Structures and Representations of Generalized Path Algebras

Shouchuan Zhang [a, b] *and Yao-Zhong Zhang* [b]
a. Department of Mathematics, Hunan University
Changsha 410082, P.R. China
b. Department of Mathematics, University of Queensland
Brisbane 4072, Australia

Abstract

It is shown that an algebra Λ can be lifted with nilpotent Jacobson radical $r = r(\Lambda)$ and has a generalized matrix unit $\{e_{ii}\}_I$ with each $\bar{e}_{ii}$ in the center of $\bar{\Lambda} = \Lambda/r$ iff Λ is isomorphic to a generalized path algebra with weak relations. Representations of the generalized path algebras are given. As a corollary, Λ is a finite algebra with non-zero unity element over perfect field k (e.g. a field with character zero or a finite field) iff Λ is isomorphic to a generalized path algebra $k(D, \Omega, \rho)$ of finite directed graph with weak relations and $dim\ \Omega < \infty$; Λ is a generalized elementary algebra which can be lifted with nilpotent Jacobson radical and has a complete set of pairwise orthogonal idempotents iff Λ is isomorphic to a path algebra with relations.

0 Introduction

It is well known that every elementary algebra is isomorphic to a path algebra of a finite directed graph with relations (see [2]). In fact, every path algebra of a finite directed graph with relations is also an elementary algebra. The results are very useful because all representations of path algebras can be obtained easily. in [3] F.U. Coelho and S.X. Liu introduced the concept of generalized path algebras to study other algebras. Recently, it is proved in [4] that some finite-dimensional algebras over a field with character 0 are isomorphic to generalized path algebras of finite directed graphs with weak relations. However, to our knowledge, representations of the generalized path algebras and infinite-dimensional generalized path algebras have been barely touched so far in the literature.

The aim of this paper is to give the structures and representations of generalized path algebras of infinite directed graphs with weak relations. We study the infinite dimensional algebras and generalized path algebras by using generalized matrix algebras introduced in [7]. In fact, every generalized path algebra is a generalized matrix algebra. In section 1, we study the structure of generalized matrix rings. We find the relations among the decomposition of a ring, the complete set of pairwise orthogonal idempotents (possibly infinite many) and generalized matrix ring. This generalizes the theory about decomposition of rings. In section 2, we study the representations of the generalized path algebras. In section 3, we characterize the generalized path algebras with weak relations by algebras which can be lifted with nilpotent Jacobson radical.

We say that an algebra Λ can be lifted, if there exists a subalgebra A of Λ such that $\Lambda = A \oplus r(\Lambda)$. By the famous Wedderburn-Malcev Theorem (see [6, Theorem 11.6 and Corollary 11.6]), for every finite dimensional algebra Λ over field k with char $k = 0$, Λ can be lifted and $r(\Lambda)$ is nilpotent. We shall see, in section 3, that every generalized path algebra with weak relations can be also lifted and its Jacobson radical is nilpotent. In that section we show that the converse also holds. That is, it is shown that an algebra Λ is isomorphic to a generalized path algebra with weak relations iff Λ can be lifted with nilpotent Jacobson radical $r(\Lambda)$ and has a complete set $\{e_{ii}\}_I$ of pairwise orthogonal idempotents with each $\bar{e}_{ii}$ in the center of $\bar{\Lambda} = \Lambda/r$. As a corollary, Λ is a finite algebra with non-zero unity element over field k iff Λ is isomorphic to a generalized path algebra $k(D, \Omega, \rho)$ of finite directed graph with weak relations and the dimension of Ω is finite; Λ is a generalized elementary algebra which can be lifted with nilpotent Jacobson radical iff Λ is isomorphic to a path algebra with relations.

Preliminaries

Let k be a field. We first recall the concepts of Γ_I-systems, generalized matrix rings (algebras) and generalized path algebras. Let I be a non-empty set. If for any $i, j, l, s \in I, A_{ij}$ is an additive group and there exists a map μ_{ijl} from $A_{ij} \times A_{jl}$ to A_{il} (written $\mu_{ijl}(x, y) = xy$) such that the following conditions hold:

(i) $(x + y)z = xz + yz, \quad w(x + y) = wx + wy;$

(ii) $w(xz) = (wx)z,$

for any $x, y \in A_{ij}$, $z \in A_{jl}, w \in A_{li}$, then the set $\{A_{ij} \mid i, j \in I\}$ is a Γ_I -system with index I.

Let A be the external direct sum of $\{A_{ij} \mid i, j \in I\}$. We define the multiplication in A as

$$xy = \{\sum_k x_{ik}y_{kj}\}$$

for any $x = \{x_{ij}\}, y = \{y_{ij}\} \in A$. It is easy to check that A is a ring (possibly without the unity element). We call A a generalized matrix ring, or a gm ring in short, written as $A = \sum\{A_{ij} \mid i,j \in I\}$. For any non-empty subset S of A and $i,j \in I$, set $S_{ij} = \{a \in A_{ij} \mid$ there exists $x \in S$ such that $x_{ij} = a\}$. If B is an ideal of A and $B = \sum\{B_{ij} \mid i,j \in I\}$, then B is called a gm ideal. If for any $i,j \in I$, there exists $0 \neq e_{ii} \in A_{ii}$ such that $x_{ij}e_{jj} = e_{ii}x_{ij} = x_{ij}$ for any $x_{ij} \in A_{ij}$, then the set $\{e_{ii} \mid i \in I\}$ is called a generalized matrix unit of Γ_I-system $\{A_{ij} \mid i,j \in I\}$, or a generalized matrix unit of gm ring $A = \sum\{A_{ij} \mid i,j \in I\}$, or a gm unit in short. It is easy to show that if A has a gm unit $\{e_{ii} \mid i \in I\}$, then every ideal B of A is a gm ideal. Indeed, for any $x = \sum_{i,j\in I} x_{ij} \in B$ and $i_0, j_0 \in I$, since $e_{i_0i_0}xe_{j_0j_0} = x_{i_0j_0} \in B$, we have $B_{i_0j_0} \subseteq B$. Furthermore, if B is a gm ideal of A, then $\{A_{ij}/B_{ij} \mid i,j \in I\}$ is a Γ_I-system and $A/B \cong \sum\{A_{ij}/B_{ij} \mid i,j \in I\}$ as rings.

If for any $i,j,l,s \in I$, A_{ij} is a vector space over field k and there exists a k-linear map μ_{ijl} from $A_{ij} \otimes A_{jl}$ into A_{il} (written $\mu_{ijl}(x,y) = xy$) such that $x(yz) = (xy)z$ for any $x \in A_{ij}$, $y \in A_{jl}, z \in A_{ls}$, then the set $\{A_{ij} \mid i,j \in I\}$ is a Γ_I- system with index I over field k. Similarly, we get an algebra $A = \sum\{A_{ij} \mid i,j \in I\}$, called a generalized matrix algebra, or a gm algebra in short.

Assume that D is a directed (or oriented) graph (D is possibly an infinite directed graph and also possibly not a simple graph) (or quiver). Let $I = D_0$ denote the vertex set of D and D_1 denote the set of arrows of D. Let Ω be a generalized matrix algebra over field k with gm unit $\{e_{ii} \mid i \in I\}$, the Jacobson radical $r(\Omega_{ii})$ of Ω_{ii} is zero and $\Omega_{ij} = 0$ for any $i \neq j \in I$. The sequence $x = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n}a_{i_n}$ is called a generalized path (or Ω-path) from i_0 to i_n via arrows $x_{i_0i_1}, x_{i_1i_2}, x_{i_2i_3}, \cdots, x_{i_{n-1}i_n}$, where $0 \neq a_{i_p} \in \Omega_{i_pi_p}$ for $p = 0,1,2,\cdots,n$. In this case, n is called the length of x, written $l(x)$. For two Ω-paths $x = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n}a_{i_n}$ and $y = b_{j_0}y_{j_0j_1}b_{j_1}y_{j_1j_2}b_{j_2}y_{j_2j_3}\cdots y_{j_{m-1}j_m}b_{j_m}$ of D with $i_n = j_0$, we define the multiplication of x and y as

$$xy = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n}(a_{i_n}b_{j_0})y_{j_0j_1}y_{j_1j_2}b_{j_1}y_{j_2j_3}\cdots y_{j_{m-1}j_m}b_{j_m}. \qquad (*)$$

