Semi-tilting modules and mutation

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Abstract

We introduce the notion of semi-tilting modules and show that the class of basic semi-tilting modules is closed under mutation. Using this, we provide a partial answer to the Wakamatsu tilting conjecture.

In this note, using the notion of mutation, we will provide a partial answer to the Wakamatsu tilting conjecture. Let R be a commutative noetherian complete local ring and A a noetherian R-algebra, i.e., A is a ring endowed with a ring homomorphism $\varphi : R \to A$ whose image is contained in the center of A and A is a finitely generated R-module. A module $T \in$ mod-A is said to be a Wakamatsu tilting module if the following conditions are satisfied: (1) $\mathrm{Ext}^i_A(T, T) = 0$ for $i \neq 0$; (2) A admits a right resolution $A \to T^\bullet$ in mod-A with $T^\bullet \in \mathcal{K}^+(\mathrm{add}(T))$ and $\mathrm{Ext}^j_A(\mathrm{Z}^i(T^\bullet), T) = 0$ for all $i, j \geq 1$ (see [24]). The Wakamatsu tilting conjecture states that proj dim $_{\mathrm{End}_A(T)}T$ = proj dim T_A for every Wakamatsu tilting module $T \in$ mod-A (see [7]). Note that if both proj dim $_{\mathrm{End}_A(T)}T$ and proj dim T_A are finite then T is a tilting module (see Definition 2.7) and proj dim $_{\mathrm{End}_A(T)}T$ = proj dim T_A. Some partial answers to the conjecture for artinian algebras were provided in [14], [17] and [25]. Unfortunately, this conjecture does not hold true for artinian rings (see [22]). This conjecture is related to the generalized Nakayama conjecture (see [3]) and a conjecture stating that inj dim $_\Lambda\Lambda$ = inj dim Λ_Λ for every left and right noetherian ring Λ. If both inj dim $_\Lambda\Lambda$ and inj dim Λ_Λ are finite then inj dim $_\Lambda\Lambda$ = inj dim Λ_Λ (see [26]). In [13] Hoshino and the author provided a partial answer to the latter conjecture.

Mutation is an operation to construct an object from another object by replacing a direct summand, which has its origin in the study of exceptional collections of vector bundles on $\mathbb{P}^n$ (see [9] and [10]). In [21] Riedtmann and Schofield introduced the method of mutation for tilting modules. In [20] Rickard introduced the notion of tilting complexes which generalizes that of tilting modules, and provided a necessary and sufficient condition for two rings to be derived equivalent. In [16] the author provided a sufficient condition to mutate tilting

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complexes. In [11] Happel and Unger showed that mutation for tilting modules is closely related to the partial order of tilting modules defined by Riedtmann and Schofield. This is also the case for semi-tilting modules. The notion of semi-tilting modules is introduced as follows.

A module $T \in \text{mod-}A$ is said to be a semi-tilting module if the following conditions are satisfied: (1) $\text{Ext}_A^i(T, T) = 0$ for $i \neq 0$; (2) A admits a right resolution $A \to T^\bullet$ in mod-A with $T^\bullet \in \mathcal{K}^{\text{b}}(\text{add}(T))$. Note that ${}_{\text{End}_A(T)}T$ is a Wakamatsu tilting module of finite projective dimension. We will show that the class of basic semi-tilting modules is closed under mutation, i.e., for a basic semi-tilting module $T = U \oplus X \in \text{mod-}A$ with X indecomposable, if X is generated by U then there exists a non-split exact sequence $0 \to Y \to E \to X \to 0$ in mod-A with Y indecomposable, $E \in \text{add}(U)$ and $U \oplus Y$ a semi-tilting module (see Theorems 3.3 and 3.4). Note that for a basic semi-tilting module $T \in \text{mod-}A$ there always exists a direct summand X of T such that $T \cong U \oplus X$ and X is generated by U unless T is projective (see Proposition 2.12). If X is generated by U, then we will denote by $\mu_X(T)$ the module $U \oplus Y$. Following [21], we will define a quiver K as follows: The vertices of K are isomorphism classes of basic semi-tilting modules and there is an arrow $V \to W$ if W and V are represented by basic semi-tilting modules T' and $\mu_{X'}(T')$ with X' a non-projective indecomposable direct summand of T', respectively. Then K contains no oriented cycles (see Proposition 3.10). We will show that if the connected component of K including a semi-tilting module T contains a tilting module then T itself is a tilting module (see Theorem 3.11), i.e., the Wakamatsu tilting conjecture holds true for such a Wakamatsu tilting module.

