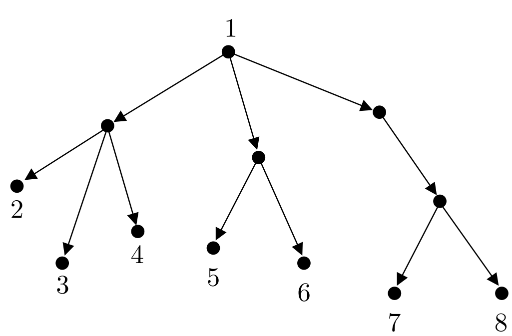
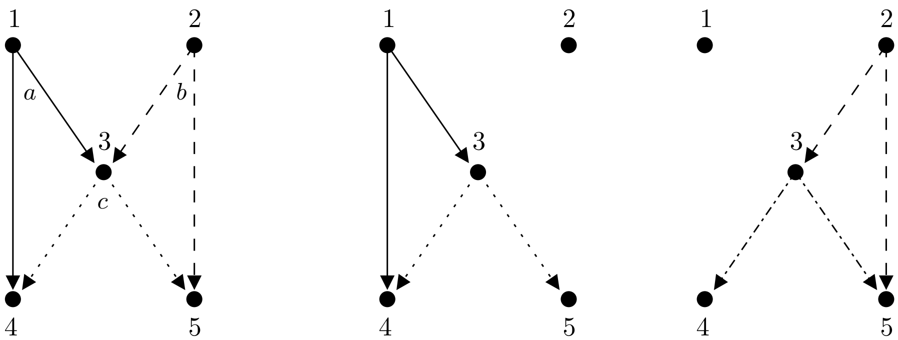


Fig. 4. A wireless binary tree. The multicast capacity is equal to $C_2(1) = 0.7324$ bit per use.

B. Wireless Butterfly

Another example for a wireless butterfly network [14] is shown in Fig. 5. Nodes 1 and 2 intend to multicast information to sink nodes 4 and 5 via both a direct link and a half-duplex constrained relay node 3. Like before, broadcast transmission and bit pipes are assumed. All nodes with two incoming arrows behave according to a collision model, i. e. received information is erased if there was a transmission on both incoming links. By means of network coding (NC), $2/3$ bit per use are achievable at the sink nodes, as is illustrated in Fig. 5 (a). The (well-known) strategy is to send in the first time slot a binary symbol u_1 via broadcast a to nodes 3 and 4, in the second time slot a binary symbol u_2 via broadcast b to nodes 3 and 5 and, subsequently, in the third time slot $u_1 \oplus u_2$ via broadcast c to both sinks. However, under the usage of timing, at least 0.7729 bit per use is achievable as is illustrated in Fig. 5 (b). This results from the fact that information originating from node 1 can be sent by means of timing at a rate of $C_1(1) = 0.7729$ bit per use (see Table V) concurrently on paths $1, (1, 3), 3, (3, 4), 4$ and $1, (1, 3), 3, (3, 5), 5$. Equivalently, information originating from node 2 can also be sent by means of timing at a rate of $C_1(1) = 0.7729$ bit per use concurrently on paths $2, (2, 3), 3, (3, 4), 4$ and $2, (2, 3), 3, (3, 5), 5$. Hence, time-sharing of both source nodes yields a multicast rate of 0.7729 bit per use. It should be noted that the direct links $(1, 4)$ and $(2, 5)$ are not necessary in order to achieve $R_1 = R_2 = 0.7729$ bit per use, which suggests that the multicast capacity is even larger.



(a) With NC (b) With Timing

Fig. 5. The wireless binary butterfly network. With network coding $2/3$ bit per use are achievable. Timing yields $C_1(1) = 0.7729$ bit per use.

VII. Conclusion

The half-duplex constraint is a property common to many wireless networks. In order to overcome the half-duplex constraint, practical transmission protocols deterministically split the time of each network node into transmission and reception periods. However, this is not optimum from an information theoretic point of view, as is demonstrated in this paper by means of noise-free relay cascades of various lengths with one or multiple sources. We show that significant rate gains are possible when information is represented by an information-dependent, non-deterministic allocation of the transmission and reception slots of the relays. Moreover, we provide a coding strategy which realizes this idea and, based on the asymptotic behavior of the strategy, we establish capacity expressions for three different scenarios. These results may be instrumental in deriving the capacity of half-duplex constrained networks with a more elaborate topology.

