# ON THE RADICAL OF A MONOMIAL IDEAL

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ABSTRACT. Algebraic and combinatorial properties of a monomial ideal and its radical are compared.

## 1. INTRODUCTION

There are simple examples of Cohen-Macaulay ideals whose radical is not Cohen-Macaulay. The first such example is probably due to Hartshorne [5], who proved that in positive characteristic the toric ring $K[s^4, s^3t, st^3, t^4]$ is a set theoretic complete intersection. With CoCoA or other computer algebra systems many other examples, also in characteristic zero, can be constructed. The following example due Conca was computed with CoCoA: let $S = K[x_1, x_2, x_3, x_4, x_5]$ and $J = (x_2^2 - x_4x_5, x_1x_3 - x_3x_4, x_3x_4 - x_1x_5) \subset S$. Then $S/J$ is a 2-dimensional Cohen-Macaulay ring, $\sqrt{J} = (x_1x_3 - x_1x_5, x_3x_4 - x_1x_5, x_2^2 - x_4x_5, x_1^2x_2 - x_1x_2x_4, x_2x_3^2 - x_2x_3x_5)$ and $S/\sqrt{J}$ is *not* Cohen-Macaulay. Indeed, the depth of $S/\sqrt{J}$ equals 1. On the other hand it is well-known that the Cohen-Macaulay property of a monomial ideal is inherited by its radical. The reason is that the radical of a monomial ideal is essentially obtained by polarization and localization. This observation, was communicated to the third author by David Eisenbud. Both operations, polarization and localization, preserve the Cohen-Macaulay property. An explicit proof of this fact can be found in [11]. The purpose of this paper is to exploit this idea and to show that many other nice properties are inherited by the radical of a monomial ideal.

## 2. THE COMPARISON

For the proof of the main result of this paper we need some preparation. We begin with the following extension [10, Theorem 1.1] of Hochster's formula [1, Theorem 5.3.8] describing the local cohomology of a monomial ideal.

Let $K$ be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring and $I \subset S$ a monomial ideal. The unique minimal monomial system of generators of $I$ is denoted by $G(I)$. For $i = 1, \dots, n$ we set

$$t_i = \max\{\nu_i(u)\colon u \in G(I)\},$$

where for a monomial $u \in S$, $u = x_1^{a_1} \cdots x_n^{a_n}$ we set $\nu_i(u) = a_i$ for $i = 1, \dots, n$.

For $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we set

$$G_a = \{i\colon 1 \le i \le n,\ a_i < 0\},$$

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and define the simplicial complex $\Delta_a(I)$ whose faces are the sets $L\setminus G_a$ with $G_a \subset L$, and such that $L$ satisfies the following condition: for all $u \in G(I)$ there exists $i \notin L$ such that $\nu_i(u) > a_i \geq 0$.

Notice that the inequality $a_i \geq 0$ in the definition of $\Delta_a(I)$ follows from the condition $i \notin L \supset G_a$. It is included only for the reader's convenience.

With the notation introduced one has

**Theorem 2.1** (Takayama [10]). *Let $I \subset S$ be a monomial ideal. Then the Hilbert series of the local cohomology modules of $S/I$ with respect to the $\mathbb{Z}^n$-grading is given by*

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(S/I), \mathbf{t}) = \sum_{F\in\Delta}\sum_{a} \dim_K \tilde{H}_{i-|F|-1}(\Delta_a(I); K)\mathbf{t}^a$$

*where $\Delta$ is the simplicial complex corresponding to the Stanley-Reisner ideal $\sqrt{I}$, and the second sum is taken over all $a \in \mathbb{Z}^n$ such that $a_i \leq t_i - 1$ for all $i$, and $G_a = F$.*

As a first application of this theorem we have

**Corollary 2.2.** *Let $I \subset S$ be a monomial ideal. Then*

$$a(S/I) \leq \sum_{i=1}^{n} t_i - n,$$

*where $a(S/I)$ is the a-invariant of $S/I$.*

*Proof.* By Theorem 2.1, we know that $H^i_{\mathfrak{m}}(R)_a = 0$ for all $i$ and for all $a \in \mathbb{Z}^n$ such that $a_i > t_i - 1$ for some $i$. Thus in particular, if $d = \dim R$, then $H^d_{\mathfrak{m}}(R)_j = 0$ for $j > \sum_{i=1}^n t_i - n$. □

We say that $S/I$ has *maximal a-invariant* if the upper bound in Corollary 2.2 is attained, that is, if $a(S/I) = \sum_{i=1}^n t_i - n$.

