GORENSTEIN DIMENSIONS OVER SOME RINGS OF THE FORM $R \oplus C$

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Abstract. Given a semidualizing module C over a commutative noetherian ring, Holm and Jørgensen [13] investigate some connections between C-Gorenstein dimensions of an R-complex and Gorenstein dimensions of the same complex viewed as a complex over the "trivial extension" $R \ltimes C$. We generalize some of their results to a certain type of retract diagram. We also investigate some examples of such retract diagrams, namely D'Anna and Fontana's amalgamated duplication [6] and Enescu's pseudocanonical cover [7].

1. Introduction

In this paper, let R be a commutative noetherian ring with identity.

As in the famous theorem of Auslander-Buchsbaum [2] and Serre [18] where projective dimension of R-modules is used to characterize regularity of R, Auslander and Bridger introduced Gorenstein dimension in [1] to characterize Gorenstein rings: a local ring R is Gorenstein if and only if every finitely generated R-module M has finite Gorenstein dimension, i.e., $\text{G-dim}_R M < \infty$. To extend similar results to non-finitely generated R-modules, Enochs and Jenda introduced Gorenstein projective dimension [8]. In particular, a local ring is Gorenstein if and only if every (finitely generated) R-module M has finite Gorenstein projective dimension, i.e., $\operatorname{Gpd}_R M < \infty$; see [4, 10]. Enochs and Jenda also studied the Gorenstein injective dimension Gid and, with Torrecillas [9], the Gorenstein flat dimension Gfd.

Semidualizing R-modules, first introduced by Foxby in [11] and later studied by Vasconcelos [20] and Golod [12], arise naturally in the study of the connection between R and its modules: a finitely generated R-module M is *semidualizing* if $R \cong \operatorname{Hom}_R(C, C)$ and $\operatorname{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$. For example, Golod introduced the G_C-dimension in [12] and proved a formula of the same type as the Auslander-Buchsbaum and Auslander-Bridger formulae.

Holm and Jørgensen extended the G_C-dimension in [13] introducing three new homological dimensions called the C-Gorenstein projective, C-Gorenstein injective and C-Gorenstein flat dimensions, denoted as $C\text{-}\operatorname{Gpd}_R(M)$, $C\text{-}\operatorname{Gid}_R(M)$, and $C\text{-}\operatorname{Gfd}_R(M)$, respectively, for an R-complex M. They also proved how these new dimensions coincide with Enochs, Jenda and Torrecillas' Gorenstein dimensions over the trivial extension $R \ltimes C$ [13, Theorem 2.16]. This means that for an R-module

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M, one has

$$
\begin{aligned}
C\text{-}\mathrm{Gpd}_R(M) &= \mathrm{Gpd}_{R\ltimes C}(M)\\
C\text{-}\mathrm{Gid}_R(M) &= \mathrm{Gid}_{R\ltimes C}(M)\\
C\text{-}\mathrm{Gfd}_R(M) &= \mathrm{Gfd}_{R\ltimes C}(M).
\end{aligned}
$$

In this paper, we generalize this from the trivial extension $R \ltimes C$ to a retract diagram

$$
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
 & \underset{\mathrm{id}_R}{\searrow} & \downarrow g \\
 & & R
\end{array}
$$

where R and S are commutative rings with the identity map id_R on R, satisfying the following properties:

(i) $C \cong \operatorname{Ker} g$,
(ii) $\operatorname{Hom}_R(S, C) \cong S$ as S-modules, and
(iii) $\operatorname{Ext}^i_R(S, C) = 0$ for all $i \geqslant 1$.

We prove the following generalized version of Holm and Jørgensen's Theorem 2.16 [13]; see Theorem 3.19.

Theorem A. *In the setting of the above retract diagram with a semidualizing C, given a homologically left-bounded R-complex M and a homologically right-bounded R-complex N, one has*

$$
\begin{aligned}
C\text{-}\mathrm{Gid}_R(M) &= \mathrm{Gid}_S(M)\\
C\text{-}\mathrm{Gpd}_R(N) &= \mathrm{Gpd}_S(N)\\
C\text{-}\mathrm{Gfd}_R(N) &= \mathrm{Gfd}_S(N).
\end{aligned}
$$

Along the way we prove the following characterization of semidualizing modules; see Theorem 3.5.

Theorem B. *In the setting of the above retract diagram with a finitely generated C, the following are equivalent:*

(a) C is semidualizing over R;
(b) R is Gorenstein projective over S and $\operatorname{Ann}_R(C) = 0$; and
(c) C is Gorenstein projective over S and $\operatorname{Ann}_R(C) = 0$.

We further show that $S = R \ltimes C$ is not the only example of a ring satisfying our generalized settings set forth in the retract diagram above. See Theorems 4.4 and 4.9, along with their corollaries.

Theorem C. *The following examples satisfy the hypotheses of Theorem A:*

(a) *D'Anna and Fontana's amalgamated duplication $S = R \bowtie C$, and*
(b) *Enescu's pseudocanonical cover $S = S(h)$, when h is a square in R.*

In particular, we recover the main result of Salimi, Tavasoli and Yassemi in [16] as the special case where $S = R \bowtie C$.

2. Preliminaries

We provide in this section some preliminary definitions and properties to be used later. We first extend a couple of results of Ishikawa [15] to our setting.

Lemma 2.1. *Let $f : R \to S$ be a ring homomorphism. Let S be finitely generated as an R-module, let M be an R-module, and let N be an injective R-module. Then the natural map*

$$\Theta_{S,M,N} : S \otimes_R \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(\operatorname{Hom}_R(S, M), N)$$

defined as $(\Theta_{S,M,N}(s \otimes_R \psi))(\phi) = \psi(\phi(s))$ for each $s \otimes_R \psi \in S \otimes_R \operatorname{Hom}_R(M, N)$ and each $\phi \in \operatorname{Hom}_R(S, M)$, is an S-module isomorphism.

Proof. By Ishikawa's Hom evaluation result [15, Lemma 1.6], $\Theta_{S,M,N}$ is an R-module isomorphism. One readily checks that $\Theta_{S,M,N}$ is also an S-module homomorphism, hence it is an S-module isomorphism. □

Lemma 2.2. *Let $f : R \to S$ be a ring homomorphism. Let S be finitely generated as an R-module, let M be an R-module, and let N be a flat R-module. Then the natural map*

$$\Omega_{S,M,N} : \operatorname{Hom}_R(S, M) \otimes_R N \to \operatorname{Hom}_R(S, M \otimes_R N)$$

defined as $\Omega_{S,M,N}(\psi \otimes_R n)(s) = \psi(s) \otimes_R n$ for each $\psi \otimes_R n \in \operatorname{Hom}_R(S, M) \otimes_R N$ and each $s \in S$, is an S-module isomorphism.

