PRIMENESS, SEMIPRIMENESS AND PRIME RADICAL OF ORE EXTENSIONS

T.Y. LAM
University of California
Berkeley, Ca. 94720
e-mail : lam@math.berkeley.edu

A. LEROY
Université d'Artois
rue Jean Souvraz
62307 Lens, France
e-mail : leroy@gat.univ-lille1.fr

J. MATCZUK
University of Warsaw
Ul. Banacha,2
02-097 Warsaw, Poland
e-mail : jmatczuk@mimuw.edu.pl

Introduction.

With the impetus of quantized derivations, renewed interest in the general Ore extension $T = R[t; S, D]$ has arisen during the last few years. The prime radical of T is currently being analyzed (e.g. [G]) and in the special cases when either $S = id$ or $D = 0$ it has been completely described ([FKM], [PS]). In the former case this description is very much like that for the ordinary polynomial ring, i.e. $\text{rad}(T) = I[t; D]$ where I is a D-ideal of R. In the latter case such a description is not possible (cf. [PS]). This difference in behavior is mainly due to the fact that the contraction of an ideal of $R[t; D]$ to R is D-stable but the analogue for $R[t; S]$ and a fortiori for $R[t; S, D]$ is false. One of our aims in Section 5 is to provide some conditions under which $\text{rad}(T)$ is of the form $I[t; S, D]$ where I is an (S, D)-ideal of R. In trying to obtain connections between the prime radical of T and that of the base ring R, different notions appear naturally. These notions (which are of interest on their own) are carefully defined, studied and compared in Sections 1 and 2.

The third section is devoted to the study of the S-nil radical, a notion already introduced in [PSW] in the context of Jacobson radical of T. This nice radical is also related to the prime radical of T and will be used in a later section (e.g. Propositions 4.15 and 4.19).

In Section 4 we are concerned with the characterization of primeness and semi-primeness of $T = R[t; S, D]$ in terms of R, S and D. The case of primeness (Theorem 4.4) is relatively straightforward, but (to the best of our knowledge) has not been mentioned in the literature before. This characterization shows that while investigating primeness of $T = R[t; S, D]$ it suffices to consider only elements from R. To be more precise : T is prime if and only if for any $a, b \in R \setminus \{0\}, aTb \neq 0$. The expected analogue for semiprimeness, however, turns out to be false. After paving the way by recalling the special cases when either $S = id_R$ or $D = 0$, we give in Theorem 4.9 a necessary and sufficient condition for the semiprimeness of T. In the rest of Section 4, we analyze other more tractable conditions that are either necessary or sufficient for T to be semiprime. Particularly, we study the case when either R satisfies some ascending chain conditions or D is quantized.

In the last section we study the prime radical of $R[t; S, D]$, the aim being to relate this radical to the radicals of R introduced in the first three sections. Once more the best results are obtained under the assumption that either R is noetherian or D is quantized.

1. Relative Prime Ideals and Prime Radicals

In this section, we shall introduce generalizations of some classical concepts and state a few results in a wide framework, leaving for the second section special settings that are of more direct interest to us.

While studying ideals I of a ring extension $T \supset R$, it is natural to consider the relationship between I and $I \cap R$. The contracted ideal $I \cap R$ has generally some extra properties related to the way T is built upon R. Let us mention the following examples:

(a) $T = R * G$, where G denotes a finite group. The elements of T are of the form $\sum r_g \overline{g}$ where $G \to U(T)$ $(g \mapsto \overline{g})$ is a map from G into $U(T)$, the unit group of T, such that $\overline{g}R = R\overline{g}$ for any $g \in G$. Here, each $\overline{g}$ induces an automorphism of R and the ideals of R stable under all such automorphisms are strongly related to the ideals of T (cf. [M], [P] or [MR:

(2.10)]).

(b) $T = R\#U(\mathcal{G})$ where R is a K-algebra, $\mathcal{G}$ is a Lie algebra over the field K, and $U(\mathcal{G})$ denotes the enveloping algebra of $\mathcal{G}$. Here, $\mathcal{G}$ acts on R as K-derivations. Once more, ideals of R which are stable under the action of $\mathcal{G}$ play a special role and are worth studying by themselves ([BMP]).

(c) $T = R[t; S, D]$ where $S \in \mathrm{Aut}(R)$ and D is an S-derivation (i.e. $D \in \mathrm{End}(R, +)$ such that for $a, b \in R$, $D(ab) = S(a)D(b) + D(a)b$). Elements of T are polynomials $\sum a_i t^i, a_i \in R$, and multiplication is based on the rule $ta = S(a)t + D(a)$. Such ring extensions, called Ore extensions, were mainly studied when either $S = id_R$ ([FKM], [FM]...) or $D = 0$ ([CFG], [PS]...). In the former case the notion of D-invariant ideals of R was crucial and in the latter the S-invariant ideals naturally come into play. Recently, some authors also considered the general case $T = R[t; S, D]$ when $S \neq id_R$ and $D \neq 0$ ([GL], [LM], [V]).

In view of the different terminology and notations used in the literature we will use the following ones which have the merit of being explicit.

Let R be a ring, $\mathrm{End}(R, +)$ the ring of additive endomorphisms of R and Φ a subset of $\mathrm{End}(R, +)$. An ideal I of R is called a Φ-ideal if $\varphi(I) \subseteq I$ for any $\varphi \in \Phi$. A Φ-ideal $P \neq R$ is a Φ -prime ideal if for any Φ-ideals I and J such that $IJ \subseteq P$, we have either $I \subseteq P$ or $J \subseteq P$. We shall use the notation $I \lhd_\Phi R$ (resp. $P \lhd'_\Phi R$) to express the fact that I is a Φ-ideal (resp. P is a Φ-prime ideal) of R. We write $\mathcal{P}_\Phi = \mathrm{Spec}(R; \Phi)$ for the set of all Φ-prime ideals of R and $\mathrm{rad}(R; \Phi) = \cap_{P \in \mathcal{P}_\Phi} P$ for the Φ-prime radical. By definition, R is Φ-prime (resp. Φ semiprime) if (0) is Φ-prime (resp. if $\mathrm{rad}(R; \Phi) = 0$).

Observe that if $\Phi \subseteq \Omega \subseteq \mathrm{End}(R, +)$ and if R is Φ-prime then R is Ω-prime; more generally any Ω-ideal which is Φ-prime is also Ω-prime. In fact we can state the following (keeping the notations $\Phi \subseteq \Omega \subseteq \mathrm{End}(R, +)$).

Proposition 1.1. *Let P be a Φ-prime ideal of R and Q the largest Ω-ideal of R contained in P. Then Q is an Ω-prime ideal. In particular, any Φ-prime ideal contains an Ω-prime ideal and* $\mathrm{rad}(R; \Omega) \subseteq \mathrm{rad}(R; \Phi)$.

Proof. Since for any $I_1 \lhd_\Omega R$ and $I_2 \lhd_\Omega R$, $I_1 + I_2 \lhd_\Omega R$, we have $Q = \sum\{I \mid I \lhd_\Omega R \text{ and } I \subseteq P\}$. Now assume $I_1 \lhd_\Omega R$ and $I_2 \lhd_\Omega R$ are such that $I_1 I_2 \subseteq Q \subseteq P$. Since P is Φ-prime, we conclude that either $I_1 \subseteq P$ or $I_2 \subseteq P$ and the definition of Q yields that either $I_1 \subseteq Q$ or $I_2 \subseteq Q$. **QED**

Obviously $I \lhd_\Phi R$ if and only if I is a $\overline{\Phi}$ -ideal where $\overline{\Phi}$ denotes the (multiplicative) semi-group (with 1) generated in $\mathrm{End}(R,+)$ by Φ. In the sequel we will additionally assume that:

(H_1) Φ is closed under composition and $id_R \in \Phi$, i.e. $\overline{\Phi} = \Phi$.

(H_2) For any $a \in R$, $\sum_{\varphi\in\Phi} R\varphi(a)R \lhd_\Phi R$.

Let us give examples of subsets $\Phi \subseteq \mathrm{End}(R,+)$ satisfying (H_1) and (H_2).

Example 1.2. Any subset of $\mathrm{End}(R,+)$ closed under composition and consisting of ring endomorphisms, i.e. any subsemigroup of $\mathrm{End}(R)$.

Example 1.3. Let σ, τ be ring endomorphisms of R. (i.e. $\sigma, \tau \in \mathrm{End}(R)$) and $\delta \in \mathrm{End}(R,+)$ be such that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)\tau(b) \quad \text{for} \quad a, b \in R$$

Such a map δ is called a (σ,τ)-derivation. If δ_i's are (σ_i,τ_i)-derivations and Φ is the subsemigroup of $End(R,+)$ generated by σ_i, τ_i and δ_i's then it is easy to check that Φ satisfies (H_1) and (H_2). If $\sigma_i, \tau_i \in \mathrm{Aut}(R)$, the same can be said about the semigroup generated by $\sigma_i, \sigma_i^{-1}, \tau_i, \tau_i^{-1}$ and the δ_i's.

Example 1.4. Let $S \in \mathrm{End}(R)$ and $D \in \mathrm{End}(R,+)$ be such that D is a (S, id_R)-derivation. We say that D is an S-derivation for short. For $0 \le l \le n$ let us denote by $f_l^n \in \mathrm{End}(R,+)$ the sum of all words composed with l letters "S" and $n-l$ letters "D" (e.g. $f_n^n = S^n, f_0^n = D^n$). It is easy to prove by induction, that for $a, b \in R$, $f_l^n(ab) = \sum_{i=l}^n f_i^n(a) f_l^i(b)$. Fix $l_0 \in \mathbb{N} = \{0,1,2,\dots\}$ and let Φ_{l_0} be the subsemigroup of $\mathrm{End}(R,+)$ generated by $\{f_i^n \mid l_0 \le i \le n, n \in \mathbb{N}\}$. Then Φ_{l_0} satisfies (H_1) and (H_2).

Example 1.5. More generally, let R and C be an algebra and a coalgebra over a field K respectively. Denote by Δ and ε the comultiplication and counit of C and put $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$. Then $Hom_K(C, R)$ has a K-algebra structure defined by $(f * g)(c) = \sum_{(c)} f(c_1)g(c_2)$ for $f, g \in Hom(C, R)$ and $c \in C$. Suppose that there exists $\psi \in Hom_K(C \otimes_K R, R)$ such that (ψ, C) measures R to R in the sense of [S, Chap. VII], i.e.

$$\psi(c \otimes ab) = \sum_{(c)} \psi(c_1 \otimes a)\psi(c_2 \otimes b) \quad \text{for} \quad a, b \in R,\ c \in C;$$

$$\psi(c \otimes 1) = \varepsilon(c)1_R$$

(This means that the map corresponding to ψ under the standard isomorphism $Hom(C \otimes_K R, R) \cong Hom(R, Hom(C, R))$ is a morphism of K-algebras). For $c \in C$, $\psi(c \otimes -)$ defines an element in $\mathrm{End}(R, +)$ denoted by $\psi_c : R \to R : r \mapsto \psi(c \otimes r)$. Let Φ be the subsemigroup of $\mathrm{End}(R, +)$ generated by $\{\psi_c \mid c \in C\}$. Since $\psi_c(ab) = \sum_{(c)} \psi_{c_1}(a)\, \psi_{c_2}(b)$, it is easy to prove that Φ satisfies (H_1) and (H_2).

Example 1.6. As a particular (but important) case of Example 1.5 we have the following. Let H be a Hopf algebra and R be an H-module algebra (over a field K). Then the map $H \otimes_K R \to R$ defined by $h \otimes a \mapsto h \cdot a$ measures R to R and the K-subalgebra of $Hom_K(R, R)$ corresponding to the action of H satisfies (H_1) and (H_2).

In the rest of this section, Φ will denote a subset of $\mathrm{End}(R, +)$ satisfying (H_1) and (H_2).

By using the condition (H_2) we have immediately the following:

Lemma 1.7. *R is Φ-prime if and only if for any $a, b \in R\backslash\{0\}$, there exist $\varphi_1, \varphi_2 \in \Phi$ such that $\varphi_1(a)R\varphi_2(b) \neq 0$.*

Definitions 1.8.

(1) A subset $M \subseteq R$ is called a Φ-m-system if for any $a, b \in M$ there exist $\varphi_1, \varphi_2 \in \Phi$ and $r \in R$ such that $\varphi_1(a)r\varphi_2(b) \in M$.

(2) A sequence $(a_0, a_1, \dots, a_n, \dots)$ of elements of R is called a Φ-m-sequence if for any $i \in \mathbb{N} = \{0, 1, 2, \dots\}$ there exist $\varphi_i, \varphi_i' \in \Phi$ and $r_i \in R$ such that $a_{i+1} = \varphi_i(a_i)r_i\varphi_i'(a_i)$.

(3) An element $a \in R$ is strongly Φ-nilpotent if every Φ-m-sequence starting with a eventually vanishes.

Remark 1.9. (1) If $\Phi = \{id_R\}$ we recover the corresponding classical notions.

(2) Our definitions of Φ-prime ideals and Φ-m-systems are symmetrical, but it is possible, and sometimes useful, to give unsymmetrical definitions (cf. 5.15 hereafter and [PS]).

Proposition 1.10. *For an ideal P of R, we have:*

(a) *A Φ-ideal P is Φ-prime if and only if $R \backslash P$ is a Φ-m-system.*

(b) *Let M be a Φ-m-system not containing 0 and P a Φ-ideal of R maximal among Φ-ideals disjoint from M. Then P is a Φ-prime ideal.*

(c) *Let $(a_0, a_1, \dots, a_n, \dots)$ be a Φ-m-sequence not containing 0 and let P be a Φ-ideal of R maximal among Φ-ideals not intersecting $\{a_0, a_1, \dots, a_n, \dots\}$ then P is a Φ-prime ideal.*

Proof. (a) Suppose $P \lhd'_\Phi R$ and $a, b \in R\backslash P$. Since $I := \sum_{\varphi\in\Phi} R\varphi(a)R \lhd_\Phi R$ and $J := \sum_{\varphi\in\Phi} R\varphi(b)R \lhd_\Phi R$, we have $IJ \not\subseteq P$, and so there exist $\varphi_1, \varphi_2 \in \Phi$ such that $\varphi_1(a)R\varphi_2(b) \not\subseteq P$.

Conversely suppose $P \lhd_\Phi R$ and $R\backslash P$ is a Φ-m-system. Let $I \lhd_\Phi R$, $J \lhd_\Phi R$ be such that $P \subset I$ and $P \subset J$. Pick $a \in I \backslash P$ and $b \in J\backslash P$. Then there exist $\varphi_1, \varphi_2 \in \Phi$ and $r \in R$ with $\varphi_1(a)r\varphi_2(b) \in R\backslash P$. Since $\varphi_1(a) \in I$ and $\varphi_2(b) \in J$, $IJ \not\subseteq P$ follows.

(b) Let P be a Φ-ideal as in the statement (b). Let $I_1 \lhd_\Phi R$ and $I_2 \lhd_\Phi R$ be such that $P \subset I_1$ and $P \subset I_2$. There exist $a_1 \in I_1 \cap M$ and $a_2 \in I_2 \cap M$. Since M is a Φ-m-system, we can also pick $\varphi_1, \varphi_2 \in \Phi$ and $r \in R$ such that $\varphi_1(a_1)\, r\, \varphi_2(a_2) \in M \cap I_1 I_2$. Hence $I_1 I_2 \not\subseteq P$.

(c) Let P be as in the statement (c) and put $\sum = \{a_0, a_1, \dots, a_n, \dots\}$. As above let $I_1 \lhd_\Phi R$ and $I_2 \lhd_\Phi R$ be such that $P \subset I_1$ and $P \subset I_2$. Then $I_1 \cap \sum \neq \emptyset$ and $I_2 \cap \sum \neq \emptyset$, so there exist $i, j \in \mathbb{N}$ such that $a_i \in I_1$ and $a_j \in I_2$. Since $\sum$ is a Φ-m-sequence, we see easily that if $s \geq max\{i,j\}$ then $a_s \in I_1 \cap I_2$. In particular, letting $k := \max\{i,j\}$, we have $a_k \in I_1 \cap I_2$, and so $a_{k+1} = \varphi_k(a_k) r_k \varphi'_k(a_k) \in I_1 I_2$ for some

$\varphi_k, \varphi'_k \in \Phi$ and $r_k \in R$. We conclude that $I_1 I_2 \not\subseteq P$. This proves that P is a Φ-prime ideal. **QED**

Part (c) of the above proposition cannot be deduced directly from part (b) since, in general, a Φ-m-sequence is not a Φ-m-system.