For our main theorem the next corollary is important.

**Corollary 2.3.** *Let $I \subset S$ be a monomial ideal. Then we have the following isomorphisms of $K$-vector spaces*

$$H^i_{\mathfrak{m}}(S/I)_a \cong H^i_{\mathfrak{m}}(S/\sqrt{I})_a$$

*for all $a \in \mathbb{Z}^n$ with $a_i \leq 0$ for $1 \leq i \leq n$.*

*Proof.* Consider the multigraded Hilbert series of $H^i_{\mathfrak{m}}(S/I)$ and $H^i_{\mathfrak{m}}(S/\sqrt{I})$. Let $a \in \mathbb{Z}^n$ be such that $a_i \leq 0$ for all $1 \leq i \leq n$. Then by Theorem 2.1, we have

$$\begin{aligned} \dim_K H^i_{\mathfrak{m}}(S/I)_a &= \dim_K \tilde{H}_{i-|F|-1}(\Delta_a(I); K), \quad \text{and} \\ \dim_K H^i_{\mathfrak{m}}(S/\sqrt{I})_a &= \dim_K \tilde{H}_{i-|F|-1}(\Delta_a(\sqrt{I}); K), \end{aligned}$$

For a monomial $u$ we set $\mathrm{supp}(u) = \{i\colon x_i \text{ divides } u\}$. Now since for every $u \in G(I)$ there exists $v \in G(\sqrt{I})$ such that $\mathrm{supp}(u) \supset \mathrm{supp}(v)$, and since for every $v \in G(\sqrt{I})$ there exists $u \in G(I)$ such that $\mathrm{supp}(v) = \mathrm{supp}(u)$, it follows that $\Delta_a(I) = \Delta_a(\sqrt{I})$. Thus we have $\dim_K H^i_{\mathfrak{m}}(S/I)_a = \dim_K H^i_{\mathfrak{m}}(S/\sqrt{I})_a$. □

Let $M$ be a graded $S$-module. For the convenience of the reader we recall the following two concepts which generalize the Cohen-Macaulay property and non-pure shellability of simplicial complexes.

The following definition is due to Stanley [9, Section II, 3.9]:

**Definition 2.4.** Let $M$ be a finitely generated graded $S$-module. The module $M$ is *sequentially Cohen-Macaulay* if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$$

of $M$ by graded submodules of $M$ such that each quotient $M_i/M_{i-1}$ is CM, and $\dim M_1/M_0 < \dim M_2/M_1 < \ldots < \dim M_r/M_{r-1}$.

It is known (see for example [6, Corollary 1.7]) that if $M$ is sequentially Cohen-Macaulay, then the filtration given in the definition is uniquely determined. We call it the *attached filtration* of the sequentially Cohen-Macaulay module $M$.

The uniqueness of the filtration is seen as follows: suppose $\operatorname{depth} M = t$, then $M_1$ is the image of the natural map $\operatorname{Ext}_S^{n-t}(\operatorname{Ext}_S^{n-t}(M, \omega_S), \omega_S) \to M$. Here $\omega_S = S(-n)$ is the canonical module of $S$. Then one notices that $M/M_1$ is again sequentially Cohen-Macaulay and uses induction on the length of the attached sequence.

