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On the finitistic global dimension conjecture for Artin algebras

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This paper is dedicated to Vlasta Dlab on the occasion of his 70 birthday.

Abstract. We find a simple condition which implies finiteness of (little) finitistic global dimension for artin algebras. As a consequence we obtain a short proof of the finitistic global dimension conjecture for radical cubed zero algebras. The same condition also holds for algebras of representation dimension less then or equal to three. Hence the conjecture holds in that case as well.

Let Λ be an Artin algebra (an algebra of finite length over a commutative Artinian ring). We will consider only finitely generated (f.g.) modules over Λ, and hence we will only address the *little finitistic global dimension conjecture* which states that there exists a uniform bound called $findim\Lambda$ for the finite projective dimensions of all f.g. (left) Λ-modules of finite projective dimension. This conjecture implies the Nakayama conjecture. Some of the known cases in which the finitistic global dimension conjecture holds are the radical cubed zero case [GZ] and the monomial relation case [GKK] (see also [IZ], [BFGZ]). The conjecture is also true in the case the category of modules of finite projective dimension is contravariantly finite in the category of all f.g. modules [AR]. However, the converse is not true [IST]. In this paper we give a short proof that $sup\{pdM \mid M \text{ f.g. with } rad^2M = 0 \text{ and } pdM < \infty\}$ is finite. This is a generalization of the radical cubed zero case since all syzygies have radical square zero in that case. A thorough overview of the state of both little and big finitistic global dimension conjectures is given by B. Huisgen-Zimmerman [H].

As another consequence of the main theorem we prove the finitistic dimension conjecture for algebras with representation dimension $repdim\Lambda \leq 3$. The notion of representation dimension was introduced by M. Auslander [A1]. O. Iyama showed

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that it is always finite [I]. Many classes of algebras are known to have $repdim\Lambda = 3$ [EHIS], [CP]. However, there is an example by Rouquier of an algebra of representation dimension 4 [R].

The proof of the main theorem uses the following well-known lemma and the function η_f defined therein.

Lemma 0.1 (Fitting's Lemma) *a) Let M be a f.g. module over a Noetherian ring R and let $f : M \to M$ be an endomorphism of M. Then for any submodule X of M there is an integer n so that f sends $f^m(X)$ isomorphically onto $f^{m+1}(X)$ for all $m \geq n$. Let $\eta_f(X)$ denote the smallest value of $n \geq 0$.*

b) If Y is a submodule of X then $\eta_f(Y) \leq \eta_f(X)$.

c) If R is an Artin algebra and $X = M$ there is a direct sum decomposition $X = Y \oplus Z$ so that $Z = ker f^m$ and $Y = im f^m$ for all $m \geq \eta_f(X)$.

Let K_0 be the abelian group generated by all symbols $[M]$, where M is a f.g. Λ-module, modulo the relations:

a) $[C] = [A] + [B]$ if $C \approx A \oplus B$.

b) $[P] = 0$ if P is projective.

Then K_0 is the free abelian group generated by the isomorphism classes of indecomposable f.g. nonprojective Λ-modules. For any f.g. Λ-module M let $L[M] = [\Omega M]$ where ΩM is the first syzygy of M. Since Ω commutes with direct sums and takes projective modules to zero this gives a homomorphism $L : K_0 \to K_0$. For every f.g. Λ-module M let $\langle addM \rangle$ denote the subgroup of K_0 generated by all the indecomposable summands of M. Let

$$\phi(M) := \eta_L \langle addM \rangle .$$

Lemma 0.2 *a) If M has finite projective dimension then $\phi(M) = pdM$.*

b) If M is indecomposable with $pdM = \infty$ then $\phi(M) = 0$.

c) $\phi(A) \leq \phi(A \oplus B)$.

d) $\phi(kM) = \phi(M)$ if $k \geq 1$.

Proof Part (c) follows from Lemma 0.1(b) since $\langle addA \rangle$ is a subgroup of $\langle add(A \oplus B) \rangle$. Part (d) follows from the fact that $add\ kM = addM$. Parts (a) and (b) are easy. □

We need one more definition. For any f.g. Λ-module M let

$$\psi(M) := \phi(M) + \sup\{pdX \mid pdX < \infty, X \text{ direct summand of } \Omega^{\phi(M)}M\}.$$

Lemma 0.3 *a) $\psi(M) = \phi(M) = pdM$ whenever $pdM < \infty$.*

b) $\psi(kM) = \psi(M)$ if $k \geq 1$.

c) $\psi(A) \leq \psi(A \oplus B)$.

d) If Z is a summand of $\Omega^n M$ where $n \leq \phi(M)$ and $pdZ < \infty$ then $pdZ + n \leq \psi(M)$.

Proof Part (a) is easy and part (b) follows from Lemma 0.2(d). Part (c) follows from (d) in the case when $M = A \oplus B, n = \phi(A)$ and from Lemma 0.2(c).

(d) Since Z is a summand of $\Omega^n M$ it follows that $\Omega^{\phi(M)-n}Z$ is a summand of $\Omega^{\phi(M)}M$. Also $pdZ < \infty$ implies that $pd(\Omega^{\phi(M)-n}Z)$ is finite. So, by definition of $\psi(M)$ it follows that

$$\phi(M) + pd(\Omega^{\phi(M)-n}Z) \leq \psi(M)$$

and hence $\phi(M) + pdZ - \phi(M) + n \leq \psi(M)$. □

Theorem 0.4 *Suppose that* $0 \to A \to B \to C \to 0$ *is a short exact sequence of f.g.* Λ*-modules and* C *has finite projective dimension. Then* $pdC \leq \psi(A \oplus B) + 1$.