Proof. The proof is similar to that of Lemma 2.1, using Ishikawa's Tensor evaluation result instead. □

We collect here some properties of injectivity, projectivity and flatness associated with restriction of scalars. A version of this result for $S = R \ltimes C$ is found in [14, Lemma 3.1].

Lemma 2.3. *Let $f : R \to S$ be a ring homomorphism.*

(a) *Each injective S-module J is a direct summand in $\operatorname{Hom}_R(S, I)$ for some injective R-module I.*
(b) *Each projective S-module Q is a direct summand in $S \otimes_R P$ for some projective R-module P.*

Proof. (a) Since J is also an R-module via f, we have an exact sequence $0 \to J \to I$ of R-modules for some injective R-module I. Applying the left-exact $\operatorname{Hom}_R(S, -)$ to this exact sequence, noting that $\operatorname{Hom}_S(S, J)$ is an S-submodule of $\operatorname{Hom}_R(S, J)$, and using Hom cancellation, we obtain the following S-module iso/mono-morphisms.

$$J \xrightarrow{\cong} \operatorname{Hom}_S(S, J) \hookrightarrow \operatorname{Hom}_R(S, J) \hookrightarrow \operatorname{Hom}_R(S, I).$$

Since J is injective over S, this composite monomorphism splits as desired.

(b) This part is proved dually. □

We next define some useful classes and resolutions.

Definition 2.4. Let M be an R-module, and let $\mathcal{A}$ be a class of R-modules. Then an *augmented $\mathcal{A}$-resolution $\underline{X}^+$ of M* is an exact sequence of R-modules of the form

$$\underline{X}^+ = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \to M \to 0$$

where $X_i \in \mathcal{A}$ for each integer $i \geqslant 0$. The R-complex

$$\underline{X} = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \to 0$$

is the associated $\mathcal{A}$-*resolution* of M.

Definition 2.5. Let N be an R-module, and let $\mathcal{B}$ be a class of R-modules. Then an *augmented* $\mathcal{B}$-*coresolution* $^+\underline{Y}$ *of* N is an exact sequence of R-modules of the form

$$^+\underline{Y} = \quad 0 \to N \to Y_0 \xrightarrow{\partial_0^Y} Y_{-1} \xrightarrow{\partial_{-1}^Y} \cdots$$

where $Y_j \in \mathcal{B}$ for each integer $j \leqslant 0$. The R-complex

$$\underline{Y} = \quad 0 \to Y_0 \xrightarrow{\partial_0^Y} Y_{-1} \xrightarrow{\partial_{-1}^Y} \cdots$$

is the associated $\mathcal{B}$-*coresolution* of N.

Definition 2.6. Let C be an R-module.

(a) Let $\mathcal{I}$ be the class of injective R-modules.
(b) Let $\mathcal{P}$ be the class of projective R-modules.
(c) Let $\mathcal{F}$ be the class of flat R-modules.
(d) Let $\mathcal{I}_C$ be the class of R-modules isomorphic to $\operatorname{Hom}_R(C, I)$ for some injective R-module I.
(e) Let $\mathcal{P}_C$ be the class of R-modules isomorphic to $C \otimes_R P$ for some projective R-module P.
(f) Let $\mathcal{F}_C$ be the class of R-modules isomorphic to $C \otimes_R F$ for some flat R-module F.

The following two classes, known collectively as *Foxby classes*, are associated with a finitely generated R-module C. The definitions can be found in [3] and [5], and they are studied in conjunction with various homological dimensions, such as the G-dimension in [22], the C-projective dimension in [19] and the Gorenstein projective dimension in [21].

Definition 2.7. Let C be a finitely generated R-module. The *Auslander class* $\mathcal{A}_C(R)$ is the class of all R-modules M such that

(a) the natural map $\gamma_M^C : M \to \operatorname{Hom}_R(C, C \otimes_R M)$, defined as $\gamma_M^C(m)(c) := c \otimes_R m$ for all $m \in M$ and $c \in C$, is an isomorphism; and
(b) $\operatorname{Tor}_i^R(C, M) = 0 = \operatorname{Ext}_R^i(C, C \otimes_R M)$ for all $i \geqslant 1$.

Definition 2.8. Let C be a finitely generated R-module. The *Bass class* $\mathcal{B}_C(R)$ is the class of all R-modules M such that

(a) the evaluation map $\xi_M^C : C \otimes_R \operatorname{Hom}_R(C, M) \to M$, defined as $\xi_M^C(c \otimes_R \psi) := \psi(c)$ for all $c \in C$ and $\psi \in \operatorname{Hom}_R(C, M)$, is an isomorphism; and
(b) $\operatorname{Ext}_R^i(C, M) = 0 = \operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, M))$ for all $i \geqslant 1$.

Definition 2.9. Let C be an R-module.

(a) A *complete* $\mathcal{I}_C\mathcal{I}$-*resolution* $\underline{X}$ of R-modules is an exact sequence of R-modules of the form

$$\underline{X} = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots$$

such that $X_i \in \mathcal{I}_C$ for each integer $i \geqslant 1$, $X_j \in \mathcal{I}$ for each integer $j \leqslant 0$, and $\operatorname{Hom}_R(A, \underline{X})$ is exact for each $A \in \mathcal{I}_C$.

(b) A *complete $\mathcal{P}\mathcal{P}_C$-resolution* $\underline{X}$ of R-modules is an exact sequence of R-modules of the form

$$\underline{X} = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots$$

such that $X_i \in \mathcal{P}$ for each integer $i \geqslant 0$, $X_j \in \mathcal{P}_C$ for each integer $j \leqslant -1$, and $\operatorname{Hom}_R(\underline{X}, A)$ is exact for each $A \in \mathcal{P}_C$.

