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Pattern Recognition Letters 25 (2004) 1231–1242
Pattern Recognition Letters
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Characterization of digital circles in triangular grid ☆

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Received 3 July 2003; received in revised form 3 February 2004
Available online 14 May 2004

Abstract

In this paper we present some former results about properties of digital circles defined by neighbourhood sequences in the triangular grid. Das and Chatterji [Inform. Sci. 50 (1990) 123] analyzed the geometric behaviour of two-dimensional periodic neighbourhood sequences. We use a more general definition of neighbourhood sequences, which does not require periodicity [Publ. Math. Debrecen 60 (2002) 405]. We study the development of wave-fronts and grow digital circles from a triangle with general neighbourhood sequences in triangular grid. We present the possible types of polygons, and characterize them by the initial part of the neighbourhood sequences. The symmetry and the convexity analysis of the digital circles is also presented.

Besides those who are interested in the underlying theory there may be readers from the pattern recognition or the image processing communities or even the geometric modelling field who could find some of the consequences of the paper of interest.

PACS: 52C99; 68U10
Keywords: Digital geometry; Neighbourhood sequences; Triangular grid; Digital circles; Region growing; Constructive geometry; Finite state automaton

1. Introduction

The classical digital—cityblock and chessboard—motions were introduced by Rosenfeld and Pfaltz (1968) in square grid. Based on these types of motions, the authors in (Rosenfeld and Pfaltz, 1968) defined three distances. The d_4 or d_8 distance of two points is the number of steps required to reach either point from the other, where only cityblock or chessboard motions can be used, respectively. The distance d_{oct} obtains a better approximation for the Euclidean distance, in this Rosenfeld and Pfaltz used both the cityblock and chessboard motions, alternate. Geometrically, the corresponding 'disks' are diamonds for the distance d_4, squares for d_8, and octagons for d_{oct}. (In Fig. 1 there are some examples.)

By allowing arbitrary mixture of cityblock and chessboard motions, Das et al. introduced the concept of periodic neighbourhood sequences in

☆ This research was supported by a grant from the Hungarian National Foundation for Scientific Research (OTKA F043090).
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doi:10.1016/j.patrec.2004.04.001

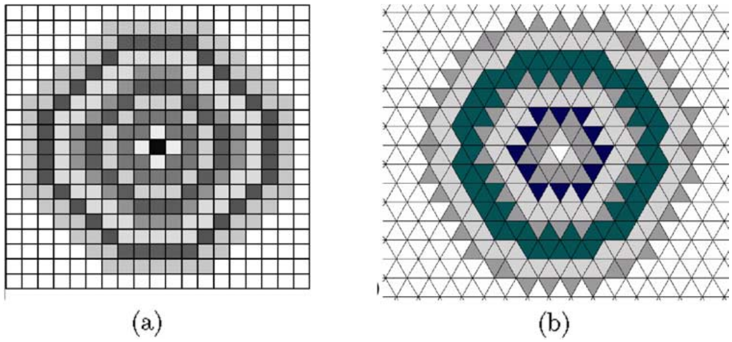

Fig. 1. Examples for growing digital circles up to radius 8. (a) Circles in square grid, (b) circles in triangular grid.

(Das et al., 1987). In that paper only periodic neighbourhood sequences were investigated. The concept of neighbourhood sequences was extended to the concept of generalised (not necessarily periodic) neighbourhood sequences in (Fazekas et al., 2002).

The main advantage of neighbourhood sequences over the classical distances d_4, d_8 is that they provide more flexibility in moving on the plane. Making use of this property, Das and Chatterji (1990) were able to determine distance functions that provide good approximation of the Euclidean distance. They analyzed the geometric properties of the octagons occupied by a neighbourhood sequence during 'spreading' on the 2D plane.

Similarly to the square grid, in triangular grid there are also more neighbourhood criteria (Deutsch, 1972). In (Nagy, 2003) we used neighbourhood sequences in triangular grid. We presented an algorithm which provides a shortest path between two arbitrary triangles by using a given neighbourhood sequences. (We used the concept neighbourhood sequences in the general way, as can be found in (Fazekas et al., 2002).)

The triangular grid is a valid concurrent plane to the rectangular one in digital geometry. It has many nice properties, but it also has some strange properties (in (Nagy, 2002), we showed that the distance based on neighbourhood sequences in triangular grid may not be symmetric). The digital circles (we show some examples in Fig. 1) of triangular grid are better approximations to the Euclidean circles than the ones in square grid (Hajdu and Nagy, 2002). This is an important reason why we study this plane in detail. The triangular grids play an increasing role in geometric modelling, many 3D-scanners produce triangulations. These grids are generally not regular, but at high enough resolution they are close to regular ones. The human retina is often modelled by a Delauney triangulation. Many algorithms of computer graphics are also given for triangular grid. Therefore we can say that the triangular grids are among the most important grids in digital geometry, in digital image processing (see Deutsch, 1972; Shimizu, 1981) and in cellular neural networks also (see Radványi (2002), where is shown how the rectangular arrays with weight functions, which are easy-usable by computers, can be used to represent other important grids).

