ON THE COMPLEXITY OF A CONCENTRATOR

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ABSTRACT

In this paper a switching network with n inputs and m outputs is considered. The network satisfies the following condition: any $k \leqslant m$ inputs can be simultaneously connected to some k outputs. Such networks are referred to as (n, m) concentrators. The problem of constructing a concentrator with a minimum possible number of crosspoints is investigated.

A concentrator with less than 29n crosspoints is constructed. Two cases are considered:

a) $\frac{m}{n} \longrightarrow 0$ for $n \longrightarrow \infty$, b) $\frac{m}{n} \longrightarrow 1$ for $n \longrightarrow \infty$.

The constructed concentrator has asymptotically no more than 3n crosspoints in the case (a) and 4n crosspoints in the case (b).

The paper is concerned with a switching network having n inputs and m outputs $(m < n)$ and satisfying the following condition: any $k \leqslant m$ inputs can be simultaneously connected to some k outputs. Such networks are referred to as (n, m) - concentrators.

The paper deals with the problem of constructing concentrators with a minimum possible number of crosspoints.

As in [1], the problem is formulated and solved in terms of the theory of graphs. It is necessary to construct an oriented n inputs m outputs graph f in which any m of n inputs can be connected to m outputs by non-intersecting paths. The number $Q(f)$ of crosspoints of a concentrator is determined as the number of edges in this graph f.

Let $F_{n,m}$ be a set of all oriented graphs f which are (n, m) - concentrators and

$$Q(n, m) = \min_{f \in F_{n,m}} Q(f) \quad (1)$$

The main results of the work can be summarized in the following theorem.

Theorem:

(a) $2n-2 \leqslant Q(n, m) < cn, \quad m \geqslant 2$, (2)

where the constant $c < 29$ is independent of n and m;

(b) $Q(n, \alpha n) \leqslant 3n(1+o(1))$, $o(1) \longrightarrow 0$ for $\alpha \to 0$, $n \longrightarrow \infty$, (3)*)

(c) $Q(n, \alpha n) \leqslant 4n(1+o(1))$, $o(1) \longrightarrow 0$ for $\alpha \longrightarrow 1$, $n \longrightarrow \infty$. (4)

The proof of the Theorem is based on several lemmas.

Let us first introduce some definitions. An oriented graph with n inputs and r outputs is called an (n, r, m) - concentrator (where $n > r > m$; n, r, m are integers) if any m of n inputs can be connected by non-intersecting paths to some m outputs. When $r = m$ an (n, r, m) - concentrator

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*) For the sake of simplicity of notions, it is assumed that αn, βn, etc. are integers.

becomes an (n, m) - concentrator. The set of all (n, r, m) concentrators will be denoted by $F_{n,r,m}$ and

$$Q(n,r,m) = \min_{f \in F_{n,r,m}} Q(f)$$

An oriented graph containing no vertices other than inputs and outputs is called an elementary graph. An elementary graph each input of which is connected to s outputs is called an s - regular graph.

Let us also consider the following operations with oriented graphs:

(a) inversion: graph $\bar{f}$ is obtained from graph f by reversing the orientation of each edge of the graph f; the outputs of the graph $\bar{f}$ are the inputs of the graph f, and vice versa;

(b) multiplication: if the number of outputs of the graph f is equal to the number of inputs of the graph g, then a graph $f \times g$ is obtained by identifying the outputs of the graph f with the inputs of the graph g (the inputs and outputs of each graph are assumed to be enumerated);

(c) addition: if the graphs f and g have no common edges and its common vertices are common inputs or common outputs of both graphs, a graph $f+g$ is obtained by joining the vertices and edges of the graphs f and g; the inputs and outputs of the graph $f+g$ are obtained by joining the inputs and outputs respectively of the graphs f and g.

Lemma 1.

$$Q(n,r,m) \leqslant sn, \quad m < r,n \tag{5}$$

for any integer s satisfying the inequalities:

$$\left[H\left(\frac{m}{n}\right) + \frac{r}{n} H\left(\frac{m}{r}\right) + \frac{1}{n}\log\frac{r(m-s)}{2\pi(r-m)^2}\right] / \left(-\frac{m}{n}\log\frac{m}{r}\right) < s <$$

$$< \frac{r}{n} C_{r-1}^{s-1} / (nm)^{1/s} \quad , \tag{6}$$

where $H(\alpha) = -\alpha\log\alpha - (1-\alpha)\log(1-\alpha)$, $0 < \alpha < 1$.