(c) A *complete $\mathcal{F}\mathcal{F}_C$-resolution* $\underline{X}$ of R-modules is an exact sequence of R-modules of the form

$$\underline{X} = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots$$

such that $X_i \in \mathcal{F}$ for each integer $i \geqslant 0$, $X_j \in \mathcal{F}_C$ for each integer $j \leqslant -1$, and $A \otimes_R \underline{X}$ is exact for each $A \in \mathcal{I}_C$.

Using complete resolutions, we next define C-Gorenstein injectivity, C-Gorenstein projectivity and C-Gorenstein flatness. We also note that these definitions are equivalent to [13, Definition 2.7].

Definition 2.10. Let C be an R-module. Then, an R-module M is

(a) *C-Gorenstein injective* if there is a complete $\mathcal{I}_C\mathcal{I}$-resolution $\underline{X}$, as in Definition 2.9(a), such that $\operatorname{Ker} \partial_0^X \cong M$.

(b) *C-Gorenstein projective* if there is a complete $\mathcal{P}\mathcal{P}_C$-resolution $\underline{X}$, as in Definition 2.9(b), such that $\operatorname{Coker} \partial_1^X \cong M$.

(c) *C-Gorenstein flat* if there is a complete $\mathcal{F}\mathcal{F}_C$-resolution $\underline{X}$, as in Definition 2.9(c), such that $\operatorname{Coker} \partial_1^X \cong M$.

When $C = R$, Definition 2.10 reduces to the definitions of Gorenstein injectivity, Gorenstein projectivity and Gorenstein flatness of Enochs, Jenda, and Torrecillas [8, 9], with complete $\mathcal{I}_C\mathcal{I}$-resolution, complete $\mathcal{P}\mathcal{P}_C$-resolution and complete $\mathcal{F}\mathcal{F}_C$-resolution becoming complete injective resolution, complete projective resolution and complete flat resolution, respectively.

Lemma 2.11. *Let C and M be R-modules. Then M is C-Gorenstein injective if and only if*

(a) *For each $A \in \mathcal{I}_C$, $\operatorname{Ext}_R^i(A, M) = 0$ for all $i \geqslant 1$.*

(b) *M admits an augmented $\mathcal{I}_C$-resolution $\underline{Y}^+$ such that $\operatorname{Hom}_R(A, \underline{Y}^+)$ is exact for each $A \in \mathcal{I}_C$.*

Lemma 2.12. *Let C and M be R-modules. Then M is C-Gorenstein projective if and only if*

(a) *For each $A \in \mathcal{P}_C$, $\operatorname{Ext}_R^i(M, A) = 0$ for all $i \geqslant 1$.*

(b) *M admits an augmented $\mathcal{P}_C$-coresolution ${}^+\underline{Y}$ such that $\operatorname{Hom}_R({}^+\underline{Y}, A)$ is exact for each $A \in \mathcal{P}_C$.*

Lemma 2.13. *Let C and M be R-modules. Then M is C-Gorenstein flat if and only if*

(a) *For each $A \in \mathcal{I}_C$, $\operatorname{Tor}_i^R(A, M) = 0$ for all $i \geqslant 1$.*

(b) *M admits an augmented $\mathcal{F}_C$-coresolution ${}^+\underline{Y}$ such that $A \otimes_R ({}^+\underline{Y})$ is exact for each $A \in \mathcal{I}_C$.*

As the injective R-complexes give rise to the injective dimension of an R-complex M, denoted $\operatorname{id}_R M$, the C-Gorenstein injective R-complexes give rise to the *C-Gorenstein injective dimension* of M: for a homologically left-bounded R-complex M, one has

$$C\text{-}\operatorname{Gid}_R M := \inf \left\{ \sup\{i \in \mathbb{Z} \mid X_{-i} \neq 0\} \,\middle|\, \begin{array}{c} X \text{ is a } C\text{-Gorenstein injective} \\ \text{resolution of } M \end{array} \right\}.$$

In the case $C = R$, the C-Gorenstein injective dimension of M is the *Gorenstein injective dimension* of M, denoted $\operatorname{Gid}_R M$. In other words, one has $R\text{-}\operatorname{Gid}_R M = \operatorname{Gid}_R M$. The *$C$-Gorenstein projective dimension*, the *Gorenstein projective dimension*, the *C-Gorenstein flat dimension*, and the *Gorenstein flat dimension* of an R-module M, denoted respectively $C\text{-}\operatorname{Gpd}_R M$, $\operatorname{Gpd}_R M$, $C\text{-}\operatorname{Gfd}_R M$, and $\operatorname{Gfd}_R M$, are defined similarly.

3. Semidualizing Modules and Gorenstein Dimensions

The main point of this section is to prove Theorem A from the introduction.

Property 3.1. Let R and S be rings, and let C be an R-module. Then the triple (R, S, C) *satisfies Property 3.1* if there is a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ & \searrow^{\operatorname{id}_R} & \downarrow g \\ & & R \end{array}$$

of ring homomorphisms with the identity map id_R on R such that $\operatorname{Hom}_R(S, C) \cong S$ as S-modules and $\operatorname{Ext}^i_R(S, C) = 0$ for all $i \geqslant 1$.

Remark 3.2. Property 3.1 implies that $\mathbf{R}\operatorname{Hom}_R(S, C) \simeq S$ in the derived category $\mathcal{D}(S)$. In other words, if $\underline{I}$ is an injective resolution of C over R, then Property 3.1 implies that $\operatorname{Hom}_R(S, \underline{I})$ is an injective resolution of the S-module S.

Property 3.3. Let R and S be rings, and let C be an R-module. Then the triple (R, S, C) *satisfies Property 3.3* if it satisfies Property 3.1 and $C \cong \operatorname{Ker} g$ as R-modules.

We here note that if (R, S, C) satisfies Property 3.3, it follows that $S \cong R \oplus C$ as R-modules. We next state and prove versions of several lemmas of Holm and Jørgensen [13, 14] in the general setting of Properties 3.1 and 3.3.

Lemma 3.4. *Let R and S be rings, and let C be an R-module. If (R, S, C) satisfies Property 3.1, then the following facts hold:*

(a) *For any R-module M, we have $\operatorname{Ext}^i_S(M, S) \cong \operatorname{Ext}^i_R(M, C)$ as S-modules for all $i \geqslant 0$.*

(b) *We also have $\operatorname{Hom}_S(R, S) \cong C$ as S-modules and $\operatorname{Ext}^i_S(R, S) = 0$ for all $i \geqslant 1$.*

Proof. (a) Argue as in [14, Lemma 3.2 (ii)] with the ring S taking the place of the trivial extension $R \ltimes C$. The essential point is to use Hom-tensor adjointness with the injective resolution $\operatorname{Hom}_R(S, \underline{I})$ of S, as described in Remark 3.2.