This note is organized as follows. In Section 1, we will recall several basic facts and definitions. In Section 2, we will introduce the notion of semi-tilting modules and show that every non-projective semi-tilting module $T \in \text{mod-}A$ admits a decomposition $T = U \oplus X$ with $X \in \text{gen}(U)$ indecomposable. In Section 3, we will show that the class of semi-tilting modules is closed under mutation and provide a partial answer to the Wakamatsu tilting conjecture.

We refer to [8], [12] and [23] for basic results in the theory of derived categories.

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1 Preliminaries

Let A be a ring. We denote by $\text{rad}(A)$ the Jacobson radical of A. We denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely generated modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation M_A (resp., ${}_AM$) to stress that the module M considered is a right (resp., left) A-module. We denote by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules. For a cochain complex $X^\bullet$ and an integer $i \in \mathbb{Z}$, we denote by $\text{Z}^i(X^\bullet)$, $\text{Z}'^i(X^\bullet)$, $\text{B}^i(X^\bullet)$ and $\text{H}^i(X^\bullet)$ the ith cycle, the ith cocycle, the ith boundary and the ith cohomology of $X^\bullet$, respectively. We denote by $\mathcal{D}(\text{Mod-}A)$ the derived category of cochain complexes over Mod-A. For an additive category $\mathcal{A}$ we denote by $\mathcal{K}(\mathcal{A})$ the homotopy category of cochain complexes over $\mathcal{A}$ and by $\mathcal{K}^+(\mathcal{A})$ (resp., $\mathcal{K}^{\mathrm{b}}(\mathcal{A})$) the full triangulated subcategory of $\mathcal{K}(\mathcal{A})$ consisting of bounded below (resp., bounded) complexes. For an object X in an additive category $\mathcal{A}$ we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of copies of X, by $\mathrm{gen}(X)$ the full subcategory of $\mathcal{A}$ consisting of epimorphic images of objects in $\mathrm{add}(X)$ and by $\mathrm{cog}(X)$ the full subcategory of $\mathcal{A}$ consisting of subobjects of objects in $\mathrm{add}(X)$. We denote by $\mathrm{Hom}^\bullet(-,-)$ (resp., $-\otimes^\bullet -$) the associated single complex of the double hom (resp., tensor) complex. Finally, we consider modules as complexes concentrated in degree zero.

Definition 1.1. An exact sequence $0 \to M \to E^0 \to E^1 \to \cdots$ in Mod-A is called a right resolution of M, which we denote by $M \to E^\bullet$. A right resolution $M \to E^\bullet$ is said to be finite if there exists $n \geq 0$ such that $E^i = 0$ for $i > n$. Dually, an exact sequence $\cdots \to E^{-1} \to E^0 \to X \to 0$ in Mod-A is called a left resolution of X, which we denote by $E^\bullet \to X$. A left resolution $E^\bullet \to X$ is said to be finite if there exists $n \geq 0$ such that $E^{-i} = 0$ for $i > n$.

Lemma 1.2. *Let $T, U \in \text{Mod-}A$ with $\mathrm{Ext}^i_A(T,T) = 0$ for $i \neq 0$ and $0 \to U \to V^0 \to V^1 \to \cdots \to V^m \to 0$ a right resolution in Mod-A. Assume that each V^i admits a finite right resolution $V^i \to T^{i\bullet}$ in Mod-A with $T^{i\bullet} \in \mathcal{K}^{\mathrm{b}}(\mathrm{add}(T))$. Then we have a finite right resolution $U \to W^\bullet$ in Mod-A such that $W^n = \oplus_{i+j=n} T^{ij}$ for all $n \geq 0$.*

Lemma 1.3 (cf. [18])**.** *Let $T, U \in \text{Mod-}A$ with $\mathrm{Ext}^i_A(T,T) = \mathrm{Ext}^i_A(U,T) = 0$ for $i \neq 0$ and $0 \to V^{-m} \to V^{-m+1} \to \cdots \to V^0 \to U \to 0$ a left resolution of U in Mod-A. Assume that each V^{-i} admits a finite right resolution $V^{-i} \to T^{-i\bullet}$ in Mod-A with $T^{-i\bullet} \in \mathcal{K}^{\mathrm{b}}(\mathrm{add}(T))$. Then we have a finite right resolution $U \to W^\bullet$ in Mod-A with $W^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathrm{add}(T))$.*