We would like to mention here that the digital (or discrete) circles (and spheres) are also well examined from different points of view. In most papers the Euclidean circles are approximated using only points of a grid. The aim is to use a digital figure which looks like a circle. (For example, how a computer can draw a circle on a digital display.) Difference equation methods and grid-following algorithms for drawing circles are frequently used. One of the basic circle-drawing algorithms is the Bresenham's algorithm (Bresenham, 1977). It has many extensions. In (McIlroy, 1983) there are algorithms for circles and arcs with non-integer radii and/or non-integer centres in the square grid. In (Kulpa and Kruse, 1983) there are discrete circles, disks and rings in the square grid. Algorithms for figure propagation and disk generation are discussed. Their 4-direction, 8-direction and octagonal propagation disks are the same digital objects as the digital circles using neighbourhood sequences in the square grid. In (Andres, 1994) an efficient incremental generation algorithm is presented for square and cube grids (an extension of Bresenham's circle called arithmetical circle is used), while in (Shimizu, 1981) a computer graphical drawing algorithm is given for the nodes of the triangular grid. Recently we introduced and investigated distance functions based on neighbourhood sequences in the triangular grid (Nagy, 2002, 2003) (we use the regions of the grid as points). In this paper we will use these digital distance functions having only integer values between any two points of the grid. Since our digital circles will be defined by these neighbourhood-based distances they are really circles (using digital distances instead of the Euclidean distance). Our main aim is not to produce digital objects in the triangular grid which look like the Euclidean circles, but to describe the digital circles based on these discrete distance functions.

The structure of this paper is as follows. In Section 2 we give our notation, and provide some properties of the concepts introduced. We present some properties, in which the triangular grid differs from the square grid. In the other sections of this paper we show some former results of Das and Chatterji (1990) on the triangular grid. Moreover, we use generalized neighbourhood sequences instead of periodic ones in our analysis. In Section 3 we analyze the changing and development of wave-fronts, and give an illustrated description of the digital circles with neighbourhood sequences in triangular grid. In Section 4 we show a characterization of digital circles. In Section 5 a short overview of possible applications is presented. Finally in Section 6 we summarize our results.

2. Basic notation and concepts

In this section we recall some definitions and notation from (Fazekas et al., 2002; Nagy, 2002, 2003) concerning neighbourhood relations and sequences.

The neighbourhood relations in triangular grid is based on the widely used relations (see Deutsch (1972)), we use three types of neighbours as Fig. 2 shows. These relations are reflexive (i.e. the pixel marked dark triangle is a 1-, 2-, and 3-neighbour of itself). In addition, all 1-neighbours of a pixel are its 2-neighbours and all 2-neighbours are 3-neighbours, as well (i.e. increasing and inclusion properties).

We will use three coordinates to represent the triangles of the triangular grid as in (Nagy, 2002) and in (Nagy, 2003) (sometimes we refer the triangles as the points of the triangular grid). Fig. 3 shows a part of the triangular grid with the associated coordinate values. We note that the points of the triangular plane have exactly the same points which have sum of coordinate value 0 or 1, we call them even and odd triangles, respectively. The coordinate axes are lines go through the origin with growing direction of the assigned coordinate value (and non-ascending of the other two values).

Let P be a triangle in triangular grid. The ith coordinate of P is indicated by p_i $(1 \leqslant i \leqslant 3)$. Let m be an integer with $1 \leqslant m \leqslant 3$. The triangles P, Q are called m-neighbours if the following two conditions hold:

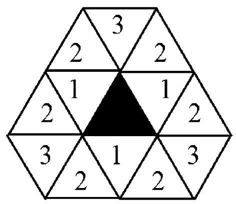

Fig. 2. Neighbourhood relations in triangular grid.

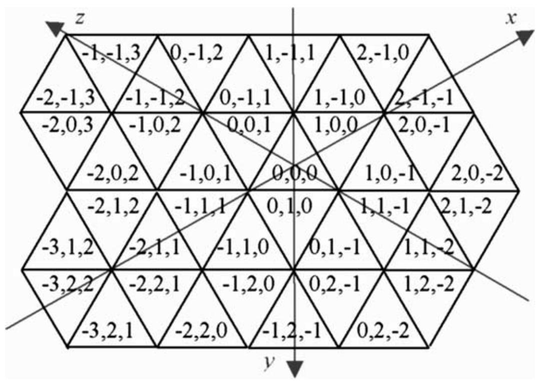

Fig. 3. Coordinate axes and values in triangular grid.

- $|p_i - q_i| \leqslant 1 \ (1 \leqslant i \leqslant 3)$,
- $\sum_{i=1}^{3} |p_i - q_i| \leqslant m$.

Note, that the neighbouring relations are reflexive and symmetric relations. It is easy to check that the formal definition above with the presented coordinate values (Fig. 3) gives the neighbourhood relations shown in Fig. 2.

The sequence $B = (b(i))_{i=1}^{\infty}$, where $1 \leqslant b(i) \leqslant 3$ for all $i \in \mathbb{N}$, is called a neighbourhood sequence for triangular grid, or (triangular) neighbourhood sequence. B is periodic, if for some $l \in \mathbb{N}$, $b(i + l) = b(i)$ $(i \in \mathbb{N})$. For a periodic neighbourhood sequence B with period l we briefly write $B = (b(1), b(2), \ldots, b(l))$. The set of all triangular neighbourhood sequences will be denoted by S.