(b) This is the special case of part (a) where $M = R$. □

The following is Theorem B from the introduction.

Theorem 3.5. *Let R and S be rings, and let C be a finitely generated R-module such that (R,S,C) satisfies Property 3.1. Then the following are equivalent:*
(a) C is semidualizing over R;
(b) R is Gorenstein projective over S and $\mathrm{Ann}_R(C)=0$; and
(c) C is Gorenstein projective over S and $\mathrm{Ann}_R(C)=0$.

Proof. To prove that (a) implies (b), we assume that C is semidualizing over R. Using Lemma 3.4, we note that

$$\mathrm{Ext}^i_S(\mathrm{Hom}_S(R,S),S) \cong \mathrm{Ext}^i_S(C,S) \cong \mathrm{Ext}^i_R(C,C).$$

This is equal to 0 for all $i \geqslant 1$ and isomorphic to R when $i=0$ because C is semidualizing over R. Again, using the Ext-vanishing from Lemma 3.4(b), this means that R is Gorenstein projective over S by [4, Proposition 2.2.2]. We also note that $\mathrm{Ann}_R(C)$ is the kernel of the homothety map $\chi^R_C : R \to \mathrm{Hom}_R(C,C)$, which is 0 because C is semidualizing over R.

To prove that (b) implies (c), we recall that $\mathrm{Hom}_S(-,S)$ preserves the class of finitely generated Gorenstein projective S-modules by [4, Observation 1.1.7]. This proves the desired implication because $C \cong \mathrm{Hom}_S(R,S)$ as S-modules by Lemma 3.4(b).

To prove that (c) implies (a), we assume that C is Gorenstein projective over S and $\mathrm{Ann}_R(C)=0$. Since C is finitely generated over R, it is also finitely generated over S. Therefore, by [4, Theorem 4.2.6], we have

$$\mathrm{Ext}^i_S(C,S) = 0 = \mathrm{Ext}^i_S(\mathrm{Hom}_S(C,S),S)$$

for all $i \geqslant 1$ and the biduality map

$$\delta^S_C : C \to \mathrm{Hom}_S(\mathrm{Hom}_S(C,S),S)$$

is an S-module isomorphism. Using Lemma 3.4, we have

$$\mathrm{Ext}^i_R(C,C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(C,C),C)$$

for all $i \geqslant 1$ and the biduality map

$$\delta^S_C : C \to \mathrm{Hom}_S(\mathrm{Hom}_S(C,S),S) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(C,C),C)$$

is an R-module isomorphism. Therefore, C is "totally C-reflexive" over R. Since $\mathrm{Ann}_R(C)=0$, it follows that C is semidualizing over R by [17, Fact 1.1]. □

The assumption $\mathrm{Ann}_R(C)=0$ is essential in Theorem 3.5; see [17, Example 1.2].

Lemma 3.6. *Let R and S be rings, let N be a finitely generated R-module, and let C be a semidualizing R-module. If (R,S,C) satisfies Property 3.1, and if N is Gorenstein projective as an S-module, then the module $\mathrm{Hom}_R(N,I)$ is Gorenstein injective over S for any injective R-module I.*

Proof. Since N is Gorenstein projective over S, the module N has a complete projective resolution $\underline{P}$ over S. Moreover, since N is finitely generated over R (hence over S as well) $\underline{P}$ can be chosen to consist of finitely generated S-modules by [4, Theorems 4.1.4 and 4.2.6]. As in the proof of [14, Lemma 3.3 (ii)], it is straightforward to show that $\mathrm{Hom}_S(\underline{P},\mathrm{Hom}_R(S,I))$ is a complete injective resolution of $\mathrm{Hom}_R(N,I)$ over S. □

We here recover a version of [14, Lemma 3.3 (ii)] for our general setting.

Proposition 3.7. *Let R and S be rings, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.1, then for any injective R-module I, the modules* $\mathrm{Hom}_R(C, I)$ *and* $\mathrm{Hom}_R(R, I) \cong I$ *are Gorenstein injective over S.*

Proof. The modules C and R are Gorenstein projective over S by Theorem 3.5. Thus, the duals $\mathrm{Hom}_R(C, I)$ and $\mathrm{Hom}_R(R, I) \cong I$ are Gorenstein injective over S by Lemma 3.6. □

Next we prove a version of [14, Lemma 3.4] in the general setting.

Lemma 3.8. *Let R and S be rings, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.3, then for any injective R-module J, we have*

$$\mathrm{Ext}^i_S(\mathrm{Hom}_R(S, J), -) \cong \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), -)$$

for $i \geqslant 0$ as functors on S-modules.

Proof. Argue as in the proof of [14, Lemma 3.4] that

$$\mathrm{Hom}_R(S, J) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(S, C), J) \cong S \otimes_R \mathrm{Hom}_R(C, J)$$

using Lemma 2.1 and the fact that (R, S, C) satisfies Property 3.3 (hence Property 3.1). If $\underline{P}$ is a projective resolution over R of $\mathrm{Hom}_R(C, J)$, one can argue that $S \otimes_R \underline{P}$ is a projective resolution over S of $S \otimes_R \mathrm{Hom}_R(C, J) \cong \mathrm{Hom}_R(S, J)$. This uses the facts that $S \cong R \oplus C$ as R-modules and $J \in \mathcal{B}_C(R)$ by [19, 1.9 (b)] (hence $\mathrm{Hom}_R(C, J) \in \mathcal{A}_C(R)$ by Foxby equivalence [11, Proposition 1.4]). Using this projective resolution over S of $\mathrm{Hom}_R(S, J)$ and Hom-tensor adjointness, one can obtain the desired isomorphism. □

As a consequence of the above lemma, we have the following proposition.