We can define a lower nil radical by transfinite induction:

$L_0 = L_0(R;\Phi) = (0)$

$$L_1 = L_1(R;\Phi) = \sum_{I \in \mathcal{N}_\Phi} I \ \text{ where } \ \mathcal{N}_\Phi = \{I \triangleleft_\Phi R \mid I \text{ is nilpotent}\}$$

($L_1 \triangleleft_\Phi R$ and any $\varphi \in \Phi$ induces an additive endomorphism of R/L_1)

$L_\alpha = L_\alpha(R;\Phi) = \{r \in R \mid r + L_\beta(R;\Phi) \in L_1(R/L_\beta(R;\Phi);\Phi)\}$ if $\alpha = \beta + 1$

$L_\alpha = L_\alpha(R;\Phi) = \cup_{\beta<\alpha} L_\beta(R;\Phi)$ if α is a limit ordinal.

There exists an ordinal β such that $L_\beta(R;\Phi) = L_{\beta+1}(R;\Phi)$ and we put $L(R;\Phi) = L_\beta(R;\Phi)$.

Proposition 1.11. *Keeping the above notations, we have:*

$$L(R;\Phi) = \mathrm{rad}(R;\Phi) = \{a \in R \mid a \textit{ is strongly } \Phi\textit{-nilpotent}\}.$$

Proof. Let us prove the first equality. Since $\mathrm{rad}(R;\Phi) = \cap\{P \mid P \triangleleft'_\Phi R\}$, it is easy to show that $L_1 \subseteq \mathrm{rad}(R;\Phi)$ and a transfinite induction gives immediately that $L(R;\Phi) \subseteq \mathrm{rad}(R;\Phi)$.

In order to prove the reverse inclusion we may factor out $L(R;\Phi)$ and suppose $L(R;\Phi) = 0$. We must then prove that $\mathrm{rad}(R;\Phi) = 0$. Assume, on the contrary, that $0 \neq a_0 \in \mathrm{rad}(R;\Phi)$ and consider $I := \sum_{\varphi \in \Phi} R\,\varphi(a_0)\,R \triangleleft_\Phi R$. Since $L(R;\Phi) = 0$, we know that $I^2 \neq 0$ and so there exist $\varphi_1, \varphi_2 \in \Phi$ such that $0 \neq \varphi_1(a_0) r \varphi_2(a_0) =: a_1 \in \mathrm{rad}(R;\Phi)$, for some $r \in R$. Repeating this process with a_1, we can construct a Φ-m-sequence M not containing 0, such that $a_0 \in M$ and Proposition 1.10(c) implies the existence of a Φ-prime ideal P of R such that $P \cap M = \emptyset$. In particular $a_0 \notin P$ and this contradicts the fact that $a_0 \in \mathrm{rad}(R;\Phi)$.

To prove the second equality : let $a \in R\backslash\mathrm{rad}(R;\Phi)$ and fix a Φ-prime ideal P such that $a \notin P$. Since, by Proposition 1.10(a), $R\backslash P$ is a Φ-m-system, it is easy to construct a Φ-m-sequence $(a = a_0, a_1, \dots)$ in $R\backslash P$. Hence a is not strongly Φ-nilpotent.

Conversely assume $(a = a_0, a_1, \dots)$ is a Φ-m-sequence not containing 0. Then Proposition 1.10(c) implies that there exists a Φ-prime ideal P of R such that $a \notin P$ and so $a \notin \mathrm{rad}(R;\Phi)$. **QED**

The next corollary is an immediate consequence of the above proposition.

Corollary 1.12. *Let Φ be a subset of $\mathrm{End}(R,+)$ satisfying (H_1) and (H_2). Then the following statements are equivalent:*

(1) *R is Φ-semiprime.*

(2) *R has no nonzero nilpotent Φ-ideals.*

(3) $L(R;\Phi) = 0$.

(4) *R does not have nonzero strongly Φ-nilpotent elements.*

Corollary 1.13. *Suppose R satisfies the ACC on Φ-ideals. Then $\mathrm{rad}(R;\Phi)$ is nilpotent.*

Proof. By the given ACC, $L_1(R;\Phi)$ is nilpotent. Now if N/L_1 is a nilpotent Φ-ideal of R/L_1 then $N \lhd_\Phi R$ is nilpotent and so $N \subseteq L_1$. We conclude that $\mathrm{rad}(R;\Phi) = L_1(R;\Phi)$ is nilpotent. **QED**

Proposition 1.1 shows that if $\Phi \subseteq \Omega \subseteq \mathrm{End}(R,+)$ then $\mathrm{rad}(R;\Omega) \subseteq \mathrm{rad}(R;\Phi)$. In certain circumstances the equality holds. This is the content of part (c) of the proposition below.

Proposition 1.14. *Suppose $\Gamma \subseteq \Phi \cap \mathrm{Aut}(R)$ is a commutative semigroup of automorphisms of R with $id_R \in \Gamma$. Let Ω be the subsemigroup of $\mathrm{End}(R,+)$ generated by Φ and $\{\gamma^{-1} \mid \gamma \in \Gamma\}$. If for any Φ-ideal I, we have $\overline{I} := \sum_{\gamma\in\Gamma} \gamma^{-1}(I) \lhd_\Omega R$, then:*

(a) *Ω satisfies H_1 and H_2.*

(b) *An Ω-ideal P is Ω-prime if and only if it is Φ-prime. In particular R is Ω-prime if and only if R is Φ-prime.*

(c) $\mathrm{rad}(R;\Omega) = \mathrm{rad}(R;\Phi)$.

In particular, if $\gamma \in \Phi \cap \mathrm{Aut}(R)$ is such that for any Φ-ideal I of R, $\sum_{i\geq 0} \gamma^{-i}(I) \lhd_\Omega R$, then the above conclusions hold for $\Gamma = \{\gamma^i \mid i \geq 0\}$ and Ω the subsemigroup of $\mathrm{End}(R,+)$ generated by Φ and γ^{-1}.

Proof. (a) For $a \in R$, let $I(a)$ stands for the Φ-ideal $\sum_{\varphi\in\Phi} R\varphi(a)R$ and put $\overline{I}(a) = \sum_{\gamma\in\Gamma} \gamma^{-1}(I(a))$. Thus, by our assumption, $\overline{I}(a) \lhd_\Omega R$, and so $\sum_{\omega\in\Omega} R\omega(a)R \subseteq \overline{I}(a)$. On the other hand, since every $\gamma \in \Gamma$ is a ring automorphism, we have also the reverse inclusion. Therefore equality holds, which shows that $\sum_{\omega\in\Omega} R\omega(a)R \lhd_\Omega R$, so Ω satisfies H_2. (The commutativity of Γ is not needed for this part.)

(b) We have already noticed that if $P \lhd_\Omega R$ and $P \lhd'_\Phi R$ then $P \lhd'_\Omega R$. Let us now prove that if $P \lhd'_\Omega R$ then $P \lhd'_\Phi R$. Let I_1, I_2 be two Φ-ideals of R not contained in P and define $\overline{I}_j := \sum_{\gamma\in\Gamma} \gamma^{-1}(I_j) \lhd_\Omega R$ for $j = 1, 2$. Since P is Ω-prime, we have $\overline{I}_1\overline{I}_2 \not\subseteq P$. Then there exist $a \in I_1$, $b \in I_2$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1^{-1}(a)\gamma_2^{-1}(b) \notin P$. Applying $\gamma_1\gamma_2 = \gamma_2\gamma_1$, we get $\gamma_2(a)\gamma_1(b) \notin \gamma_1\gamma_2(P) \subseteq P$. Hence $\gamma_2(a)\gamma_1(b) \in I_1I_2 \backslash P$. This means $I_1I_2 \not\subseteq P$ as we wanted to prove. The last statement of (b) follows easily.

(c) Part (b) above and Proposition 1.1 show that $\mathrm{rad}(R; \Phi) = \mathrm{rad}(R; \Omega)$. **QED**

The following corollary is in fact a special case of the above proposition.

Corollary 1.15. *Let Φ be a commutative semigroup of automorphisms of R. Then:*

(a) *A (Φ, Φ^{-1})-ideal P of R is Φ-prime iff P is (Φ, Φ^{-1})-prime iff P is Φ^{-1}-prime. In particular R is Φ-prime iff R is (Φ, Φ^{-1})-prime iff R is Φ^{-1}-prime.*

(b) $\mathrm{rad}(R; \Phi) = \mathrm{rad}(R; \Phi, \Phi^{-1}) = \mathrm{rad}(R; \Phi^{-1})$. *In particular, R is Φ-semiprime iff R is (Φ, Φ^{-1})-semiprime iff R is Φ^{-1}-semiprime.*

Keeping the notations as in Proposition 1.14, let us now mention a few instances in which the crucial hypothesis $I \lhd_\Phi R \Longrightarrow \overline{I} \lhd_\Omega R$ is satisfied.

Example 1.16. Assume that R satisfies the ACC on Γ-ideals. For any $I \lhd_\Phi R$ and $\gamma \in \Gamma$, consider the ascending chain $I \subseteq \gamma^{-1}(I) \subseteq \gamma^{-2}(I) \subseteq \ldots$. Since Γ is commutative, $\gamma^{-k}(I)$ is a Γ-ideal for any $k \in \mathbb{N}$, so $\gamma^{-n}(I) = \gamma^{-(n+1)}(I)$ for some $n \in \mathbb{N}$. Therefore $I = \gamma^{-1}(I) = \gamma(I)$ and hence $\overline{I} = I$. Thus, the hypothesis of Proposition 1.14 is trivially satisfied.

Example 1.17. Assume that for each $\gamma \in \Gamma$, there exists $n \geq 1$ such that γ^n is an inner automorphism. Then, for any $I \lhd_\Phi R$, we see easily that $\gamma(I) = I$. Thus again $\overline{I} = I$ and the hypothesis of Proposition 1.14 is satisfied.

Example 1.18. Suppose Γ is generated by a (commuting) family $\{\gamma_i\}_{i\in E}$ and assume that, for any $\varphi \in \Phi$ and $i \in E$, there exists $q_i(\varphi) \in R$ such that $\gamma_i\varphi = q_i(\varphi)\varphi\gamma_i$. Then, by induction, for any $\gamma \in \Gamma$ and $\varphi \in \Phi$, there exists $q \in R$ such that $\gamma\varphi = q\varphi\gamma$. Now, making use of this identity, it is easy to check that for any Φ-ideal I of R, we have

$$\varphi\gamma^{-1}(I) = \gamma^{-1}(q\varphi(I)) \subseteq R \cdot \gamma^{-1}(I) \subseteq \gamma^{-1}(I).$$

¿From this, we conclude that $\overline{I} \lhd_\Omega R$, so the hypothesis of Proposition 1.14 is again satisfied.

2. Relations between $(S, D), (S, S^{-1}, D)$... primeness and semiprimeness

In this section we will study Φ-primeness and Φ-semiprimeness in the case when Φ is generated either by an automorphism S and an S-derivation D of a ring R or by S, S^{-1} and D.

Recall that D is an S-derivation if $D \in \mathrm{End}(R, +)$ and $D(ab) = S(a)\ D(b) + D(a)b$ for all $a, b \in R$. Such a D is called a *q-quantized S-derivation* if , $SD = qDS$ for some $q \in R$. To make life easier, we shall always assume in the sequel that $S(q) = q$ and $D(q) = 0$.

Let us also introduce the following useful notations: if D is an S-derivation and $n \geq i \geq 0$, then $f_i^n \in \mathrm{End}(R, +)$ is the sum of all words composed with i maps S and $n - i$ maps D (e.g. $f_n^n = S^n, f_0^n = D^n$).

Let X be an inderminate. For any $n \in \mathbb{N}$ define

$$(n!)_X := (X^{n-1} + X^{n-2} + \cdots + 1)(X^{n-2} + X^{n-3} + \cdots + 1) \ldots (X + 1) \cdot 1.$$

It can be shown that for $0 \leq i \leq n$, the rational function

$$\binom{n}{i}_X := \frac{(n!)_X}{(i!)_X((n-i)!)_X}$$

is in fact a polynomial [A: pp. 35]. For $q \in R$ we can thus put $\binom{n}{i}_q$ to be the evaluation of $\binom{n}{i}_X$ at $X = q$.

Let us recall some properties of the symbols f_i^n and $\binom{n}{i}_q$. The proofs of these properties (by induction) will be left to the reader.

Lemma 2.1. *Let D be an S-derivation of the ring R. Using the above notations, we have:*

(a) $f_i^n = f_{i-1}^{n-1} \cdot S + f_i^{n-1} \cdot D$ *for* $1 \leq i \leq n-1$.

(b) $f_i^n(ab) = \sum_{j=i}^n f_j^n(a) f_i^j(b)$ *for* $a, b \in R$, $0 \leq i \leq n$.

(c) $D^n(ab) = \sum_{i=0}^n f_i^n(a) D^i(b)$ *for* $a, b \in R$, $n \in \mathbb{N}$.

(d) *For* $q \in R$, $1 \leq i \leq n-1$, $\binom{n}{i}_q = \binom{n-1}{i-1}_q + \binom{n-1}{i}_q q^i$.

(e) *Suppose D is a q-quantized S-derivation:* $SD = qDS$, $S(q) = q$, $D(q) = 0$. *Then* $f_i^n = \binom{n}{i}_q D^{n-i} S^i$ *for* $0 \leq i \leq n$.

In particular, for $n \in \mathbb{N}$ and $a, b \in R$ we have

$$D^n(ab) = \sum_{i=0}^n \binom{n}{i}_q D^{n-i} S^i(a) D^i(b).$$

We will now give a specific criterion for (S, S^{-1}, D)-primeness and examine in particular the case when D is a q-quantized S-derivation. For convenience we will denote by Ω the set of all words in S, S^{-1} and D.

Lemma 2.2. *Let P be an (S, S^{-1}, D)-ideal. Then:*

(a) *P is (S, S^{-1}, D)-prime iff for any $a, b \in R \backslash P$ there exists $\omega \in \Omega$ such that $aR\omega(b) \not\subseteq P$.*

(b) *P is (S, S^{-1}, D)-semiprime iff for any $a \in R \backslash P$ there exists $\omega \in \Omega$ such that $aR\omega(a) \not\subseteq P$.*

(c) *Suppose D is a q-quantized S-derivation with $SD = qDS$, where q is a unit in R. Then the following conditions are equivalent:*

  (i) *P is (S, D)-prime (resp.(S, D)-semiprime).*

(ii) *P is (S, S^{-1}, D)-prime (resp.(S, S^{-1}, D)-semiprime).*

(iii) *For any $a, b \in R\backslash P$ (resp. $a \in R\backslash P$) there exist $(k,l) \in \mathbb{N}\times\mathbb{Z}$ such that $aRD^kS^l(b) \not\subseteq P$ (resp. $aRD^kS^l(a) \not\subseteq P$).*

If, in addition, R satisfies the ACC on S-ideals, conditions (i),(ii) and (iii) are also equivalent to :

(iv) *For any $a, b \in R\backslash P$ (resp. $a \in R\backslash P$) there exists $(k,l) \in \mathbb{N}\times\mathbb{N}$ such that $aRD^kS^l(b) \not\subseteq P$ (resp. $aRD^kS^l(a) \not\subseteq P$).*

In particular, for $P = 0$, these statements give criterions for (S, S^{-1}, D)-primeness and semiprimeness of R.

Proof. (a) The sufficiency of the condition is given by Proposition 1.10(a). Conversely, suppose P is (S, S^{-1}, D)-prime and let $a, b \in R\backslash P$. By Proposition 1.10(a) again, we know that there exist $\omega_1', \omega_2' \in \Omega$ such that $\omega_1'(a)R\omega_2'(b) \not\subseteq P$. Let $\omega_1 \in \Omega$ be of minimal length such that there exists $\omega_2 \in \Omega$ with $\omega_1(a)R\omega_2(b) \not\subseteq P$. We finish by proving that $l(\omega_1) = 0$ (i.e. $\omega_1 = id_R$). Assume, on the contrary, that $l(\omega_1) > 0$. Then ω_1 is either of the form $\omega_1 = S^\varepsilon\psi_1$ or $\omega_1 = D\psi_1$ where $\varepsilon \in \{+1, -1\}$ and $\psi_1 \in \Omega$ is such that $l(\psi_1) < l(\omega_1)$.