For any $i,j \in I$, let A'_{ij} denote the vector space over field k with basis being all Ω-paths from i to j. B_{ij} is the sub-space spanned by all elements of forms:

$$a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}\cdots x_{i_{s-1}i_s}(a^{(1)}_{i_s} + a^{(2)}_{i_s} + \cdots + a^{(m)}_{i_s})x_{i_si_{s+1}}\cdots x_{i_{n-1}i_n}a_{i_n}$$
$$-\sum_{l=1}^{m} a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}x_{i_2i_3}\cdots x_{i_{s-1}i_s}a^{(l)}_{i_s}x_{i_si_{s+1}}\cdots x_{i_{n-1}i_n}a_{i_n},$$

where $i_0 = i, i_n = j, a^{(l)}_{i_s} \in \Omega_{i_si_s}, a_{i_p} \in \Omega_{i_pi_p}$, $x_{i_ti_{t+1}}$ is an arrow, $p = 0,1,\cdots,n$, $t = 0,1,\cdots,n-1$, $l = 0,1,\cdots,m$, $0 \le s \le n$, n and m are natural numbers. Let $A_{ij} = A'_{ij}/B_{ij}$, written $[\alpha] = \alpha + B_{ij}$ for any generalized path α from i to j. We can get a k-linear map from $A_{ij} \otimes A_{jl}$ to A_{il} induced by $(*)$. We write a instead of $[a]$ when $a \in \Omega$. In fact, $[\Omega_{ii}] \cong \Omega_{ii}$ as algebras for any $i \in I$. Notice that we write $e_{ii}x_{ij} = x_{ij}e_{jj} = x_{ij}$ for any arrow x_{ij} from i to j. It is clear that $\{A_{ij} \mid i, j \in I\}$ is a Γ_I-system with gm unit $\{e_{ii} \mid i \in I\}$. The gm algebra $\sum\{A_{ij} \mid i, j \in I\}$ is called the generalized path algebra, or Ω-path algebra, written as $k(D, \Omega)$ (see, [2, Chapter 3] and [3]). Let J denote the ideal generated by all arrows in D of $k(D, \Omega)$. If ρ is a non-empty subset of $k(D, \Omega)$ and the ideal (ρ) generated by ρ satisfies $J^t \subseteq (\rho) \subseteq J^2$, then $k(D, \Omega)/(\rho)$ is called generalized path algebra with relations. If $J^t \subseteq (\rho) \subseteq J$, then $k(D, \Omega)/(\rho)$ is called generalized path algebra with weak relations. If $\Omega_{ii} = ke_{ii}$ for any $i \in I$, then $k(D, \Omega)$ is called a path algebra, written as kD. If D_0 and D_1 are finite sets, then D is called a finite directed graph.

Let $r(\Lambda)$ denote the Jacobson radical of ring Λ. Let $|S|$ denote the number of elements in set S. Let δ_{ij} denote the Kronecker δ-function. Rings and algebras are possible without unity elements.

1 Decomposition of generalized matrix rings

In this section, we study the structure of generalized matrix rings. We find the relations among the decomposition of a ring, the complete set of pairwise orthogonal idempotents (possible infinite many) and generalized matrix rings. This generalizes the theory of direct sum decomposition of rings in [1].

Definition 1.1 *If A is an algebra and $\{e_{ii} \mid i \in I\} \subseteq A$ such that the following conditions are satisfied (i) $e_{ii}e_{jj} = \delta_{ij}e_{ii}$ for any $i, j \in I$; (ii) for any $x \in A$, there exists a finite subset F of I such that $(\sum_{i\in F} e_{ii})x = x(\sum_{i\in F} e_{ii}) = x$; (iii) $e_{ii} \neq 0$ for any $i \in I$, then $\{e_{ii} \mid i \in I\}$ is called the complete set of pairwise orthogonal idempotents of A with index I. Moreover, if each e_{ii} is a primitive idempotent (i.e. it can not be written as a sum of two non-zero orthogonal idempotents), then $\{e_{ii} \mid i \in I\}$ is called a complete set of pairwise orthogonal primitive idempotents of A with index I*

Remark : (i) Let $\{e_{ii} \mid i \in I\}$ be a complete set of pairwise orthogonal idempotents of A. Assume that $x \in A$ and finite subset $F \subseteq I$ such that $x = (\sum_{i\in F} e_{ii})x = x(\sum_{i\in F} e_{ii}) = x$. If F' is a finite subset of I and $F \subseteq F'$, then $x = (\sum_{i\in F'} e_{ii})x = x(\sum_{i\in F'} e_{ii}) = x$. Indeed,

$$\begin{aligned}(\sum_{i\in F'} e_{ii})x &= (\sum_{i\in F'} e_{ii})((\sum_{i\in F} e_{ii})x) \\ &= ((\sum_{i\in F'} e_{ii})(\sum_{i\in F} e_{ii}))x \\ &= (\sum_{i\in F} e_{ii})x \\ &= x.\end{aligned}$$

Similarly, $x(\sum_{i\in F'} e_{ii}) = x$.

(ii) Let I be a non-empty set and A a ring with additive sub-groups A_{ij} for any $i,j \in I$. If $A = \sum_{i,j\in I} A_{ij}$ as additive groups and $A_{ij}A_{st} \subseteq \delta_{js}A_{it}$ for any $i,j,s,t \in I$, then $\{A_{ij}, \mid i,j \in I\}$ is a Γ_I -system. Let A' denote the gm ring $\sum\{A_{ij} \mid i,j \in I\}$ of Γ_I-system $\{A_{ij}, \mid i,j \in I\}$. Moreover, if A_{ii} has a non-zero unity element e_{ii} for any $i \in I$, then A is the inner direct sum of $\{A_{ij}, \mid i,j \in I\}$ as additive groups and A' is isomorphic to A under canonical isomorphism ϕ by sending $\{x_{ij}\}$ to $\sum_{i,j\in I} x_{ij}$ for any $\{x_{ij}\} \in A'$. In this case, A is called the inner gm ring of Γ_I-system $\{A_{ij}, \mid i,j \in I\}$, also written $A = \sum\{A_{ij}, \mid i,j \in I\}$. If we view each element in A_{ij} as one in $\sum\{A_{ij} \mid i,j \in I\}$, then every gm ring can be viewed as an inner gm ring. Similarly, every inner gm ring can be viewed as a gm ring.

Theorem 1.2 *A has a complete set $\{e_{ii} \mid i \in I\}$ of pairwise orthogonal idempotents with index I iff $A = \sum\{A_{i,j} \mid i,j \in I\}$ is a gm ring with gm unit $\{e_{ii} \mid i \in I\}$ and $A_{ij} = e_{ii}Ae_{jj}$ for any $i,j \in I$.*

Proof. The sufficiency is obvious. We now prove the necessity. Assume that A has a complete set $\{e_{ii} \mid i \in I\}$ of pairwise orthogonal idempotents with index I. Let $A_{ij} = e_{ii}Ae_{jj}$ for any $i,j \in I$. It is easy to check $A_{ij}A_{st} \subseteq \delta_{js}A_{it}$ for any $i,j,s,t \in I$. Thus A is an inner gm ring of $\{A_{i,j} \mid i,j \in I\}$ with gm unit $\{e_{ii} \mid i \in I\}$. □

This theorem implies that an algebra A has a complete set of pairwise orthogonal idempotents iff A is a gm ring with gm unit.

Proposition 1.3 *(i) If A has the non-zero unity element u then A has a complete set $\{e_{ii} \mid i \in I\}$ of pairwise orthogonal idempotents with finite index I and $\sum_{i\in I} e_{ii} = u$.*

(ii) If ring A has the non-zero unity element u and a complete set $\{e_{ii} \mid i \in I\}$ of pairwise orthogonal idempotents with index I, then I is a finite set and $\sum_{i\in I} e_{ii} = u$.

(iii) If A is a finite dimensional algebra over field k, then A has the non-zero unity element iff A has gm unit.

Proof. (i) Let $I = \{1\}$ and $u = e_{11}$.

(ii) Since A has a gm unit $\{e_{ii}\}_I$, by Theorem 1.2, $A = \sum\{A_{ij} \mid i,j \in I\}$ is a gm ring with gm unit $\{e_{ii}\}_I$ and $A_{ij} = e_{ii}Ae_{jj}$ for any $i.j \in I$. Let $u = \sum_{i,j\in F} u_{ij}$ with finite subset F of I and $u_{ij} \in A_{ij}$ for any $i,j \in F$. Since u is the unity element of A, $A_{ij} = 0$ for any $i \notin F$ or $j \notin F$. Thus $F = I$ since $e_{ii} \neq 0$ for any $i \in I$. For any $s \in I$ and $x_{ss} \in A_{ss}$, since $ux_{ss} = x_{ss}$ and $x_{ss}u = x_{ss}$, we have $u_{ss}x_{ss} = x_{ss}$ and $x_{ss}u_{ss} = x_{ss}$. This implies $u_{ss} = e_{ss}$ for any $s \in F$. Next we show $u_{ij} = 0$ when $i \neq j$. On the one hand, $u_{ii}u = u_{ii}$. On the other hand, $u_{ii}u = \sum_{s\in I} u_{ii}u_{is}$. Consequently, $u_{ij} = 0$ for any $i \neq j$.

