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THE CONVERGENCE OF A CLASS OF QUASIMONOTONE REACTION–DIFFUSION SYSTEMS

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Abstract

It is proved that every solution of the Neumann initial-boundary problem

$$\begin{cases} \partial u_i/\partial t = d_i \Delta u_i + F_i(u) & t>0,\ x\in\Omega, \\ \partial u_i/\partial n(t,x) = 0 & t>0,\ x\in\partial\Omega,\ i=1,2,\dots,n, \\ u_i(x,0)=u_{i,0}(x)\geqslant 0 & x\in\bar{\Omega}, \end{cases}$$

converges to some equilibrium, if the system satisfies (i) $\partial F_i/\partial u_j \geqslant 0$ for all $1\leqslant i\neq j\leqslant n$, (ii) $F(u*g(s))\geqslant h(s)*F(u)$ whenever $u\in\mathbb{R}^n_+$ and $0\leqslant s\leqslant 1$, where $x*y=(x_1y_1,\dots,x_ny_n)$ and $g,h:[0,1]\longrightarrow[0,1]^n$ are continuous functions satisfying $g_i(0)=h_i(0)=0$, $g_i(1)=h_i(1)=1$, $0<g_i(s),h_i(s)<1$ for all $s\in(0,1)$ and $i=1,2,\dots,n$, and (iii) the solution of the corresponding ordinary differential equation system is bounded in $\mathbb{R}^n_+$. We also study the convergence of the solution of the Lotka–Volterra system

$$\begin{cases} \partial u_i/\partial t = \Delta u_i + u_i\left(r_i+\sum_{j=1}^n a_{ij}u_j\right) & t>0,\ x\in\Omega, \\ \partial u_i/\partial n+\alpha u_i = 0 & t>0,\ x\in\partial\Omega,\ i=1,2,\dots,n, \\ u_i(x,0)=u_{i,0}(x)\geqslant 0 & x\in\bar{\Omega}, \end{cases}$$

where $r_i>0$, $\alpha\geqslant 0$, and $a_{ij}\geqslant 0$ for $i\neq j$.

1. *Introduction*

This article is motivated by the study of large asymptotic behaviour of solutions to the following reaction–diffusion-type initial-boundary value problem (1) for an unknown vector-valued function $u(t,x)\in\mathbb{R}^n$ of the time–space variable $(t,x)\in\mathbb{R}^1_+\times\Omega$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + F(u) & t>0,\ x\in\Omega, \\ \dfrac{\partial u}{\partial n}(t,x) = 0 & t>0,\ x\in\partial\Omega, \\ u(x,0)=u_0(x) & x\in\bar{\Omega}. \end{cases} \tag{1}$$

Here $D=\operatorname{diag}(d_1,d_2,\dots,d_n)$ is a diagonal matrix with $d_i>0$ on the diagonal. Ω is an open bounded domain in $\mathbb{R}^k$ with a smooth boundary $\partial\Omega$. $\partial/\partial n$ denotes differentiation in the direction of the outward normal to $\partial\Omega$ and $F(u):\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is a co-operative C^2-vector field, that is, $\partial F_i/\partial u_j\geqslant 0$ whenever $1\leqslant i\neq j\leqslant n$, for all $u\in\mathbb{R}^n$.

Of course, Δ denotes the Laplacian.

Given any initial distribution $u_0\in C(\bar{\Omega},\mathbb{R}^n)$, in order to guarantee global existence

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and boundedness of a *classical* solution $u \in C(\mathbb{R}^1_+ \times \bar{\Omega}, \mathbb{R}^n) \cap C^{1,2}((0,\infty) \times \bar{\Omega})$, we make the following additional hypothesis.

(H): $F(0) \geqslant 0$ and the system of ordinary differential equations

$$\begin{cases} \dfrac{dv}{dt} = F(v) \quad t > 0, \\ v(0) = v_0, \end{cases} \tag{2}$$

possesses a bounded global solution $v \in C^1(\mathbb{R}^1_+, \mathbb{R}^n_+)$, for every $v_0 \in \mathbb{R}^n_+$.

Let $X = \Pi_{i=1}^n X_i$ be the Banach space with the norm $\|\phi\| = \sum_{i=1}^n |\phi_i|$, where $X_i = C(\bar{\Omega}, \mathbb{R})$ are Banach spaces with the supremum norm.

For each $I = \{j_1, j_2, \ldots, j_m\} \subset N = \{1, 2, \ldots, n\}$, we can induce an order in $X_I = \Pi_{k=1}^m X_{j_k}$, that is, for $u_I, v_I \in X_I$,

$$\begin{aligned} u_I &\leqslant_I v_I \Leftrightarrow u_i(x) \leqslant v_i(x) \quad \text{for all } x \in \bar{\Omega}, i \in I, \\ u_I &\neq_I v_I \Leftrightarrow u_i(x) \neq v_i(x) \quad \text{for some } x \in \bar{\Omega} \text{ and some } i \in I, \\ u_I &<_I v_I \Leftrightarrow u_I \leqslant_I v_I \quad \text{and} \quad u_I \neq_I v_I, \\ u_I &\ll_I v_I \Leftrightarrow u_i(x) < v_i(x) \quad \text{for all } x \in \bar{\Omega}, i \in I. \end{aligned}$$

Notations such as $u_I >_I v_I$ have natural meanings. If $A, B \subset X_I$ are subsets of X_I, then $A <_I B$ means $a <_I b$ for all $a \in A$, $b \in B$; and similarly for $A \leqslant_I B$, $A \ll_I B$, etc.

In case $I = N$, we omit the order subscripts and obtain the order $\leqslant (<, \ll, \neq)$ in X. In case $I = \{i\}$ is a singleton index subset of N, we denote the order of X_i by $\leqslant_i (<_i, \ll_i, \neq_i)$.

Let $X_+ = \{u \in X : u \geqslant 0\}$. Then the above quasimonotone reaction–diffusion system with initial value $u_0(x) \in X_+$ generates a monotone semiflow on X_+ and every positive semi-orbit has compact closure (cf. [**13**]). There is now an extensive literature on monotone (or order-preserving) dynamical systems, beginning with the ground-breaking work of Hirsch [**3**, **4**] for monotone semiflows. The results of Hirsch and later improvements by Matano [**8**], Smith and Thieme [**14**, **15**] and Poláčik [**9**] established that most orbits of a continuous-time, strongly monotone semiflow converge to the set of equilibria. Many researchers have tried to find some extra conditions added to strongly monotone dynamical systems to prove that every positive semi-orbit converges to an equilibrium.