If $\omega_1 = S^\varepsilon\psi_1$, then $\psi_1(a)RS^{-\varepsilon}\omega_2(b) \not\subseteq P$ and this contradicts the minimality of $l(\omega_1)$.

Assume $\omega_1 = D\psi_1$. Then, since $l(\psi_1) < l(\omega_1)$, we have $\psi_1(a)R\omega_2(b) \subseteq P$. Hence, for any $r \in R$, $D(\psi_1(a)r\omega_2(b)) \in P$. Computing $D(\psi_1(a)r\omega_2(b))$ we easily obtain

$$S(\psi_1(a)rS^{-1}D\omega_2(b)) + S(\psi_1(a)S^{-1}D(r)S^{-1}\omega_2(b)) + \omega_1(a)r\omega_2(b) \in P.$$

The minimality of $l(\omega_1)$ implies that $\psi_1(a)RS^{-1}D\omega_2(b) \subseteq P$ and also $\psi_1(a)RS^{-1}\omega_2(b) \subseteq P$. Therefore, since $S(P) \subseteq P$, we get $\omega_1(a)r\omega_2(b) \in P$ for all $r \in R$, a contradiction.

(b) Suppose that P is (S, S^{-1}, D)-semiprime, i.e. R/P is an (S^{-1}, S, D)-semiprime ring. By Example 1.3, for $a \in R\backslash P$, $I := \sum_{\omega\in\Omega} R\omega(a)R \not\subseteq P$ is an (S, S^{-1}, D)-ideal. Hence $I^2 \not\subseteq P$ and there exist $\omega_1, \omega_2 \in \Omega$ such that $\omega_1(a)R\omega_2(a) \not\subseteq P$. Now, a similar argument as the one used in (a) yields the existence of $\omega \in \Omega$ such that $aR\omega(a) \not\subseteq P$.

Conversely, assume that for any $a \in R\backslash P$ there exists $\omega \in \Omega$ such that $aR\omega(a) \nsubseteq P$. Then, for $I \lhd_{(S,S^{-1},D)} R$ and $I \nsubseteq P$, we have $I^2 \nsubseteq P$.

(c) The equivalence (i) $\leftrightarrow$ (ii) is a consequence of Proposition 1.14 and Example 1.18. The equivalence (ii) $\leftrightarrow$ (iii) is immediate from parts (a) and (b) above if we remark that, under the assumptions of (c), any word in (S, S^{-1}, D) can be written in the form $\alpha D^k S^l$ where $k \in \mathbb{N}$, $l \in \mathbb{Z}$ and $\alpha \in R$ is invertible.

Assume now that R satisfies ACC on S-ideals. Suppose that (iii) holds. Let $a, b \in R\backslash P$ and $I := \sum_{(k,l)\in\mathbb{N}^2} RD^k S^l(b)R$. Then $S(I) \subseteq I$ and, as we have remarked in Example 1.16, we get $S^{-1}(I) = I = S(I)$. Therefore $I = \sum_{(k,l)\in\mathbb{N}\times\mathbb{Z}} RD^k S^l(I)R$. By assumption, there exists $(k,l) \in \mathbb{N}\times\mathbb{Z}$ such that $aRD^k S^l(b) \nsubseteq P$. Hence $aI \nsubseteq P$ and there exists $(m,n) \in \mathbb{N}^2$ such that $aRD^m S^n(b) \nsubseteq P$. This gives $(iii) \to (iv)$. The reverse implication is a tautology. **QED**

The next lemma stresses some particularities of the (S, S^{-1})-setting (with $D = 0$).

Lemma 2.3. *Let S be an automorphism of the ring R. Then:*

(a) *An S-ideal P of R is S-prime iff there exists an S-m-system M such that P is maximal among S-ideals disjoint from M.*

(b) *An (S, S^{-1})-ideal P of R is (S, S^{-1})-prime iff there exists an S-m-system M such that P is maximal among (S, S^{-1})-ideals disjoint from M.*

(c) $\mathrm{rad}(R; S) = \mathrm{rad}(R; S, S^{-1}) = \{a \in R \mid$ *if* $(a_0 = a, a_1, a_2, \dots)$ *is such that* $a_{i+1} \in a_i RS^{l_i}(a_i)$ *for all* $i \geq 0$ *and some* $l_i \in \mathbb{Z}$*, then there exists* $n \in \mathbb{N}$ *for which* $a_n = 0$.

(d) *For any* $n \geq 1$, $\mathrm{rad}(R; S) = \mathrm{rad}(R; S^n)$.

Proof. We say that a sequence $(a_0, a_1, \dots)$ of elements from R is a $(1, S)$-sequence if for any $i \geq 0$ there exists $l_i \in \mathbb{Z}$ such that $a_{i+1} \in a_i RS^{l_i}(a_i)$.

(a) This is a particular case of Proposition 1.10 (a) and (b).

(b) This is left as an easy exercise for the reader (note that the characterization of (S, S^{-1})-primeness here is via S-m-system, not (S, S^{-1})-m-system).

(c) The first equality is a particular case of Corollary 1.15. Put $E := \{a \in R \mid \text{such that every } (1,S)\text{-sequence starting with } a \text{ eventually vanishes}\}$

Let $a \in \operatorname{rad}(R;S)$ and $(a_0 = a, a_1, a_2, \dots)$ be a $(1,S)$-sequence. Then the sequence is also an (S, S^{-1})-m-sequence. Hence, since $a \in \operatorname{rad}(R;S) = \operatorname{rad}(R;S,S^{-1})$, we have that $a_n = 0$ for some $n \in \mathbb{N}$ and so $a \in E$. This shows that $\operatorname{rad}(R;S) \subseteq E$.

Assume now that $a \in E$ and let $(a_0 = a, a_1, a_2, \dots)$ be an (S, S^{-1})-m-sequence, i.e. for any $i \in \mathbb{N}$, there exist $k_i, l_i \in \mathbb{Z}$ such that $a_{i+1} \in S^{k_i}(a_i)RS^{l_i}(a_i)$. Let us define $a'_0 := a$ and $a'_i := S^{-(k_{i-1}+\dots+k_0)}(a_i)$ for all $i \geq 1$. It is standard to check that $a'_{i+1} \in a'_i RS^{l_i-k_i}(a'_i)$ for any $i \in \mathbb{N}$. This means that the sequence $(a'_0 = a, a'_1, \dots)$ is a $(1,S)$-sequence. Therefore, as $a \in E$, there exists $n \in \mathbb{N}$ such that $a'_n = 0$. Then obviously $a_n = 0$, i.e. any $a \in E$ is strongly (S, S^{-1})-nilpotent and, by Proposition 1.11, $E \subseteq \operatorname{rad}(R;S,S^{-1})$ follows.

(d) The inclusion $\operatorname{rad}(R;S) \subseteq \operatorname{rad}(R;S^n)$ is obvious (cf. e.g. Proposition 1.1). Part (c) above shows that $\operatorname{rad}(R;S) = \{a \in R \mid \text{every } (1,S)\text{-sequence starting with } a \text{ eventually vanishes}\}$. Notice that if $(a_0, a_1, \dots)$ is a $(1,S)$-sequence then $(a_{n_0}, a_{n_1}, \dots)$ is also a $(1,S)$-sequence if $n_0 < n_1 < \dots$.

We claim that any $(1,S)$-sequence contains a $(1,S^n)$-subsequence. Indeed, let $(a_0, a_1, \dots)$ be $(1,S)$-sequence. Then for any $i \geq 0$ there is $l_i \in \mathbb{Z}$ such that $a_{i+1} \in a_i RS^{l_i}(a_i)$. Now, there exist $r \in \{0, 1, \dots, n-1\}$ and an infinite increasing sequence of natural numbers $(n_0, n_1, \dots)$ such that, for any $i \in \mathbb{N}$, $\sum_{j=0}^{n_i-1} l_j = q_i n + r$ for some suitable $q_i \in \mathbb{Z}$. It is easy to see that for any $0 \leq k < m$, $a_m \in a_k RS^{l_k+\dots+l_{m-1}}(a_k)$. Therefore, the choice of the sequence $(n_0, n_1, \dots)$ yields that $a_{n_{i+1}} \in a_{n_i} RS^{n(q_{i+1}-q_i)}(a_{n_i})$ for any $i \geq 0$. This shows that the sequence $(a_{n_0}, a_{n_1}, \dots)$ is an $(1,S^n)$-sequence as claimed.

Now let $a \in \operatorname{rad}(R;S^n)$ and $(a = a_0, a_1, \dots)$ be a $(1,S)$-sequence. As above we can construct a $(1,S^n)$-subsequence $(a_{n_0}, a_{n_1}, \dots)$. Since $a = a_0 \in \operatorname{rad}(R;S^n)$, a_{n_0} also belongs to $\operatorname{rad}(R,S^n)$ and thus the $(1,S^n)$-

sequence $(a_{n_0}, a_{n_1}, \dots)$ must eventually vanish. But this means that the original sequence $(a_0, a_1, \dots)$ also eventually vanishes. Therefore, using the characterization of $\mathrm{rad}(R;S)$ obtained in (c), we get that a belongs to $\mathrm{rad}(R;S)$ as required. **QED**

Corollary 2.4. *Suppose that one of the following conditions is satisfied:*

(a) *Some nonzero power of S is inner.*
(b) *R satisfies the ACC on ideals.*

Then $\mathrm{rad}(R) = \mathrm{rad}(R;S) = \mathrm{rad}(R;S,S^{-1})$.

Proof. If S is an inner automorphism of R then $\mathrm{rad}(R;S) = \mathrm{rad}(R)$ as every ideal is S-invariant. Now the assertion follows from Lemma 2.3(d) and (c).

Suppose (b) holds. Then $\mathrm{rad}(R)$ is nilpotent, so $\mathrm{rad}(R;S) = \mathrm{rad}(R)$. **QED**

Before going on with our comparison of different notions of primeness we need the following technical lemma.

Lemma 2.5.

(a) *Let I be an S-ideal and D an S-derivation then $I + D(I)$ is an ideal of R.*
(b) *Let D be a q-quantized S-derivation satisfying $SD = qDS$ with $q \in Z(R)_{S,D} = \{a \in R \mid a$ belongs to the center $Z(R)$ of R and $S(a) = a,\ D(a) = 0\}$. Assume moreover, that for any $n \in \mathbb{N}$, $1 + q + \cdots + q^n$ is regular in R. Then, for any nilpotent (S,S^{-1})-ideal I of R, $J = I + D(I)$ is a nilpotent S-ideal such that $qS^{-1}(J) \subseteq J$.*
(c) *Suppose $q \in Z(R)$ is regular and $S(q) = q$. If J is a nilpotent S-ideal of R satisfying $qS^{-1}(J) \subseteq J$, then $\overline{J} = \{a \in R \mid$ there is $k \in \mathbb{N}$ such that $q^k a \in J\}$ is a nilpotent (S,S^{-1})-ideal containing J.*
(d) *Suppose R is S-semiprime (or equivalently R is (S,S^{-1})-semiprime). Then the right annihilator of an (S,S^{-1})-ideal is a (S,S^{-1},D)-ideal for any S-derivation D.*

Proof. (a) This is standard and easy.

(b) We claim that for any $a_1, \dots, a_n \in I$ we have

$$D^n(S^{-n+1}(a_1)S^{-n+2}(a_2)\dots S^{-1}(a_{n-1})a_n) - (n!)_q D(a_1)\dots D(a_n) \in I.$$

We prove this by induction on $n \geq 1$. If $n = 1$ the result is clear. For $n \geq 1$ we have

$$D^{n+1}(S^{-n}(a_1)\dots S^{-1}(a_n)a_{n+1}) = \sum_{k=0}^{n+1} \binom{n+1}{k}_q D^{n+1-k}S^k(S^{-n}(a_1)\dots S^{-1}(a_n))D^k(a_{n+1}).$$

Since $D^k(I^n) \subset I$ if $n > k$ and $a_{n+1} \in I$ the above equality implies that

$$D^{n+1}(S^{-n}(a_1)S^{-n+1}(a_2)\dots S^{-1}(a_n)a_{n+1}) - \binom{n+1}{1}_q D^n(S^{1-n}(a_1)\dots S^{-1}(a_{n-1})a_n)D(a_{n+1}) \in I$$

Now, the inductive hypothesis applied to the second term yields:

$$D^{n+1}(S^{-n}(a_1)S^{-n+1}(a_2)\dots a_{n+1}) - ((n+1)!)_q D(a_1)\dots D(a_n)D(a_{n+1}) \in I$$

This proves the claim.

Suppose now that $I^n = 0$. Then, by the above, for any $a_1, \dots a_n \in I$ we have
$(n!)_q D(a_1)\dots D(a_n) \in I$, i.e. $(n!)_q D(I)^n \subseteq I$. Since $(n!)_q \in Z(R)$ is regular we get $D(I)^{n^2} = 0$ and this leads easily to the desired conclusion that $J = I + D(I)$ is nilpotent. Since $SD = qDS$, it easy to see that $S(J) \subseteq J$ and $qS^{-1}(J) \subseteq J$.

(c) Since $S(q) = q$, S can be extended to the localization $R\mathcal{A}^{-1}$ where $\mathcal{A} = \{1, q, q^2, \dots\}$. Then $J\mathcal{A}^{-1}$ is a nilpotent (S, S^{-1})-ideal of $R\mathcal{A}^{-1}$ and $\overline{J} = R \cap J\mathcal{A}^{-1}$. This gives (c).

(d) Let I be an (S, S^{-1})-ideal then $D(I^2) \subseteq I$ and if we put $J := rann_R(I)$ we get $D(I^2)J = 0$ and $0 = D(I^2J) = I^2D(J)$. So $I^2S^kD(J) = 0$ for all $k \in \mathbb{Z}$. The S-ideal $\overline{J} = \sum_{k=-\infty}^{+\infty} IS^kD(J)R$ is such that $\overline{J}^2 = 0$.

Hence, since R is S-semiprime, $\overline{J} = 0$ and, in particular, $ID(J) = 0$, i.e. $D(J) \subseteq J$. This proves the claim since we clearly also have $S(J) = J$, . **QED**

The next theorem will provide some relations between the different notions of primeness of R which appear in the context of S-derivations. Such relations have been studied by various authors. In particular, D.A. Jordan remarked (cf. [J], remarks after Lemma 2.1) that if R is a semiprime D-prime ring (D being a derivation) then R is in fact prime. Goodearl [G] gave a nice description of the relationships between (S, D)-primeness, D-primeness and S-primeness when R is noetherian.

Theorem 2.6. *Let R be a ring, S an automorphism and D an S-derivation of R. Suppose that (R, S, D) satisfies one of the hypothesis:*

(I_1) *: R is S-semiprime*

(I_2) *: R satisfies ACC on S-ideals and D is q-quantized : $SD = qDS$ where $q \in Z(R)_{S,D}$ is such that for any $n \geq 1$, $1+q+\cdots+q^n \neq 0$.*

Then the following statements are equivalent:

(a) *R is S-prime*

(b) *R is (S, S^{-1})-prime*

(c) *R is (S, D)-prime*

(d) *R is (S, S^{-1}, D)-prime*

Proof. Let us first remark that in case D is q-quantized, any of the conditions (a), (b), (c) or (d) and the fact that $1 + \cdots + q^n \neq 0$ imply that $1 + q + \cdots + q^n$ is regular in R. We know that (a) $\leftrightarrow$ (b), and (a) $\rightarrow$ (c) $\rightarrow$ (d). Thus, it remains to prove that (d) $\rightarrow$ (b).

Suppose that (I_1) is satisfied and let I, J be nonzero (S, S^{-1})-ideals of R such that $IJ = 0$. Then $lann_R(rann_R(I)) \cdot rann_R(I) = 0$ is a product of (S, S^{-1}, D)-ideals thanks to Lemma 2.5(d) above (since R is S-semiprime). If R is (S, S^{-1}, D)-prime, then either $lann_R(rann_R(I)) = 0$ or $rann_R(I) = 0$. On the other hand, $0 \neq J \subseteq rann_R(I)$ and $0 \neq I \subseteq lann_R(rann_R(I))$. This contradiction completes the proof of (b) $\rightarrow$ (d), in the case when (I_1) is satisfied.

