Benchmark problems in stability and design of switched systems*

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Abstract

A switched system is a hybrid dynamical system consisting of a family of continuous-time subsystems and a rule that governs the switching between them. This paper surveys recent developments in three basic problems regarding stability and design of switched systems. These problems are: stability for arbitrary switching sequences, stability for certain useful classes of switching sequences, and construction of stabilizing switching sequences. We also provide motivation for studying these problems by discussing how they arise in connection with various questions of interest in control theory and applications.

1 Introduction

By a switched system we mean a hybrid dynamical system consisting of a family of continuous-time subsystems and a rule that orchestrates the switching between them. Many systems encountered in practice exhibit switching between several subsystems that is dependent on various environmental factors. Some examples of such systems are discussed in [7, 54]. Another source of motivation for studying switched systems comes from the rapidly developing area of intelligent control. The methods of intelligent control design are based on the idea of switching between different controllers. These control techniques have been applied extensively in recent years, particularly in the adaptive context, where they have been shown to achieve stability and improve transient response (see, among many references, [21, 31, 34]). The importance of such control methods also stems in part from the existence of systems that cannot be asymptotically stabilized by a single smooth feedback control law [5].

Mathematically, a *switched system* can be described by a differential equation of the form

$$\dot{x} = f_\sigma(x) \tag{1}$$

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where $\{f_p : p \in \mathcal{P}\}$ is a family of sufficiently regular functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is parameterized by some index set $\mathcal{P}$, and $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant *switching signal*. In specific situations, the value of σ at a given time t might just depend on t or $x(t)$ or both, or may be generated using more sophisticated techniques such as hybrid feedback with memory in the loop. The set $\mathcal{P}$ is typically a compact (often finite) subset of a normed linear vector space.

In the particular case when all the individual subsystems are linear, we obtain a *switched linear system*

$$\dot{x} = A_\sigma x. \tag{2}$$

This class of systems is the one most commonly treated in the literature. In this paper, whenever possible, problems will be formulated and discussed in the more general context of the switched system (1).

The first basic problem that we will consider can be formulated as follows.

Problem A. *Find conditions that guarantee that the switched system (1) is asymptotically stable for any switching signal.*

Clearly, a necessary condition for (asymptotic) stability under arbitrary switching is that all of the individual subsystems are (asymptotically) stable. Indeed, if the p-th system is unstable, the switched system will be unstable if we set $\sigma(t) \equiv p$. To see that this condition is not sufficient, consider two second-order asymptotically stable systems whose trajectories are sketched in the top row of Figure 1. Depending on a particular switching signal, the trajectories of the switched system might look as shown in the bottom left corner (asymptotically stable) or as shown in the bottom right corner (unstable).

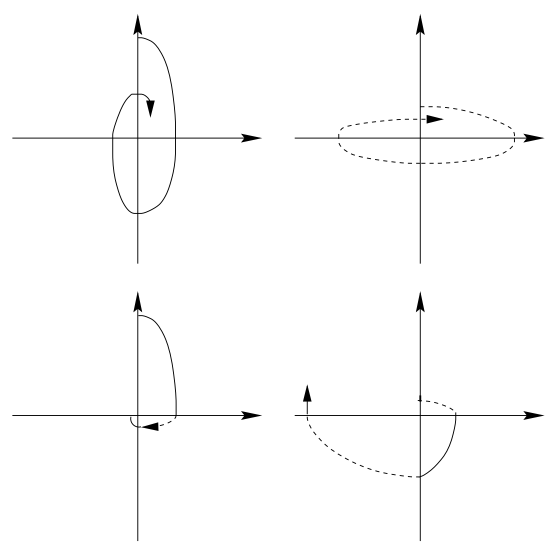

Figure 1: Possible trajectories of a switched system

The above example shows that Problem A is not trivial in the sense that it is possible to get instability by switching between asymptotically stable systems.[1] If this happens, one may ask

[1]However, there are certain limitations as to what kind of instability one can have in this case. For example, it is easy to see that the trajectories of such a switched system cannot escape to infinity in finite time.

whether the switched system will be asymptotically stable for certain useful classes of switching signals. This leads to the following problem.

Problem B. *Identify those classes of switching signals for which the switched system (1) is asymptotically stable.*

Since it is usually unreasonable to exclude constant switching signals of the form $\sigma(t) \equiv p$, Problem B will be considered under the assumption that all the individual subsystems are asymptotically stable. However, in many applications it is difficult to ensure that this assumption is satisfied. A more realistic goal might be to find a particular switching signal that drives the state of the system to zero in spite of the fact that some (or all) of the individual subsystems are unstable. In other words, we can formulate the following problem.

Problem C. *Construct a switching signal that makes the switched system (1) asymptotically stable.*

Of course, if at least one of the individual subsystems is asymptotically stable, the above problem is trivial (just keep $\sigma(t) \equiv p$ where p is the index of this stable system). Therefore, in the context of Problem C it will be understood that none of the individual subsystems are asymptotically stable.

The last problem is more of a design problem than a stability problem, but the above discussion illustrates that all three problems are closely related. In what follows, we will give an exposition of recent results that address these problems. We will also try to motivate the study of these problems by discussing how they are related to different areas of control theory and applications. To make the paper more accessible, we present many ideas and results on the intuitive level and refer the reader to the literature for technical details. Open questions are pointed out throughout. The Appendix provides proofs of two results whose sources were not readily available at the time when this paper was written.

2 Stability for arbitrary switching

One situation in which Problem A is of great importance is when a given process is being controlled by means of switching among a family of stabilizing controllers, each of which is designed for a specific task. Stability of the switched system can usually be ensured by keeping each controller in the loop for a long enough time, so as to allow the transient effects to dissipate (cf. Section 3 below). However, modern computer-controlled systems are capable of very fast switching rates, which creates the need to be able to test stability of the switched system for arbitrarily fast switching signals.

It is well known that if the family of systems

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P} \tag{3}$$

has a common Lyapunov function, then the switched system (1) is asymptotically stable for any switching signal σ. Hence, one possible approach to Problem A is to find conditions under which there exists a common Lyapunov function for the family (3).

In the next two subsections we discuss various results on common Lyapunov functions and stability for arbitrary switching. The last subsection is devoted to converse Lyapunov theorems. Our discussion throughout the paper is restricted to state space methods. For some frequency domain results the reader may consult [10, Chapter 3] where it is shown that if a linear process and a family of linear controllers are given by their transfer matrices, then there always exist realizations such that the family of feedback connections of the process with the controllers possesses a quadratic common Lyapunov function.