Takáč [**16**] replaced the concavity conditions in [**3**] and [**11**] by the sublinear hypotheses to generalize the convergent results. His hypothesis is that the system (1) is quasimonotone (co-operative) and irreducible and F is sublinear in the sense that

$$F(\alpha u) \geqslant \alpha F(u) \quad \text{for all } (\alpha, u) \in [0,1] \times X_+. \tag{3}$$

Together with some technical hypotheses, he proved that every positive semi-orbit with compact closure converges to an equilibrium. The set E of all equilibria of (1) is either $\{O\}$, or $\{O, p\}$ with $p \gg O$, or $\{O\} \cup L$, where L is a simply ordered line segment. Therefore all equilibria are simply ordered by the usual order relation $\ll$.

The above-mentioned results apply only to strongly monotone dynamical systems. Recently, Jiang has studied the asymptotic behaviour of finite dimensional continuous-time order-preserving dynamical systems without the irreducible assumption. In [**7**], he studied the system (2) and proved that every positive semi-orbit converges to an equilibrium if every positive semi-orbit is compact and F satisfies the following property.

(P): $F(u * g(s)) \geqslant h(s) * F(u)$ whenever $u \in \mathbb{R}^n_+$ and $0 \leqslant s \leqslant 1$, where $x * y =$

$(x_1y_1, \ldots, x_ny_n)$ and $g, h : [0,1] \longrightarrow [0,1]^n$ are continuous functions satisfying $g_i(0) = h_i(0) = 0$, $g_i(1) = h_i(1) = 1$, $0 < g_i(s)$, $h_i(s) < 1$ for all $s \in (0,1)$ and $i = 1, 2, \ldots, n$.

The main goal of this paper is to study the asymptotic behaviour of the solution with initial value in X_+ in the above-mentioned system (1) with the hypotheses (H) and (P). The infinite-dimensional dynamical system generated by our reaction–diffusion system (1) is only monotone. Therefore, we are unable to directly use any of the convergent results mentioned in the previous paragraphs since none of them combines monotonicity with infinite dimension. Without the strong assumption on order, we prove that Jiang's results in [**7**] also hold in the infinite-dimensional dynamical system generated by (1). Note that hypothesis (P) is weaker than (3), thus we generalize Takáč's result. An example is also given to show that the set of equilibria is not simply ordered and, therefore, is not included in the type of Takáč [**16**], but we can still obtain the convergence of the semi-orbits by our result.

In this paper, we also study the following Lotka–Volterra system:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \Delta u_i + u_i \left(r_i + \displaystyle\sum_{j=1}^{n} a_{ij} u_j \right) & t > 0,\ x \in \Omega, \\ \dfrac{\partial u_i}{\partial n} + \alpha u_i = 0 & t > 0,\ x \in \partial\Omega\ i = 1, 2, \ldots, n, \\ u_i(x, 0) = u_{0,i}(x) \geqslant 0 & x \in \bar{\Omega}, \end{cases} \tag{4}$$

where $r_i > 0$, $\alpha \geqslant 0$, $a_{ii} < 0$ and $a_{ij} \geqslant 0$ for $i \neq j$. In case $\alpha = 0$, our result has shown that every compact semi-orbit of (4) is convergent to an equilibrium. Furthermore, if $A = (a_{ij})_{n \times n}$ is stable (cf. [**12**]), then there is a positive equilibrium which is globally asymptotically stable. In case $\alpha > 0$, under some hypotheses of the coefficients, we also prove the same convergent result. Roughly speaking, if r_i and $|a_{ii}|$ are large enough, every solution of (4) is bounded and convergent to a unique positive equilibrium.

2. *Notations and preliminaries*

In this section, we first agree on some notations, give important definitions and state some known results which will be important in our proofs.

Let $N = \{1, 2, \ldots, n\}$. If $I \subset N$, then $C(I) = N \backslash I$. $H_I = \{u \in X : u_i(x) = 0$ for all $x \in \bar{\Omega},\ i \in C(I)\}$, $H_I^+ = H_I \cap X_+$ and $\operatorname{Int} H_I^+ = \{u \in H_I^+ : u_i \gg_i 0,\ i \in I\}$. At each point $p \in X$, there is a non-negative cone defined by

$$p + X_+ = \{u \in X : u \geqslant p\}.$$

Definition 2.1. A semiflow $\Phi : \mathbb{R}_+^1 \times X_+ \longrightarrow X_+; (t, u) \longmapsto \Phi(t, u)$ is called *monotone* (respectively, *strictly monotone*) if $u, v \in X_+$ and $u \leqslant v$ ($u < v$) implies that $\Phi(t, u) \leqslant \Phi(t, v)$ (respectively, $\Phi(t, u) < \Phi(t, v)$) for all $t \geqslant 0$. The semiflow is called *strongly monotone* if $\Phi(t, u) \ll \Phi(t, v)$ whenever $u < v$ and $t > 0$.

Definition 2.2. A monotone semiflow is said to be of *type K* if any subset $I \subset N$ and $u < v$ with $u_i <_i v_i$ for all $i \in I$, one has $\Phi(t, u) < \Phi(t, v)$ with $\Phi_i(t, u) \ll_i \Phi_i(t, v)$ for all $i \in I$ and $t > 0$.

From Definitions 2.1 and 2.2, we immediately obtain the following relationship:

strongly monotone ⇒ Type-K monotone ⇒ strictly monotone ⇒ monotone.

However, it is easy to see that every converse relationship does not hold.

We denote by $O^+(u)$ the *positive semi-orbit* from $u \in X_+$. A subset $A \subset X_+$ is called *positive invariant*, if $u \in A$ implies that $O^+(u) \subset A$. A point p is called an *equilibrium* if $p \equiv \Phi(t, p)$ for all $t \geqslant 0$. The set of all equilibria of (1) is denoted by E and the set of equilibria of (2) by E_o. Obviously, $E_o \subset E$. p is called *Liapunov stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $u \in X_+$ and $\|u-p\| < \delta$ implies that $\|\Phi(t, u)-p\| < \varepsilon$ for all $t \geqslant 0$. The *ω-limit set* of u is defined by $\omega(u) = \{y \in X_+ : \Phi(t_n, u) \longrightarrow y$ for some sequence $t_n \to \infty\}$. Notice that if $O^+(u)$ is relatively compact in X_+, then $\omega(u) \neq \emptyset$ and it is connected and totally invariant, that is, $\Phi(t, \omega(u)) = \omega(u)$ for all $t \geqslant 0$.

The following proposition shows that (1) with hypothesis (H) can generate a type-K monotone semiflow in the Banach space X_+.