Suppose now that (I_2) is satisfied. Let N be the unique largest nilpotent

S-ideal of R. Then $S(N) = N$ and, by Lemma 2.5(b), $N + D(N)$ is a nilpotent S-ideal. Thus $N + D(N) \subseteq N$ and $D(N) \subseteq N$ follows. If R is (S, S^{-1}, D)-prime, we conclude $N = 0$. Thus R is S-semiprime and we are back in the case when (I_1) is satisfied.

QED

The implication (c) $\rightarrow$ (a) above was given by Goodearl ([G], Proposition 6.5), in the case when I_2 is satisfied. The proof in the Goodearl 's paper was different from the above and made use of the fact that $\mathrm{rad}(R; S, S^{-1})$ is D-stable. We can also obtain this fact using Lemma 2.5(b) and the equality $\mathrm{rad}(R; S, S^{-1}) = L(R; S, S^{-1})$.

Corollary 2.7. *Let D be a q-quantized S-derivation satisfying $SD = qDS$ where $q \in Z(R)_{S,D}$ is such that both q and $q^n + q^{n-1} + \cdots + q + 1$ are regular in R for all $n \in \mathbb{N}$. Then $\mathrm{rad}(R; S, S^{-1})$ is an (S, S^{-1}, D)-ideal.*

Proof. In view of Proposition 1.11 and the construction of $L(R; S, S^{-1})$ preceding it, we need only to show that $D(L_1(R; S, S^{-1})) \subseteq L_1(R; S, S^{-1})$ where $L_1(R; S, S^{-1})$ is the sum of all nilpotent (S, S^{-1})-ideals. Let $a \in L_1(R; S, S^{-1})$. Then a belongs to some nilpotent (S, S^{-1})-ideal I. Now Lemma 2.5(b) and (c) imply that $I + D(I)$ is contained in a nilpotent (S, S^{-1})-ideal. This shows that $I + D(I) \subseteq L_1(R; S, S^{-1})$. Therefore $D(a) \in L_1(R; S, S^{-1})$. **QED**

Proposition 2.8. *Let R be a ring, $S \in \mathrm{Aut}(R)$ and D an S-derivation of R. Suppose that R satisfies ACC on ideals and that D is a q-quantized S-derivation where $q \in Z(R)_{S,D}$ is such that $1 + q + \cdots + q^n$ is regular in R for any $n \geq 1$. Then*

$$\begin{aligned}\mathrm{rad}(R; S, S^{-1}, D) = \mathrm{rad}(R; S, D) &= \mathrm{rad}(R; S) = \mathrm{rad}(R; S, S^{-1}) \\ &= \mathrm{rad}(R; D) = \mathrm{rad}(R).\end{aligned}$$

Proof. Using Proposition 1.1 and Corollary 2.4, we get

$$\mathrm{rad}(R; S, S^{-1}, D) \subseteq \mathrm{rad}(R; S, D) \subseteq \mathrm{rad}(R; S) = \mathrm{rad}(R; S, S^{-1}) = \mathrm{rad}(R).$$

Since R satisfies ACC on ideals, $\mathrm{rad}(R)$ is nilpotent and Lemma 2.5(b) implies that $\mathrm{rad}(R) + D(\mathrm{rad}(R)) \subseteq \mathrm{rad}(R)$. This means that $\mathrm{rad}(R)$ is an

(S, S^{-1}, D)-ideal and we conclude that $\mathrm{rad}(R) \subseteq \mathrm{rad}(R; S, S^{-1}, D)$. Now, inclusions $\mathrm{rad}(R; S, D) \subseteq \mathrm{rad}(R; D) \subseteq \mathrm{rad}(R)$ complete the proof. **QED**

Let us pause in order to give a few examples.

Example 2.9. Let K be a field of characteristic 2 and D be a derivation of $R := \dfrac{K[x]}{(x^2)}$ induced by the standard derivation $\frac{\partial}{\partial x}$ of $K[x]$. Then R is D-prime (in fact D-simple), but R is not semiprime.

Example 2.10. Let K be a field and S a K-automorphism of $R := \dfrac{K[X,Y]}{(XY)}$ defined by $S(X) = Y$ and $S(Y) = X$. It is easy to show that R is semiprime and S-prime but not prime.

Example 2.11. This example comes from [G] (cf. also [GL] Example 3.1) and will also be used later (cf. 4.16) in this paper. Hence we shall present it in detail for the convenience of the reader. We shall construct a ring R an automorphism S and an S-derivation D of R such that R is (S, D)-simple but R is neither S-prime nor D-prime.

Let K be a field and τ be the left shifting automorphism of K^3 : $\tau(a, b, c) = (b, c, a)$ for $(a, b, c) \in K^3$. Let $R = \dfrac{K^3[X; \tau]}{(X^3)}$. The canonical image of X in R will be denoted by x. τ^2 can be extended to $K^3[X; \tau]$ by putting $\tau^2(X) = X$, and this induces an automorphism S of R. We then define an S-derivation D of R by

$$D((a, b, c,)x^n) = \begin{cases} 0 & n = 0 \\ (0, -a, b) & n = 1 \\ (-c, 0, b)x & n = 2 \end{cases}$$

(In fact D is induced by the restriction to $K^3[X; \tau]$ of the inner τ^2-derivation determined on $K^3[X, X^{-1}, \tau]$ by $(0, 0, 1)X^{-1}$).

Here, R is a 9-dimensional K-algebra which is not semiprime. Since $D((1, 0, 0)x^2) = 0$, the set $K(1, 0, 0)x^2$ is a nilpotent D-ideal and so R is not D-prime. To prove that R is (S, D)-simple assume there exists $0 \neq u \in I \lhd_{(S,D)} R$. Then either u, ux or ux^2 is of the form $(a, b, c)x^2 \in$

$I\backslash\{0\}$. Replacing u by $S(u)$ or $S^2(u)$ if necessary, we may assume that $c \neq 0$ and $\omega = (0,0,c^{-1})u = (0,0,1)x^2 \in I\backslash\{0\}$. But then $D(\omega) = (-1,0,0)x, D^2(\omega) = (0,1,0) \in I\backslash\{0\}$. Since $S(0,1,0), S^2(0,1,0) \in I$. We conclude that $I = R$.

3. Upper S-nil radical

In Section 5 we shall study the relationship between the prime radical of an Ore extension $R[t;S,D]$ and that of the base ring R. In this study a certain radical generalizing the classical upper nil radical will emerge and prove to be particularly helpful. This radical was first introduced in [PSW]. We adopt a somewhat different approach here and show that our definition agrees with the one given in [PSW]. We then study this radical, paving the way to future sections.

Definitions 3.1. Let R be a ring and $S \in \mathrm{Aut}(R)$

(a) For $a \in R,\ n \in \mathbb{N} \setminus \{0\}$ $N_n^S(a) := aS(a)\dots S^{n-1}(a)$.
(b) An element $r \in R$ is S-nilpotent if and only if for any $l \in \mathbb{N} \setminus \{0\}$ there exists $n_l \in \mathbb{N} \setminus \{0\}$ such that $N_{n_l}^{S^l}(r) = 0$.
(c) An S-nil (left, right or two-sided) ideal is a (left, right, two-sided) ideal such that all its elements are S-nilpotent.

Remarks 3.2. (1) The notion of an S-nilpotent element was introduced in [JJ] and appears naturally in connection with the nilpotency of $rt^l \in R[t;S]$. This notion was also used in [R].

(2) Another reason for adopting this definition of S-nilpotency is that in $R[t,S^{-l}]$ we have $t^n = (t-r)q(t) + N_n^{S^l}(r)$, for some $q(t) \in R[t;S^{-l}]$. This shows that $N_n^{S^l}(r)$ is the analogue of r^n when considering evaluation of polynomials in $R[t;S^{-l}]$.

Lemma 3.3. *Let R be a ring and $S \in$Aut(R). Then:*

(a) $N_{n+m}^S(a) = N_n^S(a)S^n(N_m^S(a))$ *for any $a \in R$ and $n,m \in \mathbb{N} \setminus \{0\}$.*
(b) $N_{n+1}^{S^l}(xay) = xN_n^{S^l}(ayS^l(x))S^{nl}(ay)$ *for any $a,x,y \in R$ and $l,n \in \mathbb{N} \setminus \{0\}$.*
(c) $N_j^{S^{li}}(N_i^{S^l}(a)) = N_{ij}^{S^l}(a)$ *for any $a \in R,\ i,j,l \in \mathbb{N} \setminus \{0\}$.*

Proof. These identities can be proved by straightforward calculations. As a sample, we give the proof for (c).

$$
\begin{aligned}
N_j^{S^{li}}(N_i^{S^l}(a)) &= N_i^{S^l}(a)S^{li}(N_i^{S^l}(a))S^{2li}(N_i^{S^l}(a))\dots S^{li(j-1)}(N_i^{S^l}(a)) \\
&= aS^l(a)\dots S^{l(i-1)}(a)S^{li}(a)\dots S^{li(j-1)+l(i-1)}(a) \\
&= aS^l(a)\dots S^{l(i-1)}(a)\dots S^{l(ij-1)}(a) \\
&= N_{ij}^{S^l}(a)
\end{aligned}
$$

QED

Proposition 3.4. *Let I_1, I_2 be two S-nil ideals of the ring R with $S(I_2) \subseteq I_2$. Then $I_1 + I_2$ is an S-nil ideal.*

Proof. Let $a_1 \in I_1$, $a_2 \in I_2$ and $a = a_1 + a_2$. We want to prove that for any $j \in \mathbb{N}\setminus\{0\}$ there exists $l_j \in \mathbb{N}\setminus\{0\}$ such that $N_{l_j}^{S^j}(a) = 0$. Assume $N_n^{S^j}(a_1) = 0$. From

$$
\begin{aligned}
N_n^{S^j}(a) &= (a_1 + a_2)S^j(a_1 + a_2)\dots S^{j(n-1)}(a_1 + a_2) \\
&= N_n^{S^j}(a_1) + \alpha, \quad \text{where } \alpha \in I_2,
\end{aligned}
$$

we have $N_n^{S^j}(a) = \alpha \in I_2$. Since I_2 is S-nil, there exists $r \in \mathbb{N}\setminus\{0\}$ such that $N_r^{S^{jn}}(\alpha) = 0$ and so $0 = N_r^{S^{jn}}(N_n^{S^j}(a)) = N_{rn}^{S^j}(a)$ (by using formula (c) of Lemma 3.3). **QED**

Theorem 3.5. *Let R be a ring and $S \in$Aut(R). Then R contains a unique maximal S-nil S-ideal N. The automorphism S induces an automorphism on R/N and R/N has no nonzero S-nil S-ideals.*

Proof. This is an immediate consequence of Lemma 3.3(c), and Proposition 3.4. (We just take N to be the sum of all S-nil S-ideals.) **QED**

The above theorem leads to the first of the following.

Definitions 3.6.

(a) The unique maximal S-nil S-ideal of Theorem 3.5 is called the S-upper nil radical of R and is denoted by Nil$(R; S)$.

(b) [PSW] A one-sided (or two-sided) ideal is S-n-nil ($n \in \mathbb{N} \setminus \{0\}$) if all its elements are S^m-nilpotent for any $m \geq n$.

In [PSW] $N_S(R) := \sum\{I \lhd_{SS^{-1}} R \mid I$ is S-n-nil for some $n \in \mathbb{N}\}$ was introduced. In fact this radical coincides with our radical, as the following proposition shows.

Proposition 3.7. *Let R be a ring, $S \in$Aut(R) and I be a right ideal of R. Using the above notations and definitions, we have:*

(a) *I is S-nil if and only if I is S-n-nil for some $n \geq 1$.*

(b) $\mathrm{Nil}(R; S) = N_S(R)$.

Proof. (a) Obviously if I is S-nil then it is S-1-nil. Assume now that I is S-n-nil. Let $i \in \mathbb{N} \setminus \{0\}$ and $x \in I$. Then $N_n^{S^i}(x) \in I$ and so there exists an $s \in \mathbb{N} \setminus \{0\}$ such that$N_s^{S^{ni}}(N_n^{S^i}(x)) = 0$. By using formula (c) of Lemma 3.3 we get $N_{sn}^{S^i}(x) = 0$.

(b) This is a direct consequence of Theorem 3.5 and part (a) above, in view of the fact that $\mathrm{Nil}(R, S)$ is an (S, S^{-1})-ideal . **QED**

Lemma 3.8. *Let R be a ring and $S \in$Aut(R). For any $l \in \mathbb{Z}$ and $a \in R$ the set $M_l(a) := \{N_n^{S^l}(a) \mid n \in \mathbb{N}\backslash\{0\}\}$ is an S-m-system.*

Proof. This is obvious from Lemma 3.3(a). **QED**

Proposition 3.9. *Let R be a ring and $S \in$Aut(R). Denote by $\mathcal{P}$ the set of all (S, S^{-1})-prime ideals $P \subset R$ such that* $\mathrm{Nil}(R/P; S) = 0$. *Then*

$$\mathrm{Nil}(R; S) = \cap\{P \mid P \in \mathcal{P}\}.$$

Proof. Clearly $\mathrm{Nil}(R; S) \subset P$ for any $P \in \mathcal{P}$. Hence, it remains to prove that $\mathrm{Nil}(R; S) \supseteq \cap\{P \mid P \in \mathcal{P}\}$. For this, it is enough to show that if $a \in R$ is not S-nilpotent, then there exists $P \in \mathcal{P}$ such that $a \notin P$. Indeed from this we can conclude that $\cap\{P \mid P \in \mathcal{P}\}$ is an S-nil ideal and hence is contained in $\mathrm{Nil}(R; S)$. Thus assume $a \in R$ is not S-nilpotent. Then there exists $l \in \mathbb{N} \setminus \{0\}$ such that $N_n^{S^l}(a) \neq 0$ for any $n \in \mathbb{N} \setminus \{0\}$. Put, as above, $M_l(a) := \{N_n^{S^l}(a) \mid n \in \mathbb{N} \setminus \{0\}\}$. Lemma 3.8 shows that $M_l(a)$ is an S-m-system. By Zorn's Lemma there exists a maximal (S, S^{-1})-ideal P disjoint from $M_l(a)$ and Lemma 2.3(b) implies that P is in fact (S, S^{-1})-prime. We claim that $P \in \mathcal{P}$. For this, we need only show that $\mathrm{Nil}(R/P; S) = 0$. Assume on the contrary that $\mathrm{Nil}(R/P; S) \neq 0$. Then there exists an (S, S^{-1})-ideal Q strictly containing P such that Q is S-nil modulo P. By definition of P we have $N_n^{S^l}(a) \in Q \cap M_l(a)$ for some $n \in \mathbb{N} \backslash \{0\}$. Since Q is S-nil modulo P we conclude that there exists $m \in \mathbb{N} \backslash \{0\}$ such that $N_m^{S^{ln}}(N_n^{S^l}(a)) \in P$, i.e. using formula (c) of Lemma 3.3, $N_{mn}^{S^l}(a) \in P$ and so $P \cap M_l(a) \neq \emptyset$, a contradiction. This shows that $\mathrm{Nil}(R/P; S) = 0$, i.e. $P \in \mathcal{P}$ as claimed. Since $a = N_1^{S^l}(a) \in M_l(a)$, we get $a \notin P$. This finishes the proof. **QED**

Corollary 3.10. *Using the above notations we have*

$$\mathrm{rad}(R; S) \subseteq \mathrm{Nil}(R; S)$$

.

Proof. Using Proposition 3.9 and the definition of $\mathrm{rad}(R; S, S^{-1})$ we obtain that $\mathrm{Nil}(R; S) \supseteq \mathrm{rad}(R; S, S^{-1})$. Corollary 1.15(b) implies that $\mathrm{rad}(R; S, S^{-1}) = \mathrm{rad}(R; S)$, so we obtain $\mathrm{Nil}(R; S) \supseteq \mathrm{rad}(R; S)$. **QED**

We will soon find other connections between $\mathrm{Nil}(R; S)$, $\mathrm{rad}(R; S)$ and $\mathrm{rad}(R)$ but let us first give a definition and state a related result.

Definition 3.11. An automorphism S of R is locally of finite inner order if, for any $a \in R$ there exist $n = n(a) \in \mathbb{N} \backslash \{0\}$ and an invertible element $u = u(a) \in R$ such that for every $x \in RaR$, $S^n(x) = uxu^{-1}$.