Definition 1.4 ([1], [2])**.** A homomorphism $f : E \to X$ in Mod-A is said to be right minimal if every $h \in \mathrm{End}_A(E)$ with $fh = f$ is an isomorphism. Dually, a homomorphism $f : X \to E$ in Mod-A is said to be left minimal if every $h \in \mathrm{End}_A(E)$ with $hf = f$ is an isomorphism.

Note that an epimorphism $P \to X$ in Mod-A with P projective is right minimal if and only if it is a superfluous epimorphism and that a monomorphism $X \to I$ in Mod-A with I injective is left minimal if and only if it is an essential monomorphism.

Lemma 1.5 ([1])**.** *Let $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ be a non-split exact sequence in Mod-A with $\mathrm{End}_A(X)$ local. Then μ is left minimal.*

Lemma 1.6 ([1])**.** *Let $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ be a non-split exact sequence in Mod-A with $\mathrm{End}_A(Y)$ local. Then ε is right minimal.*

Definition 1.7. A left resolution $E^\bullet \to X$ in Mod-A is said to be minimal if the epimorphism $E^i \to \mathrm{Z}'^i(E^\bullet)$ is right minimal for all $i \geq 0$. Dually, a right resolution $X \to E^\bullet$ in Mod-A is said to be minimal if the monomorphism $\mathrm{Z}^i(E^\bullet) \to E^i$ is left minimal for all $i \geq 0$.

Definition 1.8 ([4])**.** Let $\mathcal{C}$ be a full subcategory of Mod-A closed under isomorphisms and direct summands. A homomorphism $f : E \to X$ in Mod-A with $E \in \mathcal{C}$ is said to be a right $\mathcal{C}$-approximation of X if $\mathrm{Hom}_A(E', f)$ is an epimorphism for all $E' \in \mathcal{C}$. A right $\mathcal{C}$-approximation $f : E \to X$ is said to be a minimal right $\mathcal{C}$-approximation of X if f is right minimal. Dually, a homomorphism $f : X \to E$ in Mod-A with $E \in \mathcal{C}$ is said to be a left $\mathcal{C}$-approximation of X if $\mathrm{Hom}_A(f, E')$ is an epimorphism for all $E' \in \mathcal{C}$. A left $\mathcal{C}$-approximation $f : X \to E$ is said to be a minimal left $\mathcal{C}$-approximation of X if f is left minimal.

The next lemma is due essentially to Auslander [2].

Lemma 1.9. *Let $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ be an exact sequence in* Mod-A *with μ a minimal left* $\mathrm{add}(E)$*-approximation of Y, ε a minimal right* $\mathrm{add}(E)$*-approximation of X. Then* $\mathrm{End}_A(X)$ *is a local ring if and only if so is* $\mathrm{End}_A(Y)$*.*

Lemma 1.10. *Let $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ and $\cdots \to T^{i-1} \xrightarrow{d^{i-1}} T^i \oplus (\oplus^n X) \to T^{i+1} \to \cdots$ be exact sequences in* Mod-A*. Assume that $\mathrm{Hom}_A(T^{i-1}, \varepsilon)$ is an epimorphism. Then there exists an exact sequence $\cdots \to T^{i-1} \oplus (\oplus^n Y) \to T^i \oplus (\oplus^n E) \to T^{i+1} \to \cdots$ in* Mod-A*.*

Lemma 1.11. *Let $0 \to Y \xrightarrow{\mu} E \to X \to 0$ and $\cdots \to T^{i-1} \to T^i \oplus (\oplus^n Y) \to T^{i+1} \to \cdots$ be exact sequences in* Mod-A*. Assume that $\mathrm{Hom}_A(\mu, T^{i+1})$ is an epimorphism. Then there exists an exact sequence $\cdots \to T^{i-1} \to T^i \oplus (\oplus^n E) \to T^{i+1} \oplus (\oplus^n X) \to \cdots$ in* Mod-A*.*