PROPOSITION 2.1. *Suppose that* (H) *holds. Then for each $u \in X_+$,* (1) *has a unique classical solution $u(t) \in X_+$ defined on $[0, +\infty)$ satisfying $u(0) = u$. Furthermore, $\Phi(t, u) = u(t)$ is a type-K monotone semiflow on X_+ and the positive semi-orbit $O^+(u)$ of any u has compact closure in X_+.*

Proof. The existence of a unique classical solution of (1), belonging to X_+, is a consequence of [**13**], Theorem 7.3.1. It follows from hypothesis (H) that the solution can be continued to a maximal interval of existence $[0, +\infty)$. Then [**13**], Theorem 7.3.1 and Corollary 7.3.5 show that $\Phi(t, u)$ is a monotone semiflow and every positive semi-orbit has compact closure. That the semiflow is of type-K follows from maximum principle (cf. [**10**], Theorem 3.15) and its proof can be found in [**17**]. □

For system (2), we have the following propositions.

PROPOSITION 2.2. *Let $p < q$ be order-related equilibria of* (2). *If there does not exist any other equilibrium of* (2) *between p and q, then there is a monotone orbit connecting p and q.*

This proposition is due to Dancer and Hess and is adapted from [**1**], where the strict monotonicity follows from the well-known Kamke theorem.

PROPOSITION 2.3 (Jiang [**7**]). *Suppose that* (H) *and* (P) *hold. Then for system* (2), *we have the following:*

- (i) *If there is a positive equilibrium p in E_o, then p is Liapunov stable and every solution of* (2) *is bounded.*
- (ii) *Every solution of* (2) *converges to an equilibrium in E_o.*

3. *Main result and its proof*

The main result in this section is as follows.

THEOREM 3.1. *Suppose that* (P) *and* (H) *hold. Then every positive semi-orbit of* (1) *converges, that is, $\omega(u)$ is a singleton for each $u \in X_+$.*

REMARK 3.1. Without the hypothesis of irreducibility of the nonlinearity $F(u)$, we conclude the convergent result from Theorem 3.1, which was not obtained by the papers mentioned in the introduction. This result also generalizes the corresponding result in ordinary differential equation systems by Jiang [**7**] to partial differential equation systems (1). Furthermore, under hypothesis (P) which is much weaker than condition (3), we generalize the convergent result obtained by Takáč [**16**] and Hess [**2**]. The example below shows that there exists a new limit-sets distribution which violates the characteristics of limit-sets obtained by Takáč [**16**].

EXAMPLE 3.1. Consider the partial differential equation system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1 - u_1(u_1 - u_2), \\ \dfrac{\partial u_2}{\partial t} = \Delta u_2 + u_2(u_1 - u_2) \quad t > 0, \ x \in \Omega, \\ \dfrac{\partial u_3}{\partial t} = \Delta u_3 + u_3^2(1 - u_3), \\ \dfrac{\partial u_i}{\partial n} = 0, \ t > 0, \ x \in \partial\Omega; \ \text{and } u_i(0,x) = u_{i,0}(x) \quad x \in \bar{\Omega}. \end{cases} \tag{5}$$

Then hypotheses (P) and (H) can be easily checked. Therefore, we can apply Theorem 3.1 to (5) and conclude that every solution of (5) with initial value in X_+ tends to an equilibrium as $t \longrightarrow \infty$. Note that we are unable to directly use any of the results on the monotone semiflow mentioned in the introduction to obtain the same result. Furthermore,

$$\{(a,a,c) \in \mathbb{R}^3 \subset X : a \geqslant 0, \ c = 0 \text{ or } 1\} \subset E \cap X_+.$$

Therefore system (5) violates the characteristics of E obtained by Takáč in the introduction, that is, there are new types of dynamic behaviour of system (1) without the hypothesis of irreducibility.

Inspired by the methods provided in [**6**], we first present some lemmas which are of great importance in the proof of our results.

LEMMA 3.1. *Suppose that* (H) *and* (P) *hold. Then for every* $u \in X_+$, *there exist two constant equilibria* $p \leqslant q$ *satisfying* $p \leqslant \omega(u) \leqslant q$, $\{r \in E_o : r \leqslant \omega(u)\} \leqslant p$ *and* $\{s \in E_o : s \geqslant \omega(u)\} \geqslant q$.

Proof. Let $A = \{r \in E_0 : r \leqslant \omega(u)\}$, $B = \{s \in E_o : s \geqslant \omega(u)\}$. It follows from Proposition 2.1 that $\omega(u)$ is compact. Then there exist two points p_0, $q_0 \in \mathbb{R}^n_+$ such that $p_0 \leqslant \omega(u) \leqslant q_0$. By the monotonicity, the uniqueness of solution of (1) and Proposition 2.3 imply that we can find $r, s \in E_o$ satisfying $r \leqslant \omega(u) \leqslant s$. Thus $A \neq \emptyset$, $B \neq \emptyset$. Let $\tilde{p}_i = \sup_{r\in A}\{r_i\}$ and $\tilde{q}_i = \inf_{s\in B}\{s_i\}$ for $i \in N$. Then $\tilde{p} \leqslant \omega(u) \leqslant \tilde{q}$. It follows from Proposition 2.3 that $\omega(\tilde{p}) = p \in E_o$, $\omega(\tilde{q}) = q \in E_o$ and $p \leqslant \omega(u) \leqslant q$. It is obvious that $A \leqslant p$ and $B \geqslant q$. This completes the proof. □

Given $u \in X_+$, we define

$$L(u,p) = \max\{\sigma_p(y) : y \in \omega(u)\}, \quad \text{where } \sigma_p(y) = \#\{i : y_i >_i p_i\},$$

where # denotes the cardinality of a set. Obviously, $L(u,p) = 0$ implies that $\omega(u) = p$.