2.1 Lie-algebraic conditions

Let us start by considering the family of linear systems

$$\dot{x} = A_p x, \qquad p \in \mathcal{P} \tag{4}$$

such that the matrices A_p are stable (i.e., with eigenvalues in the open left half of the complex plane) and the set $\{A_p : p \in \mathcal{P}\}$ is compact in $\mathbb{R}^{n\times n}$. If all the systems in this family share a quadratic common Lyapunov function, then the switched linear system (2) is globally uniformly exponentially stable (the word "uniform" is used here to describe uniformity with respect to switching signals). This means that if there exist two symmetric positive definite matrices P and Q such that we have

$$A_p^T P + P A_p \leq -Q \qquad \forall p \in \mathcal{P}$$

then there exist positive constants c and μ such that the solution of (2) for any initial state $x(0)$ and any switching signal σ satisfies

$$\|x(t)\| \leq c e^{-\mu t}\|x(0)\| \qquad \forall t \geq 0. \tag{5}$$

In this subsection we present sufficient conditions for the existence of a quadratic common Lyapunov function that involve the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ generated by the individual matrices A_p. First we recall some definitions. Given a Lie algebra g, the sequence $g^{(k)}$ is defined inductively as follows: $g^{(1)} := g$, $g^{(k+1)} := [g^{(k)}, g^{(k)}] \subset g^{(k)}$. If $g^{(k)} = 0$ for k sufficiently large, then g is called *solvable*. Similarly, one defines the sequence g^k by $g^1 := g$, $g^{k+1} := [g, g^k] \subset g^k$, and calls g *nilpotent* if $g^k = 0$ for k sufficiently large. For example, if g is a Lie algebra generated by two matrices A_1 and A_2, i.e., $g = \{A_1, A_2\}_{LA}$, then we have: $g^{(1)} = g^1 = g = \text{span}\{A_1, A_2, [A_1, A_2], [A_1, [A_1, A_2]], \dots\}$, $g^{(2)} = g^2 = \text{span}\{[A_1, A_2], [A_1, [A_1, A_2]], \dots\}$, $g^{(3)} = \text{span}\{[[A_1, A_2], [A_1, [A_1, A_2]]], \dots\} \subset g^3 = \text{span}\{[A_1, [A_1, A_2]], [A_2, [A_1, A_2]], \dots\}$, and so on. Every nilpotent Lie algebra is solvable, but the converse is not true.

The simplest case is when $\mathcal{P}$ is a finite set (say, $\mathcal{P} = \{1, \dots, m\}$) and the matrices in the family (4) commute pairwise, i.e., the Lie bracket $[A_p, A_q] := A_p A_q - A_q A_p$ equals zero for all $p, q \in \mathcal{P}$. It is well known and easy to show that in this case the system (2) is asymptotically stable for any switching signal σ. An explicit construction of a quadratic common Lyapunov function for a finite commuting family of linear systems is given in [33].

Proposition 1 [33] *Let $P_1, \dots, P_m$ be the unique symmetric positive definite matrices that satisfy the Lyapunov equations*

$$\begin{aligned} A_1^T P_1 + P_1 A_1 &= -I, \\ A_p^T P_p + P_p A_p &= -P_{p-1}, \qquad p = 2, \dots, m. \end{aligned}$$

Then the function $V(x) := x^T P_m x$ is a common Lyapunov function for the systems $\dot{x} = A_i x$, $i = 1, \dots, m$.

The matrix P_m is given by the formula

$$P_m = \int_0^\infty e^{A_m^T t_m} \dots \left(\int_0^\infty e^{A_1^T t_1} e^{A_1 t_1} dt_1 \right) \dots e^{A_m t_m} dt_m$$

Since the matrices A_i commute, for each $i \in \{1, \dots, m\}$ we can rewrite this in the form

$$P_m = \int_0^\infty e^{A_i^T t_i} \, Q_i \, e^{A_i t_i} \, dt_i$$

with $Q_i > 0$, which makes the statement of Proposition 1 obvious.

The connection between asymptotic stability of a switched linear system and the properties of the corresponding Lie algebra was apparently discussed for the first time by Gurvits in [9]. That paper is concerned with the discrete-time counterpart of (2) which takes the form

$$x(k+1) = A_{\sigma(k)} x(k). \tag{6}$$

where σ is a function from nonnegative integers to a finite index set $\mathcal{P}$. Gurvits conjectured that if the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ is nilpotent then (6) is asymptotically stable for any such switching signal σ. He used the Baker-Campbell-Hausdorff formula to prove this conjecture for the particular case when $\mathcal{P} = \{1, 2\}$, the matrices A_1 and A_2 are nonsingular, and their third-order Lie brackets vanish: $[A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$.

It was recently shown in [22] that if the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the family (4) possesses a quadratic common Lyapunov function. One can derive the corresponding statement for the discrete-time case in a similar fashion, thereby confirming and directly generalizing the above conjecture because every nilpotent Lie algebra is solvable. The proof of the result given in [22] is based on the following well known fact that can be found in most textbooks on the theory of Lie algebras (see, e.g., [43]).

Proposition 2 (Lie's Theorem) *Let g be a solvable Lie algebra over an algebraically closed field, and let ρ be a representation of g on a vector space V of finite dimension n. Then there exists a basis $\{v_1, \dots, v_n\}$ of V such that for each $X \in g$ the matrix of $\rho(X)$ in that basis takes the upper-triangular form*

$$\begin{pmatrix} \lambda_1(X) & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n(X) \end{pmatrix}$$

($\lambda_1(X), \dots, \lambda_n(X)$ being its eigenvalues).

In our context this means that if $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, then there exists a nonsingular complex matrix T such that for each $p \in \mathcal{P}$ we have $A_p = T^{-1} B_p T$ where B_p is a complex upper-triangular matrix. It is a relatively simple matter to show that a family of linear systems with stable upper-triangular matrices possesses a quadratic common Lyapunov function—see, e.g., [22, 46, 47] for details (in particular, one can construct a common Lyapunov function that takes the form $x^T D x$ where D is a diagonal matrix). We thus obtain the following result which incorporates the ones mentioned before as special cases.

Theorem 3 [22] *If $\{A_p : p \in \mathcal{P}\}$ is a compact set of stable matrices and the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the switched linear system (2) is globally uniformly exponentially stable.*

Note that while it is a nontrivial matter to find a basis in which all matrices take the triangular form or even decide whether such a basis exists, the Lie-algebraic condition given by Theorem 3 is formulated *in terms of the original data* and can always be checked in a finite number of steps if $\mathcal{P}$ is a finite set. We now briefly discuss implications of this result for switched nonlinear systems of the form (1). Consider, together with the family (3), the corresponding family of linearized systems

$$\dot{x} = F_p x, \quad p \in \mathcal{P}$$

where $F_p = \frac{\partial f_p}{\partial x}(0)$. Assume that the matrices F_p are stable, that $\mathcal{P}$ is a compact set, and that $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p for each $x \in D$. A straightforward application of Theorem 3 and the Lyapunov's first method gives the following result.