(iii) If A has gm unit $\{e_{ii}\}_I$, then I is finite since A is finite dimensional. It is clear that $u = \sum_{i\in I} e_{ii}$ is the unity element of A. The converse follows from (i). □

Proposition 1.4 *If A is a left (or right) artinian or noetherian ring with gm unit $\{e_{ii}\}_I$, then I is finite and $\sum_{i\in I} e_{ii}$ is the unity element of A.*

Proof. By Theorem 1.2, $A = \sum\{A_{ij} \mid i, j \in I\}$ with $A_{ij} = e_{ii}Ae_{jj}$ for any $i, j \in I$. If I is infinite, then there exists a infinite sequence $i_1, i_2, \cdots, i_n, \cdots$ in I. Let $A_1 = Ae_{i_1 i_!}$, $A_2 = A_1 + Ae_{i_2 i_2}, \cdots, A_{n+1} = A_n + A_{i_{n+1} i_{n+1}}, \cdots$. Obviously $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ is an ascending chain of left ideals of A. Let $B_1 = \sum_{j\in I, j\neq i_1} Ae_{jj}, B_2 = \sum_{j\in I, j\neq i_1, i_2} Ae_{jj}, \cdots, B_{n+1} = \sum_{j\in I, j\neq i_1, i_2, \cdots, i_{n+1}} Ae_{jj}$ for any natural number n. Obviously, $B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots$ is an descending chain of left ideals of A. We get a contradiction. Consequently, I is finite. □

Let $\mathcal{A}\Gamma_I$ denote the category of all Γ_I -systems with gm unit, the morphism of two objects from $\{A_{ij} \mid i, j \in I\}$ with gm unit $\{e_{ii}\}_I$ to $\{B_{ij} \mid i, j \in I\}$ with gm unit $\{e'_{ii}\}_I$ is a set $\{f_{ij}\}_I$, where f_{ij} is an additive group homomorphism from A_{ij} to B_{ij} with $f_{ij}(xy) = f_{is}(x)f_{sj}(y)$ and $f_{ii}(e_{ii}) = e'_{ii}$ for any $i, j, s \in I, x \in A_{is}, y \in A_{sj}$. Let $\mathcal{GM}_I$ denote the category of all generalized matrix algebras with index I and gm unit, the morphism between the two objects is gm homomorphism. A gm homomorphism of two objects from $A = \sum\{A_{ij} \mid i, j \in I\}$ with gm unit $\{e_{ii}\}_I$ to $B = \sum\{B_{ij} \mid i, j \in I\}$ with gm unit $\{e'_{ii}\}_I$ is a ring homomorphism $f : A \to B$ such that $f(A_{ij}) \subseteq B_{ij}$ and $f(e_{ii}) = e'_{ii}$ for any $i, j \in I$.

Proposition 1.5 *$\mathcal{A}\Gamma_I$ and $\mathcal{GM}_I$ are two equivalent categories.*

Proof. Let $H : \mathcal{A}\Gamma_I \to \mathcal{GM}_I$ by $H(\{A_{ij}\}_I) = \sum\{A_{ij} \mid i, j \in I\}$, $H(\{f_{ij}\}_I) = \oplus_{i,j\in I} f_{ij}$ for any morphism $\{f_{ij}\}_I$ from $\{A_{ij} \mid i, j \in I\}$ to $\{B_{ij} \mid i, j \in I\}$. Let $G : \mathcal{GM}_I \to \mathcal{A}\Gamma_I$ by $G(\sum\{A_{ij} \mid i, j \in I\}) = \{A_{ij}\}_I$ and $G(f) = \{f_{ij}\}_I$ with $f_{ij} = f|_{A_{ij}}$ for any $i, j \in I$. Obviously, $HG = id$ and $GH = id$. □

2 Representations of generalized path algebras

In this section, we study representations of the generalized path algebras.

Definition 2.1 *Let $\{A_{ij} \mid i, j \in I\}$ be an Γ_I-system with gm unit $\{e_{ii}\}_I$. For any $i, j \in I, M_i$ is an additive group and there exists a map ϕ_{ij} from $A_{ij} \times M_j$ to M_i (written $\phi_{ij}(a, x) = ax$) such that the following conditions are satisfied:*

(i) $a(x + y) = ax + ay$ and $(a + b)x = ax + bx$.

(ii) $(ca)x = c(ax)$.

(iii) $e_{jj}x = x$

For any $x, y \in M_j, a, b \in A_{ij}, c \in A_{si}$, then $\{M_i \mid i \in I\}$ is called an $\{A_{ij}\}_I$- module system.

Let Rep $\{A_{ij}\}_I$ denote the category of $\{A_{ij}\}_I$-module systems. The morphism of two objects $\{M_i\}_I$ and $\{N_i\}_I$ is a collection $\{f_i\}_I$ such that f_i is an additive group homomorphism from M_i to N_i with $f_i(a_{ij}x_j) = a_{ij}f_j(x_j)$ for any $a_{ij} \in A_{ij}, x_j \in M_j$.

An A-module is called a local unitary A-module if for any $x \in M$ there exists $u \in A$ such that $ux = x$.

Lemma 2.2 *If A is a gm ring with gm unit $\{e_{ii}\}_I$, then M is a local unitary A-module iff M is an A-module with $AM = M$.*

Proof. Assume $AM = M$. For any $x \in M$, there exist $a^{(p)} \in A, x^p \in M$ such that $x = \sum_{p=1}^{n} a^{(p)}x^{(p)}$. There exists a finite subset F of I such that $a^{(p)} \in \sum_{i,j\in F} A_{ij}$ for $p = 1, 2, \cdots, n$. Let $u = \sum_{i\in F} e_{ii}$. We have that $ux = u(\sum_{p=1,2,\cdots,n} a^{(p)}x^{(p)}) = \sum_{p=1,2,\cdots,n} a^{(p)}x^{(p)} = x$. Therefore, M is a local unitary A-module. Conversely, it is clear that $AM = M$ when M is a local unitary A-module. $\Box$

Lemma 2.3 *Let A be a gm ring with gm unit $\{e_{ii}\}_I$.*

(i) If M is a local unitary A-module, then $\{M_i \mid i \in I\}$ is an $\{A_{ij}\}_I$-module system with $e_{ii}M = M_i$.

(ii) If $\{M_i\}_I$ is an $\{A_{ij}\}_I$-module system, then the external direct sum M of $\{M_i\}_I$ becomes a local unitary A-module under module operation $ax = \{\sum_{s\in I} a_{is}x_s\}_I$ for any $a = \{a_{ij}\}_I \in A$, $x = \{x_i\}_I \in M$.

Proof. (i) If M is a local unitary A-module. Set $e_{ii}M = M_i$ for any $i \in I$. It is clear that $\{M_i\}_I$ is an $\{A_{ij}\}_I$-module system. Indeed, for any $x, y \in M_j, a, b \in A_{ij}$ and $c \in A_{si}$, we have that $a(x + y) = ax + ay$, $(a + b)x = ax + bx$, $(ca)x = c(ax)$ and $e_{jj}x = x$.

(ii) It is clear. Indeed, for any $a = \{a_{ij}\}_I, b = \{b_{ij}\}_I \in A$ and $x = \{x_i\}_I \in M$, it is easy to check $(ab)x = a(bx)$. Since there exists finite subset F of I such that $x = \sum_{i\in F} x_i$, we have that $(\sum_{i\in F} e_{ii})x = x$. Thus M is a local unitary A-module. $\Box$

Let ${}_A\mathcal{M}LU$ denote the category of local unitary A-modules. every morphism of two objects M and N is a homomorphism of A- modules.

Theorem 2.4 *Let $A = \sum\{A_{ij} \mid i, j \in I\}$ be a gm ring with gm unit. Then Rep $\{A_{ij}\}_I$ and ${}_A\mathcal{M}LU$ are equivalent.*

Proof. Let H : Rep $\{A_{ij}\}_I \to {}_A\mathcal{M}LU$ by $H(\{M_i\}_I) = \sum\{M_i \mid i \in I\}$, $H(\{f_i\}_I) = \oplus_{i\in I} f_i$ for any morphism $\{f_i\}_I$ between two objects $\{M_i\}_I$ and $\{N_i\}_I$. Let $G : {}_A\mathcal{M}LU \to$ Rep $\{A_{ij}\}_I$ by $G(M) = \{M_i\}_I$ with $M_i = e_{ii}M$ for any $i \in I$. $G(f) = \{f_i\}_I$ with $f_i = f|_{M_i}$ for any morphism f between two objects M and N. It is clear $HG = id$ and $GH = id$. $\Box$

If $A = \sum\{A_{ij} \mid i, j \in I\}$ is a gm algebra over field k with gm unit $\{e_{ii}\}_I$, we can similarly define $\{A_{ij}\}_I$-module systems as follows.