LEMMA 3.2. *If $L(u,p) = m \leqslant n$, then there exist $y \in \omega(u)$ and $I \subset N$, $\#I = m$ such that $p_i \ll_i y_i(t)$ for all $i \in I$ and $t \geqslant 0$, $p_j \equiv y_j(t)$ for all $j \in C(I)$ and $t \geqslant 0$.*

Proof. It follows from the definition that there exist $\bar{y} \in \omega(u)$ and $I \subset N$, $\#I = m$ such that $p_i <_i \bar{y}_i$ for all $i \in I$ and $p_j \equiv \bar{y}_j$ for $j \in C(I)$. We denote by $\bar{y}(t)$ $(t \geqslant 0)$ the solution of (1) with initial value $\bar{y}$. Let $y = \bar{y}(\delta)$ for some $\delta > 0$. Then $y \in \omega(u)$. By the type-K condition and $p \in E_o \subset E$, we obtain $p_i \ll_i y_i(t)$ for all $i \in N$ and $t \geqslant 0$. The fact that $p_j \equiv y_j(t)$ for all $j \in C(I)$ and $t \geqslant 0$ is a direct result of the maximal property of m and the type-K condition. □

PROPOSITION 3.1. *Suppose that system* (1) *has at least one positive constant equilibrium, that is, $r \in E_o$ and $r \gg 0$. Then $\omega(u)$ is a singleton for any $u \gg 0$.*

Proof. Given $u \gg 0$, we can find $s \in (0,1)$ such that $0 \ll g(s) * r \leqslant u$. From [**7**, Lemma 3.1], we obtain that $g(s) * r$ is a sub-solution of (1). Then $(g(s) * r)(t)$ is a nondecreasing solution which tends to $\bar{p} \in E_o$ as $t \to \infty$. Since $\bar{p} \leqslant \omega(u)$, Lemma 3.1 implies that $\bar{p} \leqslant p$. Therefore,

$$0 \ll g(s) * r \leqslant p \leqslant \omega(u) \leqslant q.$$

It follows from Proposition 2.3(i) that every constant equilibrium in $[p,q]$ is Liapunov stable according to system (2). If $\omega(u) \cap E_o \neq \emptyset$, then $\omega(u)$ contains a Liapunov stable equilibrium. By [**1**, Lemma 2.1], $\omega(u)$ is a singleton, thus we have obtained the result. Therefore, we assume that

$$\omega(u) \cap E_o = \emptyset \tag{6}$$

below.

Claim 1. $L(u,p) \leqslant n - 1$.

Suppose that $L(u,p) = n$. Then it follows from Lemma 3.2 that there exists some $y \in \omega(u)$ such that $p_i \ll_i y_i$ for all $i \in N$, that is, $p \ll y \leqslant q$. We assert that p is not an isolated equilibrium of system (2) from above. Otherwise, it is easy to prove that $B := ([p,q] \cap E_o)\backslash\{p\}$ is compact. By Zorn's lemma, B contains a minimal element w which implies that there exists no equilibrium of (2) between p and w. From the paragraph above, p and w are Liapunov stable according to system (2), contradicting Lemma 2.2. This shows our assertion is true. Therefore, we can find $w \in E_o$ such that $p < w \ll y \leqslant q$. It follows from $y \in \omega(u)$ that there exists a $t_0 > 0$ such that $w \ll u(t_0)$. Thus $w \leqslant \omega(u)$. By Lemma 3.1, $w \leqslant p$, contradicting $p < w$. This proves Claim 1.

Assume that $L(u,p) = m$. Then there exist $u^1 \in \omega(u)$ and $I \subset N$, $\#I = m$ such that $p_I \ll_I u^1_I(t)$ and $p_J \equiv u^1_J(t)$, $J = C(I)$ for all $t \geqslant 0$. Without loss of generality, we assume that $I = \{1, 2, \ldots, m\}$. By Lemma 3.1, there exist p^1, $q^1 \in E_o$ such that $p^1 \leqslant \omega(u^1) \leqslant q^1$. Therefore, $0 \ll p \leqslant p^1 \leqslant \omega(u^1) \leqslant q^1 \leqslant q$ and

$$p_J \equiv p^1_J \equiv u^1_J(t) \quad \text{for all } t \geqslant 0. \tag{7}$$

Claim 2. $L(u^1, p^1) \leqslant m - 1$.

If $L(u^1, p^1) = m$, then there exists $z \in \omega(u^1)$ such that $p^1_I \ll_I z_I$ and $p^1_J \equiv z_J$. It is easy to see that $u_I^1(t)$ $(t \geqslant 0)$ is the solution of system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i \Delta u_i + F_i(u_1, \ldots, u_m;\ p_{m+1}^1, \ldots, p_n^1) & t > 0,\ x \in \Omega, \\ \dfrac{\partial u_i}{\partial n} = 0 & t > 0,\ x \in \partial\Omega, \\ u_(x,0) = u_{0,i}(x) \geqslant 0 & x \in \bar{\Omega},\ i = 1,2,\ldots,m. \end{cases} \tag{8}$$

In order to show that every solution of (8) is bounded, we consider the following partial differential equation system:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i \Delta u_i + G_i(u) & t > 0,\ x \in \Omega, \\ \dfrac{\partial u_i}{\partial n} = 0 & t > 0,\ x \in \partial\Omega,\ i = 1,2,\ldots,n, \\ u_i(x,0) = u_{0,i}(x) \geqslant 0 & x \in \bar{\Omega}, \end{cases} \tag{9}$$

where $G_i = F_i$ for $i \in I$, $G_j \equiv 0$ for $j = m+1, \ldots, n$. It is easy to see that G satisfies hypothesis (P). Then we can apply Proposition 2.3(i) to the corresponding ordinary differential equation system of (9) and conclude that every solution of this ordinary differential equation system is bounded. Therefore, every solution of (9) is bounded. Obviously, $u_I^1(t)$ is the solution of (8) if and only if $(u_I^1(t), p_J^1)$ is the solution of (9). Hence, every solution of (8) is bounded. System (8) has a corresponding ordinary differential equation system

$$\frac{du_i}{dt} = F_i(u_1, \ldots, u_m;\ p_{m+1}^1, \ldots, p_n^1) \quad t > 0,\ i = 1,2,\ldots,m. \tag{10}$$

Noticing that p_I^1 is an equilibrium of (10), we assert that p_I^1 is not an isolated equilibrium of system (10) from above.

Proof of the assertion. Since $\omega(z) \subset \omega(u^1) \subset \omega(u)$ and (6), we obtain $\omega(z) \cap E_o = \emptyset$, which implies that $p^1 < \omega(z)$. Given any $w = (w_1, \ldots, w_n) \in \omega(z)$. By $\omega(z) \subset \omega(u^1)$ and (7), we have $w_j \equiv z_j \equiv p_j^1 \equiv p_j$ for $j = m+1, \ldots, n$. Let $z_I(t) = (z_1(t), \ldots, z_m(t))$ be the solution of (8) with the initial value $z_I = (z_1, \ldots, z_m)$ and $\omega_I(z_I)$ the ω-limit set of $z_I(t)$. Then $\omega_I(z_I) = \{w_I : w \in \omega(z)\}$, hence,