Corollary 4 [22] *If the Lie algebra $\{F_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the system (1) is locally uniformly exponentially stable*[2].

Recent work reported in [44] directly generalizes the result and the proof technique of [33] to switched nonlinear systems. Namely, suppose that $\mathcal{P} = \{1, \dots, m\}$, and denote by $\varphi_p(t, z)$ the solution of the system $\dot{x} = f_p(x)$ starting at a point z when $t = 0$. If all these systems are exponentially stable and the corresponding vector fields commute pairwise, i.e., $[f_p, f_q](x) = 0$ for all $p, q \in \mathcal{P}$, then a common Lyapunov function can be constructed by the following iterative procedure:

$$\begin{aligned} V_1(x) &:= \int_0^T |\varphi_1(s, x)|^2 ds, \\ V_p(x) &:= \int_0^T V_{p-1}(\varphi_p(s, x)) ds, \qquad p = 2, \dots, m. \end{aligned}$$

The function V_m is a common Lyapunov function for the family (3) locally in a neighborhood of zero (unless all the functions f_p are globally Lipschitz in which case V_m is a global common Lyapunov function). One therefore has the following statement.

Proposition 5 [44] *If all the systems in the family (3) are exponentially stable, $\mathcal{P}$ is a finite set, and the vector fields $f_p(x)$, $p \in \mathcal{P}$ commute pairwise, then the switched system (1) is locally asymptotically stable for any switching signal σ.*

Note that, unlike the condition of Corollary 4, the above commuting condition is formulated in terms of the Lie algebra generated by the original nonlinear vector fields, which opens interesting new possibilities. It remains to be seen whether Lie-algebraic sufficient conditions for *global* asymptotic stability under arbitrary switching can be found in the general case.

Finally, we comment on the issue of robustness. Both exponential stability and existence of a quadratic common Lyapunov function are robust properties in the sense that they are not destroyed by sufficiently small perturbations of the systems' parameters. Regarding perturbations of upper-triangular matrices, one can obtain explicit bounds that have to be satisfied by the elements below the diagonal so that the quadratic common Lyapunov function for the unperturbed systems remains a common Lyapunov function for the perturbed ones [30]. Unfortunately, the condition of Theorem 3 is not robust, which indicates one direction in which one might try to improve it.

[2]This is to say that the estimate (5) holds for all trajectories starting in a certain neighborhood of the origin.

2.2 Matrix pencil conditions

We now turn to some recently obtained sufficient, as well as necessary and sufficient, conditions for the existence of a quadratic common Lyapunov function for a pair of second-order asymptotically stable linear systems

$$\dot{x} = A_i x, \qquad A_i \in \mathbb{R}^{2\times 2}, \quad i = 1, 2. \tag{7}$$

These conditions, presented in [45, 46], are given in terms of eigenvalue locations of suitable linear combinations of the matrices A_1 and A_2. We refer the reader to [35, 36] for some sufficient conditions for the existence of a quadratic common Lyapunov function that involve Lyapunov matrix operators.

Given two matrices A and B, the *matrix pencil* $\gamma_\alpha(A, B)$ is defined as the one-parameter family of matrices $\alpha A + (1-\alpha)B$, $\alpha \in [0, 1]$. One has the following result.

Proposition 6 [45] *If A_1 and A_2 have real distinct eigenvalues and all the matrices in $\gamma_\alpha(A_1, A_2)$ have negative real eigenvalues, then the pair of linear systems (7) has a quadratic common Lyapunov function.*

In [46] Shorten and Narendra considered, together with the matrix pencil $\gamma_\alpha(A_1, A_2)$, the matrix pencil $\gamma_\alpha(A_1, A_2^{-1})$. This allowed them to obtain, apparently for the first time, a *necessary and sufficient* condition for the existence of a common Lyapunov function.

Proposition 7 [46] *The pair of linear systems (7) has a quadratic common Lyapunov function if and only if all the matrices in $\gamma_\alpha(A_1, A_2)$ and $\gamma_\alpha(A_1, A_2^{-1})$ are stable.*

The above results are limited to a pair of second-order linear systems. It is interesting to note that the conditions of Propositions 6 and 7 are in general robust in the sense specified at the end of Section 2.1. Indeed, the property that all eigenvalues of a matrix have negative real parts is preserved under sufficiently small perturbations. Moreover, if these eigenvalues are real, they will remain real under small perturbations, providing that they are distinct (because eigenvalues of a real matrix come in conjugate pairs).

2.3 Converse Lyapunov theorems

In the preceding subsections we have relied on the fact that the existence of a common Lyapunov function implies asymptotic stability for arbitrary switching signals. The question arises whether the converse holds. A converse Lyapunov theorem for differential inclusions proved by Molchanov and Pyatnitskiy in [29] gives a positive answer to this question. Their result can be adapted to the present setting as follows.

Theorem 8 [29] *If the switched linear system (2) is asymptotically stable for all switching signals, then the family of linear systems (4) has a strictly convex, homogeneous (of second order) common Lyapunov function of a quasi-quadratic form*

$$V(x) = x^T L(x) x$$

where $L(x) = L^T(x) = L(\tau x)$ for all nonzero x and τ.

The construction of such a Lyapunov function given in [29] (see also [7]) proceeds in the same spirit as the classical one that is used to prove standard converse Lyapunov theorems (cf. [19, Theorem 4.5]), except that supremum over all indices $p \in \mathcal{P}$ needs to be taken. It is also shown in [29] that one can find a common Lyapunov function that takes the piecewise quadratic form

$$V(x) = \max_{1 \le i \le k} \langle l_i, x \rangle^2$$

where l_i, $i = 1, \dots, k$ are constant vectors.

Interestingly, a *quadratic* common Lyapunov function does not always exist. In [7] Dayawansa and Martin give an example of two second-order linear systems which do not share any quadratic common Lyapunov function, yet the switched system is asymptotically stable for arbitrary switching. They also generalize Theorem 8 to a class of switched nonlinear systems as follows.

Theorem 9 [7] *If the switched system (1) is globally asymptotically stable and in addition locally exponentially stable for all switching signals, then the family (3) has a common Lyapunov function.*

Some technical properties of this common Lyapunov function are discussed in [7]. However, the problem of determining the specific form of this function remains largely open. The paper [7] announces that the above converse Lyapunov theorem is actually valid without the exponential stability assumption, although the proof is more involved.