Lemma 1.12 (cf. [5])**.** *Let $X, Y \in$* Mod-A *with $\mathrm{End}_A(X)$ local, X finitely generated over $\mathrm{End}_A(X)^{\mathrm{op}}$ and $\mathrm{Hom}_A(Y, X)$ finitely generated over $\mathrm{End}_A(Y)$. Assume that there exist $f_i : X \to X$ for $i = 1, \cdots, n$ and $f_0 : Y \to X$ such that $f = (f_0, f_1, \cdots, f_n) : Y \oplus (\oplus^n X) \to X$ is a non-split epimorphism. Then $X \in \mathrm{gen}(Y)$.*

Let $T \in$ Mod-A and set $B = \mathrm{End}_A(T)$. We denote by $\mathbf{R}\mathrm{Hom}^\bullet_A(-, T)$ (resp., $\mathbf{R}\mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(-, T)$) the right derived functor of $\mathrm{Hom}^\bullet_A(-, T) : \mathcal{K}(\text{Mod-}A) \to \mathcal{K}(\text{Mod-}B^{\mathrm{op}})$ (resp., $\mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(-, T) : \mathcal{K}(\text{Mod-}B^{\mathrm{op}}) \to \mathcal{K}(\text{Mod-}A)$).

Lemma 1.13 ([15])**.** *We have*

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet, \mathbf{R}\mathrm{Hom}_{B^{\mathrm{op}}}(Y^\bullet, T)) \cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}B^{\mathrm{op}})}(Y^\bullet, \mathbf{R}\mathrm{Hom}_A(X^\bullet, T))$$

for $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ and $Y^\bullet \in \mathcal{D}(\text{Mod-}B^{\mathrm{op}})$.

Proof. See [15, Lemma 2.3] the proof of which remains to work in our setting. □

Lemma 1.14. *Let $U, V \in \text{Mod-}A$. Assume that $\text{Ext}^i_A(U,T) = \text{Ext}^i_A(U,V) = 0$ for $i \neq 0$, and that $\text{Ext}^i_{B^{\text{op}}}(\text{Hom}_A(V,T),T) = 0$ for $i \neq 0$ and the canonical homomorphism $V \to \text{Hom}_{B^{\text{op}}}(\text{Hom}_A(V,T),T)$ is an isomorphism. Then $\text{Ext}^i_{B^{\text{op}}}(\text{Hom}_A(V,T), \text{Hom}_A(U,T)) = 0$ for $i \neq 0$.*

Definition 1.15 ([6])**.** A family of idempotents $\{e_\lambda\}_{\lambda \in \Lambda}$ in a ring A is said to be orthogonal if $e_\lambda e_\mu = 0$ unless $\lambda = \mu$. An idempotent $e \in A$ is said to be primitive if eA_A is indecomposable and to be local if $eAe \cong \text{End}_A(eA)$ is local. A ring A is said to be semiperfect if $1 = e_1 + \cdots + e_n$ in A with the e_i orthogonal local idempotents.

Let R be a commutative noetherian ring. In this note, a ring A is said to be a noetherian R-algebra if A is a ring endowed with a ring homomorphism $\varphi : R \to A$ whose image is contained in the center of A and A is a finitely generated R-module.

Lemma 1.16. *Assume that R is a complete local ring. Then every noetherian R-algebra A is semiperfect, so that the Krull-Schmidt theorem holds in* mod-A.

Proof. This is well known but for the benefit of the reader we include a proof. Let $\mathfrak{m}$ be the maximal ideal of R and I an injective envelope of $R/\mathfrak{m}$ in Mod-R. Since A is right noetherian, we have $A = e_1 A \oplus \cdots \oplus e_n A$ with the e_i orthogonal primitive idempotents. Since $e_i A \cong \text{Hom}_R(\text{Hom}_R(e_i A, I), I)$ canonically, $\text{Hom}_R(e_i A, I) \in \text{Mod-}A^{\text{op}}$ is indecomposable injective. Also, we have a ring isomorphism $\text{End}_A(e_i A) \cong \text{End}_{A^{\text{op}}}(\text{Hom}_R(e_i A, I))^{\text{op}}$ with $\text{End}_{A^{\text{op}}}(\text{Hom}_R(e_i A, I))$ local. Thus every e_i is local, so that A is semiperfect. Since $\text{End}_A(X)$ is a noetherian R-algebra for $X \in \text{mod-}A$, the Krull-Schmidt theorem holds in mod-A. □

2 Semi-tilting modules

Throughout the rest of this note, R is a commutative complete local ring and A is a noetherian R-algebra. Note that $\text{Hom}_A(T,X)$ is finitely generated over $\text{End}_A(T)$ for $T, X \in \text{mod-}A$ and that by Lemma 1.16 every $X \in \text{mod-}A$ admits a minimal projective resolution.