$$p_I^1 < \omega_I(z_I). \tag{11}$$

We will show p_I^1 is not the unique equilibrium of (10) in $p_I^1 + \mathbb{R}_+^m$. If this is not the case, by Jiang [**5**], we conclude that p_I^1 is globally asymptotically stable in $p_I^1 + \mathbb{R}_+^m$ according to (10). Then we can find an m-dimensional constant vector m_I such that $z_I \leqslant_I m_I$, thus $p_I^1 \leqslant_I z_I(t) \leqslant m_I(t)$ for all $t \geqslant 0$. Therefore, $\omega_I(z_I) = p_I^1$, contradicting (11). We have proved that p_I^1 is not the unique equilibrium of (10) in $p_I^1 + \mathbb{R}_+^m$. Then there exists an equilibrium w_I of (10) satisfying $p_I^1 <_I w_I$. Suppose that p_I^1 is an isolated equilibrium of system (10) from above. Then we can find an equilibrium of (10), still denoted by w_I, such that there exists no equilibrium of (10) between p_I^1 and w_I. It is easy to see that $w = (w_I, p_J^1)$ is a constant equilibrium of (9). Therefore, there exists no constant equilibrium of (9) between p^1 and w. Applying Proposition 2.3(i) to the corresponding ordinary differential equation system of (9), we obtain that every constant equilibrium of (9) in $[p^1, w]$ is Liapunov stable, which contradicts Proposition 2.2. We have proved the assertion.

From the assertion, we can find an equilibrium w_I of (10) satisfying $p_I^1 < w_I \ll_I z_I$. Let $w = (w_I, p_J)$, then $p^1 < w < z$. It follows from the monotonicity that $p^1 < w(t) < z(t)$ for all $t \geqslant 0$. Since $z_j(t) \equiv p_j^1$ for all $t \geqslant 0$ and $j = m+1, \ldots, n$, $w_j(t) \equiv p_j^1$ for all $t \geqslant 0$ and $j = m+1, \ldots, n$. This shows that $w_I(t)$ is the solution of (8) with initial value w_I. Noticing that w_I is an equilibrium of (8), we have $w_I(t) \equiv w_I$ for all $t \geqslant 0$. Thus $w \in E_o$. Since $z \in \omega(u^1)$, $w_I \ll_I z_I$ and (7) holds, we conclude that $w \leqslant u^1(t)$ for all sufficiently large t, which implies that $w \leqslant \omega(u^1)$. Then $w \leqslant p^1$, a contradiction. Thus we have proved Claim 2.

We can use the method in the proof of Claim 2 repeatedly to obtain a sequence $u^i \in \omega(u)$ such that

$$L(u^{i+1}, p^{i+1}) < L(u^i, p^i) < L(u, p) \leqslant n - 1$$

for $i = 1, 2, \ldots$, where $p^i \in E_o$ which are obtained by Lemma 3.1. Since $L(u^i, p^i)$ is a positive integer not more than $n-1$ for every i, $L(u^i, p^i) = 0$ for sufficiently large i. Then $\omega(u) \cap E_o \neq \emptyset$, contradicting (6). Therefore, $\omega(u)$ is a singleton. This completes the proof. □

PROPOSITION 3.2. *$\omega(u)$ is a singleton for any $u \gg 0$.*

Proof. Let $m = m(E_o) = \max\{\sigma(p) : p \in E_o\}$, where $\sigma(p) = \#\{i : p_i >_i 0_i\}$. It follows from the type-K property that $\sigma(p) = \#\{i : p_i \gg_i 0_i\}$. If $m = n$, then the conclusion is obtained by Proposition 3.1. Therefore, we assume that $m < n$. Then there exist $p = (p_1, p_2, \ldots, p_n) \in E_o$ and an index subset $I \subset N$, $\#I = m$ such that $p_i \gg_i 0_i$ for all $i \in I$, $p_j \equiv 0$ for $j \in C(I)$. Without loss of generality, we assume that $I = \{1, 2, \ldots, m\}$. Then $p = (p_1, \ldots, p_m; 0, \ldots, 0)$. It is easy to prove that $E_o \subset H_I^+$.

Claim A. For any $u \gg 0$, $\omega(u) \subset E_o$.

It follows from Lemma 3.1 that there exist $r, q \in E_o$ such that $r \leqslant \omega(u) \leqslant q$. Therefore, $\omega(u) \subset H_I^+$. Since $u \gg 0$, there is $s > 0$ sufficiently small such that $0 < z := g(s) * p \ll u$. By Jiang [**7**], we obtain $z(t) \geqslant z$ for all $t \geqslant 0$. Therefore, $z + X_+$ is positive invariant and $\omega(u) \subset z + X_+$. Noticing that $z_I \gg_I 0$, we now obtain $\omega(u) \subset \operatorname{Int} H_I^+$. If Claim A is false, we can find $y \in \omega(u)$ such that $y \notin E_o$. Let $y(t)$ ($t \geqslant 0$) be the solution of (1) with the initial value y. Then $y(t) = (y_1(t), \ldots, y_m(t); 0, \ldots, 0)$ and $y_i(t) \geqslant_i z_i \gg_i 0$ for all $t \geqslant 0$ and $i = 1, \ldots, m$. Let $p_{m+1}^1 \equiv \ldots \equiv p_n^1 \equiv 0$ in partial differential equation system (8). Then $y_I(t) = (y_1(t), \ldots, y_m(t))$ is a solution of (8). Since (8) has a positive constant equilibrium $p = (p_1, \ldots, p_m)$ and system (8) satisfies hypotheses (P) and (H), we can apply Proposition 3.1 to (8) and conclude that every solution of (8) with the positive initial value converges to a constant equilibrium of (8).

It follows from the total invariance of $\omega(u)$ that there exist $y^{-t} \in \omega(u)$ such that $y^{-t}(t) = y$ for all $t \geqslant 0$. By the compactness of $\omega(u)$, we may assume that $y^{-n} \to w \in \omega(u)$ as $n \to \infty$, where the convergence is the uniform convergence in $\bar{\Omega}$. Therefore, $y_I^{-n}(n) = y_I$ and $y_I^{-n} \to w_I$ ($n \to \infty$). Obviously, $w \in \omega(u) \subset \operatorname{Int} H_I^+$, then $w_I \gg_I 0$, hence $w_I(t) \to v_I \gg_I 0$ ($t \to +\infty$) by the result obtained in the previous paragraph, where v_I is a constant equilibrium of (8). It follows from Proposition 2.3(i) that v_I is Liapunov stable. Since $y \notin E_o$, we get $\|y_I - v_I\| = \mu > 0$. On the other hand, given any $\varepsilon > 0$, $\|w_I(t_m) - v_I\| < \varepsilon/2$ for t_m sufficiently large. For each $t_m > 0$, it follows from the dependence on initial conditions that we can find n_m ($> t_m$)

sufficiently large such that $\|y_I^{-n_m}(t_m) - w_I(t_m)\| < \varepsilon/2$. Therefore,

$$\|y_I^{-n_m}(t_m) - v_I\| < \varepsilon \tag{12}$$

for $n_m > t_m$ and $m = 1, 2, \ldots$.