We call a step $b(i)$-step when we move from a triangle P to a triangle Q and they are $b(i)$-neighbours. Let P, Q be triangles and $B \in S$. The triangle sequence $P = P_0, P_1, \ldots, P_k = Q$, in which we move from P_{i-1} to P_i by a $b(i)$-step $(1 \leqslant i \leqslant k)$, is called a B-path from P to Q of length k. The B-distance $d(P, Q; B)$ from P to Q is defined as the length of the shortest B-path(s), respectively.

In this paper we investigate the way a neighbourhood sequence spreads in the digital space starting from a triangle of the triangular grid. This spreading is translation-invariant among the triangles of the same parity and it is central-symmetric concerning triangles with different parities. So, for simplicity we may choose the origin $\mathbf{0}$ as the starting triangle.

Let B be a triangular neighbourhood sequence. For $k \in \mathbb{N}$, let

$$C_B^k = \{P : d(\mathbf{0}, P; B) \leqslant k\}.$$

So C_B^k is the region (digital circle) occupied by B after k steps.

Similarly, in square grid we will use the

$$O_A^k = \{N : d(\mathbf{0}, N; A) \leqslant k\}$$

notations for the occupied regions where N is a square in the square grid $(\mathbb{Z}^2)$, $A = (a(i))_{i=1}^{\infty}$ is a neighbourhood sequence for square grid, i.e., $1 \leqslant a(i) \leqslant 2$ for all i, and k is a natural number.

In the following we summarize some simple observations about the digital circles. We underline some properties which are different for the digital circles in square grid and in triangular grid.

In square grid the region O_A^k occupied by k steps of a neighbourhood sequence A is independent of the ordering of the first k element of A. A proof is based on the permutability of steps to a 1-neighbour and to a 2-neighbour (see Hajdu and Nagy (2002) for more details, where we approximate the Euclidean circle by digital ones).

Lemma 1. *Contrary to the case of square grid, it is possible for a neighbourhood sequence B and for a $k \in \mathbb{N}$, that the region C_B^k does depend on the order of the first k elements of B.*

Proof. We will show an example. Assume that $B_1 = (1, 3)$ and $B_2 = (3, 1)$ then our regions $\langle B2 \rangle = C_{B_1}^2$ and $\langle D1 \rangle = C_{B_2}^2$ differ as Fig. 4 shows in the margins of row 3. □

This strange property of triangular neighbourhood sequences occurs if the distances defining by them are not symmetric (it can happen that $d(P, Q; B) \neq d(Q, P; B)$ for some triangles P, Q and a neighbourhood sequence B). In (Nagy, 2002) we presented the necessary and sufficient condition for a neighbourhood sequence to generate a symmetric and/or triangular distance function (it can happen that a B-distance is non-triangular, i.e., there are points P, Q, R such that $d(P, Q; B) + d(Q, R; B) < d(P, R; B)$). (A distance based on a triangular neighbourhood sequence satisfies the metric properties iff it is triangular and symmetric.)

In square grid for any neighbourhood sequence A the regions O_A^k and O_A^l are in the following relation: $O_A^k \supsetneq O_A^l$ if and only if $k > l$.

The triangular neighbourhood sequences have the similar property as the next remark claims.

Remark 2. For any $B \in S$, the sequence of regions $(C_B^k)_{k=1}^{\infty}$ is a strictly monotone increasing sequence. That is, $k > l$ implies $C_B^k \supsetneq C_B^l$.

In square grid it is impossible for $k \neq l$ and any two neighbourhood sequences A_1 and A_2 that $O_{A_1}^k = O_{A_2}^l$. This statement follows from the fact, that for any neighbourhood sequence A the point $N(0, k) \in O_A^l$ if and only if $l \geqslant k$.