3 Stability for slow switching

We have seen above that a switched system might become unstable for certain switching signals even if all the individual subsystems are asymptotically stable. Thus, if the goal is to achieve stability of the switched system, one often has to restrict the class of admissible switching signals. This leads us to Problem B posed in the Introduction. As we already mentioned, one way to address this problem is to make sure that the intervals between consecutive switching times are long enough. Such *slow switching* assumptions greatly simplify the stability analysis and are, in one form or another, ubiquitous in the switching control literature[3] (see, e.g., [15, 37, 51]).

Below we discuss multiple Lyapunov function tools that are useful in analyzing stability of slowly switched systems. We then present stability results for such systems. Some of these results parallel the more familiar ones on stability of slowly time-varying systems (cf. [48] and references therein).

3.1 Multiple Lyapunov functions

In Section 2 we discussed various situations in which asymptotic stability of a switched system for arbitrary switching signals can be established by means of showing that the family of individual subsystems possesses a common Lyapunov function. We also know (Section 2.3) that the existence of a common Lyapunov function is necessary for asymptotic stability under arbitrary switching. However, if the class of switching signals is restricted, this converse result might not hold. In other words, the properties of admissible switching signals can sometimes be used to prove asymptotic stability of the switched system even in the absence of a common Lyapunov function.

[3] Another reason for slow switching is to avoid chattering.

One tool for proving stability in such cases employs *multiple* Lyapunov functions [39, 3, 4, 16]. Fix a switching signal σ with switching times $t_0 < t_1 < \dots$ and assume for concreteness that it is continuous from the right everywhere: $\sigma(t_i) = \lim_{t\to t_i^+} \sigma(t)$ for each i. Since the individual members of the family (3) are assumed to be asymptotically stable, there is a family of Lyapunov functions $\{V_p : p \in \mathcal{P}\}$ such that the value of V_p decreases on each interval where the p-th subsystem is active. If the value of V_p at the end of each such interval exceeds the value at the beginning of the next one (see Figure 2), then the switched system can be shown to be asymptotically stable.

Lemma 10 [39] *If there exists a constant* $\rho > 0$ *such that*

$$V_{\sigma(t_{i+1})}(x(t_{i+2})) - V_{\sigma(t_i)}(x(t_{i+1})) \le -\rho |x(t_{i+1})|^2, \qquad i = 0, 1, \dots \tag{8}$$

then the switched system (1) is globally asymptotically stable.

To see why this is true, observe that the sequence $V_{\sigma(t_i)}(x(t_{i+1}))$, $i = 0, 1, \dots$ is decreasing and positive, and therefore has a limit $L \ge 0$ as $i \to +\infty$. We have

$$\begin{aligned} 0 = L - L &= \lim_{i\to+\infty} V_{\sigma(t_{i+1})}(x(t_{i+2})) - \lim_{i\to+\infty} V_{\sigma(t_i)}(x(t_{i+1})) \\ &= \lim_{i\to+\infty} [V_{\sigma(t_{i+1})}(x(t_{i+2})) - V_{\sigma(t_i)}(x(t_{i+1}))] \le \lim_{i\to+\infty} [-\rho |x(t_{i+1})|^2] \le 0 \end{aligned}$$

which implies that $x(t_i)$ converges to zero. As pointed out in [4], Lyapunov stability should and can be checked via a separate argument.

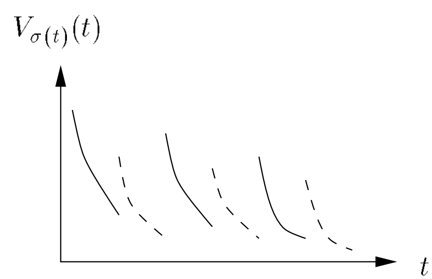

Figure 2: Two Lyapunov functions

Some variations and generalizations of this result are discussed in [3, 4, 16, 40], while the basic idea seems to go back at least to [38]. We will return to multiple Lyapunov function techniques in Section 4. A closely related problem of computing such Lyapunov functions numerically by means of LMIs is addressed in [18, 40].

3.2 Dwell time

The simplest way to specify slow switching is to introduce a number $\tau > 0$ and restrict the class of admissible switching signals to signals with the property that the interval between any two consecutive switching times is no smaller than τ. This number τ is sometimes called the *dwell time* (because σ "dwells" on each of its values for at least τ units of time). It is well known that when all the linear systems in the family (4) are asymptotically stable, the switched linear system (2) is globally exponentially stable if the dwell time τ is large enough. In fact, the required lower bound on τ can be explicitly calculated from the parameters of the individual subsystems. For details, see [31, Lemma 2] or [17, Theorem 3.3].

What is perhaps less well known is that under suitable assumptions a sufficiently large dwell time guarantees asymptotic stability of the switched system in the nonlinear case as well. Arguably the best way to prove most general results of this kind is by using multiple Lyapunov functions. We will not discuss the precise assumptions that are needed here (in fact, there is considerable work still to be done in that regard) but will present the general idea instead. Assume for simplicity that all the systems in the family (3) are globally exponentially stable. Then for each $p \in \mathcal{P}$ there exists a Lyapunov function V_p that for some positive constants a_p, b_p and c_p satisfies

$$a_p|x|^2 \leq V_p(x) \leq b_p|x|^2 \tag{9}$$

and

$$\nabla V_p(x) f_p(x) \leq -c_p|x|^2 \tag{10}$$

(see, e.g., [19, Theorem 4.5]). Combining (9) and (10), we obtain

$$\nabla V_p(x) f_p(x) \leq -\lambda_p V_p(x), \qquad p \in \mathcal{P}$$

where $\lambda_p = c_p/b_p$. This implies that

$$V_p(x(t_0+\tau)) \leq e^{-\lambda_p \tau} V_p(x(t_0)) \tag{11}$$

providing that $\sigma(t) = p$ for almost all $t \in [t_0, t_0+\tau]$. To simplify the next calculation, let us consider the case when $\mathcal{P} = \{1,2\}$ and σ takes on the value 1 on $[t_0, t_1)$ and 2 on $[t_1, t_2)$, where $t_{i+1} - t_i \geq \tau$, $i = 0, 1$. From the above inequalities one has

$$V_2(t_1) \leq \frac{b_2}{a_1} V_1(t_1) \leq \frac{b_2}{a_1} e^{-\lambda_1 \tau} V_1(t_0)$$

and furthermore

$$V_1(t_2) \leq \frac{b_1}{a_2} V_2(t_2) \leq \frac{b_1}{a_2} e^{-\lambda_2 \tau} V_2(t_1) \leq \frac{b_1 b_2}{a_1 a_2} e^{-(\lambda_1+\lambda_2)\tau} V_1(t_0).$$

We see that $V_1(t_2) < V_1(t_0)$ if τ is large enough. In fact, it is not hard to compute an explicit lower bound on τ that ensures that the hypotheses of Lemma 10 are satisfied, which means that the switched system is globally asymptotically stable.