Since v_I is Liapunov stable according to (8), for $\mu/2 > 0$, we can choose $\varepsilon > 0$ sufficiently small such that (12) holds and

$$\|(y_I^{-n_m}(t_m))(t) - v_I\| < \mu/2 \quad \text{for all } t \geqslant 0$$

where $(y_I^{-n_m}(t_m))(t)$ denotes the solution of (8) with the initial value $y_I^{-n_m}(t_m)$. Therefore,

$$\|y_I^{-n_m}(t_m + t) - v_I\| < \mu/2 \quad \text{for all } t \geqslant 0.$$

Let $t = n_m - t_m > 0$, we obtain $\|y_I^{-n_m}(n_m) - v_I\| < \mu/2$, that is, $\|y_I - v_I\| < \mu/2$, a contradiction. Thus we have proved Claim A.

Claim B. For any $u \gg 0$, $\omega(u)$ is a singleton.

If this is not the case, it follows from Claim A that there exist two distinct points y, $z \in \omega(u) \subset E_o$. Without loss of generality, we can assume that $I_0 = \{i : y_i <_i z_i\} \neq \emptyset$. Let $\tau = \max\{y_i/z_i : i \in I_0\}$. Then $0 < \tau < 1$, hence there exists $\tau_0 \in (0, 1)$ such that $g_i(s) > \tau$ for all $s \in (\tau_0, 1)$ and $i \in N$. Let $v = z * g(s)$, then we have

(i) $v + \mathbb{R}_+^n$ is positive invariant;

(ii) $y \notin v + \mathbb{R}_+^n$;

(iii) $z_I \in v_I + \operatorname{Int} \mathbb{R}_+^m$, where $I = \{i : z_i >_i 0\}$ and $m = \#I$.

It follows from $z \in \omega(u)$ and (iii) that there exists t_0 sufficiently large satisfying $u(t_0) \geqslant v$. Therefore, by (i), $\omega(u) \subset v + \mathbb{R}_+^n$, contradicting (ii). This completes the proof. □

Proof of Theorem 3.1. Ifss there exists $t_0 > 0$ such that $u(t_0) \gg 0$, then we can apply Proposition 3.2 to obtain the theorem. Therefore, we assume that $u(t) \in \partial X_+$. Let $m = \max\{\sigma(u(t)) : t \geqslant 0\}$, where $\sigma(y) = \#\{i : y_i >_i 0_i\}$. Then $m < n$. Hence there exist some $t_0 > 0$ and $I \subset N$, $\#I = m$ such that $u_i(t_0) >_i 0_i$ for $i \in I$, $u_j(t_0) \equiv 0$ for $j \in C(I)$. Given any $\bar{t} > t_0$, it follows from the definition of m and the type-K condition that $u_I(t) \gg_I 0_I$, $u_J(t) \equiv 0$ for all $t \geqslant \bar{t}$. Therefore, $u_I(t)$ $(t \geqslant \bar{t})$ is the solution of the partial differential equation system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i \Delta u_i + F_i(u_1, \ldots, u_m; 0, \ldots, 0) & t > \bar{t},\ x \in \Omega, \\ \dfrac{\partial u_i}{\partial n} = 0 & t > \bar{t},\ x \in \partial\Omega,\ i = 1, 2, \ldots, m, \\ u_i(x, \bar{t}) \gg_i 0 & x \in \bar{\Omega}. \end{cases} \tag{13}$$

Since (13) satisfies hypotheses (H) and (P), we apply Proposition 3.2 to (13) and conclude that $u_I(t) \to v_I$ $(t \to +\infty)$, where v_I is a constant equilibrium of (13). Therefore, $u(t) = (u_I(t), 0_J) \to (v_I, 0_J)$ as $t \to +\infty$ and this completes the proof. □

4. *Lotka–Volterra systems*

In this section, we will discuss the following Lotka–Volterra systems

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \Delta u_i + u_i\left(r_i + \displaystyle\sum_{j=1}^{n} a_{ij}u_j\right) & t > 0,\ x \in \Omega, \\ \dfrac{\partial u_i}{\partial n} + \alpha u_i = 0 & t > 0,\ x \in \partial\Omega,\ i = 1,2,\dots,n, \\ u_i(x,0) = u_{i,0}(x) \geqslant 0 & x \in \bar{\Omega}, \end{cases} \tag{14}$$

where $r_i > 0$, $\alpha \geqslant 0$, and $a_{ij} \geqslant 0$ for $i \neq j$.

In case $\alpha = 0$, (14) has very simple asymptotic behaviour. More precisely, we have the following theorem.

THEOREM 4.1. *For the Lotka–Volterra system* (14) *with* $\alpha = 0$, *we have the following:*

(i) *If* $A = (a_{ij})_{n\times n}$ *is a stable matrix* (*that is, for* A *to have all its eigenvalues in the open left half plane*), *then every solution with the initial value in* X_+ *converges to some constant equilibrium. Furthermore, there exists a unique positive constant equilibrium attracting all the solution* $u(t)$ *with initial value* $u_i >_i 0_i$ *for all* $i = 1,2,\dots,n$.

(ii) *If* A *is an unstable matrix, then every solution* $u(t) \to +\infty$ *as* $t \to \eta(u)^-$ *for every* u *satisfying* $u_i >_i 0_i$ *for all* $i = 1,2,\dots,n$, *where* $[0,\eta(u))$ *is the maximal existence interval of* $u(t)$.

Proof. (i) If A is a stable matrix, then it follows from [**12**] that hypothesis (H) is satisfied. Let $g(s) = s$ and $h(s) = s^2$ in hypothesis (P). Then it is easy to see that (14) satisfies hypothesis (P) and hence the convergent result follows from Theorem 3.1.

In this case, Smith [**12**] showed that, for the corresponding ordinary differential equation system of (14), there exists a unique equilibrium $p \gg 0$ which is globally (with respect to positive initial conditions) asymptotically stable. Thus if the initial value u of (14) satisfys $u_i >_i 0_i$ for all $i = 1,2,\dots,n$, then the type-K condition implies that $u(\delta) \gg 0$ for any given $\delta > 0$ and hence there exist positive vectors M and V such that $0 \ll M \leqslant u(\delta) \leqslant V$, so by the monotonicity and the result of Smith [**12**], $u(t) \to p$ as $t \to \infty$.