The proof of Lemma 1 and its discussion are given in the Appendix.

Lemma 2. For $2m \geqslant n \geqslant m$

$$Q(n,m) \leqslant m + Q(2(n-m),\ n-m) + Q(m,n-m) \tag{7}$$

Proof of Lemma 2.

Let us construct a graph f, which can be represented in the form

$$f = f_1 + f_2, \quad f_2 = f_3 \times \bar{f}_4, \tag{8}$$

where f_1 is an elementary graph with m inputs, m outputs and m edges; the j-th edge connects the j-th input to the j-th output; f_3 and f_4 are arbitrary $(2(n-m),\ n-m)$ and $(m,n-m)$ concentrators; graphs f_1 and f_2 have m common outputs and $n-m$ common inputs. It is obvious that in the graph f_2 any $n-m$ of $2(n-m)$ inputs can be connected by non-intersecting paths to any $n-m$ of m inputs.

Let us show that $f \in F_{n,m}$, then (7) follows from (8). Let A be an arbitrary set of m inputs of the graph f. Let us now divide the set A into two disjoint subsets A_1 and A_2; A_2 contains $n-m$ vertices of the graph f_2, including all the vertices from A which are not inputs of the graph f_1, while A_1 contains $2m-n$ vertices from A which are the inputs of the graph f_1. To obtain the desired paths, we connect the vertices of A_1 in the graph f_1 to $2m-n$ outputs by non-intersecting paths and the vertices of A_2 in the graph f_2 to the remaining non-occupied $n-m$ outputs.

Lemma 2 is proved.

Lemma 3. For $m < r < n$

$$Q(n,m) \leqslant Q(n,r,m) + Q(r,m) \tag{9}$$

$$Q(r,m) \leqslant Q(n,m) \tag{10}$$

Proof of Lemma 3.

Relation (9) follows from the fact that the graph f, representable in the form $f = f_1 \times f_2$, where $f_1 \in F_{n,r,m}$, $f_2 \in F_{r,m}$, is a (n,m) - concentrator.

Inequality (10) is obvious, since considering r inputs of a concentrator with n inputs and m outputs, we obtain a concentrator with r inputs and m outputs.

Lemma 3 is proved.

Proof of the Theorem.

Let us show, first of all, that

$$Q\left(n, \frac{5}{6}n\right) \leqslant 22n. \tag{11}$$

If $n \leqslant 26$, then connecting each input with each of $\frac{5}{6}n$ outputs by edges we obtain a concentrator f for which $Q(f) < 22n$; it proves (11). Let now $n > 26$. From (7) and (10) we obtain the following results:

$$Q\left(n, \frac{5}{6}n\right) \leqslant \frac{5}{6}n + Q\left(\frac{1}{3}n, \frac{1}{6}n\right) + Q\left(\frac{5}{6}n, \frac{1}{6}n\right);$$

$$Q\left(\frac{5}{6}n, \frac{1}{6}n\right) \leqslant Q\left(\frac{5}{6}n, \frac{1}{3}n, \frac{1}{6}n\right) + Q\left(\frac{1}{3}n, \frac{1}{6}n\right);$$

$$Q\left(\frac{1}{3}n, \frac{1}{6}n\right) \leqslant Q\left(\frac{1}{3}n, \frac{1}{5}n, \frac{1}{6}n\right) + Q\left(\frac{1}{5}n, \frac{1}{6}n\right).$$

Hence,

$$Q\left(n, \frac{5}{6}n\right) \leqslant \frac{5}{6}n + 2Q\left(\frac{1}{3}n, \frac{1}{5}n, \frac{1}{6}n\right) + 2Q\left(\frac{1}{5}n, \frac{1}{6}n\right) + Q\left(\frac{5}{6}n, \frac{1}{3}n, \frac{1}{6}n\right). \tag{12}$$

By (5)

$$Q\left(\tfrac{1}{3}n, \tfrac{1}{5}n, \tfrac{1}{6}n\right) \leqslant 11 \cdot \tfrac{1}{3}n = 3\tfrac{2}{3}\, n;$$

$$Q\left(\tfrac{5}{6}n, \tfrac{1}{3}n, \tfrac{1}{6}n\right) \leqslant 6 \cdot \tfrac{5}{6}\, n = 5n.$$

Substituting the right sides of these relations into (12), the following result is obtained:

$$Q\left(n, \tfrac{5}{6}n\right) \leqslant \tfrac{5}{6}n + 7\tfrac{1}{3}n + 2Q\left(\tfrac{1}{5}n, \tfrac{1}{6}n\right) + 5n = 13\tfrac{1}{6}n + 2Q\left(\tfrac{1}{5}n, \tfrac{1}{6}n\right),$$

from which it follows that inequality (11) holding for $n \leqslant 26$ remains valid for any integer $n > 26$.