Appendix

Proof of Theorem 2: It is first shown that $\|p_{X_{i-1}X_i} - p_{X_iX_{i+1}}\|_\infty \to 0$ if $m \to \infty$ for all $2 \le i \le m-1$. The capacity series $(C_m(q))_{m\in\mathbb{N}}$ is bounded (e. g. by 0 and $C_1(q)$) and monotonically decreasing what follows from the fact that each new relay causes an additional constraint in the corresponding convex program (V.3). Hence, $(C_m(q))_{m\in\mathbb{N}}$ is convergent, i. e. for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$|C_m(q) - C_{m+1}(q)| < \epsilon \tag{A.1}$$

for all $m \ge N$.

Further, the capacity achieving input pmf $p_{X_0\ldots X_m}$ in (IV.10) yields $H(Y_i|X_i) = H(Y_j|X_j)$ for all $1 \le i, j \le m$. Assume this is wrong, i. e. there exist indices i, j such that $H(Y_i|X_i) > H(Y_j|X_j)$. However, $H(Y_i|X_i)$, i.e. the transmission rate from node $i-1$ to node i, can be decreased without forcing any of the remaining nodes to decrease their transmission rates. Simply reduce the fraction of a block used by node $i-1$ for encoding until the desired rate is achieved and fill the remaining slots of the block with quiet symbols N.

Hence, assuming the capacity achieving input pmf, we have for instance $C_m(q) = H(X_m)$ and $C_{m+1}(q) = H(X_{m+1})$. Then, by (A.1)

$$|H(X_m) - H(X_{m+1})| < \epsilon \tag{A.2}$$

for all $m \ge N$ where

$$H(X_k) = -p_k \log p_k - (1-p_k)\log\frac{1-p_k}{q}, \qquad k = m, m+1 \tag{A.3}$$

(see Fig. 2 for a plot of $H(X_k)$ for $q = 2$). Two cases can appear in (A.2) when ϵ approaches zero: p_m and p_{m+1} are arbitrarily close to a common point p or p_m and p_{m+1} are arbitrarily close to two distinct points p', p''.

We note that if the second case occurs, $p' + p''$ is not allowed to be smaller than one since otherwise negative probability masses would result (see Table IV). However, $p' + p''$ is always smaller than one what can be seen as follows. First, note that the maximum of $H(X_k)$ is at $1/(q+1)$. Hence, without restriction we can assume that $p' < 0.5$ and $p'' > 0.5$ (otherwise $p' + p'' < 1$ a priori). Since the first derivative of $H(X_k)$ is point symmetric with respect to $(0.5, -\log q)$, we have $0.5 - p' > p'' - 0.5$ what yields $p' + p'' < 1$.

Hence, only the first case is valid, i. e. $|p_m - p_{m+1}| \to 0$ as $m \to \infty$. But this implies $\|p_{X_{i-1}X_i} - p_{X_iX_{i+1}}\|_\infty \to 0$ for all $i \in \{2, \dots, m-1\}$ and, thus, $|H(X_{i-1}|X_i) - H(X_{j-1}|X_j)| \to 0$ for all $2 \le i, j \le m-1$ as $m \to \infty$.

In the final step of the proof, we show that the capacity $C_\infty(q)$ is equal to the maximum in p of

$$H(X_{i-1}|X_i) = -(1-p)\log\frac{1-p}{qp} - (2p-1)\log\frac{2p-1}{p}. \tag{A.4}$$

where $i > 1$. Elementary calculus yields

$$\max_p H(X_{i-1}|X_i) = \log\left(\frac{1+\sqrt{4q+1}}{2}\right) \tag{A.5}$$

achieved at

$$p^\star = \frac{1}{2}\left(1 + \frac{1}{\sqrt{4q+1}}\right). \tag{A.6}$$

It remains to show that $H(X_0|X_1)$ evaluated at $p^\star$, i. e.