(ii) If u satisfying $u_i >_i 0_i$ for all $i = 1,2,\dots,n$ is such that $\eta(u) < \infty$, then clearly $u(t) \to +\infty$ as $t \to \eta(u)^-$. Let u satisfy $u_i >_i 0_i$ for all $i = 1,2,\dots,n$ be such that $\eta(u) = \infty$. Then we can argue as in the previous paragraph to obtain $0 \ll M \leqslant u(\delta)$ and hence $\eta(M) = \infty$. It follows from Smith [**12**] that $M(t) \to +\infty$ as $t \to \infty$. Hence $u(t) \to +\infty$ as $t \to \infty$ by the monotonicity. This completes the proof. □

REMARK 4.1. Since our hypotheses are such that each subsystem of (14), obtained by setting some $u_I \equiv 0$ $(I \subset N)$ inherits the property (H) and (P), this result can be applied to each subsystem. Therefore, it turns out that the asymptotic behaviour of a solution with initial condition lying on a boundary of X_+, can be completed described as well.

In case $\alpha > 0$, we cannot argue as in the proof of Theorem 4.1, because (14) cannot have the corresponding ordinary differential equation system any more. Hence we need new methods to study the asymptotic behaviour of (14) with $\alpha > 0$.

In (14), it is biologically and mathematically reasonable to assume that $a_{ii} < 0$ $(i \in N)$ and $a_{ii} > 0$ $(i \neq j)$. Furthermore, let

$$F(u) = \operatorname{diag}(u_1, \ldots, u_n)(r + Au),$$

where $A = (a_{ij})$, $r = (r_1, \ldots, r_n)^{\mathrm{T}}$ and $u = (u_1, \ldots, u_n)^{\mathrm{T}}$.

LEMMA 4.1. *Suppose that* $|a_{ii}| > \sum_{j\neq i} a_{ij}$. *Then every solution of* (14) *with initial value in* X_+ *is bounded and can be continued to infinity. Therefore, the closure of every semi-orbit is precompact.*

Proof. Let $b = \max_{1\leqslant i\leqslant n}\{r_i/(|a_{ii}| - \sum_{j\neq i} a_{ij})\}$. Then $B = (b, \ldots, b) \gg 0$. For any $l \geqslant 1$, we assert that $[0, lB]$ is positive invariant. Indeed, whenever $u \in [0, lB]$ satisfies $u_i = lb$, then

$$\begin{aligned} F_i(u) &= lb\left(r_i + \sum_{j\neq i} a_{ij}u_j - l|a_{ii}|b\right) \\ &\leqslant lb\left[r_i + lb\left(\sum_{j\neq i} a_{ij} - |a_{ii}|\right)\right] \leqslant 0. \end{aligned}$$

Hence the assertion follows from [**13**, Theorem 7.3.1].

Now given any $u \in X_+$, there exists some $l \geqslant 1$ such that $0 \leqslant u \leqslant lB$. Hence the positive invariance of $[0, lB]$ implies that $0 \leqslant u(t) \leqslant lB$ for all $t \geqslant 0$. That is, $u(t)$ is bounded and can be continued to infinity. Also, by [**13**, Theorem 7.3.1(iv)], we obtained that the closure of every semi-orbit is precompact. This completes the proof. □

Let $\lambda_0 < 0$ be the negative principal eigenvalue of the system

$$\begin{cases} \lambda w = \Delta w & x \in \Omega, \\ \dfrac{\partial w}{\partial n} + \alpha w = 0 & x \in \partial\Omega, \end{cases}$$

and w be the normalized positive eigenfunction to λ_0. Then we have $W = (w, \ldots, w) \gg 0$ and the following lemma.

LEMMA 4.2. *Suppose that* $\lambda_0 + \min_{1\leqslant i\leqslant n}\{r_i\} > 0$. *Then there exists* $\varepsilon_0 > 0$ *such that* εW *is a strict sub-solution of* (14) *for any* $0 < \varepsilon \leqslant \varepsilon_0$.

Proof. We only need to compute

$$\begin{aligned} \Delta(\varepsilon W) + F(\varepsilon W) &= \varepsilon\Delta W + DF(0)\cdot \varepsilon W + o(\varepsilon)w \\ &= \varepsilon(\Delta W + DF(0)W) + o(\varepsilon)w \\ &= \varepsilon[\operatorname{diag}(\lambda_0 + r_1, \ldots, \lambda_0 + r_n)W] + o(\varepsilon)w. \end{aligned}$$

Observe that if $\lambda_0 + \min_{1\leqslant i\leqslant n}\{r_i\} > 0$, then it follows that $\Delta(\varepsilon W) + F(\varepsilon W) > 0$ if $\varepsilon > 0$ is sufficiently small, which completes the proof. □

LEMMA 4.3. *Let the hypotheses of Lemma* 4.1 *and Lemma* 4.2 *be satisfied. Then there exist two equilibria* p, q *satisfying* $0 \ll \varepsilon_0 W \leqslant p \leqslant q \leqslant B$ *such that* $(\varepsilon W)(t) \to p$ *as* $t \to \infty$ *for any* $0 < \varepsilon \leqslant \varepsilon_0$, $(lB)(t) \to q$ *as* $t \to \infty$ *for any* $l \geqslant 1$.

Proof. First we assert that lB $(l \geqslant 1)$ is a strict super-solution of (14). Indeed,

$$\Delta(lb) + F_i(lB) = lb\left(r_i + lb\sum_{j\neq 1} a_{ij} - l|a_{ii}|b\right) < 0 \quad i \in N,\ l \geqslant 1,$$

and

$$\frac{\partial(lB)}{\partial n} + \alpha(lB) = \alpha lB > 0,$$

which implies the assertion. Therefore, it follows from [**13**, Corollary 7.3.6] and Lemma 4.1 above that $(\varepsilon_0 W)(t) \to p$ $(t \to \infty)$ and $(lB)(t) \to q_l$ as $t \to \infty$ for any $l \geqslant 1$, where p, q_l are the equilibria of (14) and $0 \ll p \leqslant q_l$. Since the semi-flow in $p + X_+$ is strongly monotone, $(lB) \gg q_l \geqslant p$ for all $l \geqslant 1$.