Let $\{A_{ij} \mid i,j \in I\}$ be a Γ_I -system over field k with gm unit $\{e_{ii}\}_I$. If for any $i,j \in I, M_i$ is a vector space and there exists k-linear map ϕ_{ij} from $A_{ij} \otimes M_j$ to M_i (written $\phi_{ij}(a,x) = ax$) such that the following conditions are satisfied:

(i) $(ca)x = c(ax)$.

(ii) $e_{jj}x = x$,

for any $x \in M_j, a \in A_{ij}, c \in A_{si}$, then $\{M_i \mid i \in I\}$ is called an $\{A_{ij}\}_I$- module system. We still use the two notations Rep $\{A_{ij}\}_I$ and ${}_A\mathcal{M}LU$ to denote the corresponding categories.

Theorem 2.5 *Let $A = \sum\{A_{ij} \mid i,j \in I\}$ be a gm algebra with gm unit. Then Rep $\{A_{ij}\}_I$ and ${}_A\mathcal{M}LU$ are equivalent.*

For a generalized path algebra $k(D,\Omega,\rho)$ with weak relations, let $P = k(D,\Omega)$, $N = (\rho)$ and $Q = P/N$. It is clear that the generalized path algebra $k(D,\Omega,\rho)$ with weak relations is a gm algebra, so its representation corresponds to $\{Q_{ij}\}_I$-module system. That is, Rep $\{Q_{ij}\}_I$ and ${}_Q\mathcal{M}LU$ are equivalent. However, we have a simpler category.

A representation of (D,Ω) is a set $(V,f) =: \{V_i, f_\alpha \mid V_i$ is an unitary Ω_{ii}-module, $f_\alpha : V_i \to V_j$ is a k-linear map, $i,j \in I$, α is an arrow from j to $i\}$. A morphism $h : (V,f) \to (V',f')$ between tow representations of (D,Ω) is the collection $\{h_i\}_I$ such that $h_i : V_i \to V_i'$ is a k-linear map and $h_j f_\alpha = f'_\alpha h_i$ for any arrow $\alpha : i \to j$ and $i,j \in I$. Let Rep (D,Ω) denote the category of representations of (D,Ω).

Lemma 2.6 *Let $P = k(D,\Omega)$ and $Q = k(D,\Omega,\rho)$.*

(i) If (V,f) is an object in Rep (D,Ω), then $\{V_i\}_I$ is a $\{P_{ij}\}_I$-module system under operation $\alpha \cdot v_{i_n} = a_{i_0} \cdot f_{x_{i_0 i_1}}(a_{i_1} \cdot (f_{x_{i_1 i_2}} \cdots f_{x_{i_{n-1} i_n}}(a_{i_n} \cdot v_{i_n})))$ for any Ω-path $\alpha = a_{i_0} x_{i_0 i_1} a_{i_1} x_{i_1 i_2} \cdots x_{i_{n-1} i_n} a_{i_n}$ from i_0 to i_n and $v_{i_n} \in V_{i_n}$.

(ii) If $\{V_i\}_I$ is a $\{P_{ij}\}_I$-module system, then (V,f) is an object in Rep (D,Ω) under operation $f_{x_{ij}}(v_j) = x_{ij} \cdot v_j$ for any arrow $x_{ij} \in P_{ij}$ and $v_j \in V_j$.

Proof. (i) It is sufficient to show that

$$(\alpha\beta) \cdot v_{j_m} = \alpha \cdot (\beta \cdot x_{j_m}) \qquad (*)$$

for two Ω- paths $\alpha = a_{i_0} x_{i_0 i_1} a_{i_1} x_{i_1 i_2} a_{i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n} a_{i_n}$ and $\beta = b_{j_0} y_{j_0 j_1} b_{j_1} y_{j_1 j_2} b_{j_2} y_{j_2 j_3} \cdots y_{j_{m-1} j_m} b_{j_m}$ of D with $i_n = j_0$.

When $\alpha\beta \neq 0$, i.e. $a_{i_n} b_{j_0} \neq 0$, $\alpha\beta$ is an Ω-path. By definition, (*) holds. When $\alpha\beta = 0$, i.e. $a_{i_n} b_{j_0} = 0$, $\alpha\beta$ is not an Ω-path. Obviously the left side of (*) =0.

$$\begin{aligned}
\text{The right side of } (*) &= \alpha \cdot (b_{j_0} \cdot f_{y_{j_0 j_1}}(b_{j_1} \cdot f_{y_{j_1 j_2}}(\cdots f_{y_{y_{m-1} j_m}}(b_{i_m} \cdot v_{i_m})))) \\
&= a_{i_0} \cdot f_{x_{i_0 i_1}}(a_{i_1} \cdot f_{x_{i_1 i_2}}(\cdots \\
&\quad f_{x_{i_{n-1} i_n}}((a_{i_n} b_{j_0}) \cdot f_{y_{j_0 j_1}}(b_{j_1} \cdot f_{y_{j_1 j_2}}(\cdots f_{y_{y_{m-1} j_m}}(b_{i_m} \cdot v_{i_m})))))) \\
&= 0.
\end{aligned}$$

Consequently, (*) holds.

(ii) It is obvious. □

Combining Lemma 2.6 and Theorem 2.5, we have

Theorem 2.7 *Rep* (D, Ω) *and* ${}_{k(\Gamma,\Omega)}\mathcal{M}LU$ *are equivalent.*

For a representation (V, f) in Rep (D, Ω) and any element $\sigma \in k(D, \Omega)$, by Lamma 2.6 and Theorem 2.5, (V, f) can be viewed as $k(D, \Omega)$-module, so for any $\sigma \in k(D, \Omega)$, we write $f_\sigma : V \to V$ by sending x to $\sigma \cdot x$ for any $x \in V$. Let Rep (D, Ω, ρ) denote the full subcategory of Rep (D, Ω) whose objects are (V, f) with $f_\sigma = 0$ for each $\sigma \in \rho$.

Lemma 2.8 *Let* $P = k(D, \Omega)$ *and* $Q = k(D, \Omega, \rho)$.

(i) If (V, f) *is an object in Rep* (D, Ω, ρ), *then* $\{V_i\}_I$ *is a* $\{Q_{ij}\}_I$*-module system under operation induced by operation of* $\{P_{ij}\}$*- module system in Lemma 2.6.*

(ii) If $\{V_i\}_I$ *is a* $\{Q_{ij}\}_I$*-module system, then* (V, f) *is an object in Rep* (D, Ω, ρ) *under operation* $f_{x_{ij}}(v_j) = x_{ij} \cdot v_j$ *for any arrow* $x_{ij} \in P_{ij}$ *and* $v_j \in V_j$.

Theorem 2.9 *(i) Rep* (D, Ω, ρ) *and* ${}_{k(\Gamma,\Omega,\rho)}\mathcal{M}LU$ *are equivalent.*

(ii) If D is finite (i.e. I is finite and the number of arrows between any two vertexes is finite), then f.d.Rep (D, Ω, ρ) *and f.d.*${}_{k(\Gamma,\Omega,\rho}\mathcal{M}LU$ *are equivalent. Here, f.d.Rep* (D, Ω, ρ) *and f.d.*${}_{k(\Gamma,\Omega,\rho)}\mathcal{M}LU$ *denote the full subcategories of finite dimensional objects in the corresponding categories, respectively.*

3 Generalized path algebras

In this section, we characterize the generalized path algebras with weak relations by some algebras which can be lifted with nilpotent Jacobson radical.

If $V = U \oplus W$ as vector spaces and $x \in V$, then there exist $a \in U$ and $b \in W$ such that $x = a + b$. For convenience, we denote a and b by x_U and x_W, respectively.

Lemma 3.1 *Let* Λ *be an algebra and* N *an ideal of* Λ. *Then the following conditions are equivalent:*

(i) There exists a subalgebra A *of* Λ *such that* $\Lambda = A \oplus N$ *as vector spaces.*

(ii) The canonical homomorphism $\pi : \Lambda \to \Lambda/N$ *is split in the category of algebras, i.e. there exists an algebra homomorphism* $\xi : \Lambda/N \to \Lambda$ *such that* $\pi\xi = id_{\Lambda/N}$.

Proof. (i) $\Rightarrow$ (ii). Define $\xi : \Lambda/N \to \Lambda$ by sending $\xi(x + N) = x_A$ for any $x = x_A + x_N \in \Lambda$ with $x_A \in A, x_N \in N$. It is clear that ξ is an algebra homomorphism and $\pi\xi = id$.