We do not discuss possible extensions and refinements here because a more general result will be stated in the next subsection. Note, however, that the exponential stability assumption is not necessary; for example, the above reasoning would still be valid if the quadratic estimates in (9) and (10) were replaced by, say, quartic ones. In essence, all we used was the fact that

$$\mu := \sup \left\{ \frac{V_p(x)}{V_q(x)} : x \in \mathbb{R}^n,\ p, q \in \mathcal{P} \right\} < \infty. \tag{12}$$

If this inequality does not hold globally in the state space, only local asymptotic stability can be established.

3.3 Average dwell time

For each $T > 0$, let $N(T)$ denote the number of discontinuities of a given switching signal σ on the interval $[0, T)$. We will say that σ has the *average dwell time* property if there exist two nonnegative numbers a and b such that for all $T > 0$ we have $N(T) \leq a + bT$. This terminology is prompted by the observation that, if we discard the first a switchings, the average time between consecutive switchings is at least $1/b$. Dwell time switching signals considered in the previous subsection satisfy this definition with $a = 0$ and $b = 1/\tau$.

Now consider the family of nonlinear systems (3), and assume that all the systems in this family are globally asymptotically stable. Then for each $p \in \mathcal{P}$ there exist positive definite, radially unbounded C^1 functions V_p and D_p such that $\nabla V_p(x) f_p(x) \leq -D_p(x)$ for all x. As explained in [42], there is no loss of generality in taking $D_p(x) = \lambda_p V_p(x)$ for some $\lambda_p > 0$ (changing V_p if necessary). In addition, we need the following mild technical assumption[4]:

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that if } |V_p(x)| < \delta \text{ for some } p \in \mathcal{P} \text{ then } |x| < \epsilon. \tag{13}$$

When $\mathcal{P}$ is a finite set, (13) is automatic. When $\mathcal{P}$ is infinite but compact, (13) is easily achieved if the family $\{V_p : p \in \mathcal{P}\}$ is continuously parameterized. In either of these cases we can also assume that the numbers λ_p are the same for all $p \in \mathcal{P}$, so that we have

$$\nabla V_p(x) f_p(x) \leq -\lambda V_p(x), \qquad \lambda > 0. \tag{14}$$

The following result was recently proved by Hespanha in [11] with the help of Lyapunov function techniques similar to the ones we alluded to in the previous subsection. The complete proof is given in the Appendix.

Theorem 11 [11] *If (12)–(14) hold, then the switched system (1) is globally asymptotically stable for any switching signal that has the average dwell time property with $b < \lambda / \log \mu$.*

The study of average dwell time switching signals is motivated by the following considerations. Stability problems for switched systems arise naturally in the context of intelligent control. Switching control techniques employing a dwell time have been successfully applied to linear systems with imprecise measurements or modeling uncertainty (cf. [31, 6, 15, 37]). However, in the nonlinear setting these methods are often unsuitable because of the possibility of finite escape time. Namely, if a "wrong" controller has to remain in the loop with an imprecisely modeled system for a specified amount of time, the solution to the system might escape to infinity before we switch to a different controller (of course, this will not happen if all the controllers are stabilizing, but when the system is not completely known such an assumption is not realistic).

An alternative to dwell time switching for intelligent control of nonlinear systems is provided by the so-called *hysteresis* switching proposed in [32] and its scale-independent versions which were recently introduced and analyzed in [10, 13] and applied to control of uncertain nonlinear systems in [12, 14]. When the uncertainty is purely parametric and there is no measurement noise, switching signals generated by scale-independent hysteresis have the property that the switching stops in finite time, while in the presence of noise under suitable assumptions they can be shown to have the average dwell time property. Thus Theorem 11 opens the door to provably correct stabilization algorithms for uncertain nonlinear systems corrupted by noise, which is the subject of ongoing research efforts.

[4] If *exponential* stability of the switched system is desired, one needs to replace (13) by more specific growth bounds on the functions V_p.

4 Stabilizing switching signals

Since some switching signals lead to instability, it is natural to ask, given a family of systems, whether it is possible to find a switching signal that renders the switched system asymptotically stable. Such stabilizing switching signals may exist even in the extreme situation when all the individual subsystems are unstable. For example, consider two second-order systems whose trajectories are sketched in Figure 3, left and Figure 3, center. If we switch in such a way that the first system is active in the 2nd and 4th quadrants while the second one is active in the 1st and 3rd quadrants, then the switched system will be asymptotically stable (see Figure 3, right).

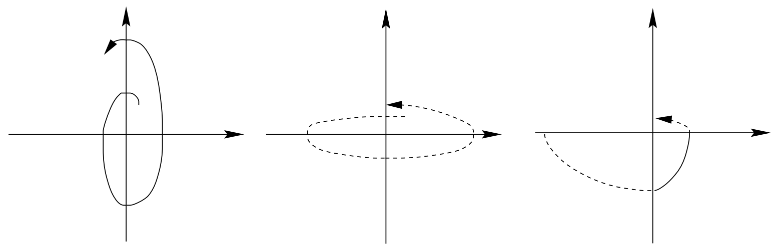

Figure 3: A stabilizing switching signal

In this section we present various methods for constructing stabilizing switching signals in the case when none of the individual subsystems are asymptotically stable (Problem C). We also discuss how these ideas apply to the problem of stabilizing a linear system with finite-state hybrid output feedback. Although we only address stabilizability here, there are other interesting questions such as attainability and optimal control via switching (cf. [25, 26]).

4.1 Single Lyapunov function techniques

In this subsection and the next one we assume that $\mathcal{P} = \{1, 2\}$ and that we are switching between two linear systems

$$\dot{x} = A_1 x \tag{15}$$

and

$$\dot{x} = A_2 x \tag{16}$$

of arbitrary dimension n. When the number of individual subsystems is larger than two, one would expect the stabilizing switching signals to be easier to construct. Interestingly, however, we are not aware of any explicit results that employ switching signals with more than two distinct values.

As demonstrated by Wicks, Peleties and DeCarlo in [53, 54], one assumption that leads to an elegant construction of a stabilizing switching signal is the following one.

Assumption 1. The matrix pencil $\gamma_\alpha(A_1, A_2)$ contains a stable matrix.