Given any $l > 1$, we define

$$Q = \{a : 0 \leqslant a \leqslant l-1 \text{ and } (l-a)B \geqslant q_l\}.$$

Obviously, $0 \in Q$ and q is a close subset of $\mathbb{R}_+$ satisfying the property that $[0,a] \subset Q$ whenever $a \in Q$. Therefore, $Q = [0, a^*]$. Furthermore, if $a \in [0, a^*]$ then $(l-a)B \geqslant q_l$, which implies that $((l-a)B)(t) \geqslant q_l$ for all $t \geqslant 0$. Hence $q_{l-a} \geqslant q_l$. On the other hand, $q_{l-a} \leqslant q_l$. Thus we obtain

$$q_{l-a} = q_l \quad \text{whenever } a \in Q = [0, a^*]. \tag{15}$$

Now we claim that $a^* = l-1$. Suppose that $a^* < l-1$. It follows that $(l-a^*)B \gg q_{l-a^*} = q_l$, where the second equality follows from (15). Hence if $\delta > 0$ is sufficiently small, then $a^* + \delta \in Q$. This produces a contradiction since $Q = [0, a^*]$.

We have now shown that $Q = [0, l-1]$. Therefore it follows from (15) that $q_1 = q_{l-(l-1)} = q_l$. That is, $(lB)(t) \to q_1$ as $t \to \infty$. By the arbitraryness of l, we have $(lB)(t) \to q_1$ as $t \to \infty$ for all $l \geqslant 1$. Thus we have found $q = q_1 = \omega(B)$.

Similarly, we can also argue as in the previous paragraphs to show the existence of p such that $(\varepsilon W)(t) \to p$ as $t \to \infty$ for any $0 < \varepsilon \leqslant \varepsilon_0$. This completes the proof. □

LEMMA 4.4. *Suppose that $|a_{ii}|$ is sufficiently large for all $i = 1, \dots, n$. Then $p \equiv q$.*

Proof. Let $v = p - q$. Then $v \leqslant 0$. On the other hand, since p, q are the equilibria of (14), they satisfy

$$\Delta p_i + p_i\left(r_i - |a_{ii}|p_i + \sum_{j\neq i} a_{ij}p_j\right) = 0,$$

$$\Delta q_i + q_i\left(r_i - |a_{ii}|q_i + \sum_{j\neq i} a_{ij}q_j\right) = 0$$

$(i = 1, 2, \dots, n)$ and the corresponding boundary condition. Therefore, we have

$$\begin{aligned} 0 &= \Delta v_i + r_i v_i + |a_{ii}|(q_i^2 - p_i^2) + p_i\sum_{j\neq i} a_{ij}p_j - q_i\sum_{j\neq i} a_{ij}q_j \\ &= \Delta v_i + r_i v_i - |a_{ii}|(p_i + q_i)v_i + v_i\sum_{j\neq i} a_{ij}p_j + q_i\sum_{j\neq i} a_{ij}v_j. \end{aligned}$$

Hence

$$-\Delta v_i + \left[-r_i + |a_{ii}|(p_i + q_i) - \sum_{j\neq i} a_{ij} p_j \right] v_i - q_i \sum_{j\neq i} a_{ij} v_j = 0 \quad i = 1, 2, \ldots, n. \quad (16)$$

Note that if

$$-r_i + |a_{ii}|(p_i + q_i) - \sum_{j\neq i} a_{ij} p_j - q_i \sum_{j\neq i} a_{ij} > 0 \quad \text{for all } i = 1, 2, \ldots, n, \quad (17)$$

then, by the maximum principle (cf. [**10**, Theorem 3.15]), we obtain $v_i \geqslant 0$ for all $i = 1, 2, \ldots, n$. Hence $p \equiv q$. In order to prove that (17) holds, it is sufficient to show that

$$2|a_{ii}|\varepsilon_0 w - r_i - 2b \sum_{j\neq i} a_{ij} > 0 \quad \text{for all } i = 1, 2, \ldots, n. \quad (18)$$

Thus it follows from the definition of b that if $|a_{ii}|$ is sufficiently large for all $i \in N$, then (18) holds. This completes the proof. □

THEOREM 4.2. *Suppose that* $\lambda_0 + \min_{1\leqslant i\leqslant n}\{r_i\} > 0$, $|a_{ii}| > \sum_{j\neq i} a_{ij}$ *and* $|a_{ii}|$ *is sufficiently large for all* $i = 1, \ldots, n$. *Then there exists a unique positive equilibrium attracting all the solution* $u(t)$ *with initial value* $u_i >_i 0_i$ *for all* $i = 1, 2, \ldots, n$.

Proof. We can argue as in the proof of Theorem 4.1 that $u(\delta) \gg 0$ for any given $\delta > 0$. Therefore, we can find some $\varepsilon > 0$ and $l > 1$ such that $0 \ll \varepsilon W \leqslant u(\delta) \leqslant lB$. By the monotonicity and Lemmas 4.3 and 4.4, we have $u(\delta) \to p \equiv q$ as $t \to +\infty$. This completes the proof. □

The following theorem shows that if r_i is sufficiently small, then 0 is globally asymptotically stable.

THEOREM 4.3. *Suppose that* $|a_{ii}| > \sum_{j\neq i} a_{ij}$. *Let* $\phi \in C(\bar{\Omega}, \mathbb{R})$ *be the solution of the boundary problem*

$$\begin{cases} -\Delta\phi = \lambda\phi + 1 & x \in \Omega, \\ \dfrac{\partial\phi}{\partial n} + \alpha\phi = 0 & x \in \partial\Omega, \end{cases}$$

where $0 < \lambda < -\lambda_0$. *Then if* $\max_{1\leqslant i\leqslant n}\{r_i\} \leqslant 1/\|\phi\|_\infty$, *then the zero solution of* (14) *is globally asymptotically stable.*

Proof. Let $\Phi = (\phi, \ldots, \phi)$. Since it is well known that ϕ is a positive function, we have $\Phi \gg 0$. Let us consider $\tilde{u}(x,t) = ce^{-\lambda t}\Phi$, where $c > 0$ and $0 < \lambda < -\lambda_0$. Thus we have

$$\frac{\partial \tilde{u}_i(x,t)}{\partial t} - \Delta\tilde{u}_i(x,t) - F_i(\tilde{u}(x,t)) = ce^{-\lambda t}\left\{1 - \phi\left[r_i - \left(|a_{ii}| - \sum_{j\neq i} a_{ij}\right)ce^{-\lambda t}\phi\right]\right\}$$
$$\geqslant ce^{-\lambda t}(1 - \phi r_i) \geqslant 0,$$

for all $i = 1, 2, \ldots, n$. Thus we have shown that $\tilde{u}(x,t)$ is a super-solution of (14). For any initial value $u \in X_+$, there exists $c > 0$ such that $0 \leqslant u \leqslant c\Phi = \tilde{u}(x, 0)$. By [**13**, Theorem 7.3.4], $0 \leqslant u(t) \leqslant \tilde{u}(x,t) = ce^{-\lambda t}\Phi$. Let $t \to +\infty$, we obtain $u(t) \to 0$. Thus we complete the proof. □

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