Proposition 3.9. *Let R and S be rings, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.3 and M is an R-module, then for each $i \geqslant 0$, we have* $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), M) = 0$ *for all J injective over R if and only if* $\mathrm{Ext}^i_S(U, M) = 0$ *for all U injective over S.*

Proof. As in [13, Corollary 2.3 (1)], this follows from Lemmas 2.3(a) and 3.8. □

Lemma 3.10. *Let R and S be rings, and let C be a semidualizing R-module. If the triple (R, S, C) satisfies Property 3.3 and M is an R-module that is Gorenstein injective over S, then there exists a short exact sequence of R-modules*

$$0 \to M' \to \mathrm{Hom}_R(C, I) \to M \to 0$$

for some injective R-module I such that

(1) M' is Gorenstein injective over S
(2) the above sequence is $\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), -)$*-exact for any injective R-module J.*

Proof. The proof begins similarly to that of [14, Lemma 4.1].

Since M is Gorenstein injective over S, it has a complete injective resolution. From this, we can construct the following short exact sequence of S-modules

$$0 \to N \to K \to M \to 0$$

where K is injective over S, N is Gorenstein injective over S and the sequence is $\mathrm{Hom}_S(L, -)$-exact for each L injective over S, particularly for $L = \mathrm{Hom}_R(S, J)$ with any J injective over R.

As in the proof of [14, Lemma 4.1], we can use Lemma 2.3(a) to assume without loss of generality that the above sequence is of the form

$$0 \to N \xrightarrow{\epsilon} \operatorname{Hom}_R(S, I) \xrightarrow{\eta} M \to 0 \tag{3.10.1}$$

for some injective R-module I.

We here note that we cannot make use of a specific ring structure of S as in the proof of [14, Lemma 4.1], so we use Lemma 2.1 instead. Since $S \cong \operatorname{Hom}_R(S, C)$ as S-modules by Property 3.1, we have

$$\operatorname{Hom}_R(S, I) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(S, C), I) \cong S \otimes_R \operatorname{Hom}_R(C, I)$$

as S-modules, where the second isomorphism is by Lemma 2.1. We note here that I is injective over R, and $S \cong R \oplus C$ as R-modules, hence S is finitely generated over R. Therefore, we can replace $\operatorname{Hom}_R(S, I)$ in (3.10.1) with $S \otimes_R \operatorname{Hom}_R(C, I)$ to obtain the top row of the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\epsilon'} & S \otimes_R \operatorname{Hom}_R(C, I) & \xrightarrow{\eta'} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi \circ \epsilon'} & & \downarrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & M' := \operatorname{Ker} \phi & \hookrightarrow & \operatorname{Hom}_R(C, I) & \xrightarrow{\phi} & M & \longrightarrow & 0 \end{array} \tag{3.10.2}$$

The maps ψ and ϕ are defined as follows. For any $s \otimes_R \beta \in S \otimes_R \operatorname{Hom}_R(C, I)$, set $\psi(s \otimes_R \beta) := s\beta$, where the scalar multiplication is afforded by the S-module structure on the R-module $\operatorname{Hom}_R(C, I)$. For any β in $\operatorname{Hom}_R(C, I)$, set $\phi(\beta) := \eta'(1_S \otimes_R \beta)$. It is routine to check that both ψ and ϕ are well-defined S-module homomorphisms and that the diagram (3.10.2) is commutative.

As in [14, Lemma 4.1], we can show that the bottom row of the diagram (3.10.2) satisfies the desired properties. □

Lemma 3.11. *Let R and S be rings, and let C be an R-module such that (R, S, C) satisfies Property 3.3. Let M be an R-module that is C-Gorenstein injective over R. Then there exists a short exact sequence of S-modules*

$$0 \to M' \to U \to M \to 0$$

where U is injective over S, M' is C-Gorenstein injective over R and the above sequence is $\operatorname{Hom}_S(V, -)$-exact for any V injective over S.

Proof. The proof is similar to [13, Lemma 2.11], using Lemma 2.1 as in the previous result. □

Using the lemmas proved above in the general setting of the retract diagram, we can claim similar propositions and theorems as in [14] and [13].

Proposition 3.12. *Let R and S be rings, and let C be a semidualizing R-module, such that the triple (R, S, C) satisfies Property 3.3. Then, for any R-module M, M is C-Gorenstein injective over R if and only if M is Gorenstein injective over S.*

Proof. This is proved similarly as in [13, Proposition 2.13 (1)]. □

We need the dual versions of Lemma 3.8, Proposition 3.9, Lemma 3.10 and Lemma 3.11 to prove the projective and flat versions of Proposition 3.12. They are stated next for the sake of completeness.

Lemma 3.13. *Let R and S be rings, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.3, then for any projective R-module Q, we have*

$$\operatorname{Ext}^i_S(-, S \otimes_R Q) \cong \operatorname{Ext}^i_R(-, C \otimes_R Q)$$

for all $i \geqslant 0$ as functors on S-modules.

Proof. This is the dual of the proof of Lemma 3.8 using Lemma 2.2 and $\operatorname{Hom}_R(S, \underline{I})$ as the injective resolution over S of $\operatorname{Hom}_R(S, C \otimes_R Q)$ where $\underline{I}$ is an injective resolution of $C \otimes_R Q$. □

Proposition 3.14. *Let R and S be rings, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.3 and M is an R-module, then for each $i \geqslant 0$, we have $\operatorname{Ext}^i_R(M, C \otimes_R P) = 0$ for all P projective over R if and only if $\operatorname{Ext}^i_S(M, V) = 0$ for all V projective over S.*

Proof. This is the dual of Proposition 3.9. □

Lemma 3.15. *Let R and S be rings, and C be a semidualizing R-module. If the triple (R, S, C) satisfies Property 3.3 and M is an R-module that is Gorenstein projective over S, then there exists a short exact sequence of R-modules*

$$0 \to M \to C \otimes_R P \to M' \to 0$$

for some projective R-module P such that

(1) M' is Gorenstein projective over S
(2) the above sequence is $\operatorname{Hom}_R(-, C \otimes_R Q)$-exact for any projective R-module Q.

Proof. This is the dual of Lemma 3.10, using Lemma 2.2 instead. □

Lemma 3.16. *Let R and S be rings, and let C be an R-module such that (R, S, C) satisfies Property 3.3. Let M be an R-module that is C-Gorenstein projective over R. Then there exists a short exact sequence of S-modules*

$$0 \to M \to W \to M' \to 0$$

where W is projective over S, M' is C-Gorenstein projective over R and the above sequence if $\operatorname{Hom}_S(-, Y)$-exact for any Y projective over S.

Proof. This is the dual of Lemma 3.11. □

Using the above results, one can prove the injective version of Proposition 3.12.