$$H(X_0|X_1)|_{p^\star} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{4q+1}}\right)\log(q+1) \tag{A.7}$$

is always greater or equal to (A.5). This is satisfied if

$$(q+1)^{\frac{1}{2}\left(1+\frac{1}{\sqrt{4q+1}}\right)} \ge \frac{1+\sqrt{4q+1}}{2} \tag{A.8}$$

or, more strictly,

$$(q+1)^{\frac{1}{2}\left(1+\frac{1}{2\sqrt{q+1}}\right)} \ge \frac{1+2\sqrt{q+1}}{2}. \tag{A.9}$$

Using the substitution

$$\tilde{q} = \frac{1}{2\sqrt{q+1}} \tag{A.10}$$

in (A.9) yields

$$(2\tilde{q})^{-\tilde{q}} \ge \tilde{q} + 1. \tag{A.11}$$

(A.11) is satisfied for all $\tilde{q} \in [0, 0.2]$ what can be seen as follows. First note that (A.11) is satisfied for $\tilde{q} = 0$ and $\tilde{q} = 0.2$. Since $(2\tilde{q})^{-\tilde{q}}$ is concave due to a non-positive second derivative in the considered domain, (A.11) is valid for all $\tilde{q} \in [0, 0.2]$. Thus, (A.8) is true for all $q > 5$. The validity of (A.8) for the remaining $q \in \{1, \dots, 5\}$ can be easily checked. ■

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References

[1] T. Lutz, C. Hausl, and R. Kötter. Coding Strategies for Noise-Free Relay Cascades with Half-Duplex Constraint. In *IEEE International Symposium on Information Theory (ISIT)*, pages 2385–2389, Toronto, Ontario, Canada, July 2008.

[2] A. Host Madsen and J. Zhang. Capacity Bounds and Power Allocation for the Wireless Relay Channel. *IEEE Trans. Inform. Theory*, 51(6):2020–2040, June 2005.

[3] M. A. Khojastepour and A. Sabharwal and B. Aazhang. On the Capacity of 'Cheap' Relay Networks. In *Proc. 37th Annual. Conf. Information Sciences and Systems (CISS), (Baltimore, MD)*, March 12-14, 2003.

[4] S. Toumpis and A. J. Goldsmith. Capacity Regions for Wireless Ad Hoc Networks. *IEEE Trans. Wireless Commun.*, 2(4):736–748, July 2003.

[5] G. Kramer. Communication Strategies and Coding for Relaying. *Wireless Communications*, vol. 143 of the IMA Volumes in Mathematics and its Applications:163–175, Springer: New York, 2007.

[6] G. Kramer. Models and Theory for Relay Channels with Receive Constraints. In *Proc. 42nd Annual Allerton Conf. Commun., Control, and Computing, (Monticello, IL)*, Sept. 29 - Oct. 1 2004.

[7] T. M. Cover and A. A. El Gamal. Capacity Theorems for the Relay Channel. *IEEE Trans. Inform. Theory*, 25:572–584, Sept. 1979.

[8] V. Anantharam and S. Verdú. Bits Through Queues. *IEEE Trans. Inform. Theory*, 42:4–18, Jan. 1996.

[9] A. A. Bedekar and M. Azizoğlu. The Information-Theoretic Capacity of Discrete-Time Queues. *IEEE Trans. Inform. Theory*, 44:446–461, Mar. 1998.

[10] I. Csiszár. The Method of Types. *IEEE Trans. Inform. Theory*, 44:2505–2523, Oct. 1998.

[11] J. H. van Lint. *Introduction to Coding Theory*. Springer, 1999.

[12] T. M. Cover and J. Thomas. *Elements of Information Theory*. New York.

[13] P. Vanroose and E. C. van der Meulen. Uniquely Decodable Codes for Deterministic Relay Channels. *IEEE Trans. Inform. Theory*, 38(4):1203–1212, July 1992.

[14] R. Ahlswede, N. Cai, S. R. Li, and R. W. Yeung. Network Information Flow. *IEEE Trans. Inform. Theory*, 46(4):1204–1216, July 2000.