In case $M$ is a cyclic module, say, $M = S/I$, with attached filtration $0 = M_0 \subset M_1 \subset \cdots$, each of the the modules $M_i$ is an ideal in $S/I$, and hence is of the form $I_i/I$ for certain (uniquely determined) ideals $I_i \subset S$. Thus $S/I$ is sequentially Cohen-Macaulay, if and only of there exists a chain of graded ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_r = S$$

such that each factor module $I_{i+1}/I_i$ is Cohen-Macaulay with

$$\dim I_{i+1}/I_i < \dim I_{i+2}/I_{i+1}$$

for $i = 0, \ldots, r-2$. Moreover if this property is satisfied, then this chain of ideals is uniquely determined.

In the particular case that $I$ is a monomial ideal, the natural map

$$\operatorname{Ext}_S^{n-t}(\operatorname{Ext}_S^{n-t}(S/I, \omega_S), \omega_S) \to S/I$$

is a homomorphism of multigraded $S$-modules. This implies that the attached chain of ideals of the sequentially Cohen-Macaulay module $S/I$ is a chain of monomial ideals.

Now let us briefly describe the other concept which was introduced by Dress [4]:

**Definition 2.5.** Let $M$ be a finitely generated graded $S$-module. A filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$$

of $M$ by graded submodules of $M$ is called *clean* if for all $i = 1, \ldots, r$ there exists a minimal prime ideal $P_i$ of $M$ such that $M_i/M_{i-1} \cong S/P_i$. The module $M$ is called *clean* if it has a clean filtration.

Again, if $M = S/I$ is cyclic, then $S/I$ is clean if there exists a chain of ideals $I = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_{r-1} \subset I_r = S$ such that $I_{i+1}/I_i \cong S/P_i$ with $P_i$ a minimal prime ideal of $I$. In other words, for all $i = 0, \ldots, r-1$ there exists $f_{i+1} \in I_{i+1}$ such that $I_{i+1} = (I_i, f_{i+1})$ and $P_i = I_i : f_{i+1}$. In case $I$ is a monomial ideal we require that all $f_i$ are monomials.

Dress [4] shows that a Stanley-Reisner ideal $I_\Delta$ is clean if and only the simplicial complex $\Delta$ is non-pure shellable in the sense of Björner and Wachs [3].

In the proof of our main theorem we use polarization, as indicated in the introduction. Let $I = (u_1, \ldots, u_m)$ with $u_i = x_1^{a_{i1}} \cdots x_n^{a_{in}}$. We fix some number $i$ with $1 \leq i \leq n$, introduce a new variable $y$, and set $v_k = x_1^{a_{k1}} \cdots x_i^{a_{ki}-1} y \cdots x_n^{a_{kn}}$ if $a_{ki} > 1$,and $v_k = u_k$ otherwise. We call $J = (v_1, \ldots, v_m)$ the 1-*step polarization of $I$ with respect to the variable $x_i$*. The element $y - x_i$ is regular on $S[y]/J$ and $(S[y]/J)/(y - x_i)(S[y]/J) \cong S/I$, see [1, Lemma 4.2.16].

Let as above $t_i = \max\{\nu_i(u_j) \colon j = 1, \ldots, m\}$, and set $t = \sum_{i=1}^n t_i - n$. Then it is clear that if we apply $t$ suitable 1-step polarizations, we end up with a squarefree monomial ideal $I^p$, which is called the *complete polarization of $I$*.

Now we are ready to present the main result of this section.

**Theorem 2.6.** *Let $K$ be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring over $K$, and $I \subset S$ a monomial ideal. Suppose that $S/I$ satisfies one of the following properties: $S/I$ is* (i) *Cohen-Macaulay,* (ii) *Gorenstein,* (iii) *sequentially Cohen-Macaulay,* (iv) *generalized Cohen-Macaulay,* (v) *Buchsbaum,* (vi) *clean, or* (vii) *level and has maximal a-invariant. Then $S/\sqrt{I}$ satisfies the corresponding property.*

*Proof.* We first use the trick, mentioned in the introduction, to show that the Betti-numbers $\beta_i(I)$ of $I$ do not increase when passing to $\sqrt{I}$.