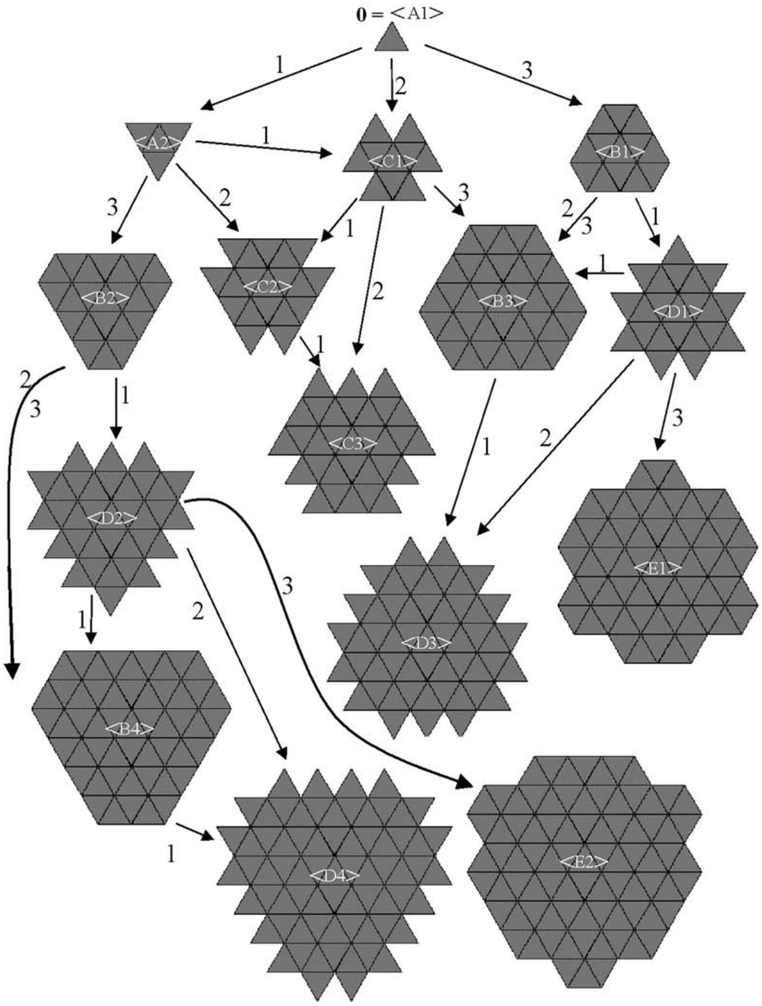

Fig. 4. Basic digital circles and their hierarchy. We will refer them by their signed names. The labelled directed paths from $\langle A1\rangle$ correspond to the initial part of the neighbourhood sequences resulting the given circles.

Lemma 3. *Contrary to the square grid in the triangular grid it is possible for* $B_1, B_2 \in S$ *that* $C^k_{B_1} = C^l_{B_2}$ *with* $k \neq l$.

Proof. We present an example. Let $B_1 = (1)$ and $B_2 = (2)$ then $C^2_{(1)} = C^1_{(2)}$ (see the ways to get $\langle C1\rangle$ from $\langle A1\rangle$ in Fig. 4). $\square$

We will use the concept of minimal equivalent neighbourhood sequences to our investigations, therefore we recall it from (Nagy, 2002).

Let $B(b(1), b(2), \ldots)$ and $B'(b'(1), b'(2), \ldots)$ be two neighbourhood sequences in the triangular grid. B' is called the minimal equivalent neighbourhood sequence of B if the following conditions hold:

- $d(P, Q; B) = d(P, Q; B')$ for all grid triangles P, Q, and
- for each neighbourhood sequence B'', if $d(P, Q; B) = d(P, Q; B'')$ for all grid triangles P, Q, then $b'(i) \leqslant b''(i)$ for all $i \in \mathbb{N}$.

In (Nagy, 2002) we proved the following lemma about the minimal equivalent neighbourhood sequence.

Lemma 4. *The minimal equivalent neighbourhood sequence B' of B is uniquely determined, and is given by*

- $b'(i) = b(i)$, *if* $b(i) < 3$,
- $b'(i) = 3$, *if* $b(i) = 3$ *and there is no* $j < i$ *such that* $b'(j) = 3$,
- $b'(i) = 3$, *if* $b(i) = 3$ *and there is some* $b'(l) = 3$ *with* $l < i$, *and* $\sum_{k=j+1}^{i-1} b'(k)$ *is odd, where* $j = \max\{l \mid l < i, b'(l) = 3\}$,
- $b'(i) = 2$, *otherwise.*

In the triangular grid, for certain neighbourhood sequences, it can happen that a 3-step is equivalent to a 2-step for our investigations (i.e. the digital circles in triangular grid have the following property).

Remark 5. According to the definition of the minimal equivalent neighbourhood sequence we obtain the same digital circles using them with the original ones, i.e., ($C_{B'}^k = C_B^k$) for any $k \in \mathbb{N}$ and minimal equivalent neighbourhood sequence B' of B.

In Fig. 4 we present some simple digital circles obtained in a few steps (small radii). We are starting from the origin, we call this 'circle' $\langle A1 \rangle$. Using a 1-step we get $\langle A2 \rangle$ or using a 3-step $\langle B1 \rangle$. The 'circle' $\langle C1 \rangle$ can be given by a 2-step from the origin, or a 1-step from $\langle A2 \rangle$. The other circles on the figure are obtained from the previous ones by the signed steps.

3. Wave-fronts in triangular grid

In (Das et al., 1987) the authors examined the wave-front sets of neighbourhood sequences in square grid. In this part we present the types and the development of wave-fronts in the triangular plane. Das and Chatterji (1990) showed that for every 2D periodic neighbourhood sequence A, O_A^k is always an octagon. In rectangular grid we have two kinds of sides: 'straight' and 'stair'-types as shown in Fig. 1. The vertical and horizontal edges are 'straight', the other four edges are 'stair'-type. In the next cases the octagon is degenerate, i.e., it is a square with only one type of edges. With the neighbourhood sequence $A_1 = (1)$ we get only four 'stair'-type edges, while using the neighbourhood sequence $A_2 = (2)$ we get a square with only 'straight' edges. In the case when we use both 1-step and 2-step our result is a non-degenerated octagon.