Proposition 3.17. *Let R and S be rings, and let C be a semidualizing R-module, such that the triple (R, S, C) satisfies Property 3.3. Then, for any R-module M, M is C-Gorenstein projective over R if and only if M is Gorenstein projective over S.*

Proof. Argue similarly as in the proof of Proposition 3.12 using Lemmas 3.13, 3.15, 3.16 and Proposition 3.14 instead. □

For the flat version of Proposition 3.12, please see [13, Proposition 2.15], whose proof can be adapted for our general setting.

Proposition 3.18. *Let R and S be rings, and let C be a semidualizing R-module, such that the triple (R, S, C) satisfies Property 3.3. Then, for any R-module M, M is C-Gorenstein flat over R if and only if M is Gorenstein flat over S.*

Proof. Argue as in the beginning of the proof of [13, Proposition 2.15], using Hom-tensor adjointness, that for any faithfully-injective R-module E, the module M is C-Gorenstein flat if and only if the module $\mathrm{Hom}_R(M, E)$ is C-Gorenstein injective.

Since $\mathrm{Hom}_R(S, E)$ is faithfully injective over S for any E faithfully injective over R, one has $\mathrm{Gfd}_S M = \mathrm{Gid}_S(\mathrm{Hom}_S(M, \mathrm{Hom}_R(S, E)))$ by [4, Theorem 6.4.2]. Moreover, since $\mathrm{Hom}_S(M, \mathrm{Hom}_R(S, E)) \cong \mathrm{Hom}_R(M, E)$ by Hom-tensor adjointness and tensor cancellation, we have $\mathrm{Gfd}_S M = \mathrm{Gid}_S(\mathrm{Hom}_R(M, E))$.

The above two facts, combined with Proposition 3.12, give the desired result. □

The last result of this section is Theorem A.

Theorem 3.19. *Let R and S be rings, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.3, then for any homologically left-bounded R-complex M and any homologically right-bounded R-complex N, one has*

$$C\text{-}\mathrm{Gid}_R M = \mathrm{Gid}_S M$$
$$C\text{-}\mathrm{Gpd}_R N = \mathrm{Gpd}_S N$$
$$C\text{-}\mathrm{Gfd}_R N = \mathrm{Gfd}_S N$$

Proof. This follows from Propositions 3.12, 3.17 and 3.18 as in [13, Theorem 2.16]. □

4. Examples

It is routine to show that Nagata's trivial extension $R \ltimes C$ satisfies Property 3.3, hence we can recover [13, Theorem 2.16] as a special case of Theorem 3.19. The rest of this section is devoted to two similar constructions. In particular, we prove in this section Theorem C from the introduction.

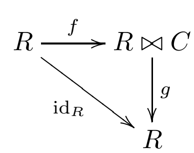
4.1. **Amalgamated Duplication of a Ring along an Ideal.**

The following construction is due to D'Anna and Fontana [6].

Definition/Notation 4.1. *Let R be a ring, and let C be an ideal in R. Then define a multiplication structure on $R \oplus C$ as follows: for each (r, c) and (r', c') in $R \oplus C$, we define $(r, c)(r', c') = (rr', rc' + r'c + cc')$. The group $R \oplus C$ with this multiplication structure is a ring with $(1_R, 0)$ as the multiplicative identity* [6]. *We denote this ring as $R \bowtie C$.*

It is routine to check that we have a retract diagram similar to the one in Property 3.1. We collect this information in the following lemma.

Lemma 4.2. *Let R be a ring, and let C be an ideal in R. Then the diagram*



where $f(r) := (r, 0)$ and $g(r, c) := r$ for each $r \in R$ and $c \in C$, is a commutative diagram of ring homomorphisms such that $\mathrm{Ker}\, g \cong C$ over R.

We prove next that the ring $R \bowtie C$ satisfies Property 3.1.

Lemma 4.3. *Let R be a ring, and let C be an ideal in R. If C is semidualizing over R, then $\operatorname{Hom}_R(R \bowtie C, C) \cong R \bowtie C$ as $R \bowtie C$-modules, and $\operatorname{Ext}^i_R(R \bowtie C, C) = 0$ for all $i \geqslant 1$.*

Proof. We first note that the $R \bowtie C$-module structure of $\operatorname{Hom}_R(R \bowtie C, C)$ comes from $R \bowtie C$ in the first slot. Specifically, for any (r, c) and (s, d) in $R \bowtie C$, and for any R-module homomorphism φ from $R \bowtie C$ to C, we have $((r, c)\varphi)(s, d) = \varphi((r, c)(s, d)) = \varphi(rs, rd + sc + cd)$. Since $R \bowtie C \cong R \oplus C$ as R-modules, we know that $\operatorname{Hom}_R(R \bowtie C, C) \cong \operatorname{Hom}_R(C, C) \oplus C$ as R-modules.

Since C is assumed to be semidualizing over R, we have $\operatorname{Hom}_R(C, C) \cong R$ as R-modules, hence $\operatorname{Hom}_R(R \bowtie C, C) \cong R \bowtie C$ as R-modules. Tracing all the natural isomorphisms involved, we see that the natural R-module isomorphism $\Theta : R \bowtie C \to \operatorname{Hom}_R(R \bowtie C, C)$ sends $(r, c) \mapsto \phi^{(r,c)}$, where $\phi^{(r,c)}$ is defined for any $(r'', c'') \in R \bowtie C$ as $\phi^{(r,c)}(r'', c'') = rc'' + r''c$.

However, unlike in the case of $R \ltimes C$, this natural R-module isomorphism Θ is *not* an $R \bowtie C$-module isomorphism. We therefore construct a new map Φ from $R \bowtie C$ to $\operatorname{Hom}_R(R \bowtie C, C)$, and we prove that Φ is indeed an $R \bowtie C$-module isomorphism.

Define $\Phi : R \bowtie C \to \operatorname{Hom}_R(R \bowtie C, C)$ as $\Phi(r, c) := \varphi_{(r,c)}$ for any $(r, c) \in R \bowtie C$, where $\varphi_{(r,c)}$ maps $(r'', c'') \mapsto rc'' + r''c + cc''$. It is routine to check that Φ is indeed an $R \bowtie C$-module homomorphism with respect to the module structures noted above.