(ii) $\Rightarrow$ (i). Obviously $\Lambda = A \oplus N$ with $A = Im\xi$. □

We say that an algebra Λ can be lifted if $\Lambda = A \oplus r(\Lambda)$ with subalgebra A.

Lemma 3.2 *Let Λ be an algebra, N an ideal of Λ and A a subalgebra of Λ. If $\Lambda = A \oplus N$, then $\Lambda/B = (A+B)/B \oplus (N+B)/B$ for any ideal B of Λ with $B \subseteq A$ or $B \subseteq N$.*

Proof. For any $x = x_A + x_N \in \Lambda$ with $x_A \in A$ and $x_N \in N$, $\bar{x} = x + B = (x_A+B)+(x_N+B) \in \Lambda/B$ with $(x_A+B) \in (A+B)/B, (x_N+B) \in (N+B)/B$. This implies that $\Lambda/B = (A+B)/B + (N+B)/B$. Assume $B \subseteq A$. then $(A/B) \cap ((N+B)/B) = 0$ and $\Lambda/B = A/B \oplus (N+B)/B$. Similarly, when $B \subseteq N$, $\Lambda/B = (A+B)/B \oplus N/B$. $\square$

Lemma 3.3 *Let Λ be an algebra, N a nilpotent ideal of Λ and A a subalgebra of Λ. Assume $\Lambda = A \oplus N$ as vector spaces. If $\{e_{ii}\}_I$ is a complete set of pairwise orthogonal idempotents of Λ, then $\{e_{ii}\}_I \subseteq A$.*

Proof. We first show that if e is idempotent in Λ with $e = e_A + e_N$ and $e_A \in A, e_N \in N$, then e_A is idempotent. Indeed, since $ee = e$ and N is an ideal of Λ, we have $e_Ae_A + (e_Ae_N + e_Ne_N + e_Ne_A) = e_A + e_N$, which implies that $e_Ae_A = e_A$.

Next we show that if e and f are pairwise orthogonal idempotents of Λ, then so are e_A and f_A. Indeed, since $ef = 0$, i.e. $e_Af_A + (e_Af_N + e_Nf_A + e_Nf_N) = 0$, we have $e_Af_A = 0$. Similarly, $f_Ae_A = 0$.

We now show that each $e_{ii} \in A$ by induction for m, where $N^m = 0$.

When $m = 1$, $N = 0$. In this case, $(e_{ii})_A = e_{ii} \in A$ for any $i \in I$.

Assume now that the claim holds when $m \leq l$ and we show that the claim also holds when $m = l + 1$. Let $\bar{\Lambda} = \Lambda/N^l$. By Lemma 3.2, $\bar{\Lambda} = (A + N^l)/N^l \oplus N/N^l$. It is clear $\{\bar{e}_{ii}\}_I$ is a complete set of pairwise orthogonal idempotents of Λ/N^l. By the inductive assumption, $\bar{e}_{ii} \in \bar{A}$, i.e. $(e_{ii})_N \in N^l$ for any $i \in I$.

For any $x \in \Lambda$, there exists a finite subset F of I such that

$$x = (\sum_{i\in F} e_{ii})x \qquad \text{and} \qquad x_A = (\sum_{i\in F} e_{ii})x_A. \tag{1}$$

By (1),

$$0 = (\sum_{i\in F}(e_{ii})_N)x_A \quad \text{and} \quad x_A = (\sum_{i\in F}(e_{ii})_A)x_A. \tag{2}$$

Since $(\sum_{i\in F}(e_{ii})_N)x_N \in N^{l+1} = 0$, $(\sum_{i\in F}(e_{ii})_N)x_N = 0$. By (1) and (2),

$$x_N = (\sum_{i\in F}(e_{ii})_A)x_N. \tag{3}$$

Combining (2) and (3), we have that $x = (\sum_{i\in F}(e_{ii})_A)x$. Similarly, $x = x(\sum_{i\in F}(e_{ii})_A)$. Consequently, $\{(e_{ii})_A\}_I$ is a complete set of pairwise orthogonal idempotents of Λ. Since e_{ii} and $(e_{ii})_A$ are the unity element of Λ_{ii}, $e_{ii} = (e_{ii})_A \in A$ for any $i \in I$. $\square$

By Lemma 3.3, we have immediately:

Lemma 3.4 *Let Λ be an algebra with non-zero unity element u, N a nilpotent ideal of Λ and A a subalgebra of Λ. If $\Lambda = A \oplus N$ as vector spaces, then $u \in A$.*

Lemma 3.5 *Let A be a subalgebra of Λ and $\Lambda = A \oplus r$ with nilpotent Jacobson radical $r = r(\Lambda)$. Let $B = \{r_u \mid u \in U\} \subseteq r$. If $\bar{B} = \{\bar{r}_u \mid u \in U\}$ generates r/r^2 as Λ/r-modules, then $A \cup B$ generates Λ as algebras.*

Proof. Since r nilpotent, there is m such that $r^m = 0$. We use induction on m. It is obvious that $r = 0$ and $\Lambda = A$ when $m = 1$. When $m = 2$, we have that $r^2 = 0$ and $r = r/r^2$. Thus $\bar{B} = B$ generates r as Λ/r-modules. That is, $r = \sum_{u\in U} \Lambda r_u = \sum_{u\in U} Ar_u$ and $\Lambda = A + r = A + \sum_{u\in U} Ar_u$. This proves our claim for $m = 2$.

Assume now that the claim holds when $m \leq l$ (where $l \geq 2$) and we show that the claim also holds when $m = l + 1$. Let W denote the subalgebra generated by $A \cup B$ as algebras in Λ. For $\bar{\Lambda} = \Lambda/r^l$, by Lemma 3.2, $\bar{\Lambda} = (A + r^l)/r^l \oplus r/r^l$. It is clear $r(\Lambda/r^l) = r/r^l$. Indeed, obviously $r/r^l \subseteq r(\Lambda/r^l)$. Since $(\Lambda/r^l)/(r/r^l) \cong \Lambda/r$, $r(\Lambda/r^l) \subseteq r/r^l$. Thus $r(\Lambda/r^l) = r/r^l$. Let $\phi : \Lambda/r^2 \to (\Lambda/r^l)/(r^2/r^l)$ be the canonical isomorphism, i.e. $\phi(x + r^2) = (x + r^l) + (r^2/r^l)$ for any $x \in \Lambda$. See

$$\begin{aligned}(r/r^l)/(r^2/r^l) &= \phi(r/r^2) \\ &= \phi(\sum_{u\in U}(\Lambda r_u) + r^2) \quad \text{by assumption} \\ &= (\sum_{u\in U}(\Lambda r_u + r^l) + (r^2/r^l).\end{aligned}$$

Therefore, $\{r_u + r^l \mid u \in U\}$ generates $(r/r^l)/(r^2/r^l)$ as $(\Lambda/r^l)/(r/r^l)$-modules. By induction assumption, we have $\Lambda/r^l = (W + r^l)/r^l$.

Let $x \in \Lambda$. There is $y \in W$ and $z \in r^l$ such that $x - y = z$. Since $l \geq 2$, there exist $\alpha_i \in r^{l-1}$, $\beta_i \in r$ for $i = 1, 2, \cdots, n$ such that $z = \sum \alpha_i\beta_i$. Again using $\Lambda/r^l = (W + r^l)/r^l$, we have that there are $a_i, b_i \in W, u_i, v_i \in r^l$ such that $\alpha_i = a_i + u_i$ and $\beta_i = b_i + v_i$, so $a_i = \alpha_i - u_i \in r^{l-1}$ and $b_i = \beta_i - v_i \in r$ for any $i = 1, 2, \cdots, n$. By computation and $r^{l+1} = 0$, we have $x - y \in W$ and $x \in W$. We complete the proof. □

Recall that J is the ideal generated by all arrows in D of $k(D, \Omega)$ and $\bar{J}$ is the ideal $J/(\rho)$ of $k(D, \Omega, \rho)$.

Lemma 3.6 *If $J^t \subseteq (\rho)$ for some t, then $r(k(D, \Omega, \rho)) = \bar{J}$.*

Proof. Let $P = k(D, \Omega)$ and $Q = k(D, \Omega, \rho)$. Obviously $Q/\bar{J} \cong P/J \cong \sum\{P_{ij}/J_{ij} \mid i, j \in I\}$. It is clear that $P_{ij} = J_{ij}$ when $i \neq j$ and $P_{ii}/J_{ii} \cong \Omega_{ii}$. Thus $r(k(D, \Omega, \rho)) \subseteq \bar{J}$. Conversely, since $J^t \subseteq (\rho)$ for some t, $\bar{J}$ is nilpotent and $\bar{J} \subseteq r(k(D, \Omega, \rho))$. □

Lemma 3.7 *Let Λ be an algebra.*

(i) If f is an algebra homomorphism from $k(D,\Omega)$ to Λ, then $f|_\Omega$ is an algebra homomorphism and $f(x_{ij}) = f(e_{ii})f(x_{ij}) = f(x_{ij})f(e_{jj})$ for any arrow x_{ij} from i to j and $i, j \in I$.