In triangular plane we have three kinds of possible 'limit lines' (edges). These are the 'straight', the 'hilly' and the 'sawtooth'. They change to each other by using a step from C_B^k to C_B^{k+1}. In Fig. 5, there is the diagram of changing them by a step using different neighbourhood criteria. (The used growing direction is bottom-up.) In the first rows we can see how modifying the 'sawtooth' with various length; in middle rows the 'hilly' and in the last rows the 'straight' edges after different type of steps. (We used steps by neighbouring criteria 1, 2 and 3 at columns (b) (c) and (d) respectively.)

The diagram of the changing of types of edges is given by the Table 1 and by Fig. 6.

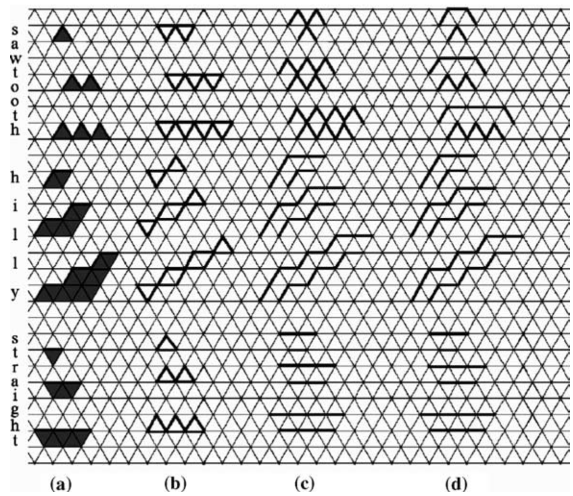

Fig. 5. Changing the edges ('sawtooth' in the top, 'hilly' in middle and 'straight' edges in the bottom rows) after a step in upward direction: (a) original edges; (b) after a 1-step; (c) after a 2-step; (d) after a 3-step.

Table 1
State transition table of edges by taking a step

Original edge type	After a 1-step	After a 2-step	After a 3-step
‘sawtooth’	‘straight’	‘sawtooth’	‘straight’
‘hilly’	‘hilly’	‘hilly’	‘hilly’
‘straight’	‘sawtooth’	‘straight’	‘straight’

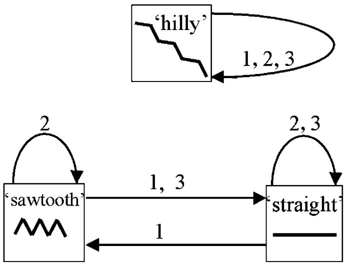

Fig. 6. State transition diagram of type of the edges of digital circles.

In the next statements we summarize our experiences.

Proposition 6
- *After a* 3-*step our straight and sawtooth edges go to straight lines*,
- *the* 2-*step do not change the type of the edges*,
- *the ‘hilly’ edge cannot change into another type edge*.

In the previous part, in Fig. 4 we showed the basic digital circles and here we analyzed the edges. In next part we analyze how the possible vertices i.e., the connections of the possible type of edges change in growing steps. In Table 3 we show what kinds of corners occur in different digital circles. Fig. 7 shows all of the cases of changing vertices by a step in upward direction, because all corners occurring at basic digital circles are in the figure (see Table 2), and all possible evolving corners are in the figure, as well. (See also Table 3 and Fig. 8.)

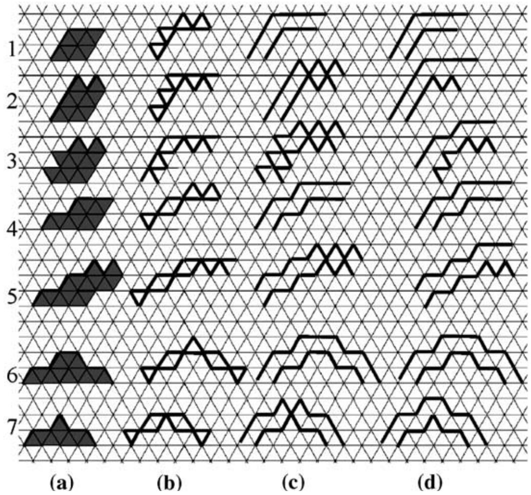

Fig. 7. Changing the corners after a step: (a) original ones (type 1–7); (b) after a 1-step; (c) after a 2-step; (d) after a 3-step.

Table 2 gathers the types of corners which occur in basic digital circles (we refer to the basic circles of Fig. 4 by their name). We use here all digital circles occurring in Fig. 4, for which the figure does not contain all three possible growing steps.

Based on Table 3 we summarize how the vertices change via the growing procedure.

Proposition 7
- *There are two types of vertices between ‘hilly’ edges, as we used type* 6 *and* 7, *the difference can be seen at their corner*: *in type* 7 *there is a peak*, *while in type* 6 *there is a plateau*.
- *The following corners can start a new type of edges, as we show in the table by two values*: *between two ‘sawtooth’ edges with* 3-*step we get a*

Table 2
Vertex-types of basic digital circles in triangular grid

Name and sign of corner-type	‘straight’–‘straight’ (1)	‘straight’–‘sawtooth’ (2)	‘sawtooth’–‘sawtooth’ (3)	‘straight’–‘hilly’ (4)	‘sawtooth’–‘hilly’ (5)	‘hilly’–‘hilly’ (6)	‘hilly’–‘hilly’ (7)
Occurrence in basic circles	$\langle B3\rangle$ $\langle B4\rangle$	$\langle C1\rangle$ $\langle C3\rangle$	$\langle D3\rangle$ $\langle D4\rangle$	$\langle E1\rangle$ $\langle E2\rangle$	–	$\langle E1\rangle$ $\langle E2\rangle$	–