To get the right inequality in (2), let us consider three cases: (1) $\frac{5}{6} \leqslant \frac{m}{n} \leqslant 1$, (2) $\frac{1}{3} \leqslant \frac{m}{n} \leqslant \frac{5}{6}$, (3) $0 < \frac{m}{n} < \frac{1}{3}$.

(1) From (10) and (11) we have

$$Q(n,m) \leqslant Q\left(\tfrac{6}{5}\, m, m\right) \leqslant 22 \cdot \tfrac{6}{5}\, m \leqslant 26.2n\,. \qquad (13)$$

(2) From (9) we obtain the inequality

$$Q(n,m) \leqslant Q\left(n, \tfrac{6}{5}m, m\right) + Q\left(\tfrac{6}{5}\, m, m\right),$$

from which, using (11) and (5), follows

$$Q(n,m) \leqslant sn + 22 \cdot \tfrac{6}{5}\, m = sn + 26.2m < 29n$$

(3) By (9)

$$Q(n,m) \leqslant Q\left(n, \tfrac{2}{3}n, m\right) + Q\left(\tfrac{2}{3}\, n, m\right). \qquad (14)$$

From (5), $Q(n, \frac{2}{3}n, m) \leqslant 6n$. Substitution of the right side of this inequality into (14) gives

$$Q(n,m) \leqslant 6n + Q\left(\tfrac{2}{3}n, m\right)$$

and the inequality $Q(n,m) < 29n$ follows obviously from validity of this inequality for $\frac{m}{n} \geqslant \frac{1}{3}$.

Thus, the right inequality in (2) is proved.

Let us turn now to deriving inequality (3). From (9), we obtain:

$$Q(n, \alpha n) \leqslant Q(n, \beta n, \alpha n) + Q(\beta n, \alpha n). \qquad (15)$$

It can be seen from (5) and (6) that for sufficiently small α, β and $-\frac{\alpha}{\beta}\log \beta$, $(\alpha < \beta)$

$$Q(n, \beta n, \alpha n) \leqslant 3n.$$

Therefore

$$Q(n, \alpha n) \leqslant 3n + 29\beta n = 3n(1 + o(1)), \quad o(1) \to 0$$

for $\alpha \to 0$, $n \to \infty$.

Finally, to derive (4), let us use (7) and (3):

$$Q(n, \alpha n) = \alpha n + Q(2(n - \alpha n), n - \alpha n) + Q(\alpha n, n - \alpha n) \leqslant$$

$$\leqslant \alpha n + 29.2(n - \alpha n) + 3\alpha n(1 + o(1)) = 4\alpha n(1 + o(1)) =$$

$$= 4n(1 + o(1)), \quad o(1) \to 0 \quad \text{for } \alpha \to 1,\ n \to \infty.$$

It now remains to prove the left inequality in (2). Two cases are possible:

(1) At least two different paths issue from each input of the concentrator $f \in F_{n,m}$. Let us assign to each input vertex a_i, $i = \overline{1,n}$, a vertex c_i where the paths issuing from a_i branch off for the first time. It is easy to see that different a_i and a_j are associated with different c_i and c_j, otherwise it would be impossible to connect the vertices a_i and a_j to output vertices with non-intersecting paths, which would imply that f is not a concentrator.

Since at least two edges issue from each vertex c_i, the following holds in the case under consideration:

$$Q(f) \geqslant 2n \qquad (16)$$

(2) Assume that a single path issues from a certain vertex a_i and ends at an output vertex b_i. Then some input different from a_i can be connected to the output b_i if b_i is the end of the edge not included in the path. Therefore, examining the concentrator obtained from f by removing the vertices a_i and b_i, we find that

$$Q(n,m) \geqslant 2 + Q(n - 1, m - 1). \qquad (17)$$

The number of input vertices l from which only one path issues cannot exceed the number of outputs m, therefore $Q(n,2) \geqslant 2n-2$.