(ii) If f is a map from $\Omega \oplus D_1$ to Λ and $f|_\Omega$ is an algebra homomorphism with $f(x_{ij}) = f(e_{ii})f(x_{ij}) = f(x_{ij})f(e_{jj})$ for any arrow x_{ij} from i to j and $i, j \in I$, then there exists (unique) algebra homomorphism $\bar{f} : k(D,\Omega) \to \Lambda$ such that $\bar{f}|_{\Omega\oplus D_1} = f$.

Proof. (i) It is obvious.

(ii) Let P denote the generalized path algebra $k(D,\Omega)$. For any $i, j \in I$ and generalized path $\alpha = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}\cdots a_{i_{n-1}}x_{i_{n-1}i_n}a_{i_n}$ from $i_0 = i$ to $i_n = j$, define $f_{ij}(\alpha) = f(a_{i_0})f(x_{i_0i_1})f(a_{i_1})f(x_{i_1i_2})\cdots f(a_{i_{n-1}})f(x_{i_{n-1}i_n})f(a_{i_n})$. We get a k-linear map f_{ij} from P_{ij} to Λ. Now we show

$$f_{is}(\alpha\beta) = f_{ij}(\alpha)f_{js}(\beta) \qquad (*)$$

for two Ω- paths $\alpha = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n}a_{i_n}$ and $\beta = b_{j_0}y_{j_0j_1}b_{j_1}y_{j_1j_2}b_{j_2}y_{j_2j_3}\cdots y_{j_{m-1}j_m}b_{j_m}$ of D with $i_n = j_0 = j, i_0 = i$ and $j_m = s$. When $\alpha\beta \neq 0$, i.e. $a_{i_n}b_{j_0} \neq 0$, $\alpha\beta$ is an Ω-path. By definition, (*) holds. When $\alpha\beta = 0$, i.e. $a_{i_n}b_{j_0} = 0$, $\alpha\beta$ is not an Ω-path. Obviously the left side of (*) =0.

$$\begin{aligned}\text{The right side of } (*) &= f(a_{i_0})f(x_{i_0i_1})f(a_{i_1})f(x_{i_1i_2})\cdots f(a_{i_{n-1}})f(x_{i_{n-1}i_n})f(a_{i_n}) \\ &\quad f(b_{j_0})f(y_{j_0j_1})f(b_{j_1})f(y_{j_1j_2})f(b_{j_2})f(y_{j_2j_3})\cdots f(y_{j_{m-1}j_m})f(b_{j_m}) \\ &= 0\end{aligned}$$

Consequently, holds. For any $i, j \in I$, f_{ij} naturally becomes a k-linear map from P_{ij} to Λ with $f_{ij}(x_{is}y_{sj}) = f_{is}(x_{is})f_{sj}(y_{sj})$ and $f(x_{is}) = f(e_{ii})f(x_{is}) = f(x_{is})f(e_{ss})$ for any $x_{is} \in P_{is}$ and $y_{sj} \in P_{sj}$ and $i, s, j \in I$. Let $\bar{f} = \oplus_{i,j\in I}f_{ij}$. This $\bar{f}$ fulfills our requirement. □

Now we give our main theorem.

Theorem 3.8 *Algebra Λ can be lifted with nilpotent Jacobson radical $r = r(\Lambda)$ and has gm unit $\{e'_{ii}\}_I$ with each $\overline{e'_{ii}}$ in the center of $\bar{\Lambda} = \Lambda/r$ iff Λ is isomorphic to a generalized path algebra with weak relations.*

Proof. Assume that $\Lambda = A \oplus r$ with nilpotent Jacobson radical $r = r(\Lambda)$ and subalgebra A. By Lemma 3.3, $e'_{ii} \in A$ for any $i \in I$. Let $e_{ii} = \overline{e'_{ii}} = e'_{ii} + r$ in Λ/r for any $i \in I$. By Lemma 3.1, we have that $\pi\xi = id$, where $\pi : \Lambda \to \Lambda/r(\Lambda)$ is the canonical homomorphism and $\xi : \Lambda/r(\Lambda) \to \Lambda$ is an algebra homomorphism by defining $\xi(x + r) = x_A$ for any $x = x_A + x_r \in \Lambda$ with $x_A \in A$ and $x_r \in r$. Let $\Omega_{ii} = e_{ii}(\Lambda/r)e_{ii}$.

Obviously $\{e_{ii}\}_I$ is gm unit of Ω and $r(\Omega) = 0$. For any $i, j \in I$, let $B_{ij} \subseteq e'_{ii} r e'_{jj} = r_{ij}$ such that $\bar{B}_{ij} =: \{\bar{x} = x + r^2 \mid x \in B_{ij}\} \subseteq r/r^2$ is the k-basis of $\overline{e'_{ii}}(r/r^2)\overline{e'_{jj}} = e_{ii}(r/r^2)e_{jj}$.

We now construct a generalized path algebra $k(D, \Omega)$. Let I be the vertex set of D and B_{ij} all of arrows from i to j. Next we define an algebra homomorphism $\varphi : k(D, \Omega) \to \Lambda$ by $\varphi|_\Omega = \xi$ and $\varphi(x) = x$ for any arrow x from i to j. Indeed, since $\xi(e_{ii}) = e'_{ii}$, we have $\varphi(x_{ij}) = x_{ij}$ and $\varphi(e_{ii})\varphi(x_{ij}) = \xi(e_{ii})\varphi(x_{ij}) = e'_{ii}x_{ij} = x_{ij}$, so $\varphi(x_{ij}) = \varphi(e_{ii})\varphi(x_{ij})$ for any arrow x_{ij} from i to j and $i, j \in I$. Similarly, $\varphi(x_{ij}) = \varphi(x_{ij})\varphi(e_{jj})$ for any arrow x_{ij} from i to j and $i, j \in I$. By Lemma 3.7, φ can become an algebra homomorphism from $k(D, \Omega)$ to Λ. Since $\bar{B}_{ij}$ is a k-basis of $e_{ii}(r/r^2)e_{jj}$ for any $i, j \in I$ and $r/r^2 = \sum_{i,j \in I} e_{ii}(r/r^2)e_{jj}$, r/r^2 is generated by $\cup_{i,j \in I} \bar{B}_{ij}$ as Λ/r-modules. By Lemma 3.5, Λ is generated by $A \cup (\cup_{i,j \in I} B_{ij})$ as algebras. This proves that φ is surjective.

We now consider $N =: ker\varphi$. Assume $r^t = 0$. Since $\varphi(J) \subseteq r$, $\varphi(J^t) = 0$. Thus $J^t \subseteq N$. For any $x \in ker\varphi$, obviously, there exist $a \in \Omega$ and $\alpha \in J$ such that $x = a + \alpha$. Thus $0 = \varphi(x) = \varphi(a) + \varphi(x) = \xi(a) + \varphi(x)$. Considering $\varphi(J) \subseteq r$ and $\Lambda = A \oplus r$, we have $a = 0$. $J^t \subseteq N \subseteq J$ has been proved.

Conversely, assume that Λ is a generalized path algebra $k(D, \Omega, \rho)$ with weak relations. Let $P = k(D, \Omega), Q = k(D, \Omega, \rho)$ and $N = (\rho)$. Since $P = \Omega \oplus J$ and $(\rho) \subseteq J$, by Lemma 3.2, we have that $Q = P/(\rho) = \Omega/(\rho) \oplus J/(\rho)$. By Lemma 3.6, the Jacobson radical $r(Q) = \bar{J}$. Thus Q can be lifted. $r(Q)^t = \bar{J}^t = 0$ since $J^t \subseteq N$. Since $\{e_{ii}\}_I$ is a complete set of pairwise orthogonal idempotents of P, $\{e_{ii} + N\}_I$ is a complete set of pairwise orthogonal idempotents of Q. Obviously, $\Omega \overset{\phi_1}{\cong} P/J \overset{\phi_2}{\cong} Q/\bar{J}$ as algebras and $\phi_2\phi_1(e_{ii}) = (e_{ii} + N) + \bar{J}$ for any $i \in I$. Since e_{ii} is in the center of Ω, $(e_{ii} + N) + \bar{J}$ is in center of $Q/\bar{J}$ for any $i \in I$. □

Corollary 3.9 *Λ can be lifted with nilpotent Jacobson radical and with non-zero unity element iff Λ isomorphic to a generalized path algebra with one vertex and with weak relations*