Table 3
State transition table of vertices by taking a step

Original vertex type	Edges after a 1-step	Edges after a 2-step	Edges after a 3-step
'straight'–'straight' (1)	'sawtooth'–'sawtooth' (3)	'straight'–'straight' (1)	'straight'–'straight' (1)
'straight'–'sawtooth' (2)	'straight'–'sawtooth' (2)	'straight'–'sawtooth' (2)	'straight'–'straight' (1)
'sawtooth'–'sawtooth' (3)	'straight'–'straight' (1)	'sawtooth'–'sawtooth' (3)	'straight'–'hilly'–'straight' (4, 4)
'straight'–'hilly' (4)	'sawtooth'–'hilly' (5)	'straight'–'hilly' (4)	'straight'–'hilly' (4)
'sawtooth'–'hilly' (5)	'straight'–'hilly' (4)	'sawtooth'–'hilly' (5)	'straight'–'hilly' (4)
'hilly'–'hilly' (6)	'hilly'–'hilly' (7)	'straight'–'hilly'–'straight' (4, 4)	'straight'–'hilly'–'straight' (4, 4)
'hilly'–'hilly' (7)	'straight'–'hilly'–'straight' (4, 4)	'hilly'–'sawtooth'–'hilly' (5, 5)	'hilly'–'hilly' (6)

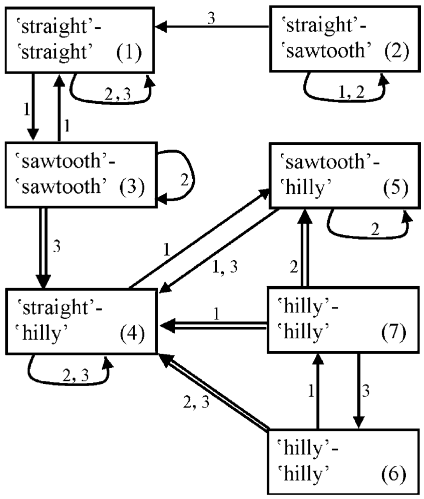

Fig. 8. State transition diagram of vertices by growing digital circles.

new 'hilly' edge between the 'straight' ones; between two 'hilly' edges we get a new 'straight' one (in case 6 *with* 2*-step or* 3*-step) and we get a 'straight' and a 'sawtooth' one using* 1*-step and* 2*-step respectively in case* 7.

We draw the state transition diagram of the vertices in Fig. 8. The double arrows mean that using these transitions we get two vertices (of the same type).

As we can see our edge-types and vertex-types are in closed sets, i.e., we cannot step out from the above used sets by the growing steps. One can check also, that all kinds of vertices and edges occur in digital circles.

Using the basic digital circles and our growing tables we get all possible digital circles of the triangular grid. In Section 4 we will list their types.

4. Characterizing digital circles

In this section—based on our previous experience—we characterize the digital circles with neighbourhood sequences in the triangular plane.

Since neighbourhood sequences spread in an 'isotropic' way, the occupied regions are somehow symmetric objects. More precisely, we have the following lemma and theorem.

Lemma 8. *Let $B \in S$ and $k \in \mathbb{N}$. If a triangle P with coordinates (p_1, p_2, p_3) belongs to C_B^k, then the triangles with coordinates $(p_{i_1}, p_{i_2}, p_{i_3})$ also belong to C_B^k. Here (i_1, i_2, i_3) is an arbitrary permutation of $(1, 2, 3)$.*

Proof. There is not a special coordinate, each of them plays equal role. Therefore permutating them we get triangles also with the same distance from the origin. □

Using the above results we know that the lines—for which the regions occupied by neighbourhood sequences are symmetric—are the coordinate axes. Moreover the digital circles are invariant for the rotation with $2k\pi/3$ for all $k \in \mathbb{Z}$. (In general, those 6 triangles have the same coordinate values with permutation.)

Theorem 9. *The digital circles in triangular grid are axial symmetric for* 3 *lines with $\pi/3$ angles between any two of them.*

Proof. It is evident by Lemma 8. The lines are exactly the coordinate axes for the circles growing from the origin. □

In the following, based on Lemma 4 and Remark 5 we will use the minimal equivalent neighbourhood sequence B' instead of the original sequence B. In Table 4 and in Fig. 9 we show the types of the possible digital circles C_B^k. In the figure we can see the state transition diagram of the them.

Theorem 10. *Table* 4 *contains all possible digital circles, and each of them is in the correct place, respectively.*

For proving this theorem we will use the following facts.

Lemma 11. *Table* 4 *contains the digital circles for all possible neighbourhood sequences.*

Proof. It is evident—that using for steps the equivalent neighbourhood sequence B' instead of B (using Remark 5)—all initial part of all neighbourhood sequences occur in the second column. □

One can check that the following statement is true.

Proposition 12. *All basic digital circles in Fig.* 4 *occur in Table* 4, *and their types are correct.*

Now for proving Theorem 10 we will use induction.