In this section, we will introduce the notion of semi-tilting modules and show that every non-projective semi-tilting module $T \in \text{mod-}A$ admits a decomposition $T = U \oplus X$ with $X \in \text{gen}(U)$ indecomposable.

Lemma 2.1. *For any $T \in \text{mod-}A$ and $X \in \text{gen}(T)$ we have a epic minimal right* add(T)*-approximation $\varepsilon : E \to X$.*

Lemma 2.2. *For any $T \in \text{mod-}A$ and $Y \in \text{cog}(T)$ we have a monic minimal left* add(T)*-approximation $\mu : Y \to E$.*

Definition 2.3 ([24])**.** A module $T \in \text{mod-}A$ is said to be a Wakamatsu tilting module if the following conditions are satisfied:

(1) $\mathrm{Ext}^i_A(T,T)=0$ for $i\neq 0$.

(2) A admits a right resolution $A\to T^\bullet$ in mod-A with $T^\bullet\in\mathcal{K}^+(\mathrm{add}(T))$ and $\mathrm{Ext}^j_A(\mathrm{Z}^i(T^\bullet),T)=0$ for $i,j\geq 1$.

Definition 2.4. A module $T\in$ mod-A is said to be a semi-tilting module if the following conditions are satisfied:

(1) $\mathrm{Ext}^i_A(T,T)=0$ for $i\neq 0$.

(2) A admits a right resolution $A\to T^\bullet$ in mod-A with $T^\bullet\in\mathcal{K}^{\mathrm{b}}(\mathrm{add}(T))$.

Note that a semi-tilting module $T\in$ mod-A is a Wakamatsu tilting module with proj dim ${}_{\mathrm{End}_A(T)}T<\infty$ (see Lemma 2.5 below).

If $T\in$ mod-A is a semi-tilting module, then any finite right resolution with terms in $\mathrm{add}(T)$ can be chosen to be minimal.

The following lemma is a slight generalization of [19, Proposition 1.4 (2)].

Lemma 2.5. *Let $T\in$ mod-A with $\mathrm{Ext}^i_A(T,T)=0$ for $i\neq 0$ and $B=\mathrm{End}_A(T)$. Then for any $M\in$ mod-A the following are equivalent.*

(1) There exists a right resolution $0\to M\to T^0\to T^1\to\cdots\to T^m\to 0$ in mod-A with $T^i\in\mathrm{add}(T)$ for $0\leq i\leq m$.

(2) proj dim ${}_B\mathrm{Hom}_A(M,T)<\infty$, $\mathrm{Ext}^i_{B^{\mathrm{op}}}(\mathrm{Hom}_A(M,T),T)=0$ for $i\neq 0$ and $M\xrightarrow{\sim}\mathrm{Hom}_{B^{\mathrm{op}}}(\mathrm{Hom}_A(M,T),T)$ canonically.

Remark 2.6. Let $T\in$ mod-A be a semi-tilting module and $0\to A\to T^0\to T^1\to\cdots\to T^m\to 0$ a minimal right resolution in mod-A with $T^i\in\mathrm{add}(T)$ for $0\leq i\leq m$. Then the following hold.

(1) $\mathrm{Ext}^i_A(T,A)=0$ for $i>m$ and $\mathrm{Ext}^m_A(T,A)\neq 0$.

(2) If $P^\bullet\to T$ is a projective resolution in mod-A, then $\oplus_{i=0}^m P^i\in\mathcal{P}_A$ is a projective generator.

Definition 2.7 ([19])**.** A module $T\in$ mod-A is said to be a tilting module if it is a semi-tilting module and has finite projective dimension.