Proposition 3.12. *Keeping the notations as above we have:*

(a) *For any* $r \in \mathbb{N} \setminus \{0\}$, $\mathrm{Nil}(R; S) = \mathrm{Nil}(R; S^r)$.

(b) *If* S *is locally of finite inner order, then* $\mathrm{Nil}(R; S) = \mathrm{Nil}(R)$.

Proof. (a) Let $r \in \mathbb{N} \setminus \{0\}$. Then any S-nil S-ideal is also an S^r-nil S^r-ideal for any $0 \neq r \in \mathbb{N}$. This gives $\mathrm{Nil}(R; S) \subseteq \mathrm{Nil}(R; S^r)$. Let us now show that $\mathrm{Nil}(R; S^r)$ is an S-nil S-ideal. Clearly $\mathrm{Nil}(R; S^r) \triangleleft_S R$. Now, if $a \in \mathrm{Nil}(R; S^r)$ then for any $l \in \mathbb{N} \backslash \{0\}, N_r^{S^l}(a) \in \mathrm{Nil}(R; S^r)$ and so there exists $n = n(l, N_r^{S^l}(a))$ such that $N_n^{S^{rl}}(N_r^{S^l}(a)) = 0$. By using Lemma 3.3(c), $N_{nr}^{S^l}(a) = 0$ follows. This shows that $\mathrm{Nil}(R; S^r)$ is in fact an S-nil ideal and so $\mathrm{Nil}(R; S^r) \subseteq \mathrm{Nil}(R; S)$.

(b) Let $a \in \mathrm{Nil}(R; S)$. Since S is locally of finite inner order, there exist $n = n(a) \in \mathbb{N}$ and an invertible element $u = u(a) \in R$ such that $S^n(x) = uxu^{-1}$ for all $x \in RaR$. Because $au^{-1} \in \mathrm{Nil}(R; S)$, there exists $l \in \mathbb{N}\setminus\{0\}$ such that $N_l^{S^n}(au^{-1}) = 0$. Notice that $au^{-1} \in RaR$. Therefore $0 = N_l^{S^n}(au^{-1}) = a^l u^{-l}$ and we conclude that $\mathrm{Nil}(R; S)$ is a nil ideal. This shows that $\mathrm{Nil}(R; S) \subseteq \mathrm{Nil}(R)$.

In order to prove the opposite inclusion, let us first show that if $a \in \mathrm{Nil}(R)$ then RaR is S^n-nil, where n and u are such that $S^n(x) = uxu^{-1}$ for any $x \in RaR$. Let $x \in RaR \subseteq \mathrm{Nil}(R)$ and $r \in \mathbb{N}$. There exists $l \in \mathbb{N} \setminus \{0\}$ such that $(xu^r)^l = 0$. Hence $N_l^{S^{nr}}(x) = (xu^r)^l u^{-lr} = 0$. This shows that x is S^n-nil and so RaR is an S^n-nil ideal. Since RaR is clearly an S^n-ideal, this yields $RaR \subseteq \mathrm{Nil}(R; S^n) = \mathrm{Nil}(R; S)$, where the last equality is given by (a) above. **QED**

Lemma 3.13. *Let R be a ring, $S \in \mathrm{Aut}(R)$ and $a \in R$. Then Ra is S-nil if and only if aR is S-nil.*

Proof. Suppose that Ra is S-nil. Then $RS^l(a)$ is S-nil for any $l \geq 1$. Hence for any $y \in R$ there exists $n \in \mathbb{N}\setminus\{0\}$ such that $N_n^{S^l}(yS^l(a)) = 0$. Then $N_{n+1}^{S^l}(ay) = aN_n^{S^l}(yS^l(a))S^{ln}(y) = 0$. This means that aR is S-nil. The proof of the reverse implication is left to the reader. **QED**

Proposition 3.14. *Suppose R satisfies ACC either on right or left annihilators. Then:*

(a) *If R contains a nonzero S-nil one-sided ideal, then R contains a nonzero nilpotent S^{-1}-ideal.*

(b) *If R is S-semiprime, then R has no nonzero one-sided S-nil ideals.*

Proof. (a) We consider only the case when R contains a nonzero right S-nil ideal and satisfies ACC on right annihilators. The three other cases can be obtained similarly, by making use of Lemma 3.13 above. Suppose that aR is S-nil for some $0 \neq a \in R$. Put $B := \cup_{k \geq 0} S^k(a)R$ and choose

$b \in B$ such that $rann_R(b)$ is maximal among right annihilators of elements from $B\backslash\{0\}$. Since aR is S-nil so is B. Thus, for any $x \in R$ for which $bx \neq 0$ and any $k \geq 1$, there exists $n \in \mathbb{N}$ such that $N_n^{S^k}(bx) = 0 \neq N_{n-1}^{S^k}(bx)$. Hence, by putting $A := S^k(x)S^{2k}(bx)\dots S^{(n-1)k}(bx)$, we get $bxS^k(b)A = 0$. Notice also that $S^k(b)A = S^k(N_{n-1}^{S^k}(bx)) \neq 0$. This shows that $A \in rann_R(bxS^k(b))\backslash rann_R(S^k(b))$. Now, since $bxS^k(b) \in B$ and $rann_R(S^k(b)) \subseteq rann_R(bxS^k(b))$, the choice of b yields that $bxS^k(b) = 0$ for any $k \geq 1$ and $x \in R$. Thus $S^{-k}(b)Rb = 0$ for any $k \geq 1$. Let $J := \sum_{k\geq 1} RS^{-k}(b)R \lhd_{S^{-1}} R$. Then $Jb = 0$ and $rann_R(J) \subseteq rann_R(S^{-1}(J)) \subseteq \dots \subseteq rann_R(S^{-n}(J)) \subseteq \dots$. Because R satisfies ACC on right annihilators, there exists $l \in \mathbb{N}$ such that for any $k \geq 1$, $rann_R(S^{-l}(J)) = rann_R(S^{-(l+k)}(J))$. On the other hand, for any $k \geq 1$, we have $S^{-(l+k)}(J)S^{-(l+k)}(b) = 0$, so $S^{-l}(J)S^{-(l+k)}(b) = 0$ for any $k \geq 1$. This gives

$$S^{-l}(J)^2 = S^{-l}(J)(\sum_{k\geq 1} RS^{-(l+k)}(b)R) = 0.$$

This means that $S^{-l}(J)$ is a nilpotent S^{-1}-ideal and completes the proof of (a).

(b) By Corollary 1.15(b), $\mathrm{rad}(R;S) = \mathrm{rad}(R;S^{-1})$. Now (b) is a direct consequence of (a). **QED**

Corollary 3.15. *Suppose R is either left or right noetherian. Then* $\mathrm{Nil}(R;S) = \mathrm{rad}(R;S) = \mathrm{rad}(R)$.

Proof. Corollary 3.10 shows that $\mathrm{rad}(R;S) \subseteq \mathrm{Nil}(R;S)$. Therefore, factoring out $\mathrm{rad}(R;S)$, we may assume that R is S-semiprime. Now, since $\mathrm{Nil}(R;S)$ is S-nil, Proposition 3.14(b) implies that $\mathrm{Nil}(R;S) = 0$. This shows that $\mathrm{rad}(R;S) = \mathrm{Nil}(R;S)$. The last equality in the proposition follows from Corollary 2.4(b). **QED**

Let us remark that when $S = id_R$, then the above corollary is just the classical theorem of Levitzki.

Example 3.16. Let K be a field and consider $R = \oplus_{i\in\mathbb{Z}} K_i$, where $K_i = K$ for every $i \in \mathbb{Z}$. Define an automorphism S of R by "left shifting". Then $\mathrm{Nil}(R;S) = R$ but $\mathrm{Nil}(R) = 0$

Example 3.17. Let K be a field. Consider $T = K\langle X_i \mid i \in \mathbb{Z}\rangle$ and let $S \in \mathrm{Aut}_K(T)$ be defined by $S(X_i) = X_{i+1}$. Let J denote the ideal of T generated by $\{X_i \omega X_i \mid i \in \mathbb{Z}, \omega \in T\}$. Then $S(J) = J$ and S induces an automorphism on $R = T/J$. It is easy to check that $\mathrm{Nil}(R) = \sum Rx_iR$ (where $x_i = X_i + J \in R$), but $\mathrm{Nil}(R; S) = 0$.

4. Primeness and semi-primeness of $R[t; S, D]$

In this section we shall give necessary and sufficient conditions for the Ore extension $T = R[t; S, D]$ to be prime and semiprime. These criterions are not always easy to use, so we will also give sufficient conditions which are less computational. In doing so, we will use the notions introduced in the first three sections. Let us recall from Example 1.4 that $f_l^n \in \mathrm{End}(R, +)$ denotes the sum of all words composed with l letters "S" and $n - l$ letters "D" (e.g. $f_n^n = S^n, f_0^n = D^n$). It is easy to check that in $R[t; S, D]$ the following identity holds :

Lemma 4.1. *For any $n \in \mathbb{N}$ and $r \in R$ we have:*

$$t^n r = f_n^n(r)t^n + f_{n-1}^n(r)t^{n-1} + \cdots + f_1^n(r)t + f_0^n(r)$$

.

Lemma 4.2. *Let $u = u_m t^m + \cdots + u_0 \in T = R[t; S, D]$. Then:*

(a) *If $lann_R(u_m) \subseteq lann_R(u)$, then $lann_T(u) \subseteq lann_T(u_m)$.*
(b) *Suppose that $rann_R(S^{-m}(u_m)) \subseteq rann_R(u)$, then $rann_T(u) \subseteq rann_T(S^{-m}(u_m))$*

Proof. (a) Assume, on the contrary, that $lann_T(u) \nsubseteq lann_T(u_m)$ and let $g(t) = g_r t^r + \cdots g_1 t + g_0$ be of minimal degree in $lann_T(u) \backslash lann_T(u_m)$. Since $gu = 0$ we obtain $g_r S^r(u_m) = 0$, that is, $S^{-r}(g_r)u_m = 0$. Using the hypothesis in (a), we get $S^{-r}(g_r)u = 0$ and so $(g - t^r S^{-r}(g_r))u = 0$. Since $\deg(g - t^r S^{-r}(g_r)) < \deg g$, the choice of g yields that $0 = (g - t^r S^{-r}(g_r))u_m = gu_m$. This contradiction shows that (a) holds.

(b) This is proved similarly. **QED**

Corollary 4.3. *Let $I \subseteq T = R[t; S, D]$ be a left (resp. right) R-module and let $u = u_m t^m + \cdots + u_1 t + u_0$ be of miminal degree in I. Then $lann_T(u)u_m = 0$ (resp. $S^{-m}(u_m)rann_T(u) = 0$).*

Proof. Since u is of minimal degree in I we conclude $lann_R(u_m) \subseteq lann_R(u)$ (resp. $rann_R(S^{-m}(u_m)) \subseteq rann_R(u)$) and the conclusion follows immediately from Lemma 4.2. **QED**

Lemma 4.2 and Corollary 4.3 are implicit in [G, (3.2)]. They will be very useful in proving Theorem 4.9 and the following.

Theorem 4.4. *For the ring $T = R[t; S, D]$ the following conditions are equivalent:*

1. *$R[t; S, D]$ is prime.*
2. *For every nonzero $a, b \in R$, $aR[t; S, D]b \neq 0$.*
3. *For every nonzero $a, b \in R$ there exist $n \geq k \geq 0$ such that $aRf_k^n(b) \neq 0$.*

Proof. By making use of Lemma 4.1 it is easy to show that (2) and (3) are equivalent.

The implication (1) $\to$ (2) is obvious.

Let us now show that (2) $\to$ (1). Assume $R[t; S, D]$ is not prime and let A, B be nonzero ideals of $T = R[t; S, D]$ such that $AB = 0$. Without loss of generality we may assume $A = lann_T(B)$ and $B = rann_T(A)$. Let $u = u_m t^m + \cdots + u_1 t + u_0$ be of minimal degree in A. Corollary 4.3 shows that $S^{-m}(u_m)rann_T(u) = 0$ and since $B \subseteq rann_T(u)$ we have $S^{-m}(u_m)B = 0$, i.e. $S^{-m}(u_m) \in lann_T(B) = A$. This yields that $A \cap R \neq 0$. Similarly we get $B \cap R \neq 0$. This gives a contradiction to (2). **QED**

The obvious analogue for semiprimeness of $R[t; S, D]$ is however false as the following example shows.

Example 4.5. (see [BR]) For $i \in \mathbb{Z}$, let $K_i = K$ be a field and put $R = \oplus_{i \in \mathbb{Z}} K_i$. Let S be the "right shifting" automorphism, i.e. $S(\sum a_i) = \sum a_{i+1}$. Suppose that $0 \neq a = \sum a_i \in R$ is such that $a_i = 0$ for all $i \notin [m, n]$, where $m \leq n$ denote fixed integers. It is easy to check that $(atR[t;S])^{n-m+2} = 0$. Hence $R[t;S]$ is not semiprime but satisfies $bR[t;S]b \neq 0$ for every $0 \neq b \in R$.

In [PS] a criterion was given for semiprimeness of $R[t;S]$ and the prime radical of $R[t;S]$ was computed. For the convenience of the reader we offer here a necessary and sufficient condition for semiprimeness of $R[t;S]$. In the next section we will deal with the problem of computing the prime radical.

Proposition 4.6. *Let $T = R[t;S]$, $S \in \mathrm{Aut}(R)$. The following statements are equivalent:*

(1) *T is semiprime.*
(2) *For any $a \in R\backslash\{0\}$ and $n \in \mathbb{N}$, $aTS^n(a) \neq 0$*
(3) *For any $a \in R\backslash\{0\}$ and $N \in \mathbb{N}$, there exists $n \geq N$ such that $aRS^n(a) \neq 0$*

Proof. (1) $\rightarrow$ (2) Let $a \in R\backslash\{0\}$ and $n \in \mathbb{N}$. By (1) we have $t^n aTt^n a \neq 0$ and this gives $aTS^n(a) \neq 0$.

(2) $\rightarrow$ (1) Suppose $(at^n + \cdots + a_0)T(at^n + \cdots + a_0) = 0$ where $a \in R\backslash\{0\}$. Then $(at^n + \cdots + a_0)Rt^k(at^n + \cdots + a_0) = 0$ for any $k \in \mathbb{N}$. By looking at the leading coefficients we get $0 = aRt^{n+k}at^n = aRt^kS^n(a)t^{2n}$. This gives $aRt^kS^n(a) = 0$ for every $k \in \mathbb{N}$, i.e. $aTS^n(a) = 0$.

The equivalence (2) $\leftrightarrow$ (3) is clear if we notice that $aTS^n(a) \neq 0$ if and only if there exists $k \in \mathbb{N}$ such that $aRt^kS^n(a) \neq 0$. **QED**

Next we examine the case when $S = id_R$ but before we introduce a definition.

Definition 4.7. For any ideal I of $R[t;S,D]$, $M(I)$ denotes the ideal of leading coefficients of I, i.e.

$$M(I) = \{\text{leading coefficients of all polynomials from } I\}$$

.

It is standard to check that $M(I)$ is an ideal of R indeed.

Proposition 4.8. *Let $T = R[t; D]$, D a derivation of R. The following conditions are equivalent:*

(1) *T is semiprime.*
(2) *R is D-semiprime.*
(3) *For any $b \in R \setminus \{0\}$, there exists $r \in \mathbb{N}$ such that $bRD^r(b) \neq 0$.*

Proof. (1) $\leftrightarrow$ (2) Let $I \triangleleft R[t; D]$. It is standard to check that $M(I)$ is a D-ideal of R. Moreover if $I^2 = 0$ then $M(I)^2 = 0$. From this we get the required equivalence (1) $\leftrightarrow$ (2).

(2) $\leftrightarrow$ (3) This equivalence is a special case of Lemma 2.2(b). **QED**

We will now give general criterions for semiprimeness of $R[t; S, D]$ and show that these criterions generalize the ones obtained in Propositions 4.6 and 4.8.