Proof. The sufficiency follows from Theorem 3.8 and its proof. We now show the necessity. Let u be the unity element of Λ. Obviously, $\{u\}$ is a gm unit of Λ and $\bar{u}$ is in the center of $\bar{\Lambda} = \Lambda/r(\Lambda)$. By Theorem 3.8 and its proof, Λ isomorphic to a generalized path algebra $k(D, \Omega, \rho)$ with one vertex and with weak relations. □

Lemma 3.10 *Let $\Lambda = A \oplus r$ with subalgebra A and with nilpotent Jacobson radical $r = r(\Lambda)$. If Λ has the non-zero unity element u and $\{\bar{e}_{ii}\}_I$ is a complete set of pairwise orthogonal idempotents of $\bar{\Lambda} = \Lambda/r$, then $\{(e_{ii})_A\}_I$ is a complete set of pairwise orthogonal idempotents of Λ.*

Proof. Let $\xi : \Lambda/r \to \Lambda$ by sending $x + r$ to x_A for any $x \in \Lambda$. Since ξ is an algebra homomorphism, we have that $\{(e_{ii})_A\}_I$ is a set of pairwise orthogonal idempotents. By

Proposition 1.3 (ii), I is finite and $\bar{u} = \sum_{i\in I} \bar{e}_{ii}$. By Lemma 3.4, $u \in A$. Thus $u = \sum_{i\in I}(e_{ii})_A$ and $\{(e_{ii})_A\}_I$ is a complete set of pairwise orthogonal idempotents of Λ. $\Box$

It is well known that, for any algebra Λ, if $\Lambda/r(\Lambda)$ is a left (or right) artinian algebra with non-zero unity element, then, by Wedderburn-Artin Theorem, $\Lambda/r(\Lambda) = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ as algebras and B_i is a simple subalgebra of $\Lambda/r(\Lambda)$ for any $i \in I = \{1, 2, \cdots, n\}$. The number n is called the Wedderburn-Artin number of Λ, written as $n_{WA}(\Lambda)$. If $\Lambda/r(\Lambda)$ is not an artinian algebra with unity element, then we write $n_{WA}(\Lambda) = \infty$.

Corollary 3.11 *(i) If $k(D, \Omega, \rho)$ is a generalized path algebra with weak relations, then $|D_0| \leq n_{WA}(k(D, \Omega, \rho))$.*

(ii) Let Λ can be lifted with nilpotent Jacobson radical r and with non-zero unity element. If $\Lambda/r = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ as algebras and B_i is a non-zero subalgebra of $\Lambda/r(\Lambda)$ for $i \in I = \{1, 2, \cdots, n\}$, then Λ isomorphic to a generalized path algebra $k(D, \Omega, \rho)$ with weak relations and $\Omega_{ii} = B_i$ for $i \in I = D_0$.

(iii) Let Λ can be lifted with nilpotent Jacobson radical r and with non-zero unity element. If $\Lambda/r(\Lambda)$ is artinian, then for any natural number $m \leq n_{WA}(\Lambda)$, Λ isomorphic to a generalized path algebra $k(D, \Omega, \rho)$ with weak relations and $|D_0| = m$.

Proof. (i) Let $P = k(D, \Omega)$, $N = (\rho)$ and $Q = P/N$. If $Q/r(Q)$ is artinian with unity element, then, by Wedderburn-Artin Theorem, $Q/r(Q) = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ as algebras and B_i is a simple subalgebra of $Q/r(Q)$ for any $i \in \{1, 2, \cdots, n\}$. It is clear that

$$\oplus_{i\in I}\Omega_{ii} \cong B_1 \oplus B_2 \oplus \cdots \oplus B_n \quad \text{ as algebras.}$$

This implies that

$$\oplus_{i\in I}\Omega_{ii} = B'_1 \oplus B'_2 \oplus \cdots \oplus B'_n \quad \text{ as algebras ,}$$

where B'_i is a simple subalgebra of Ω for $i = 1, 2, \cdots, n$. Considering $B'_1, B'_2, \cdots, B'_n$ are simple subalgebras, we have that each Ω_{ii} is a sum of some of $\{B'_1, B'_2, \cdots, B'_n\}$. Thus $|I| = |D_0| \leq n = n_{WA}(Q)$.

If $Q/r(Q)$ is not an artinian algebra with the unity element, obviously $|D_0| \leq n_{WA}(Q)$ since $n_{WA}(Q) = \infty$.

(ii) Let $\Lambda = A \oplus r$ with subalgebra A and e_{ii} be the unity element of B_i for any $i \in I$. Obviously, $\{e_{ii}\}_I$ is a complete set of pairwise orthogonal central idempotents of Λ/r. Let $e'_{ii} \in \Lambda$ such that $\overline{e'_{ii}} = e_{ii}$ for any $i \in I$. By Lemma 3.10, $\{(e'_{ii})_A\}_I$ is a complete set of pairwise orthogonal idempotents of Λ. By Theorem 3.8 and its proof, Λ is isomorphic to $k(D, \Omega, \rho)$ with weak relations and $\Omega_{ii} = B_i$ for $i \in I = D_0$.

(iii) By Wedderburn-Artin Theorem, $\Lambda/r(\Lambda) = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ as algebras and B_i is a simple subalgebra of $\Lambda/r(\Lambda)$ for any $i \in \{1, 2, \cdots, n\}$ with $n = n_{WA}(\Lambda)$. Let $B'_i = B_i$

for $i = 1, 2, \cdots, m-1$ and $B'_m = B_m + \cdots + B_n$. Obviously, $\Lambda/r(\Lambda) = B'_1 \oplus B'_2 \oplus \cdots \oplus B'_m$ as algebras. By (ii), Λ is isomorphic to $k(D, \Omega, \rho)$ with weak relations and $|D_0| = m$. □

Corollary 3.12 *Λ is isomorphic to a generalized path algebra with weak relations when one of the following conditions holds:*

(i) Λ is a finite dimensional algebra with non-zero unity element over a perfect field k (e.g. the character of k is zero or k is a finite field).

(ii) Λ is a finite-dimensional separable algebra with non-zero unity element.

(iii) Λ is an algebra over a field k with non-zero unity element and nilpotent Jacobson radical, and sup$\{n \mid H^n_k(\Lambda, M) \neq 0$ for some Λ-bimodule $M\} \leq 1$ (see [6, Definition 11.4]).

Proof. It follows from the famous Wedderburn-Malcev Theorem (see [6, Theorem 11.6 and Corollary 11.6]) that Λ can be lifted. We complete the proof by Corollary 3.9. □

Corollary 3.13 *Let k be a perfect field.*

(i) Λ is a finite dimensional algebra with non-zero unity element iff Λ is isomorphic to a generalized path algebra $k(D, \Omega, \rho)$ of finite directed graph with weak relations and with dim $\Omega < \infty$.

(ii) If Λ is a finite dimensional algebra with non-zero unity element over field k, then Λ is isomorphic to a generalized path algebra $k(D, \Omega, \rho)$ of finite directed graph with weak relations and $\Omega_{ii} = B_i$ for any $i \in I = \{1, 2, \cdots, n\}$. Here $\Lambda/r = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ as algebras and B_i is a simple subalgebra of Λ/r for any $i \in I$.

(iii) If Λ is a finite dimensional algebra with non-zero unity element over field k, then for any natural number $m \leq n_{WA}(\Lambda)$, there exists a generalized path algebra $k(D, \Omega, \rho)$ with weak relations and $|D_0| = m$.

Proof. (i) Λ is a finite dimensional algebra with non-zero unity element over field k, then Λ is isomorphic to a generalized path algebra of finite directed graph with weak relations and $dim\ \Omega < \infty$ by corollary 3.12 and the proof of Theorem 3.8. Conversely, assume $\Lambda = k(D, \Omega, \rho)$ is a generalized path algebra of finite directed graph with weak relations. Let $P = k(D, \Omega)$, $Q = k(D, \Omega, \rho)$ and $N = (\rho)$. For any $i, j \in I$, Q_{ij} is spanned by $\{[\alpha] + N \mid \alpha$ is a generalized path from i to j with $l(\alpha) \leq t\}$ since $J^t \subseteq (\rho)$. However, $\{[\alpha] \mid \alpha$ is a generalized path from i to j with $l(\alpha) \leq t\}$ is spanned by finite elements since Ω is finite dimensional. Consequently, Q is finite dimensional.

(ii) By [6, Corollary 11.6], Λ can be lifted. Obviously the Jacobson radical r is nilpotent. By Wedderburn-Artin Theorem, $\Lambda/r = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ as algebras and B_i is a simple subalgebra of Λ for any $i \in I = \{1, 2, \cdots, n\}$. Using Corollary 3.11(ii), we complete the proof.