We proceed to show that Φ is bijective. Since Θ is an isomorphism, we have $\operatorname{Im}\Phi \subseteq \operatorname{Hom}_R(R \bowtie C, C) = \operatorname{Im}\Theta$. Moreover, we can check that for any $\varphi^{(r,c)} \in \operatorname{Im}\Theta$, we have $\varphi^{(r,c)} = \varphi_{(r-c,c)} \in \operatorname{Im}\Phi$ for each $(r'', c'') \in R \bowtie C$. Therefore, we have $\operatorname{Im}\Theta \subseteq \operatorname{Im}\Phi$ proving that $\operatorname{Im}\Phi = \operatorname{Im}\Theta = \operatorname{Hom}_R(R \bowtie C, C)$, hence Φ is surjective. We here note that we have $r - c \in R$ since $C \subseteq R$.

We here prove that Φ is injective. Let $(r, c) \in \operatorname{Ker}\Phi$. Then $\varphi_{(r,c)}(r'', c'') = 0$ for any $(r'', c'') \in R \bowtie C$, particularly $(1_R, 0) \in R \bowtie C$. This implies that $0 = r(0) + (1_R)c + c(0) = c$. Therefore, for any $(r'', c'') \in R \bowtie C$, we have $0 = \phi_{(r,c)}(r'', c'') = rc''$, implying that $r \in \operatorname{Ann}_R(C)$. Moreover, since C is semidualizing over R, we have $\operatorname{Ann}_R(C) = 0$, implying that $r = 0$. This concludes that $\operatorname{Ker}\Phi = 0$, hence Φ is injective.

Finally, we note that we already have $\operatorname{Ext}^i_R(R \bowtie C, C) \cong \operatorname{Ext}^i_R(C, C)$ as R-modules. Since C is semidualizing over R, we have $\operatorname{Ext}^i_R(C, C) \cong 0$ for all $i \geqslant 1$, hence $\operatorname{Ext}^i_R(R \bowtie C, C) \cong 0$ as well. □

The next result justifies Theorem C(a) from the introduction.

Theorem 4.4. *Let R be a ring, let C be an ideal in R, and set $S := R \bowtie C$. If C is semidualizing as an R-module, then (R, S, C) satisfies Property 3.3.*

Proof. Lemmas 4.2 and 4.3 combined provide the desired result. □

Since $(R, R \bowtie C, C)$ satisfies Property 3.3, Theorem 3.19 can be applied to imply the following.

Corollary 4.5. *Let R be a ring, and let C be an ideal in R such that C is semidualizing over R. Then, for any homologically left-bounded R-complex M and any* *homologically right-bounded R-complex N, one has*

$$\begin{aligned} C\text{-}\mathrm{Gid}_R\, M &= \mathrm{Gid}_{R\bowtie C}\, M \\ C\text{-}\mathrm{Gpd}_R\, N &= \mathrm{Gpd}_{R\bowtie C}\, N \\ C\text{-}\mathrm{Gfd}_R\, N &= \mathrm{Gfd}_{R\bowtie C}\, N \end{aligned}$$

4.2. **Pseudocanonical Cover.**

In this section, we apply Theorem 3.19 to pseudocanonical covers introduced by Enescu in [7].

Definition/Notation 4.6. Let R be a ring, let $h \in R$, and let C be an ideal in R. We define a ring structure on $R \oplus C$ by defining $(r,c)(r',c') = (rr' + cc'h, rc' + r'c)$ for each $(r,c),(r',c') \in R \oplus C$. The group $R \oplus C$ with this multiplication structure, denoted as $S(h)$, is indeed a ring with $(1_R, 0)$ as its multiplicative identity [7], and is called the pseudocanonical cover of R via h.

We construct a retract diagram similar to the one in Property 3.1.

Lemma 4.7. *Let R be a ring, let C be an ideal and let $h \in R$ such that $h = r_0^2$ for some $r_0 \in R$. Then the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\ f\ } & S(h) \\ & \searrow^{\mathrm{id}_R} & \downarrow g \\ & & R \end{array}$$

where $f(r) := (r, 0)$ and $g(r,c) := r + cr_0$ for each $r \in R$ and $c \in C$, is a commutative diagram of ring homomorphisms such that $\operatorname{Ker} g \cong C$ over R.

Proof. By construction, f and g are well-defined functions making the diagram commute. It is routine to check that f is a ring homomorphism and that g respects addition. To check that g respects multiplication as well, let $r, r' \in R$ and $c, c' \in C$. Then

$$\begin{aligned} g\left((r,c)(r',c')\right) &= g(rr' + cc'h, rc' + r'c) \\ &= rr' + cc'h + rc'r_0 + r'cr_0 \\ &= r(r' + c'r_0) + cc'r_0^2 + r'cr_0 \\ &= r(r' + c'r_0) + cr_0(c'r_0 + r') \\ &= (r + cr_0)(r' + c'r_0) \\ &= g(r,c)g(r',c') \end{aligned}$$

where we used the fact that $h = r_0^2$.

We note that $\operatorname{Ker} g$ is the R-submodule of $S(h)$ consisting of all elements of the form $(-cr_0, c)$ with $c \in C$. Therefore one can readily prove that the map from C to $\operatorname{Ker} g$ sending $c \mapsto (-cr_0, c)$ is indeed an R-module isomorphism. □

Lemma 4.8. *Let R be a ring, let C be an ideal in R, and let $h \in R$ such that $h = r_0^2$ for some $r_0 \in R$. If C is semidualizing over R, then $\mathrm{Hom}_R(S(h), C) \cong S(h)$ as $S(h)$-modules, and $\mathrm{Ext}^i_R(S(h), C) = 0$ for all $i \geqslant 1$.*

Proof. We first note that the $S(h)$-module structure of $\mathrm{Hom}_R(S(h), C)$ comes from $S(h)$ in the first slot. Since $S(h) \cong R \oplus C$ as R-modules, we know that

$$\mathrm{Hom}_R(S(h), C) \cong \mathrm{Hom}_R(C, C) \oplus C$$

as R-modules.

Since C is assumed to be semidualizing over R, we have $\mathrm{Hom}_R(C, C) \cong R$ as R-modules, hence $S(h) \cong \mathrm{Hom}_R(S(h), C)$ as R-modules. Tracing the composition of all the natural R-module isomorphisms above, we have an R-module isomorphism $\Theta : S(h) \to \mathrm{Hom}_R(S(h), C)$ sending $(r, c) \mapsto \varphi^{(r,c)}$, where $\varphi^{(r,c)}$ is defined for any $(r'', c'') \in S(h)$ as $\varphi^{(r,c)}(r'', c'') = rc'' + r''c$. It is routine to check that Θ is also an $S(h)$-module homomorphism.