We denote by $I^p$ the complete polarization of $I$. Let $T$ be the polynomial ring in the variables that are needed to polarize $I$. Then $I^p$ is a squarefree monomial ideal in $T$ with $\beta_i(I^p) = \beta_i(I)$ for all $i$. It is easy to see that if we localize at the multiplicative set $N$ generated by the *new* variables which are needed to polarize $I$, one obtains $I^p T_N = (\sqrt{I})T_N$. Since localization is an exact functor, the localized free resolution will be a possibly non-minimal free resolution of $(\sqrt{I})T_N$. Since the extension $S \to T_N$ is flat, the desired inequality follows.

Proof of (i) and (ii): The inequality $\beta_i(\sqrt{I}) \leq \beta_i(I)$ implies that $\operatorname{depth} S/\sqrt{I} \geq \operatorname{depth} S/I$. On the other hand, $\dim S/I = \dim S/\sqrt{I}$. This implies that $S/\sqrt{I}$ is Cohen-Macaulay, if $S/I$ is so.

Suppose now that $S/I$ is Gorenstein. Then $\beta_q(S/I) = 1$ where $q$ is the codimension of $I$, see [1, Theorem 3.3.7 and Corollary 3.3.9]. Therefore, $\beta_q(S/\sqrt{I}) \leq 1$. Since $I$ and $\sqrt{I}$ have the same codimension, we see that $\beta_q(S/\sqrt{I}) > 0$, and hence $\beta_q(S/\sqrt{I}) = 1$. Again using [1, Theorem 3.3.7 and Corollary 3.3.9] we conclude that $S/\sqrt{I}$ is Gorenstein. This fact follows also from [2, Corollary 3.4].

Proof of (iii): Since $S/I$ is sequentially Cohen-Macaulay there exists a chain of monomial ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_k = S$$

such that $I_{j+1}/I_j$ is Cohen-Macaulay for all $j = 0, \ldots, k-1$ and such that $\dim I_1/I_0 < \dim I_2/I_1 < \ldots < \dim I_k/I_{k-1}$.

Suppose $x_1^a$ with $a > 1$ divides a generator of $I$. Then we apply a 1-step polarization for $x_1$ to all the ideals $I_i$, and obtain a chain of ideals $J = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_k = \tilde{S}$ where $\tilde{S} = S[y]$. It follows that $y - x_1$ is $\tilde{S}/J_i$-regular and $(\tilde{S}/J_i)/(y-x_1)(\tilde{S}/J_i) \cong S/I_i$ for all $i$. Therefore $y - x_1$ is $J_{i+1}/J_i$-regular, and $(J_{i+1}/J_i)/(y-x_1)(J_{i+1}/J_i) \cong I_{i+1}/I_i$. Thus $J$ is sequentially Cohen-Macaulay.

Since the complete polarization $I_i^p$ of the ideals $I_i$ for $i = 1, \dots k$, is obtained by a sequence of 1-step polarizations, it follows that $I^p$ is sequentially Cohen-Macaulay. As $I_i^p/I_{i+1}^p$ is Cohen-Macaulay, we conclude as in the proof of (i) that $\sqrt{I_{i+1}}/\sqrt{I_i}$ is Cohen-Macaulay of the same dimension as $I_{i+1}/I_i$. This shows that $\sqrt{I}$ is sequentially Cohen-Macaulay.

Proof of (iv) and (v): Assuming that $S/I$ is generalized Cohen-Macaulay or Buchsbaum, one has that $S/I$ is equidimensional and that $H^i_{\mathfrak{m}}(S/I)_j = 0$ for all $i < \dim S/I$, and all but finitely many $j$. Since $I$ and $\sqrt{I}$ have the same minimal prime ideals, it follows that $\sqrt{I}$ is again equidimensional.