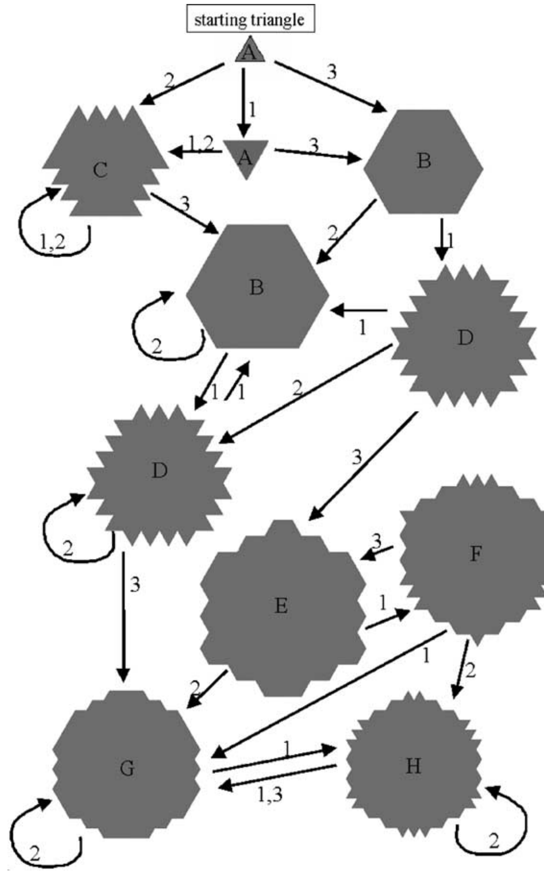

Fig. 9. State transition of types of digital circles.

Proof. By using Lemma 11 we know that all possible initial parts for the possible neighbourhood

Table 4
The possible types of digital circles

(A) Triangle	Basic: 0 (only the starting triangle) or by a 1-step
(B) Hexagon—six 'straight' edges	Only one 3-step and the others are 1-step and 2-step; the sum after the 3-step is even
(C) Hexagon—three 'straight' and three 'sawtooth' edges	With only 1-steps and 2-steps (without any 3-step); and the sum at least 2
(D) Hexagon—six 'sawtooth' edges	Only one 3-step and the others are 1-steps and 2-steps; the sum after the 3-step is odd
(E) Enneagon—six 'hilly' and three 'straight'	Only odd steps: 1-steps and 3-steps by turns (with minimum two 3-steps); without any 2-step or repetition (double 1-steps) with a 3-step at last
(F) Enneagon—six 'hilly' and three 'sawtooth'	Only 1-steps and 3-steps by turns (minimum two 3-steps) with a 1-step at last
(G) Dodecagon—six 'hilly' and six 'straight'	At least (a 2-step or repeated 1-steps) and at least two 3-steps and after the last 3-step the sum is even
(H) Dodecagon—six 'hilly' and six 'sawtooth'	At least (a 2-step or repeated 1-steps) and at least two 3-steps and after the last 3-step the sum is odd

sequences are in Table 4. Therefore we need to prove only the statement, that for all rows of the table the given digital circles are right. Our proof is by induction. From Proposition 12 we know this fact for the basic digital circles. Now we suppose that for a digital circle that it is in the correct place in Table 4. Our induction steps are based on state transitions of the wave-fronts (Tables 3 and 1 for the corners and edges respectively). Using these facts we get the state transition diagram that we show in Fig. 9. Thus Theorem 10 is proved. □

In Fig. 9 we used the minimal equivalent neighbourhood sequences to represent all elements of S. For this reason one cannot see arrows representing 3-step from the types of digital circles for which the sum of the elements after the last used 3-step must be even. For example types B, E and G are in this position. In that cases if the next element of the neighbourhood sequence B is 3 then we get the same result as we get with element 2. (Therefore we would have used both values 2 and 3 on the arrows representing 2-step if we had used the original neighbourhood sequence B.)

We can use our state transition diagram in Fig. 9 as an automaton with as starting state the starting triangle and alphabet $\{1,2,3\}$. Our terminal state(s) will be X, where X is the type of the desired polygon ($X \in \{\mathrm{A,B,C,D,E,F,G,H}\}$).

Now we analyze the convexity of the digital circles. In strict sense there are many concave occupied regions among the digital circles. It is evident that in square grid the occupied area O_A^k is convex if and only if $a(i) = 2$ for all $i < k$ in the neighbourhood sequence A. Hence we can say that only the square with 'straight' edges convex. In triangular grid we have the next theorem.