Theorem 4.9. *For the polynomial ring $T = R[t; S, D]$, the following assertions are equivalent:*

(a) *T is semiprime.*
(b) *For any $0 \neq h(t) = \sum_{i=0}^{n} a_i t^i = \sum_{i=0}^{n} t^i b_i \in T$ there exist $0 \leq i, k \leq n$ such that $b_i T a_k \neq 0$.*
(c) *For any $(b_0, \ldots, b_n) \in R^{n+1} \backslash (0, \ldots, 0)$ there exist $p, l, i, k \in \mathbb{N}$ with $0 \leq p \leq l, 0 \leq i,\ k \leq n$ such that $b_i R \sum_{j=k}^{n} f_p^l(f_k^j(b_j)) \neq 0$.*

Proof. If $b_i T a_k = 0$ for all $0 \leq i, k \leq n$, then $h(t)Th(t) = 0$. This gives (a)$\rightarrow$ (b).

(b) $\leftrightarrow$ (c) follows easily if we remark that:
(i) $\sum_{j=0}^{n} t^j b_j = \sum_{j=0}^{n} (\sum_{k=0}^{j} f_k^j(b_j) t^k) = \sum_{k=0}^{n} (\sum_{j=k}^{n} f_k^j(b_j)) t^k$, so $a_k = \sum_{j=k}^{n} f_k^j(b_j)$

and

(ii) $b_i T a_k = 0$ iff $b_i R t^l a_k = 0$ for any $l \geq 0$ iff $b_i R f_p^l(a_k) = 0$ for any $(l, p) \in \mathbb{N}^2$ such that $p \leq l$.

It remains to prove that (b) $\rightarrow$ (a). Assume T is not semiprime and among $0 \neq p \in T$ such that $pTp = 0$, choose one with minimal nonzero number of monomials, i.e. choose $p = \sum_{i=0}^{n} a_i t^i$ such that $pTp = 0$ and

$|\{i|a_i \neq 0\}|$ is minimal. Obviously we then have $lann_R(a_i) = lann_R(a_j)$ for any $(i,j) \in \mathbb{N}^2$ such that $a_i \neq 0 \neq a_j$. Using Lemma 4.2(a), we easily get $pTa_i = 0$ for any $i \in \{0, \ldots, n\}$. Let us now consider the set

$$E := \left\{ q = \sum_{j=0}^{l} \alpha_j t^j \in T \mid qT\alpha_i = 0 \quad \text{for every} \quad i \in \{0, \ldots, l\} \right\}$$

Since $p \in E$, $E \neq \{0\}$. Choose a nonzero element $h \in E$ which has minimal number of right monomials, i.e. $h = \sum t^j b_j$ and $|\{j \mid b_j \neq 0\}|$ is minimal and nonzero among elements of E. Observe that E is closed by right multiplication by elements of R. More explicitly, if $q \in E$ then $q\beta \in E$ for any $\beta \in R$. Indeed, let $q(t) = \sum_{i=0}^{l} \alpha_i t^i$ be such that $qT\alpha_i = 0$. Then $q\beta = \sum_{i=0}^{l} \alpha_i t^i \beta = \sum_{i=0}^{l} \sum_{j=0}^{i} \alpha_i f_j^i(\beta) t^j = \sum_{j=0}^{l} (\sum_{i=j}^{l} \alpha_i f_j^i(\beta)) t^j$ and $q\beta T(\sum_{i=j}^{l} \alpha_i f_j^i(\beta)) \subseteq \sum_{i=j}^{l} qT\alpha_i f_j^i(\beta) = 0$. Thus we do have $q\beta \in E$. Hence, if $h = \sum_{i=0}^{n} a_i t^i = \sum_{j=0}^{n} t^j b_j$ is chosen with a minimal number of right monomials, $h\beta \in E$ for any $\beta \in R$ and $h\beta$ can not have less number of right monomials than h. We conclude that $rann_R(b_i) = rann_R(b_j)$ for any $i, j \in \{0, \ldots, n\}$. In particular, since $b_n = S^{-n}(a_n)$, we have $rann_R(S^{-n}(a_n)) = rann_R(h)$. On the other hand, $h \in E$ so $hTa_i = 0$ and $Ta_i \subseteq rann_T(h)$ for any $i \in \{0, \ldots, n\}$. By making use of Lemma 4.2(b), we get $S^{-n}(a_n)Ta_i = 0$ for all $i \in \{0, \ldots, n\}$, i.e. $b_nTa_i = 0$ for all $i \in \{0, \ldots, n\}$. Considering $h_1 = h - t^n b_n$, we have $h_1 = t^{n-1}b_{n-1} + \cdots + tb_1 + b_0$, $rann_R(b_i) = rann_R(b_j)$ and $h_1Ta_i = 0$. Thus we conclude as above that $b_{n-1}Ta_i = 0$ for all $i \in \{0, \ldots, n\}$. Continuing this process we finally obtain $b_iTa_j = 0$ for all $i, j \in \{0, \ldots, n\}$ but $h(t) \neq 0$. **QED**

The following are special cases of the condition (c) from the above theorem:

Remarks 4.10. (1) $T = R[t; S]$ is semiprime if and only if for any $n \geq 0$ and for any $(b_0, \ldots, b_n) \in R^{n+1} \backslash (0, \ldots, 0)$ there exist $0 \leq i, k \leq n$ and $l \geq k$ such that $b_i R S^l(b_k) \neq 0$.

(2) $T = R[t; D]$ is semiprime if and only if for any $n \geq 0$ and for any $(b_0, \ldots, b_n) \in R^{n+1} \backslash (0, \ldots, 0)$ there exist $p, l, i, k \in \mathbb{N}$ with $0 \leq p \leq l$, $0 \leq i, k \leq n$ such that $b_i R \sum_{j=k}^{n} \binom{l}{p} \binom{j}{k} D^{l-p+j-k}(b_j) \neq 0$.

The proofs are left to the reader (compare those conditions with Propositions 4.6 and 4.8).

Corollary 4.11. *Suppose that either $S = id_R$ or $D = 0$, then $T = R[t; S, D]$ is semiprime if and only if for any nonzero monomial $p \in T$, $pTp \neq 0$.*

Proof. The condition is clearly necessary for semiprimeness of T.

Assume that for any monomial $0 \neq p \in T$, $pTp \neq 0$. Let $h = \sum_{i=0}^{n} a_i t^i = \sum_{i=0}^{n} t^i b_i$ be an element of T such that $a_n \neq 0$. By the hypothesis, we have $a_n t^n T a_n t^n \neq 0$.

If $S = id_R$, then $b_n = a_n$. Thus $0 \neq a_n t^n T a_n t^n \subseteq a_n T a_n t^n = b_n T a_n t^n$ and $b_n T a_n \neq 0$ follows.

If $D = 0$, then $b_n = S^{-n}(a_n)$. Thus $0 \neq a_n t^n T a_n t^n = t^n S^{-n}(a_n) T a_n t^n = t^n b_n T a_n t^n$.

In any case we conclude that $b_n T a_n \neq 0$ and Theorem 4.9(b) shows that T is semiprime. **QED**

Of course it would be nice to have simpler conditions for semiprimeness of T than the one expressed by the condition (c) in Theorem 4.9. Let us now present three conditions: the first is sufficient for the semiprimeness of $R[t; S, D]$ and the two others are necessary.

Proposition 4.12. *Let R be a ring $S \in$*Aut*(R), D an S-derivation of R and $T = R[t; S, D]$. Then:*

(a) *The following condition is sufficient for T to be semiprime: If $b \in R\backslash\{0\}$ then for any $s \in \mathbb{N}$ there exist $p, l \in \mathbb{N}$, $0 \leq p \leq l$ such that $bRf_p^l(S^s(b)) \neq 0$.*

(b) *The following condition is necessary for T to be semiprime: for any $b \in R\backslash\{0\}$ and for any $n \in \mathbb{N}$ there exists $k, p, l \in \mathbb{N}$ such that $0 \leq k \leq n$, $0 \leq p \leq l$ and $bRf_p^l(f_k^n(b)) \neq 0$.*

(c) *The following condition is necessary for T to be semiprime: for any $b \in R\backslash\{0\}$ there exists $p, l \in \mathbb{N}$, $0 \leq p \leq l$ such that $bRf_p^l(b) \neq 0$.*

Proof. (a) Let us show that the above condition implies the condition

(c) of Theorem 4.9. Suppose $(b_0, \dots, b_n) \in R^{n+1} \backslash (0, \dots, 0)$ and let $s \leq n$ be the largest index such that $b_s \neq 0$. Hence $b_j = 0$ for $j > s$ and our condition (a) above implies that there exist $p, l \in \mathbb{N}$, $0 \leq p \leq l$ such that $b_s R f_p^l(S^s(b_s)) \neq 0$. But this means that condition (c) of Theorem 4.9 is satisfied with $i = k = s$.

(b) Assume $T = R[t; S, D]$ is semiprime, and let $b \in R \backslash \{0\}$. Then condition (c) of Theorem 4.9 applied to $(0, \dots, 0, b) \in R^{n+1} \backslash (0, \dots, 0)$ shows that there exist $p, l, k \in \mathbb{N}$ such that $bRf_p^l(f_k^n(b)) \neq 0$.

(c) This is a particular case of (b) obtained by taking $n = k = 0$. This can also be deduced directly by considering the fact that T semiprime implies $bTb \neq 0$ for any $b \in R \backslash \{0\}$. **QED**

As we have seen in Example 4.5, condition (c) above is not sufficient for T to be semiprime. We don't know if one of the conditions (a) or (b) is equivalent to T being semiprime. Let us express condition (a) above in a more compact way as follows:

$$\text{for any } b \in R \backslash \{0\} \text{ and } s \in \mathbb{N}, \;\; bTS^s(b) \neq 0. \tag{4.13}$$

In the next proposition we give a few special cases in which this condition is also necessary for T to be semiprime.

Proposition 4.14. *Suppose that one of the following conditions is satisfied:*

(a) $S = id_R$*;*

(b) $D = 0$*;*

(c) R *is semiprime;*

(d) *For every* $b \in R \backslash \{0\}$ *there exists* $m \in \{-1, 0, 1, 2, \dots\}$ *such that* $bRS^m(b) \neq 0$*;*

(e) R *satisfies the ACC on two-sided* S*-ideals and* $SD = qDS$ *with* $q \in Z(R)_{S,D}$ *such that for any* $n \in \mathbb{N}$, $\sum_{i=0}^n q^i$ *is regular.*

Then $T = R[t; S, D]$ *is semiprime if and only if (4.13) holds.*

Proof. In virtue of Proposition 4.12(a), we need only to show that if T is semiprime, then (4.13) holds.

(a) If T is semiprime and $b \neq 0$ then obviously $bTb \neq 0$ and (4.13) is satisfied.

(b) This is immediate from Proposition 4.6.

(c) is a particular case of (d).

(d) Suppose that T is semiprime but let us assume that (4.13) does not hold. Let $b \in R\backslash\{0\}$ and $s \geq 1$ be such that $bTS^s(b) = 0$. Since T is semiprime we also have $S^s(b)Tb = 0$. Let $m = m(b) \geq -1$ such that $bRS^m(b) \neq 0$, then $m + s \geq 0$ and $S^s(b)Rt^{m+s}b = 0$. The leading coefficients of this last equation give us $S^s(b)RS^{m+s}(b) = 0$, i.e. $bRS^m(b) = 0$, a contradiction.

(e) Suppose that T is semiprime but let us assume that there exists $b \in R\backslash\{0\}$ and $s \in \mathbb{N}$ such that $bTS^s(b) = 0$. In particular, $bRt^lS^s(b) = 0$ for any $l \geq 0$. Hence, for any $n \geq s$, $bRS^n(b) = 0$ and Proposition 4.6 implies that the skew polynomial ring $R[t'; S]$ is not semiprime. Let $I \neq 0$ be an ideal of $R[t'; S]$ such that $I^2 = 0$. The ideal $M(I)$ of leading coefficients I is clearly an S-ideal. Since R satisfies the ACC on two-sided S-ideals, Example 1.16 shows that $M(I)$ is in fact an (S, S^{-1})-ideal and $M(I)^2 = 0$ follows. This shows that $\mathrm{rad}(R; S, S^{-1}) = \mathrm{rad}(R; S) \neq 0$ and Proposition 2.8 yields that $\mathrm{rad}(R; S, S^{-1})$ is an (S, S^{-1}, D)-ideal which is nilpotent. Hence the nonzero ideal $\mathrm{rad}(R; S, S^{-1})[t; S, D]$ of T is nilpotent but this contradicts the fact that T is semiprime. **QED**

Proposition 4.15. *Suppose R is S-semiprime and satisfies one of the following conditions:*

(a) *The ACC on two-sided S-ideals.*
(b) *The ACC on left annihilators.*
(c) *The ACC on right annihilators.*

Then the property (4.13) is satisfied and $T = R[t; S, D]$ is semiprime.

Proof. (a) Assume that there exist $b \in R\backslash\{0\}$ and $s \in \mathbb{N}$ such that $bTS^s(b) = 0$. In particular, $bRS^n(b) = 0$ for all $n \geq s$. Put $I := \sum_{n=s}^{\infty} RS^n(b)R$. Then $S(I) \subseteq I$ and the ACC on S-ideals gives us $S(I) = I$. Thus for any $l \in \mathbb{Z}$, $S^l(b)I = S^l(bI) = 0$, thus $I^2 = 0$. Since $0 \neq b \in I$ this contradicts the fact that R is S-semiprime.

(b) Assume (4.13) does not hold and let $b \in R\backslash\{0\}$ be such that $bTS^s(b) = 0$. In particular, this means that bR is S^s-nil. Using Proposition 3.14 and Lemma 2.3(d), we obtain $0 \neq \operatorname{rad}(R; S^s) = \operatorname{rad}(R; S)$. This contradicts the fact that R is S-semiprime.

(c) Is obtained similarly as (b). **QED**

Before going on giving sufficient conditions on R, S, D for T to be semiprime, let us give an example showing that even if R is very well behaved with respect to S and D, $T = R[t; S, D]$ can nevertheless fail to be semiprime.

Example 4.16. Let R, S, and D be as in Example 2.11. We claim that although R is (S, D)-simple, $T = R[t; S, D]$ is not even semiprime. We have shown in Example 2.11 that R is (S, D)-simple. Let us prove that for any $0 \leq p \leq l$ $\ (0,1,0)x^2 R f_p^l((0,1,0)x^2) = 0$. Thus, by Proposition 4.12(c), T is not semiprime.

Let us put $b := (0,1,0)x^2$ and $a := (0,0,1)x$. We have $b = xa$ and for any $0 \leq p \leq l$, $f_p^l(b) = f_p^l(xa) = \sum_{i=p}^{l} f_i^l(x) f_p^i(a)$. Hence $bRf_p^l(b) \subseteq \sum_{p \leq i \leq l} bRf_p^i(a)$. We will now show that for any $0 \leq p \leq i \leq l$, $bRf_p^i(a) = 0$ and this will lead to $bRf_p^l(b) = 0$ for any $0 \leq p \leq l$, as desired.

If $p = i$ then $f_p^i = S^i$ and $bRf_p^i(a) = 0$ since $x^3 = 0$.

If $p \leq i - 2$ then D appears at least twice in every word of f_p^i and the definition of D implies that $f_p^i(a) = 0$ and so $bRf_p^i(a) = 0$.

Suppose $p = i - 1$. Using the identity $f_i^{i+1} = Sf_{i-1}^i + DS^i$ we will show that

$$f_{i-1}^i(a) = \begin{cases} 0 & \text{if } i \equiv 0 \pmod 3 \text{ or } i \equiv 1 \pmod 3 \\ (0,-1,0) & \text{if } i \equiv 2 \pmod 3 \end{cases} \tag{4.17}$$

Notice that $f_0^1(a) = D(a) = 0$, so (4.17) holds for $i = 1$.

Suppose (4.17) holds for some i of the form $i = 3l + 1$. Then:

$$\begin{aligned} f_i^{i+1}((0,0,1)x) &= Sf_{i-1}^i((0,0,1)x) + DS^i((0,0,1)x) \\ &= DS((0,0,1)x) \qquad (\text{since } S^3 = id_R) \\ &= (0,-1,0) \end{aligned}$$

$$\begin{aligned} f_{i+1}^{i+2}((0,0,1)x) &= S((0,-1,0)) + DS^2((0,0,1)x) \\ &= (0,0,-1) + D((0,1,0)x) \\ &= 0 \end{aligned}$$

$$f_{i+2}^{i+3}((0,0,1)x = DS^{i+2}((0,0,1)x) = D((0,0,1)x) = 0$$

This shows that (4.17) holds. Hence, for proving $bRf_p^i(a) = 0$ for all $0 \leq p \leq i$, it remains to show that $bR(0,-1,0) = 0$. However $bR(0,-1,0) = (0,1,0)x^2K^3(0,-1,0) = (0,1,0)(0,0,-1)x^2K^3 = 0$. **QED**

Let us now analyze further the relationships between S or (S,D)-primeness (resp. S or (S,D)-semiprimeness) of R and primeness (resp. semiprimeness) of $T = R[t;S,D]$. Let us first make the following easy observation: if T is prime (resp. semiprime) then R is (S,D)-prime (resp. (S,D)-semiprime). Up to the end of this section we will be interested in the converse implication. We will often make some assumptions on R. For easy further references, let us give names to the ones most often used :

(C1) : R satisfies the ACC on two-sided S-ideals.
(C2) : R satisfies the ACC on left annihilators.
(C3) : A nonzero power of S is inner.