Lemma 2.8. *Let $P^\bullet\in\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $P^i=0$ unless $0\leq i\leq l$ for some integer $l\geq 1$. Assume that $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet,P^\bullet[l])=0$ and that $C(\mathrm{id}_P)\notin\mathrm{add}(P^\bullet)$ for any $P\in\mathcal{P}_A$, where $C(\mathrm{id}_P)$ is the mapping cone of the identity map of P. Then $\mathrm{add}(P^0)\cap\mathrm{add}(P^l)=\{0\}$.*

Remark 2.9. Let $M,T\in$ mod-A with $\mathrm{Ext}^i_A(M,M)=\mathrm{Ext}^i_A(T,T)=0$ for $i\neq 0$ and $B=\mathrm{End}_A(T)$. If $0\to M\to T^0\to T^1\to\cdots\to T^m\to 0$ is a minimal right resolution in mod-A with $T^i\in\mathrm{add}(T)$ for $0\leq i\leq m$, then $\mathrm{add}(T^0)\cap\mathrm{add}(T^m)=\{0\}$ unless $m=0$.

Lemma 2.10. *Let $T \in$ mod-A. Assume that there exists a non-split exact sequence $0 \to M \xrightarrow{\mu} E^0 \to E^1 \to 0$ in mod-A with μ left minimal and $E^i \in \mathrm{add}(T)$ for $i = 0, 1$. Then for any indecomposable $X \in \mathrm{add}(E^1)$, letting $T \cong U \oplus X$, we have $X \in \mathrm{gen}(U)$.*

Remark 2.11. Let $T \in$ mod-A be a semi-tilting module such that $T \cong \oplus^n X$ with $X \in$ mod-A indecomposable. Then T is projective.

Proposition 2.12. *Every non-projective semi-tilting module $T \in$ mod-A admits a decomposition $T = U \oplus X$ with $X \in \mathrm{gen}(U)$ indecomposable.*

Lemma 2.13. *Let $T = U \oplus X \in$ mod-A with X indecomposable, $X \notin \mathrm{add}(U)$ and $\mathrm{Ext}^i_A(T, T) = 0$ for $i \neq 0$. Assume that there exists an exact sequence $0 \to Y \to E \xrightarrow{\varepsilon} X \to 0$ in mod-A with ε a right $\mathrm{add}(U)$-approximation. Set $B = \mathrm{End}_A(T)$. Then $\mathrm{Hom}_A(U \oplus Y, T) \in$ mod-B^{op} is a tilting module with* proj dim ${}_B\mathrm{Hom}_A(U \oplus Y, T) = 1$.

3 Mutation

Definition 3.1. A module $M \in$ mod-A is said to be basic if $M \cong \oplus_{i=0}^m M_i$ with the M_i indecomposable and $M_i \not\cong M_j$ unless $i = j$.

Throughout this section, $T = U \oplus X \in$ mod-A is a basic semi-tilting module with X indecomposable and $B = \mathrm{End}_A(T)$. Note that $X \notin \mathrm{add}(U)$. We fix a minimal right resolution $0 \to A \to T^0 \to T^1 \to \cdots \to T^m \to 0$ in mod-A with $T^i \in \mathrm{add}(T)$ for $0 \le i \le m$, where $m =$ proj dim ${}_BT$.

We will show that the class of semi-tilting modules is closed under mutation and provide a partial answer to the Wakamatsu tilting conjecture.

Lemma 3.2. *Assume that $X \in \mathrm{add}(T^0)$. Then $X \notin \mathrm{gen}(U)$.*

Theorem 3.3. *Assume that $X \in \mathrm{gen}(U)$. Then there exists a non-split exact sequence $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ in mod-A with Y indecomposable, $E \in \mathrm{add}(U)$ and $T' = U \oplus Y$ a semi-tilting module.*

Theorem 3.4. *Assume that $X \in \mathrm{cog}(U)$. Then there exists a non-split exact sequence $0 \to X \to E' \to Z \to 0$ in mod-A with Z indecomposable, $E' \in \mathrm{add}(U)$ and $U \oplus Z$ a semi-tilting module.*

Remark 3.5. Assume that $m \ge 1$ and that there exists a non-split exact sequence $0 \to Y \to E \to X \to 0$ in mod-A with Y indecomposable, $E \in \mathrm{add}(U)$ and $T' = U \oplus Y$ a semi-tilting module. Take a minimal right resolution $0 \to A \to T'^0 \to T'^1 \to \cdots \to T'^n \to 0$ in mod-A with $T'^i \in \mathrm{add}(T')$ for $0 \le i \le n$. Then for each $1 \le i \le m$, letting $T^i \cong E^i \oplus (\oplus^{l_i} X)$ with $X \notin \mathrm{add}(E^i)$ and $T'^{i-1} \cong E'^{i-1} \oplus (\oplus^{l'_i} Y)$ with $Y \notin \mathrm{add}(E'^i)$, we have $l_i = l'_i$.