According to the definition of a matrix pencil given in Section 2.2, this means that for some $\alpha \in (0, 1)$ the convex combination $A := \alpha A_1 + (1-\alpha)A_2$ is stable (the endpoints 0 and 1 are excluded because A_1 and A_2 are not stable). Thus there exist symmetric positive definite matrices P and Q such that we have

$$A^T P + PA = -Q.$$

This can be rewritten as

$$\alpha(A_1^T P + PA_1) + (1-\alpha)(A_2^T P + PA_2) = -Q$$

or

$$\alpha x^T(A_1^T P + PA_1)x + (1-\alpha)x^T(A_2^T P + PA_2)x = -x^T Qx < 0 \qquad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Since $0 < \alpha < 1$, it follows that for each nonzero $x \in \mathbb{R}^n$ at least one of the quantities $x^T(A_1^T P + PA_1)x$ and $x^T(A_2^T P + PA_2)x$ is negative. In other words, $R^n \setminus \{0\}$ is covered by the union of two open conic regions $\Omega_1 := \{x : x^T(A_1^T P + PA_1)x < 0\}$ and $\Omega_2 := \{x : x^T(A_2^T P + PA_2)x < 0\}$. The function $V(x) := x^T Px$ decreases along solutions of the system (15) in the region Ω_1 and decreases along solutions of the system (16) in the region Ω_2. Using this property, it is possible to construct a switching signal such that V decreases along solutions of the switched system, which implies asymptotic stability. The precise result is this.

Theorem 12 [53, 54] *If Assumption 1 is satisfied, then there exists a piecewise constant switching signal which makes the switched system quadratically stable*[5]*.*

This stabilizing switching signal takes the state feedback form, i.e., the value of σ at any given time $t \geq 0$ depends on $x(t)$. An interesting observation due to Feron is that Assumption 1 is not only sufficient but also necessary for quadratic stabilizability via switching.

Proposition 13 [8] *If there exists a quadratically stabilizing switching signal in the state feedback form, then the matrices A_1 and A_2 satisfy Assumption 1.*

The proof of this result is given in the Appendix; it relies on the following well known fact (see, e.g., [2]).

Proposition 14 (S-procedure) *Let T_0, T_1 be $n \times n$ symmetric matrices. The condition*

$$T_0 - \beta T_1 > 0 \textit{ for some } \beta \geq 0 \tag{17}$$

implies that

$$x^T T_0 x > 0 \textit{ for all } x \neq 0 \textit{ such that } x^T T_1 x \geq 0. \tag{18}$$

Moreover, (18) implies (17) providing that there is some x_0 such that $x_0^T T_1 x_0 > 0$.

One can gain insight into the issue of quadratic stabilizability with the help of the following example. Take $A_1 := \begin{pmatrix} 0.1 & -1 \\ 2 & 0.1 \end{pmatrix}$ and $A_2 := \begin{pmatrix} 0.1 & -2 \\ 1 & 0.1 \end{pmatrix}$. The trajectories of the systems (15) and (16) will then look, at least qualitatively, as depicted in Figure 3, left and center, respectively. We have explained at the beginning of Section 4 how to construct a stabilizing switching signal that yields the switched system with trajectories as shown in Figure 3, right. This system is asymptotically stable, in fact, we see that the function $V(x_1, x_2) := x_1^2 + x_2^2$ decreases along solutions. However, it is easy to check that no convex combination of A_1 and A_2 is stable, and Proposition 13 tells us that the switched system cannot be quadratically stable. Indeed, on the coordinate axes (which form the set where the switching occurs) we have $\dot{V} = 0$.

The above example suggests that even when Assumption 1 does not hold and thus quadratic stabilization is impossible, asymptotic stabilization may be quite easy to achieve by using heuristic ideas that can be applied to general systems, not necessarily linear ones. This is an interesting area for future work.

[5] *Quadratic stability* means that there exists a positive ϵ such that $\dot{V} < -\epsilon x^T x$.

4.2 Multiple Lyapunov function techniques

In the previous subsection we explained how to carry out the stability analysis with the help of a single Lyapunov function that decreases along the trajectories of the switched system. There are situations when one cannot find a switching signal such that the resulting switched system possesses a quadratic Lyapunov function. In view of the results presented in Section 3.1, it might still be possible to find a stabilizing switching signal and prove stability by using multiple Lyapunov functions. Although this line of thinking does not seem to lead to such a simple and constructive procedure as the one described in [53, 54], some preliminary ideas have been explored in the literature. These are discussed next.

The method proposed in [39] is to associate to the system (15) a candidate quadratic Lyapunov function $V_1(x) = x^T P_1 x$ that decreases along solutions in an appropriate region Ω_1. This is always possible unless A_1 is a nonnegative multiple of the identity matrix. Similarly, associate to the system (16) a candidate quadratic Lyapunov function $V_2(x) = x^T P_2 x$ that decreases along solutions in an appropriate region Ω_2. If the union of the regions Ω_1 and Ω_2 covers $\mathbb{R}^n \setminus \{0\}$, then one can try to orchestrate the switching in such a way that the conditions of Lemma 10 are satisfied. The paper [39] contains an example that illustrates how this stabilizing switching strategy works.

In a more recent paper [52] this investigation is continued with the goal to put the above idea on a more solid ground, by means of formulating precise algebraic sufficient conditions for a switching strategy based on multiple Lyapunov functions to exist. Consider the situation when the following condition holds:

CONDITION 1. $x^T(P_1A_1 + A_1^T P_1)x < 0$ whenever $x^T(P_1 - P_2)x \geq 0$ and $x \neq 0$, and $x^T(P_2A_2 + A_2^T P_2)x < 0$ whenever $x^T(P_2 - P_1)x \geq 0$ and $x \neq 0$.

If this condition is satisfied, then a stabilizing switching signal can be defined by $\sigma(t) := \arg\max\{V_i(x(t)) : i = 1, 2\}$. Indeed, the function V_σ will then be continuous and will decrease along solutions of the switched system, which guarantees asymptotic stability.

Condition 1 holds if the following condition is satisfied (by virtue of the S-procedure, the two conditions are equivalent provided that there exist $x_1, x_2 \in \mathbb{R}^n$ such that $x_1^T(P_1 - P_2)x_1 > 0$ and $x_2^T(P_2 - P_1)x_2 > 0$).

CONDITION 2. There exist $\beta_1, \beta_2 \geq 0$ such that $-P_1A_1 - A_1^T P_1 + \beta_1(P_2 - P_1) > 0$ and $-P_2A_2 - A_2^T P_2 + \beta_2(P_1 - P_2) > 0$.

Alternatively, if $\beta_1, \beta_2 \leq 0$, then a stabilizing switching signal can be defined by $\sigma(t) := \arg\min\{V_i(x(t)) : i = 1, 2\}$. This leads to the following result.