Finally, we have that $\mathrm{Ext}^i_R(S(h), C) \cong \mathrm{Ext}^i_R(C, C)$ as R-modules for all $i \geqslant 1$. Since C is semidualizing over R, we have $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$, hence $\mathrm{Ext}^i_R(S(h), C) = 0$ as well. □

The next result justifies Theorem C(b) from the introduction.

Theorem 4.9. *Let R be a ring, let C be an ideal in R, let $h \in R$ such that $h = r_0^2$ for some $r_0 \in R$, and let $S(h)$ be the pseudocanonical cover of R via h. If C is semidualizing as an R-module, then (R, S, C) satisfies Property 3.3.*

Proof. Lemmas 4.7 and 4.8 combined provide the desired result. □

We can apply Theorem 3.19 to $S(h)$.

Corollary 4.10. *Let R be a ring, let C be an ideal in R such that C is semidualizing over R, and let $h \in R$ such that $h = r_0^2$ for some $r_0 \in R$. Then, for any homologically left-bounded R-complex M and any homologically right-bounded R-complex N, one has*

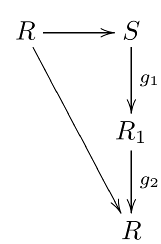
$$C - \mathrm{Gid}_R M = \mathrm{Gid}_{S(h)} M$$
$$C - \mathrm{Gpd}_R N = \mathrm{Gpd}_{S(h)} N$$
$$C - \mathrm{Gfd}_R N = \mathrm{Gfd}_{S(h)} N$$

Proof. Since $(R, S(h), C)$ satisfies Property 3.3, this is a direct application of Theorem 3.19. □

5. Counterexamples Regarding the Converse

It is natural to ask if the general settings we mentioned characterize the situation where an R-module M is C-Gorenstein injective over R if and only if M is Gorenstein injective over S. However, this fails in general, and the following is a counterexample.

Example 5.1. Let C be a semidualizing module, and set $R_1 := R \ltimes C$ and $S := R_1 \ltimes R_1$. We have the following diagram



We note that M is C-Gorenstein projective over R if and only if it is Gorenstein projective over R_1, if and only if Gorenstein projective over S by [13, Proposition 2.13]. We also note that $S \cong R \oplus (C \oplus R \oplus C)$ as R-modules. As $g = g_2 \circ g_1$, both of which are natural maps as described before, $\operatorname{Ker} g \cong R^2 \oplus C$, which is different from C.

We finally note here that the R-module structure on S in the previous example is not by accident. If we assume that a retract diagram in our general setting exists, i.e., there exists a ring homomorphism $f : R \to S$ such that $g \circ f = \mathrm{id}_R$, then g is a split surjection. This implies that $S \cong R \oplus \operatorname{Ker} g$ as R-modules as in the above example.

References

1. Maurice Auslander and Mark Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969. MR 0269685 (42 #4580)
2. Maurice Auslander and David A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405. MR 0086822 (19,249d)
3. Luchezar L. Avramov and Hans-Bjørn Foxby, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270. MR 1455856 (98d:13014)
4. Lars Winther Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, vol. 1747, Springer-Verlag, Berlin, 2000. MR 1799866 (2002e:13032)
5. ______, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 1839–1883 (electronic). MR 1813596 (2002a:13017)
6. Marco D'Anna and Marco Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6** (2007), no. 3, 443–459. MR 2337762 (2008i:13002)
7. Florian Enescu, *A finiteness condition on local cohomology in positive characteristic*, J. Pure Appl. Algebra **216** (2012), no. 1, 115–118. MR 2826424 (2012h:13010)
8. Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633. MR 1363858 (97c:16011)
9. Edgar E. Enochs, Overtoun M. G. Jenda, and Blas Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan **10** (1993), no. 1, 1–9. MR 1248299 (95a:16004)
10. Edgar E. Enochs, Overtoun M. G. Jenda, and Jin Zhong Xu, *Foxby duality and Gorenstein injective and projective modules*, Trans. Amer. Math. Soc. **348** (1996), no. 8, 3223–3234. MR 1355071 (96k:13010)
11. Hans-Bjørn Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284 (1973). MR 0327752 (48 #6094)
12. E. S. Golod, *G-dimension and generalized perfect ideals*, Trudy Mat. Inst. Steklov. **165** (1984), 62–66, Algebraic geometry and its applications. MR 752933 (85m:13011)
13. Henrik Holm and Peter Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra **205** (2006), no. 2, 423–445. MR 2203625 (2007c:13021)
14. ______, *Cohen-Macaulay homological dimensions*, Rend. Semin. Mat. Univ. Padova **117** (2007), 87–112. MR 2351787 (2008m:13022)
15. Takeshi Ishikawa, *On injective modules and flat modules*, J. Math. Soc. Japan **17** (1965), 291–296. MR 0188272 (32 #5711)
16. Maryam Salimi, Elham Tavasoli, and Siamak Yassemi, *The amalgamated duplication of a ring along a semidualizing ideal*, Rend. Semin. Mat. Univ. Padova **129** (2013), 115–127. MR 3090634
17. S. Sather-Wagstaff, *Reflexivity and connectedness*, Glasgow Mathematical Journal, to appear, `arXiv:1212.0131v1`.
18. Jean-Pierre Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955 (Tokyo), Science Council of Japan, 1956, pp. 175–189. MR 0086071 (19,119a)
19. Ryo Takahashi and Diana White, *Homological aspects of semidualizing modules*, Math. Scand. **106** (2010), no. 1, 5–22. MR 2603458 (2011a:16016)

20. Wolmer V. Vasconcelos, *Divisor theory in module categories*, North-Holland Publishing Co., Amsterdam, 1974, North-Holland Mathematics Studies, No. 14, Notas de Matemática No. 53. [Notes on Mathematics, No. 53]. MR 0498530 (58 #16637)
21. Diana White, *Gorenstein projective dimension with respect to a semidualizing module*, J. Commut. Algebra **2** (2010), no. 1, 111–137. MR 2607104 (2011d:13013)
22. Siamak Yassemi, *G-dimension*, Math. Scand. **77** (1995), no. 2, 161–174. MR 1379262 (97d:13017)

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