In the following lemma we collect a few properties of ideals $M(I)$ of leading coefficients.

Lemma 4.18. *Let I and J be ideals of $T = R[t;S,D]$. Then:*

(a) *$M(I)$ is an S-ideal of R.*
(b) *If $M(J)$ is an (S,S^{-1})-ideal then $IJ = 0$ implies $M(I)M(J) = 0$.*
(c) *If one of the conditions (C_1), (C_2), (C_3) is satisfied then $IJ = 0$ implies $M(I)M(J) = 0$.*
(d) *If $I^2 = 0$, then $M(I) \subseteq \mathrm{Nil}(R;S)$.*

Proof. (a), (b) are left as exercises.

(c) If either (C_1) or (C_3) is satisfied, then any S-ideal is in fact an (S,S^{-1})-ideal and (b) above gives the desired conclusion.

Assume now that (C_2) is satisfied and consider the descending chain of ideals

$$M(J) \supseteq S(M(J)) \supseteq S^2(M(J)) \supseteq \cdots$$

which gives rise to the ascending chain of left annihilators.

$$lann_R(M(J)) \subseteq lann_R(S(M(J))) \subseteq lann_R(S^2(M(J))) \subseteq \cdots$$

The hypothesis we are assuming on R implies that there exists $n \in \mathbb{N}$ such that $lann_R(S^n(M(J))) = lann_R(S^{n+1}(M(J)))$. Applying S^{-n} one gets $lann_R(M(J)) = lann_R(S(M(J)))$. Therefore $lann_R((M(J))) = lann_R(S^k(M(J)))$, for any $k \geq 0$. Now, if $a \in M(I)$ and $p \in I$ is of degree k having a as its leading coefficient, then by assumption, $pJ = 0$. In particular, $aS^k(M(J)) = 0$. Thus $a \in lann_R(S^k(M(J))) = lann_R(M(J))$. This yields $aM(J) = 0$ as desired.

(d) We must show that for every $a \in M(I)$ and for every $u \in \mathbb{N}\backslash\{0\}$ there exists $r \in \mathbb{N}$ such that $N_r^{S^u}(a) = 0$. Since $a \in M(I)$, there exist $n > 0$ and a polynomial $p \in I$ of degree n having a as a leading coefficient. Then $pt^{l-n}p = 0$ for any $l \geq n$, as $I^2 = 0$. Therefore $aS^l(a) = 0$ for all $l \geq n$. Let $u \in \mathbb{N}\backslash\{0\}$. Note that either $b_u = aS^u(a) \ldots S^{u(n-1)}(a) = N_n^{S^u}(a)$ is also the leading coefficient of a polynomial of degree n from I or $b_u = 0$. In any case $b_u S^l(b_u) = 0$ if $l \geq n$. In particular, $0 = b_u S^{nu}(b_u) = N_{2n}^{S^u}(a)$ as we wanted to show. **QED**

Proposition 4.19. *Suppose* $\mathrm{Nil}(R; S) = 0$*. Then* $R[t; S, D]$ *is semiprime.*

Proof. This is a direct consequence of Lemma 4.18(d). **QED**

Lemma 4.20. *If* $R[t; S]$ *is prime (resp. semiprime) then* $R[t; S, D]$ *is prime (resp. semiprime).*

Proof. This is a consequence of Theorem 4.4(3) for primeness and of Propositions 4.12(a) and 4.6 for semiprimeness.

Note that the lemma can also be seen as a consequence of the fact that $R[t; S, D]$ is filtered with graded associated ring isomorphic to $R[t; S]$. **QED**

Theorem 4.21. *The following assertions are equivalent:*

(i) *R is S-prime (resp. S-semiprime)*

(ii) *R is S^{-1}-prime (resp. S^{-1}-semiprime)*

(iii) *R is (S, S^{-1})-prime (resp. (S, S^{-1})-semiprime)*

(iv) *For any $a, b \in R\backslash\{0\}$ there exists $n \in \mathbb{Z}$ such that $aRS^n(b) \neq 0$ (resp. $a = b$)*

(v) *$R[t, t^{-1}; S]$ is prime (resp. semiprime)*

Suppose moreover that one of the conditions (C_1), (C_2), (C_3) is satisfied. Then the above assertions are also equivalent to:

(vi) *$R[t; S]$ is prime (resp. semiprime)*

(vii) *$R[t; S^{-1}]$ is prime (resp. semiprime)*

(viii) *For any $a, b \in R\backslash\{0\}$ there exists $n \in \mathbb{N}$ such that $aRS^n(b) \neq 0$ (resp. $a = b$).*

Moreover, any of the above equivalent conditions implies that $R[t; S, D]$ is prime (resp. semiprime).

Proof. The equivalences (i) $\leftrightarrow$ (ii) $\leftrightarrow$ (iii) are give by Corollary 1.15.

(iii) $\leftrightarrow$ (iv) is a special case of Lemma 2.2(a) and (b) with $D = 0$.

(i) $\leftrightarrow$ (v) is an easy consequence of the following observation. If I is an ideal of $R[t, t^{-1}; S]$, then $M(I)$ is an (S, S^{-1})-ideal of R, because $tI \subseteq I$ and $t^{-1}I \subseteq I$,.

Suppose now that one of the conditions (C_1), (C_2), (C_3) is satisfied. Then the equivalence (i) $\leftrightarrow$(vi) is a consequence of Lemma 4.18(c)

((ii) $\leftrightarrow$ (vii) is symmetric to (i) $\leftrightarrow$ (vi).

When R is prime, then (vi) $\leftrightarrow$ (viii) is a special case of (1) $\leftrightarrow$ (3) in Theorem 4.4 (since $D = 0$ implies $f_k^n = S^{n-k}$). For R semiprime, Proposition 4.6 gives (iv) $\rightarrow$ (viii) and clearly (viii) $\rightarrow$ (iv)$\leftrightarrow$(ii).

The last statement of the theorem is a direct consequence of Lemma 4.20 above.

QED

Example 4.22. Let $K_i = K$ be a field for $i \in \mathbb{Z}$ and put $R = \oplus_{i\in\mathbb{Z}} K_i$. Let S be the "left shifting" automorphism. We have seen in Example 4.5, that $R[t;S]$ is not semiprime and similarly $R[t;S^{-1}]$ is not semiprime but it is easy to check that the condition (iv) of Theorem 4.21 is satisfied and so $R[t,t^{-1};S]$ is a prime ring.

This example shows that some additional assumptions are required in order to obtain the equivalence of the eight assertions of the above theorem.

Let us finish this section with a look at the case when R satisfies the ACC on ideals and D is a q-quantized S-derivation.

Proposition 4.23. *Suppose R satisfies the ACC on two-sided ideals and let D be a q-quantized S-derivation such that $q \in Z(R)_{S,D}$ and $1 + q + \cdots + q^n$ is regular in R for all $n \in \mathbb{N}$. Then R is (S,D)-prime (resp. (S,D)-semiprime) if and only if $T = R[t;S,D]$ is prime (resp. semiprime).*

Proof. This is a simple application of Theorem 2.6 (resp. Proposition 2.8) and Theorem 4.21. **QED**

5. The prime radical of $R[t;S,D]$

In this last section we will study relations between different prime radicals of R introduced in the previous sections and the prime radical of $T = R[t;S,D]$.

Lemma 5.1. $\mathrm{rad}(R;S,D)[t;S,D] \subseteq \mathrm{rad}(R[t;S,D])$

Proof. The inclusion can be proved by transfinite induction, using the description of $\mathrm{rad}(R;S,D)$ given before Proposition 1.11, as follows:

If $p(t) \in L_1[t;S,D]$, then $p(t) \in I[t;S,D]$ for some nilpotent (S,D)-ideal. It is easy to see that $I[t;S,D]$ is itself a nilpotent ideal of $R[t;S,D]$ and so $I[t;S,D] \subseteq \mathrm{rad}(T)$. In particular $p(t) \in \mathrm{rad}(T)$ and $L_1[t;S,D] \subseteq \mathrm{rad}(T)$. Assume we have proved that $L_\alpha[t;S,D] \subseteq \mathrm{rad}(R[t;S,D])$ for all $\alpha < \beta$.

Suppose $\beta = \alpha + 1$ for some α. Then we have the following chain of isomorphisms and inclusions

$$\frac{L_\beta[t;S,D]}{L_\alpha[t;S,D]} \cong \frac{L_\beta}{L_\alpha}[t;S,D] = L_1\left(\frac{R}{L_\alpha}\right)[t;S,D]$$
$$\subseteq \operatorname{rad}\left(\frac{R}{L_\alpha}[t;S,D]\right) \cong \frac{\operatorname{rad}(T)}{L_\alpha[t;S,D]}.$$

Now, by making use of the induction hypothesis, it is easy to see that $L_\beta[t;S,D] \subseteq \operatorname{rad}(T)$.

If β is a limit ordinal, $L_\beta = \cup_{\alpha<\beta} L_\alpha$ and the induction hypothesis gives at once $L_\beta[t;S,D] \subseteq \operatorname{rad}(T)$. **QED**

Proposition 5.2. *Suppose that R satisfies the ACC on S-ideals and* $\operatorname{rad}(R;S) \lhd_D R$. *Then* $\operatorname{rad}(R[t;S,D]) = \operatorname{rad}(R;S,D)[t;S,D]$.

Proof. In virtue of Lemma 5.1, we need only prove that $\operatorname{rad}(R[t;S,D]) \subseteq \operatorname{rad}(R,S,D)[t;S,D]$. After factoring out $\operatorname{rad}(R;S,D)[t;S,D]$, we may suppose
$\operatorname{rad}(R;S,D) = 0$. Assume that I is an ideal of $R[t;S,D]$ such that $I^2 = 0$. Then Lemma 4.18(c) implies that $M(I)^2 = 0$ and hence $M(I) \subseteq \operatorname{rad}(R;S)$. Thanks to Corollary 1.13 and our hypothesis that $\operatorname{rad}(R;S) \lhd_D R$ and $\operatorname{rad}(R;S,D) = 0$ we get $\operatorname{rad}(R;S) = 0$ and so $M(I) = 0$. This shows that $I = 0$ and proves that $\operatorname{rad}(T) = 0$, as required. **QED**

Proposition 5.3. *Suppose R satisfies the ACC on ideals and D is a q-quantized S-derivation such that $q \in Z(R)_{S,D}$ and for any $n \in \mathbb{N}$, $\sum_{i=0}^n q^i$ is regular in R. Then* $\operatorname{rad}(R[t;S,D]) = \operatorname{rad}(R;S,D)[t;S,D] = \operatorname{rad}(R)[t;S,D]$.

Proof. By Proposition 2.8, $\operatorname{rad}(R;S) = \operatorname{rad}(R;D)$ is a D-ideal and this proposition together with Proposition 5.2 give the thesis. **QED**

In case $SD = DS$, the above corollary can be improved so that no assumption on $\mathbb{Z}$-torsion is required. While proving this small improvement we will need the following fact.

Lemma 5.4. *Suppose R satisfies the ACC on ideals. Then $R[t;S,D]$ also satisfies this chain condition.*

Proof. This follows the standard proof. Let I be a nonzero ideal of $R[t;S,D]$. Then, by assumption, the ideal $M(I)$ of leading coefficients is finitely generated as a two-sided ideal of R. Let $f_1,\dots,f_r$ be polynomials from I whose leading coefficients generate $M(I)$. If $p\in I$ then we can reduce the degree of p by substracting "two-sided" multiples of the f_i's, unless $\deg p$ is less than $\deg f_i$ for some i. Thus if $N=max\{\deg f_1,\dots,f_r\}$ then, by repeated substractions, we get a polynomial p' such that $\deg p'<N$ and

$$p\equiv p' \pmod{\sum_{i=1}^{r} Tf_iT} \quad \text{where} \quad T=R[t;S,D]$$

and since $f_i\in I$, $p'\in I$. Now let $g_1,\dots,g_s\in I$ be polynomials such that their leading coefficients generate I_N, where I_N denotes the ideal of R consisting of all leading coefficients of polynomials from I of degree smaller than N. Then p' can be written as a "two-sided" linear combination of the $g_i's$. Hence $I=\langle f_1,\dots,f_r,g_1,\dots,g_s\rangle$.

QED

Proposition 5.5. *Suppose R satisfies the ACC on ideals. If $SD=DS$ then* $\mathrm{rad}(R[t;S,D])=\mathrm{rad}(R;S,D)[t;S,D]$.

Proof. Due to Lemma 5.1, we can factor out $\mathrm{rad}(R;S,D)[t;S,D]$ and assume that R is (S,D)-semiprime. Since S and D commute, we can extend S and D to $T=R[t;S,D]$ by setting $S(t)=t$, $D(t)=0$. Let $I=\mathrm{rad}(T)$. Then I is an S-ideal and, since D is an inner S-derivation of T adjoint to t, I is in fact an (S,D)-ideal. Moreover Lemma 5.4 and the assumption imposed on R yield that I is a nilpotent (S,D)-ideal, so $M(I)=0$ and $\mathrm{rad}(T)=I=0$ follows. **QED**

We know that $\mathrm{rad}(R;S,D)[t;S,D]\subseteq \mathrm{rad}(R[t;S,D])$. In particular this shows that $\mathrm{rad}(R;S,D)[t;S,D]$ is nil. Let us give other examples of nil ideals of $T=R[t;S,D]$.

In order to analyze powers of the set $I[t;S,D]\subseteq T$, we need a few definitions and notations.

Let $I_0=I$ and $I_k:=\sum_{l(\omega)\le k}\omega(I)$ for any $k>0$, where $\omega\in\Omega=\{$ monomials in S and $D\}$ and $l(\omega)$ denotes the length of ω.

For $x=(x_1,\dots,x_n)\in\mathbb{N}^n$ we set $l(x)=n$ and $|x|:=x_1+\cdots+x_n$ ($0\in\mathbb{N}$),

$I_{(x)}:=I_{x_1}\dots I_{x_n}$

For $k\in\mathbb{N}$ and $0\neq n\in\mathbb{N}$, $J_{(n,k)}=\sum_{x\in\mathbb{N}^n\ |x|=k} I_{(x)}$.

These definitions are inspired by [BMP: (2.1), (2.5)].

Lemma 5.6. *With the above notations we have:*

(i) $S(I_k)\subseteq I_{k+1}$ *and* $D(I_k)\subseteq I_{k+1}$.
(ii) I_k *is an ideal of* R.
(iii)

$$D(I_{(x)})\subseteq\sum_{l(y)=l(x),\ |y|=|x|+l(x)} I_{(y)}\ \text{and}\ S(I_{(x)})\subseteq\sum_{l(y)=l(x),\ |y|=|x|+l(x)} I_{(y)}.$$

(iv) $D(J_{(n,k)})\subseteq J_{(n,k+n)}$ *and* $S(J_{(n,k)})\subseteq J_{(n,k+n)}$.
(v) $t^lJ_{(n,k)}\subseteq J_{(n,k+nl)}R[t;S,D]$.
(vi) $IJ_{(n,k)}\subseteq J_{(n+1,k)}$.
(vii) *If* $k\leq l$ *then* $J_{(n,k)}\subseteq J_{(n,l)}$.
(viii) *If* $I_{2k}^s=0$ *for some* $k\geq 0$ *and* $s\geq 1$, *then* $J_{(2s,(2s-1)k)}=0$
(ix) *If* I *is a finitely generated ideal then so is* I_k *for any* $k\geq 0$.
(x) $(It^l+\cdots+I)^n\subseteq J_{(n,(n-1)nl)}R[t;S,D]$ *for any* $l\geq 0$ *and* $n\geq 1$.