Remark 0.5 This generalizes the well known formula $pdC \leq pd(A \oplus B) + 1$. Y. Wang [W] pointed out that $pdB \leq \psi(A \oplus \Omega C) + 1$.

Proof Since pdC is finite, $\Omega^n A \approx \Omega^n B$ for some $n \geq 0$. Take n to be minimal so that $[\Omega^n A] = L^n[A] = L^n[B] = [\Omega^n B]$. Then $n \leq pdC$ and since $[A], [B] \in \langle add(A \oplus B) \rangle$ we also have $n \leq \phi(A \oplus B)$. The n-th syzygies of our short exact sequence then give an exact sequence of the form

$$0 \to X \oplus P \to X \oplus Q \to \Omega^n C \to 0$$

where P, Q are projective. Let $f : X \to X$ be the X-X component of the map $X \oplus P \to X \oplus Q$, i.e. the composition $X \to X \oplus P \to X \oplus Q \to X$. Then $X = Y \oplus Z$ so that f is an automorphism of Y plus a nilpotent endomorphism g of Z. This means there is another exact sequence:

$$0 \to Z \oplus P \to Z \oplus Q \to \Omega^n C \to 0 \tag{1}$$

where the Z-Z component of the map $Z \oplus P \to Z \oplus Q$ is g.

We claim that Z has finite projective dimension. If not, then for each k there is a module S so that $Ext^k(Z, S) \neq 0$. The endomorphism of $Ext^k(Z, S)$ induced by g is nilpotent and therefore not surjective. By the exactness of the sequence:

$$Ext^k(Z, S) \to Ext^k(Z, S) \to Ext^{k+1}(\Omega^n C, S)$$

we get that $Ext^{k+1}(\Omega^n C, S) \neq 0$ which is a contradiction.

Since pdZ is finite it follows from the exact sequence (1) that $pd\Omega^n C \leq pdZ + 1$ and that

$$pdC = n + pd\Omega^n C \leq n + pdZ + 1.$$

Furthermore, Lemma 0.3(d) implies that $pdZ + n \leq \psi(A \oplus B)$. Thus $pdC \leq \psi(A \oplus B) + 1$. □

Corollary 0.6 *If* M *is a f.g.* Λ*-module with Loewy length* 2 *and finite projective dimension then*

$$pdM \leq \psi(\Lambda/rad\Lambda \oplus \Lambda/(rad\Lambda)^2) + 1.$$

Proof There is a short exact sequence $0 \to A \to P/rad^2 P \to M \to 0$ where P is the projective cover of M, A is semisimple and $\psi(A \oplus P/rad^2 P) \leq \psi(\Lambda/rad\Lambda \oplus \Lambda/(rad\Lambda)^2)$ by Lemma 0.3. □

Corollary 0.7 *Suppose that* Λ *is an artin algebra with* $(rad\Lambda)^3 = 0$ *then*

$$findim\Lambda \leq \psi(\Lambda/rad\Lambda \oplus \Lambda/(rad\Lambda)^2) + 2$$

where $findim\Lambda = sup\{pdM | pdM < \infty$ *and* M *is f.g.*$\}$.

Proof For any Λ-module M, ΩM has Loewy length at most 2. □

Corollary 0.8 *Let* $\Lambda = End_\Gamma(P)^{op}$*, where* P *is a projective module over an artin algebra* Γ *with* $gldim\Gamma \leq 3$*. Then*

$$findim\Lambda \leq \psi((P, \Gamma)) + 3$$

where $(P, \Gamma) = Hom_\Gamma(P, \Gamma)$ *is considered as a* Λ*-module.*

Proof We use a construction due to Auslander [A2]. Any Λ-module X has a Λ-projective presentation $(P, P_1) \to (P, P_0) \to X \to 0$ with P_0, P_1 in $add(P)$. The associated sequence of projective Γ-modules $0 \to P_3 \to P_2 \to P_1 \to P_0$ induces an exact sequence of Λ-modules

$$0 \to (P, P_3) \to (P, P_2) \to (P, P_1) \to (P, P_0) \to X \to 0.$$

Hence, $pdX \leq pd(\Omega^2 X) + 2 = pd(coker((P, P_3) \to (P, P_2))) + 2 \leq$

$$\psi((P, P_3) \oplus (P, P_2)) + 3 \leq \psi((P, \Gamma)) + 3.$$

□

Corollary 0.9 *If* $repdim\Lambda \leq 3$ *then* $findim\Lambda < \infty$.

Remark 0.10 We recall that $repdim\Lambda \leq n$ if there is a f.g. module X such that $gldimEnd_\Lambda(X)^{op} \leq n$ and $addX$ contains all projective and all injective Λ-modules. Our proof does not require injectives to be in $addX$.

Proof If $repdim\Lambda \leq 3$ let X be as above, i.e., $gldimEnd_\Lambda(X) \leq 3$ and Λ is in $addX$. Then $\Lambda \approx End_\Gamma(P)^{op}$ where $\Gamma = \mathrm{End}_\Lambda(X)^{op}$ and P is the projective Γ-module $Hom_\Lambda(X, \Lambda)$. So, Corollary 0.8 applies. □

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