Proposition 15 [52] *If there exist two numbers β_1 and β_2, either both nonnegative or both nonpositive, such that the inequalities*

$$-P_1A_1 - A_1^T P_1 + \beta_1(P_2 - P_1) > 0 \tag{19}$$

and

$$-P_2A_2 - A_2^T P_2 + \beta_2(P_1 - P_2) > 0 \tag{20}$$

are satisfied for some symmetric positive definite matrices P_1 and P_2, then there exists an asymptotically stabilizing switching signal.

In [52] the hypotheses of Proposition 15 are further reformulated in terms of eigenvalue locations of certain matrix operators. Note that the algebraic matrix inequalities (19)–(20) are not LMIs since they contain products of the unknowns β_i and P_i, $i = 1, 2$.

Techniques that are quite similar to the ones described above have been developed independently in [27] in a more general, nonlinear context. That paper shows how they find application to the interesting problem of stabilizing an inverted pendulum via a switching control strategy.

4.3 Stabilization with finite-state hybrid output feedback

An interesting source of motivation for pursuing the above ideas comes from the following problem. Suppose that we are given a linear time-invariant control system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{21}$$

that is *stabilizable* and *detectable*, i.e., there exist matrices F and K such that the eigenvalues of $A + BF$ and the eigenvalues of $A + KC$ have negative real parts. Then, as is well known, there exists a continuous dynamic output feedback that asymptotically stabilizes the system. In practice, however, such a continuous dynamic feedback might not be implementable, and a suitable discrete version of the controller is often desired. Recent references [6, 17, 24, 28, 49] discuss some issues related to control of continuous plants by various types of discontinuous feedback.

In particular, in [24] it is shown that the system (21) admits a stabilizing hybrid output feedback that uses a countable number of discrete states. A logical question to ask next is whether it is possible to stabilize (21) by using a hybrid output feedback with only a *finite* number of discrete states. Artstein explicitly raised this question in [1] and discussed it in the context of a simple example (cf. below). This problem seems to require a solution that is significantly different from the ones mentioned above because a finite-state stabilizing hybrid feedback is unlikely to be obtained from a continuous one by means of any discretization process.

One approach to the problem of stabilizing (21) via finite-state hybrid output feedback is prompted by the following observation. Suppose that we are given a collection of gain matrices $K_1, \ldots, K_m$ of suitable dimensions. Setting $u = K_i y$ for some $i \in \{1, \ldots, m\}$, we obtain the system

$$\dot{x} = (A + BK_iC)x.$$

Thus the stabilization problem for the original system (21) will be solved if we can orchestrate the switching between the systems in the above form in such a way as to achieve asymptotic stability. Denoting $A + BK_iC$ by A_i for each $i \in \{1, \ldots, m\}$, we are led to the following question: using the measurements of the output $y = Cx$, can we find a switching signal σ such that the switched system $\dot{x} = A_\sigma x$ is asymptotically stable? The value of σ at a given time t might just depend on t and/or $y(t)$, or a more general hybrid feedback may be used. We are assuming, of course, that none of the matrices A_i are stable, as the existence of a matrix K such that the eigenvalues of $A + BKC$ have negative real parts would render the problem trivial.

First of all, observe that the existence of a stable convex combination $A := \alpha A_i + (1 - \alpha)A_j$ for some $i, j \in \{1, \ldots, m\}$ and $\alpha \in (0, 1)$ would imply that the system (21) can be stabilized by the linear static output feedback $u = Ky$ with $K := \alpha K_i + (1 - \alpha)K_j$, contrary to the assumption that we just made. In view of Proposition 13, this implies that a quadratically stabilizing switching signal does not exist. However, it might still be possible to construct an asymptotically stabilizing switching signal and even base a stability proof on a single Lyapunov function (cf. remarks at the end of Section 4.1).

As an example that illustrates this point, we discuss a modified version of the stabilizing switching strategy for the harmonic oscillator with position measurements described in [1]. Consider the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = x_1$$

Although this system is both controllable and observable, it cannot be stabilized by (even discontinuous) output feedback. On the other hand, it can be stabilized by hybrid output feedback; several ways to do this were presented in [1]. We will now sketch one possible stabilizing strategy. Letting $u = -y$ we obtain the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{22}$$

while letting $u = \frac{1}{2}y$ we obtain the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{23}$$

Define $V(x) := x_1^2 + x_2^2$. This function decreases along the solutions of (22) when $x_1 x_2 > 0$ and decreases along the solutions of (23) when $x_1 x_2 < 0$. Therefore, if the system (22) is active in the 1st and 3rd quadrants while the system (23) is active in the 2nd and 4th quadrants, we will have $\dot{V} < 0$ whenever $x_1 x_2 \neq 0$, hence the switched system is asymptotically stable. (This situation is similar to the one shown in Figure 3, except that here the individual subsystems are critically stable.) It is important to notice that, since both systems being switched are linear time-invariant, the times at which the state trajectory crosses the x_1-axis can be explicitly calculated from the times at which it crosses the x_2-axis. This means that the above switching strategy can be implemented via hybrid feedback based just on the measurements of the output; see [1, 23] for details.

Another possibility, exploited in [23], is to employ multiple Lyapunov functions. Assume for simplicity that $m = 2$, so that we are only given two matrices, $A_1 = A + BK_1C$ and $A_2 = A + BK_2C$. Note that in the present context the given data is not the matrices A_1 and A_2, but rather the matrices A, B and C. The problem will be solved if we can find output feedback gains K_1 and K_2 such that the resulting matrices $A_1 = A + BK_1C$ and $A_2 = A + BK_2C$ satisfy the hypotheses of Proposition 15. We can rewrite the inequalities (19)–(20) as

$$-P_1A - A^TP_1 + \beta_1(P_2 - P_1) - P_1BK_1C - C^TK_1^TB^TP_1 > 0$$

and

$$-P_2A - A^TP_2 + \beta_2(P_1 - P_2) - P_2BK_2C - C^TK_2^TB^TP_2 > 0.$$

Using the procedure for elimination of matrix variables described in [2, Section 2.6.2], one can show that the above inequalities are satisfied if and only if for some $\gamma_1, \gamma_2 \in \mathbb{R}$ we have

$$\begin{aligned} &-P_1A - A^TP_1 + \beta_1(P_2 - P_1) - \gamma_1 P_1BB^TP_1^T > 0 \\ &-P_1A - A^TP_1 + \beta_1(P_2 - P_1) - \gamma_1 C^TC > 0 \end{aligned} \tag{24}$$

and

$$\begin{aligned} &-P_2A - A^TP_2 + \beta_2(P_1 - P_2) - \gamma_2 P_2BB^TP_2^T > 0 \\ &-P_2A - A^TP_2 + \beta_2(P_1 - P_2) - \gamma_2 C^TC > 0 \end{aligned} \tag{25}$$

Thus we have the following statement.