If D *is a* q*-quantized* S*-derivation and* $S(I)\subseteq I$, *then*

(i') $S(I_k)\subseteq I_k$.
(iii')

$$D(I_{(x)})\subseteq\sum_{l(y)=l(x),\ |y|=|x|+1} I_{(y)}\ \text{and}\ S(I_{(x)})\subseteq I_{(x)}$$

(iv') $D(J_{(n,k)})\subseteq J_{(n,k+1)}$ *and* $S(J_{(n,k)})\subseteq J_{(n,k)}$.
(x') $(It^l+\cdots+I)^n\subseteq J_{(n,(n-1)l)}R[t;S,D]$ *for any* $l\geq 0$ *and* $n\geq 1$.

Proof. (i) This is obvious.

(ii) The proof, by induction on k, is easy and left to the reader.

The statements (iii) $\div$ (vii) are immediate.

(viii) Assume $I_{2k}^s = 0$. Let $x = (x_1, \ldots, x_{2s}) \in \mathbb{N}^{2s}$ with $|x| = (2s-1)k$. If $\#\{x_i | x_i > 2k\} \geq s$ we would have $|x| > 2ks > (2s-1)k$. Since this is impossible we conclude that $\#\{x_i | x_i > 2k\} < s$. This implies that $\#\{x_i \mid x_i \leq 2k\} \geq s$ and shows $I_{(x)} = 0$. Therefore $J_{(2s,(2s-1)k)} = 0$.

(ix) Let $I = Ra_1R + \cdots + Ra_nR$. Then $I_1 = I + S(I) + D(I) = \sum_{i=1}^n (Ra_iR) + \sum_{i=1}^n RS(a_i)R + \sum_{i=1}^n RD(a_i)R$ is finitely generated. Since $I_{k+1} = I_k + S(I_k) + D(I_k)$, the desired conclusion follows by induction.

Both (x) and (i') $\div$ (x') are easy exercises. **QED**

Proposition 5.7. *Suppose that for any finitely generated ideal J of R, the S-ideal generated by J is also finitely generated and D is a q-quantized S-derivation. Let N be an (S, D)-ideal of R such that any finitely generated ideal contained in N is nilpotent. Then $N[t; S, D]$ is nil.*

Proof. Take $p(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in N[t; S, D]$ and put $J := \sum_{i=0}^n Ra_iR$. By the assumption imposed on S, the S-ideal I of R generated by J is also finitely generated. Lemma 5.6 (ix) implies that for any $k \geq 0$, I_k is finitely generated. Thus, by the assumption imposed on N, each ideal I_k is nilpotent. In particular $I_{2n}^s = 0$ for some $s \geq 1$. Now applying (viii) and (x') of Lemma 5.6 we get $p(t)^{2s} = 0$. This completes the proof of the proposition. **QED**

The following example shows that the above proposition does not hold without the additional assumption on S.

Example 5.8. Let K be a field and S a K-automorphism of $R = K\langle x_i \mid x_ix_j = x_jx_i, x_i^2 = 0 \text{ for all } i, j \in \mathbb{Z}\rangle$ defined by $S(x_i) = x_{i+1}$ for any $i \in \mathbb{Z}$. Then clearly $N = (x_i, i \in \mathbb{Z})$ is an S-ideal such that every finitely generated ideal contained in N is nilpotent. However $(x_0t)^n = x_0 \cdots x_{n-1}t^n \neq 0$ for any $n \in \mathbb{N}$.

We will now analyze further the case when D is a q-quantized S-derivation. Recall that $P \triangleleft'_D R$ means that P is a D-prime ideal of R.

Lemma 5.9. *Suppose R satisfies the ACC on ideals (resp. S-ideals) and* $\mathrm{rad}(R)$ *(resp.* $\mathrm{rad}(R; S)$*) is a D-ideal, then:*
$\mathrm{rad}(R[t; S, D]) = \mathrm{rad}(R)[t; S, D]$ *(resp.* $\mathrm{rad}(R[t; S, D]) = \mathrm{rad}(R; S)[t; S, D]$*).*

Proof. Under our assumptions $\mathrm{rad}(R)$ (resp. $\mathrm{rad}(R;S)$) is a nilpotent (S,D)-ideal. Hence $\mathrm{rad}(R)[t;S,D]$ (resp. $\mathrm{rad}(R;S)[t;S,D]$) is nilpotent and so $\mathrm{rad}(R)[t;S,D] \subseteq \mathrm{rad}(R[t;S,D])$ (resp. $\mathrm{rad}(R;S)[t;S,D] \subseteq \mathrm{rad}(R[t;S,D])$).

In order to prove the reverse inclusion we first factor out $\mathrm{rad}(R)[t;S,D]$ (resp. $\mathrm{rad}(R;S)[t;S,D]$). Thus we may assume that R is semiprime (resp. S-semiprime) and hence also S-semiprime. Then Theorem 4.21 shows that $R[t;S,D]$ is semiprime, as desired. **QED**

Theorem 5.10. *Suppose that D is a q-quantized S-derivation of R, where $q \in Z(R)_{S,D}$ is such that both q and $\sum_{i=0}^{n} q^i$ are regular in R for any $n \in \mathbb{N}$. If R satisfies the ACC on S-ideals then* $\mathrm{rad}(R[t;S,D]) = \mathrm{rad}(R;S)[t;S,D]$

Proof. By Corollary 2.7 $\mathrm{rad}(R;S)$ is an (S,D)-ideal. Now, Lemma 5.9 gives the thesis. **QED**

In case when R is commutative noetherian Goodearl [G] gave a complete classification of prime ideals of $R[t;S,D]$. Using one of his results, it is easy to get the following.

Theorem 5.11. *Suppose R is commutative and noetherian then*

$$\mathrm{rad}(R[t;S,D]) = \mathrm{rad}(R;S,D)[t;S,D].$$

Proof. It is enough to prove that $\mathrm{rad}(R[t;S,D]) \subseteq \mathrm{rad}(R;S,D)[t;S,D]$. Let $I \in \mathrm{Spec}(R;S,D)$. Then Theorem 3.3 of [G] shows that $IT = I[t;S,D] \in \mathrm{Spec}(R[t;S,D])$. Hence

$$\mathrm{rad}(R;S,D)[t;S,D] = \bigcap_{I \in \mathrm{Spec}(R;S,D)} I[t;S,D] \supseteq \mathrm{rad}(R[t;S,D]).$$

QED

In general we have $\mathrm{rad}(R;S,D)[t;S,D] \subseteq \mathrm{rad}(R[t;S,D])$ and we have seen that in some cases the reverse inclusion was true. In particular ascending chain conditions were useful specially when dealing with q-quantized derivations. One feature of q-quantized derivation is the fact that both S and D can be extended to the Ore extension $T = R[t; S, D]$ itself, by setting $S(t) = qt$ and $D(t) = (1 - q)t^2$. Then D becomes an inner S-derivation adjoint to t. After extending S and D to T we can consider $\mathrm{rad}(T; S)$ and $\mathrm{rad}(T; S, D)$. It turns out that $\mathrm{rad}(R; S, D)[t; S, D]$ is characterized by these radicals, more precisely :

Theorem 5.12. *Suppose that $SD = DS$ and let $T = R[t; S, D]$. Then:*

(a) *T is (S, D)-semiprime if and only if R is (S, D)-semiprime.*
(b) $\mathrm{rad}(R; S, D)[t; S, D] = \mathrm{rad}(T; S, D) = \mathrm{rad}(T; S)$.
(c) *If a nonzero power of S is inner then* $\mathrm{rad}(T) = \mathrm{rad}(R; S, D)[t; S, D]$.

Proof. (a) Assume R is (S, D)-semiprime but T is not. Then there exists $f = a_n t^n + \cdots + a_0 \in T$, with $a_n \neq 0$, such that $fTS^kD^l(f) = 0$ for all $(k, l) \in \mathbb{Z} \times \mathbb{N}$ (see e.g. Lemma 2.2(c)). Then $a_n RS^kD^l(a_n) = 0$ and the fact that R is (S, D)-semiprime gives $a_n = 0$, a contradiction.

Conversely assume T is (S, D)-semiprime and let $I \triangleleft_{(S,D)} R$ be such that $I^2 = 0$ then $I[t; S, D] \triangleleft_{(S,D)} T$ and $(I[t; S, D])^2 = 0$. Thus $I[t; S, D] = 0$ and finally $I = 0$. This shows that R is (S, D)-semiprime.

(b) Since D is an inner S-derivation of T, we have $\mathrm{rad}(T; S, D) = \mathrm{rad}(T; S)$.

We shall now show that $\mathrm{rad}(R; S, D)[t; S, D] = \mathrm{rad}(T; S, D)$. Let$I$ stands for $\mathrm{rad}(R; S, D)$. Then R/I is (S, D)-semiprime and, by using (a) above, $R/I[t; S, D]$ is (S, D)-semiprime. Hence $R[t; S, D]/I[t; S, D]$ is (S, D)-semiprime and $I[t; S, D] \supseteq \mathrm{rad}(T; S, D)$.

Conversely if P is a (S, D)-prime ideal of $T = R[t; S, D]$, then $P \cap R$ is an (S, D)-prime ideal of R and so $\mathrm{rad}(R; S, D) \subseteq P \cap R$. Therefore $\mathrm{rad}(R; S, D) \subseteq \mathrm{rad}(T; S, D)$.

(c) If suffices to apply Lemma 2.3(d) and (b) above. **QED**

Remark 5.13. (1) If we assume that R satisfies the ACC on ideals and $SD = DS$, then Corollary 2.4(b) and Theorem 5.12(b) give back Proposition 5.5: $\mathrm{rad}(R; S, D)[t; S, D] = \mathrm{rad}(T)$.

(2) If D is a q-quantized S-derivation, then one can show that T is (S,D)-semiprime implies R is (S,D)-semiprime and $\mathrm{rad}(R;S,D)[t;S,D] \subseteq \mathrm{rad}(T;S) = \mathrm{rad}(T;S,D)$.

The ideal $\mathrm{rad}(R,S,D)[t;S,D]$ of $T = R[t;S,D]$ can be seen, according to Lemma 5.1, as a lower bound for $\mathrm{rad}(T)$. Let us now end the paper with an attempt to get an "upper bound". For this we introduce a definition of (S,D)-primeness which is unsymmetric and imitates the one given in [PS].

Definitions 5.14. (1) Let P be an (S,S^{-1},D)-ideal. P is right (S,D)-prime if for any $a,b \in R\backslash P$, there exists $n \geq k \geq 0$ such that $aRf_k^n(b) \not\subseteq P$, where $f_k^n \in \mathrm{End}(R,+)$ are defined as in Example 1.4.

(2) The set of right (S,D)-prime ideals will be denoted $\mathrm{Spec}_{(S,D)}(R)$ and $P_{(S,D)}(R)$ will stand for the intersection of all these right (S,D)-prime ideals.

In the following lemma we will compare these new notions with the analogue ones introduced in earlier sections.

Lemma 5.15. *Keeping the notations as above, we have:*

(a) *$P \in \mathrm{Spec}_{(S,D)}$ if and only if $P \triangleleft_{(S,S^{-1},D)} R$ and*

$$\forall_{A \triangleleft R} \, \forall_{B \triangleleft_{(S,D)} R} \text{ , if } AB \subseteq P \text{ then either } A \subseteq P \text{ or } B \subseteq P \quad (5.16)$$

(b) $\mathrm{Spec}_{(S,D)}(R) \subseteq \mathrm{Spec}(R;S,D)$ *and* $\mathrm{rad}(R;S,D) \subseteq P_{(S,D)}(R)$

(c) *If $S = id_R$, then* $\mathrm{Spec}_D(R) = \mathrm{Spec}(R;D)$ *and* $\mathrm{rad}(R;D) = P_D(R)$

(d) *Let $q \in R$ be such that both q and $\sum_{i=0}^{n} q^i$ are invertible in R for any $n \in \mathbb{N}$. If D is q-quantized and R satisfies ACC on S-ideals then* $\mathrm{Spec}_{(S,D)}(R) = \mathrm{Spec}(R;S,D)$ *and* $P_{(S,D)}(R) = \mathrm{rad}(R;S,D)$.

Proof. (a) Let $P \in \mathrm{Spec}_{(S,D)}(R)$ be a right (S,D)-prime ideal and assume $AB \subseteq P$ for some $A \triangleleft R$ and $B \triangleleft_{(S,D)} R$. If $A \not\subseteq P$, let $a \in A\backslash P$ then $aB \subseteq P$ so $aRB \subseteq P$ and since B is an (S,D)-ideal $aRf_k^n(B) \subseteq P$ for any $0 \leq k \leq n$. Now, since $a \in A\backslash P$, Definition 5.14(1) shows that $B \subseteq P$.

Conversely, let $P \triangleleft_{(S,S^{-1},D)} R$ such that (5.16) is satisfied. If $a,b \in R$ are such that $aRf_k^n(b) \subseteq P$ for any $0 \leq k \leq n$, then $RaR \sum_{n \geq k \geq 0} Rf_k^n(b)R \subseteq$

P. However $\sum_{n\geq k\geq 0} Rf_k^n(b)R \triangleleft_{(S,D)} R$ and (5.16) implies that either $RaR \subseteq P$ or $\sum Rf_k^n(b)R \subseteq P$. In particular, $a \in P$ or $b \in P$.

(b) The first inclusion is clear thanks to (a) and the second then follows immediately.

(c) In view of (b) above, we only need to show $\mathrm{Spec}(R, D) \subseteq \mathrm{Spec}_D(R)$. Let $P \in \mathrm{Spec}(R, D)$ be a D-prime ideal in R. If $a, b \in R$ are such that $aRD^n(b) \subseteq P$, for any $n \geq 0$, then $(RaR)(\sum_{n\geq 0} RD^n(b)R) \subseteq P$. Put $I := \sum_{n\geq 0} RD^n(b)R \triangleleft_P R$, then we have $aI \subseteq P$. By applying powers of D, we obtain $D(a)I \subseteq P, \ldots, D^i(a)I \subseteq P$. This means that $(\sum RD^i(a)R)(\sum RD^i(b)R) \subseteq P$. Since $P \in \mathrm{Spec}(R, D)$, we see that either $\sum_{i\geq 0} RD^i(a)R \subseteq P$ or $\sum_{i\geq 0} RD^i(b)R \subseteq P$. In particular either $a \in P$ or $b \in P$ and this shows $P \in \mathrm{Spec}_D(R)$ as desired.

(d) Since R satisfies ACC on S-ideals, every S-ideal is in fact an (S, S^{-1})-ideal. Therefore any ideal $P \in \mathrm{Spec}(R, S, D)$ is also an (S, S^{-1}, D)-ideal. Due to the assumption imposed on q, $\binom{n}{i}_q$ is invertible in R for any $n \in \mathbb{N}$ and $0 \leq i \leq n$. Now the statement (d) can be obtained easily using Lemma 2.2(c)(iv). **QED**

Theorem 5.17. *For $T = R[t; S, D]$, the following inclusions hold:*

$$\mathrm{rad}(R; S, D)[t; S, D] \subseteq \mathrm{rad}(T) \subseteq P_{(S,D)}(R)[t; S, D].$$

In particular, if $\mathrm{rad}(R; S, D) = P_{(S,D)}(R)$, then $\mathrm{rad}(T) = \mathrm{rad}(R; S, D)[t; S, D]$.

Proof. The first inclusion was proved in Lemma 5.1. To prove the second one it is enough to show that for any $P \in \mathrm{Spec}_{(S,D)}(R)$, $P[t; S, D]$ is a prime ideal of T. Equivalently we must show that

$$\frac{T}{P[t; S, D]} \cong \frac{R}{P}[t; S, D]$$

is a prime ring. But this is an obvious consequence of Theorem 4.4(3). **QED**

The above proposition together with Lemma 5.15(c) and (d) give the following.

Corollary 5.18.

(1) *If $S = id_R$ then rad*$(T) = \mathrm{rad}(R; D)[t; D]$

(2) *Let $q \in R$ be such that both q and $\sum_{i=0}^{n} q^i$ are invertible in R for any $n \in \mathbb{N}$. If D is a q-quantized S-derivation and R satisfies the ACC on S-ideals then* $\mathrm{rad}(T) = \mathrm{rad}(R, S, D)[t; S, D]$.

Let us mention that (1) above was obtained in [FKM] and (2) is in the same vein as those in Theorem 5.10.

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