Definition 3.6. If $X \in \mathrm{gen}(U)$, then we will denote by $\mu_X(T)$ the module $T' = U \oplus Y$ in Theorem 3.3.

Remark 3.7. Let $T \cong U' \oplus X'$ with X' indecomposable. If $X \in \text{gen}(U)$ and $X' \in \text{gen}(U')$ then $X \cong X'$ if and only if $\mu_X(T) \cong \mu_{X'}(T)$.

Lemma 3.8. *Assume that $X \in \text{gen}(U)$. Then $\mu_X(T)$ is a tilting module if and only if so is T.*

Following [21], we will define a quiver K as follows: The vertices of K are isomorphism classes of basic semi-tilting modules and there is an arrow $V \to W$ if W and V are represented by basic semi-tilting modules T' and $\mu_{X'}(T')$ with X' a non-projective indecomposable direct summand of T', respectively.

Definition 3.9. A vertex V in K is said to be a predecessor of T in K if there exists a path $V_0 \to V_1 \to \cdots \to V_n$ in K such that $V_0 = V$ and V_n is the isomorphism class of T, and T is said to have only finitely many predecessors in K if the number of predecessors of T in K is finite.

Proposition 3.10 ([21])**.** *The quiver K contains no oriented cycles.*

Proof. For the benefit of the reader we include a proof. Suppose to the contrary that K contains an oriented cycle $V_0 \to V_1 \to \cdots \to V_n \to V_0$. By definition there exists a non-split exact sequence $0 \to Y_0 \to E_0 \to X_0 \to 0$ in mod-A such that X_0 and Y_0 are indecomposable, V_0 and V_n are represented by basic semi tilting modules $U_0 \oplus X_0$ and $U_0 \oplus Y_0$, respectively, and $E_0 \in \text{add}(U_0)$. Also, there exists a non-split exact sequence $0 \to Y_1 \to E_1 \to X_1 \to 0$ in mod-A such that X_1 and Y_1 are indecomposable, V_1 and V_0 are represented by basic semi-tilting modules $U_1 \oplus X_1$ and $U_1 \oplus Y_1$, respectively, and $E_1 \in \text{add}(U_1)$. Since $U_0 \oplus X_0 \cong U_1 \oplus Y_1$, applying $\text{Hom}_A(X_0, -)$ to the exact sequence $0 \to Y_1 \to E_1 \to X_1 \to 0$, we have $\text{Ext}^i_A(X_0, X_1) = 0$ for $i \neq 0$ and hence $\text{Ext}^i_A(X_0, U_1 \oplus X_1) = 0$ for $i \neq 0$. Similarly, we have $\text{Ext}^i_A(X_0, U_j \oplus X_j) = 0$ for $i \neq 0$ and $1 \leq j \leq n$, where $U_j \oplus X_j$ is a representative of V_j with X_j indecomposable. Then, since $U_n \oplus X_n \cong U_0 \oplus Y_0$, we have $\text{Ext}^i_A(X_0, Y_0) = 0$ for $i \neq 0$, which contradicts that the exact sequence $0 \to Y_0 \to E_0 \to X_0 \to 0$ does not split. □

Theorem 3.11. *If the connected component of K including T contains a tilting module then T itself is a tilting module.*

Corollary 3.12. *If T has only finitely many predecessors in K, then T is a tilting module.*

Assume that R is a Cohen-Macaulay ring, A is a maximal Cohen-Macaulay R-module and T is a maximal Cohen-Macaulay R-module. We will denote by $\mathcal{L}(^{\perp}T)$ the full subcategory of mod-A consisting of modules M which are maximal Cohen-Macaulay R-modules and $\text{Ext}^i_A(M, T) = 0$ for $i \neq 0$.

Corollary 3.13. *Assume that $\mathcal{L}(^{\perp}T)$ contains only a finite number of non-isomorphic indecomposable modules. Then T is a tilting module.*

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