Theorem 13. *In triangular grid the digital circle C_B^k is convex if and only if it is one of the following types: A (triangle) or B (hexagon with 'straight' edges).*

Proof. It is evident that a region is not convex if it has 'hilly' or 'sawtooth' edge. Therefore the statements follows. □

5. Practical examples

In this paper, we were growing regions in the triangular grid using three kinds of neighbouring relations in various neighbourhood sequences. In image processing the region growing is an often used method for analyzing pictures (find a connected region etc. see Gonzalez and Woods (1992)). In our method with growing digital circles we did not care about other properties of the picture. In practice, starting from a point of an image a variation of our method can be used. We unite only those new points of the wave-front set to our region, which satisfy another desired property. We can finish the method when our region does not change, getting the result, which may depend on the used neighbourhood sequence. (Using a neighbourhood sequence B our result is B-connected, i.e., the definition of connectedness and therefore the result picture depends on B. In Fig. 10 a tree can be seen (original image, (a)). The image in the triangular grid is also shown (b). The next three figures show the maximal (3)—con-

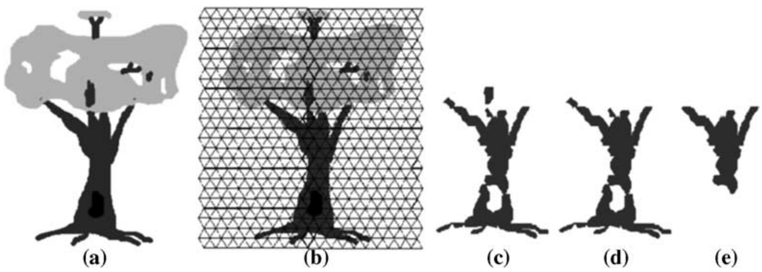

Fig. 10. Examples of different types of connected regions (a) original.

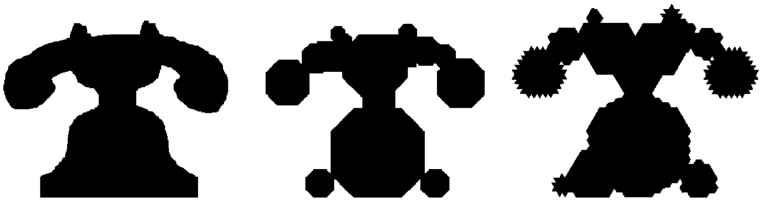

Fig. 11. Approximation of the shape with digital circles: (a) original figure; (b) in the square; (c) in the triangular grid.

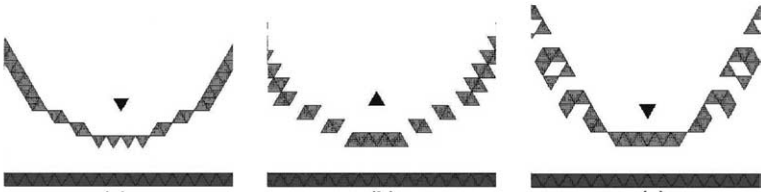

Fig. 12. Digital parabolas in the triangular grid with neighbourhood sequences. (a) $B = (1)$, (b) $B = (2)$, (c) $B = (1, 2, 3)$.

nected, the maximal (2)—connected and a maximal (1)—connected region of the trunk, (c), (d) and (e), respectively.)

In the following 'figure propagation' example we approximate the shape of the telephone by 15 overlapping digital circles both in the square and the triangular grid. Fig. 11 shows the original figure and the results as well.

We analyzed the spreading of wave-fronts originating from a point. Other kinds of sources of wave-fronts can be examined as well. For example, the so-called lanes (investigated in (Nagy, 2003)) which include all points with a fixed coordinate. It is easy to show that in this case the wave-fronts can be only of the types 'straight' and 'sawtooth'. The meeting points of a blowing digital circle and a wave-front starting from a lane form a (digital) parabola. In Fig. 12 some parabolas can be seen in the triangular grid. The parameters of the parabolas are also marked (in each figure a point and a lane).

As we mentioned in the introduction our aim was not to approximate the Euclidean circles, but discovering and using digital circles defined by digital distance functions based on neighbourhood sequences in the triangular grid. The B-distances have only non-negative integer values therefore we dealt only with circles with integer radii. We would like to mention here, that in the case of the triangular grid the approximation of the Euclidean circles is not so trivial as in the square grid. The triangular grid has some strange properties which we mentioned before. For example, the radius of the circle $\langle C3 \rangle$ can be 2 (with neighbourhood sequence (2)), can be 3 (for instance using $(1, 2, 1)$) or it can also be 4 (using (1)). Therefore the approximation is highly dependent on the 'goodness' measure and on the formal aim of the approximation. Intuitively it is trivial (and it is easy to prove formally as well), that generally with our dodecagons we can have better approximations than with the octagons which are the circles on the square grid using neighbourhood sequences.

6. Conclusion

In this paper we presented some results about triangular neighbourhood sequences, namely we characterize the digital circles. We gave the possible types of edges and vertices of these digital polygons and studied their development in growing procedure. We listed the types of the digital circles occupied by neighbourhood sequences, and performed the symmetry and convexity analysis of these regions. Moreover we presented state transition diagrams, which can work as automata.

The development of the wave-fronts and so, the digital circles can be drawn by a simple algorithm.

In cellular neural network universal machine structures, in some image processes there are effective algorithms which are based on morphological procedures of waves on binary pictures. It can be an interesting further research to analyze these algorithms in the triangular grid. In practice, it would also be interesting to examine the development of the wave-front sets in the case of 'barrels'. Another possible direction of future research is the further analysis of meeting waves, etc. It would be interesting if one mixed our method of region growing with the methods used in practice (Chen et al., 1986; Gonzalez and Woods, 1992). Extensions to non-regular grids can be topics of further research, as well.

Acknowledgements

The author would like to thank M. Arató and A. Radványi for their help. The author also wishes to thank the referees for their comments.

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