Hence, from (16) and (17) follows

$$Q(n,m) \leqslant 2l + Q(n-1, m-1)$$

$$\leqslant 2l + 2(n-1) = 2n;\ m-1 \geqslant 2;$$

$$Q(n,m) \leqslant 2(m-2) + Q(n-m+2, 2) = 2(m-2) + 2(n-m+2) - 2 =$$

$$= 2n-2,\ m-1 < 2.$$

This proves the Theorem.

<u>Remark.</u> The estimate $c < 29$ is a rough one; it can be somewhat improved if to use (5) more accurately.

APPENDIX

<u>Proof of Lemma 1.</u>

An auxiliary concept is introduced. An elementary graph f with n inputs and r outputs is called an m expanding graph if any set of $k \leqslant m$ inputs is connected by edges to at least k outputs.

According to the König's theorem [2], if an elementary graph is m-expanding, then any $k \leqslant m$ inputs can be connected to the outputs in such a way that different outputs would correspond to different inputs. Therefore, to prove Lemma 1 it is sufficient to show that for s satisfying (6) there exists an elementary s - regular m - expanding graph with n inputs and r outputs.

Let us relate each elementary graph f to its incidence matrix, i.e. a matrix in which the number of rows is equal to the number of inputs and the number of columns to the number of outputs, with 1's at the intersections of rows and columns if the corresponding inputs and outputs are connected and 0 otherwise. An elementary graph is m-expanding if and only if its incidence matrix contains no minor of the dimension $k \times (r-k+1)$, $k \leq m$, equal to zero. There exist $(C_r^s)^n$, $s \leq r$, incidence matrices of elementary s-regular graphs with n inputs and r outputs, i.e. $n \times r$ matrices with s ones in each row. Among them there are no more than $(C_{k-1}^s)^k (C_r^s)^{n-k}$ matrices whose fixed minor of the dimension $k \times (r-k+1)$ is equal to zero. Totally there are no more than

$$C_n^k \; C_r^{k-1} (C_{k-1}^s)^k (C_r^s)^{n-k}$$

$n \times r$ matrices, where each row contains s ones and one of $k \times (r-k+1)$ minors is equal to zero. Therefore, when

$$\sum_{k=s+1}^{m} C_n^k \, C_r^{k-1} (C_{k-1}^s)^k \, (C_r^s)^{n-k} < (C_r^s)^n \qquad (A1)$$

there exists an elementary s-regular m - expanding graph. Let us rewrite the relation (A1) in the form

$$\sum_{k=s+1}^{m} C_n^k \, C_r^{k-1} (C_{k-1}^s)^k \, / \, ({C_r}^s)^k < 1 \qquad (A2)$$

It is easy to see that either the last or the first term in (A2) is the largest and consequently to fulfil (A1) it is sufficient to fulfil each of the inequalities

$$(m-s) \, {C_n}^{s+1} \, / \, (C_r^s)^s < 1 \quad ; \qquad (A3)$$

$$(m-s) \, C_n^m \, {C_r}^{m-1} (C_{m-1}^s)^m \, / \, (C_r^s)^m < 1 \qquad (A4)$$

We have

$$(m-s) \, C_n^{s+1} \, / \, (C_r^s)^s \leqslant nm \left(\frac{n}{r} \right)^s s^s / (C_{r-1}^{s-1})^s =$$

$$= nm \left(\frac{sn}{r} / \, C_{r-1}^{s-1} \right)^s \qquad (A5)$$

and using the inequality (see [3], § 7.1)

$$\log C_n^l < n \, H\left(\frac{l}{n} \right) - \frac{1}{2} \log(2\pi) - \frac{1}{2} \log \left(l \left(1 - \frac{l}{n}\right)\right) \qquad (A6)$$

we derive

$$\log \left[(m-s) C_n^m {C_r}^{m-1} ({C_{m-1}}^s)^m / ({C_r}^s)^m \right] \leqslant$$

$$\leqslant \log \frac{r(m-s)}{(r-m)^2} + nH\left(\frac{m}{n} \right) + rH\left(\frac{m}{n} \right) - \log(2\pi) +$$

$$+ ms \log \frac{m}{r} \, . \qquad (A7)$$

Comparing (A3) with (A5) and (A4) with (A7) we find that to fulfil (6) it is sufficient to fulfil inequality (A1).
Lemma 1 is proved.

Remark. There is only one non-constructive part in the proof of the Theorem - the proof of existence of an expanding graph in Lemma 1. Recently, G.A.Margulis [4] has get regular methods for constructing such graphs.

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