(iii) It follows from Corollary 3.11(iii) and [6, Corollary 11.6]. □

An algebra Λ over field k is called a generalized elementary algebra if $\Lambda/r(\Lambda) \cong \oplus_{i\in I} B_{ii}$ as algebras with $B_{ii} = k$ for any $i \in I$. A finite dimensional generalized elementary algebra with unity element is called an elementary algebra.

Corollary 3.14 *Λ is a generalized elementary algebra which can be lifted with nilpotent Jacobson radical $r = r(\Lambda)$ and has a complete set of pairwise orthogonal idempotents iff Λ is isomorphic to a path algebra with relations.*

Proof. The sufficiency follows from Theorem 3.8. We now show the necessity. Assume that $\Lambda = A \oplus r$ and $\Lambda/r = \oplus_{i\in I} k\bar{e}_{ii}$ as algebras, where A is a subalgebra of Λ and r is the Jacobson radical of Λ. Obviously, $\{\bar{e}_{ii}\}_I$ is a complete set of pairwise orthogonal central idempotents of $\bar{\Lambda} = \Lambda/r$. Let $\xi : \Lambda/r \to \Lambda$ by sending $x + r$ to x_A for any $x \in \Lambda$. Since ξ is an algebra homomorphism by Lemma 3.1, we have that $\{(e_{ii})_A\}_I$ is a set of pairwise orthogonal idempotents. However, $\Lambda = (\sum_{i\in I} k(e_{ii})_A) + r$. For any $x \in (\sum_{i\in I} k(e_{ii})_A) \cap r$, there exist $\alpha_i \in k$ such that $x = \sum_{i\in I} \alpha_i (e_{ii})_A$. Since $0 = \bar{x} = \sum_{i\in I} \alpha_i \overline{(e_{ii})_A}$, we have $\alpha_i = 0$ for any $i \in I$. This implies $x = 0$ and $\Lambda = (\sum_{i\in I} k(e_{ii})_A) \oplus r$. Since $(\sum_{i\in I} k(e_{ii})_A) \subseteq A$, $\sum_{i\in I} k(e_{ii})_A = A$.

Let $\{e'_{ii}\}_I$ be a complete set of pairwise orthogonal idempotents of Λ. By Lemma 3.3, $\{e'_{ii}\}_I \subseteq A = \sum_{i\in I} k(e_{ii})_A$. Since $\{e'_{ii}\}_I$ is a complete set then so is $\{(e_{ii})_A\}_I$. By Theorem 3.8, Λ is isomorphic to a path algebra with weak relations.

It remains to show $ker\varphi \subseteq J^2$, where φ is the same as in the proof of Theorem 3.8. For any $x \in ker\varphi$, obviously, there exist $y \in J$, $y \notin J^2$ and $z \in J^2$ such that $x = y + z$. Thus $0 = \varphi(x) = \varphi(y) + \varphi(z)$ and $\varphi(z) \in r^2$. Thus $\varphi(y) \in r^2$. Since $y \in J$ and $y \notin J^2$, there are mutually different arrows $x_1, x_2, \cdots, x_n$ such that $y = \sum_{p=1}^n \alpha_p x_p$ with $\alpha_p \in k$ for $p = 1, 2, \cdots, n$. Notice $x_1, x_2, \cdots, x_n \in \cup_{i,j\in I} B_{ij}$, where B_{ij} is the same as in the proof of Theorem 3.8. See that $0 = \overline{\varphi(y)} = \sum_{p=1}^n \alpha_p \bar{x}_p$ in r/r^2. However, $\{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n\}$ is independent, so $\alpha_p = 0$ for $p = 1, 2, \cdots, n$. This implies $y = 0$. Consequently, $ker\varphi \subseteq J^2$. □

There exist generalized elementary algebras whose Jacobson radicals are not nilpotent.

Example 3.15 *Let D be a directed graph with vertex set $I = \mathbf{N}$ of natural numbers and only one arrow from i to $i+1$ for any $i \in I$. Path algebra kD is an elementary algebra since its Jacobson radical $r(kD)$ is J. However, $r(kD)$ is not nilpotent.*

It immediately follows from Corollary 3.14 that

Corollary 3.16 *Λ is an elementary algebra which can be lifted iff Λ is isomorphic to a path algebra of finite directed graph with relations.*

Remark: In the above corollary, we require the condition that Λ can be lifted, but this was not mentioned explicitly in [2, Theorem 1.9]. Assume that $\Lambda/r = \oplus_{i=1,2,\cdots,n} k\bar{e}_{ii}$ as algebras. It is clear that there exists a complete set $\{e'_{ii} \mid i = 1, 2, \cdots, m\}$ of pairwise orthogonal primitive idempotemts of Λ. In the proof of [2, Theorem 1.9], the condition $m = n$ was used without proof. However, this condition implies that Λ can be lifted. Indeed, since e'_{ii} is non-zero idempotent, $e'_{ii} \notin r$ for any $i = 1, 2, \cdots, n$. Thus $\{\bar{e'_{ii}} \mid i = 1, 2, \cdots, n\}$ is linear independent in $\bar{\Lambda} = \Lambda/r$. Consequently, $\Lambda/r = \oplus_{i=1,2,\cdots,n} k\bar{e}_{ii} = \oplus_{i=1,2,\cdots,n} k\bar{e'_{ii}}$. It is easy to check $\Lambda = (\oplus_{i=1,2,\cdots,n} ke'_{ii}) \oplus r$ and $(\oplus_{i=1,2,\cdots,n} ke'_{ii})$ is a subalgebra of Λ. That is, Λ can be lifted. $\square$

Finally we give gradations of gm algebras and generalized path algebras.

Proposition 3.17 *(see [8, Proposition 2.1]) Let $A = \sum\{A_{ij} \mid i, j \in I\}$ be a gm algebra and G an abelian group. If there exists a bijective map $\phi : I \to G$, then A is an algebra graded by G with $A_g = \sum_{\phi(i)=\phi(j)+g} A_{ij}$ for any $g \in G$. In this case, the gradation is called a generalized matrix gradation, or gm gradation in short.*

Proof. For any $g, h \in G$, see that

$$\begin{aligned} A_g A_h &= (\sum_{\phi(i)=\phi(j)+g} A_{ij})(\sum_{\phi(s)=\phi(t)+h} A_{st}) \\ &\subseteq \sum_{\phi(i)=\phi(t)+h+g} A_{i,\phi^{-1}(\phi(t)+h))} A_{\phi^{-1}(\phi(t)+h),t} \\ &\subseteq A_{g+h}. \end{aligned}$$

Thus $A = \sum\{A_{ij} \mid i, j \in I\} = \sum_{g \in G} A_g$ is a G-grading algebra. $\square$

Proposition 3.18 *(i) If $A = \sum\{A_{ij} \mid i, j \in I\}$ is a gm algebra, then there exists an abelian group G with the same cardinality as I such that A has a gm gradation by G.*

(ii) Let $Q = k(D, \Omega, \rho)$ be a generalized path algebra with weak relations. If D_0 is finite, then Q has a gm gradation by $\mathbf{Z}_m$ when $m \leq D_0$.

(iii) Assume that Λ can be lifted with nilpotent Jacobson radical r and with non-zero unity element. If $\Lambda/r(\Lambda)$ is artinian, then for any natural number $0 \neq m \leq n_{WA}(\Lambda)$, Λ has a gm gradation by $\mathbf{Z}_m$.

(iv) If Λ is a finite dimensional algebra with non-zero unity element over perfect field k, then for any natural number $m \leq n_{WA}(\Lambda)$, Λ has a gm gradation by $\mathbf{Z}_m$.

Proof. (i) Let $G_i = 0$ for any $i \in I$ and $G = \oplus_{i \in I} G_i$. Obviously, I and G have the same cardinality. By Proposition 3.17, we complete the proof.

(ii) Assume $D_0 = \{1, 2, \cdots, n\}$. Let $e'_{ii} = e_{ii}$ for $i = 1, 2, \cdots m-1$, $e'_{mm} = e_{mm} + \cdots + e_{nn}$. It is clear that $\{e'_{ii}\}$ is a complete set of pairwise orthogonal idempotents of Q with

$\bar{e}'_{ii}$ in the center of $Q/r(Q)$ since $\bar{e}_{jj}$ is in the center of $Q/r(Q)$ for any $i = 1, 2, \cdots m$ and $j = 1, 2, \cdots, n$. By Theorem 3.8, Q can be lifted. It follows from Theorem 3.8 that Q is isomorphic to a generalized path algebra with weak relations and with m vertexes. By Proposition 3.17, Q has a gm gradation by $\mathbf{Z}_m$.

(iii) It follows from Proposition 3.17 and Corollary 3.11 (iii).

(iv) It follows Corollary 3.13 and Proposition 3.17. □

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