Proposition 16 [23] *If there exist two numbers β_1 and β_2, either both nonnegative or both nonpositive, and two numbers γ_1 and γ_2 such that the inequalities (24)–(25) are satisfied for some symmetric positive definite matrices P_1 and P_2, then the system (21) can be asymptotically stabilized by using hybrid output feedback with two discrete states.*

The fact that the individual gains can be chosen as part of the design introduces considerable flexibility into the problem and is to be explored further. Note that when $\beta_1 = \beta_2 = 0$, we recover LMIs that express conditions for stabilizability of (21) by static output feedback (and are equivalent to the ones given, e.g., in [50, Theorem 3.8]). It would be interesting to compare the above bilinear matrix inequalities with the ones obtained in [41] as a characterization of stabilizability via switched *state* feedback.

5 Concluding remarks

We have surveyed recent developments in three basic problems regarding switched dynamical systems: stability for arbitrary switching signals, stability for slow switching signals, and construction of stabilizing switching signals. We have aimed at providing an overview of general results and ideas involved. For technical details, the reader may consult the references cited below. These references also address many issues that are relevant to switched systems but fell outside the scope of this survey. In spite of a number of interesting results presented here, it is safe to say that the subject is still largely unexplored. Various open questions, some of which we have mentioned in the paper, remain to be investigated.

The three problems studied here are very general and address fundamental issues concerning stability and design of switched systems. As we have pointed out throughout the paper, special cases of these problems arise frequently in various contexts associated with control design. In such situations, specific structure of a problem at hand can sometimes be utilized to obtain satisfactory results even in the absence of a general theory. Examples of results that use such additional structure include the conditions for existence of a common Lyapunov function given in [20] which exploit positive realness and the so-called Switching Theorem proved in [31] which plays a role in the supervisory control of uncertain linear systems . It is our opinion that to make significant further progress one needs to stay in close contact with particular applications that motivate the study of switched systems.

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We would like to thank João Hespanha for illuminating discussions on many topics related to the material of this paper.

Appendix

Proof of Theorem 11. Let $t_0 := 0,\ t_1,\ \ldots,\ t_{N(T)}$ be the discontinuities of σ on $[0,T)$. Define the function

$$W(t) := e^{\lambda t} V_{\sigma(t)}(x(t)).$$

This function is piecewise differentiable along solutions to (1), and on any interval $[t_i, t_{i+1})$ we have

$$\dot{W} = \lambda W + e^{\lambda t} \nabla V_\sigma(x) f_\sigma(x) \leq 0$$

by virtue of (14). Therefore,

$$W(t) \leq W(t_i), \qquad t \in [t_i, t_{i+1}),\ i \in \{0, 1, \ldots, N(T)-1\}. \tag{26}$$

This together with (12) implies

$$\begin{aligned} W(t_{i+1}) &= e^{\lambda t_{i+1}} V_{\sigma(t_{i+1})}(x(t_{i+1})) \leq \mu e^{\lambda t_{i+1}} V_{\sigma(t_i)}(x(t_{i+1})) \\ &= \mu \lim_{t \to t_{i+1}^-} W(t) \leq \mu W(t_i). \end{aligned}$$

Iterating this inequality from $i = 0$ to $i = N(T) - 1$, we have

$$W(t_{N(T)}) \leq \mu^{N(T)} W(0)$$

and, using (26),

$$\lim_{t \to T^-} W(t) \leq \mu^{N(T)} W(0).$$

It follows from the definition of W that

$$e^{\lambda T} V_{\sigma(T^-)}(x(T)) \leq \mu^{N(T)} V_{\sigma(0)}(x(0))$$

which according to the hypotheses can be rewritten as

$$V_{\sigma(T^-)}(x(T)) \leq e^{-\lambda T + (a+bT)\log\mu} V_{\sigma(0)}(x(0)) = e^{a \log \mu} e^{(b \log \mu - \lambda)T} V_{\sigma(0)}(x(0)) = c e^{-\bar{\lambda} T} V_{\sigma(0)}(x(0))$$

where $c := e^{a \log \mu}$ and $\bar{\lambda} := \lambda - b \log \mu$. Note that $\log \mu > 0$ because $\mu > 1$ in view of the interchangeability of p and q in (12). Thus $c > 0$, and $\bar{\lambda} > 0$ by virtue of the inequality $b < \lambda / \log \mu$. Now it is easy to conclude from (13) that $x(T)$ converges to zero as $T \to +\infty$. □

Proof of Proposition 13. Suppose that the switched system is quadratically stable, i.e., there exists a Lyapunov function $V(x) = x^T P x$ whose derivative along solutions of the switched system satisfies $\dot{V} < -\epsilon x^T x$ for some $\epsilon > 0$. Since the switching signal takes the state feedback form, this implies that for any nonzero x we must have either

$$x^T (A_1^T P + P A_1) x < -\epsilon x^T x$$

or

$$x^T (A_2^T P + P A_2) x < -\epsilon x^T x.$$

We can restate this as follows:

$$x^T(-A_1^T P - P A_1 - \epsilon I)x > 0 \text{ whenever } x^T(A_2^T P + P A_2 + \epsilon I) x \geq 0 \tag{27}$$

and

$$x^T(-A_2^T P - PA_2 - \epsilon I)x > 0 \text{ whenever } x^T(A_1^T P + PA_1 + \epsilon I)x \geq 0. \tag{28}$$

If $x^T(A_1^T P + PA_1 + \epsilon I)x \leq 0$ for all $x \neq 0$, then the matrix A_1 is stable and there is nothing to prove. Similarly, if $x^T(A_2^T P + PA_2 + \epsilon I)x \leq 0$ for all $x \neq 0$, then A_2 is stable. Discarding these trivial cases, we can apply the S-procedure to one of the last two conditions, say, to (27), and conclude that for some $\beta \geq 0$ we have

$$A_1^T P + PA_1 + \beta(A_2^T P + PA_2) < -(1+\beta)\epsilon x^T x$$

or, equivalently,

$$\frac{(A_1 + \beta A_2)^T}{1+\beta} P + P\frac{(A_1 + \beta A_2)}{1+\beta} < -\epsilon x^T x.$$

Therefore, the matrix $(A_1 + \beta A_2)/(1+\beta) \in \gamma_\alpha(A_1, A_2)$ is stable, and Assumption 1 is satisfied. □

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