Let $\mathbb{Z}^n_-$ be the set of all $a \in \mathbb{Z}^n$ such that $a_i \leq 0$ for $i = 1, \dots, n$. By Corollary 2.3, $H^i_{\mathfrak{m}}(S/I)_a = H^i_{\mathfrak{m}}(S/\sqrt{I})_a$ for all $a \in \mathbb{Z}^n_-$. Moreover, by Hochster's formula, $H^i_{\mathfrak{m}}(S/\sqrt{I})_a = 0$ for all $a \notin \mathbb{Z}^n_-$. Therefore, $\dim_K H^i_{\mathfrak{m}}(S/\sqrt{I})_j \leq \dim_K H^i_{\mathfrak{m}}(S/I)_j$ for all $j \leq 0$ and $H^i_{\mathfrak{m}}(S/J)_j = 0$ for $j > 0$. It is known [8] that a squarefree monomial ideal is Buchsbaum if and only if it is generalized Cohen-Macaulay. Thus (iv) and (v) follow.

Proof of (vi): Assuming that $S/I$ is clean, there exists a chain of monomial ideals $I = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_{r-1} \subset I_r = S$ such that $I_{i+1}/I_i \cong S/P_i$ with $P_i$ a minimal prime ideal of $I$. We claim that $\sqrt{I_{i+1}}/\sqrt{I_i} = S/P_i$, if $\sqrt{I_{i+1}} \neq \sqrt{I_i}$. This then implies that $S/\sqrt{I}$ is clean, since the prime ideals $P_i$ are also minimal prime ideals of $\sqrt{I}$.

In order to prove this claim we introduce some notation: let $u = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ and $v = x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ be two monomials. Then we set

$$u : v = \prod_{i=1}^{n} x_i^{\max\{a_i - b_i, 0\}}, \quad \text{and} \quad u_{red} = \prod_{\substack{i \\ a_i > 0}} x_i.$$

We then have

$$(u : v)_{red} = (u_{red} : v_{red}) \prod_{\substack{i, \\ a_i > b_i > 0}} x_i. \tag{1}$$

Note that if $I$ is a monomial ideal with monomial generators $u_1, \dots, u_m$, then

$$\sqrt{I} = ((u_1)_{red}, \dots, (u_m)_{red}) \quad \text{and} \quad I : v = (u_1 : v, \dots, u_m : v).$$

Back to the proof of our claim, our assumption implies that for all $i = 0, \dots, r-1$ there exists a monomial $v_{i+1} \in I_{i+1}$ such that $I_{i+1} = (I_i, v_{i+1})$ and $P_i = I_i : v_{i+1}$. Suppose $P_i = (x_{i_1}, \dots, x_{i_s})$. Then $P_i = I_i : v_{i+1}$ if and only if

(a) for all $j = 1, \dots, s$ there exists $u \in I_i$ such that $u : v_{i+1} = x_{i_j}$, and

(b) for all monomial generators $w \in I_i$ there exists an integer $j$ with $1 \leq j \leq s$ such that $x_{i_j}|(w : v_{i+1})$.

We need to show that $P_i = \sqrt{I_i} : (v_{i+1})_{red}$, if $(v_{i+1})_{red} \notin \sqrt{I_i}$, and prove this by checking (a) and (b) for the pair $\sqrt{I_i}$ and $(v_{i+1})_{red}$.

Let $j$ be an integer with $1 \leq j \leq s$. Then there exists $u \in I_i$ such that $u : v_{i+1} = x_{i_j}$. Suppose $u = \prod_{k=1}^n x_k^{a_k}$ and $v_{i+1} = \prod_{k=1}^n x_k^{b_k}$, then (1) implies that $x_{i_j} = (u : v_{i+1})_{red} = (u_{red} : (v_{i+1})_{red})w$ where $w = \prod_{k,\, a_k > b_k > 0} x_k$. Suppose $x_{i_j}$ divides $w$, then $u_{red} : (v_{i+1})_{red} = 1$. This implies that $(v_{i+1})_{red} \in \sqrt{I_i}$, a contradiction. Therefore $u_{red} : (v_{i+1})_{red} = x_{i_j}$, and this proves (a). The argument also shows that $b_{i_j} = 0$ for $j = 1, \ldots, s$.

For the proof of (b), let $w \in I_i$ be a monomial generator. Then there exists an integer $j$ with $1 \leq j \leq s$ such that $x_{i_j}|(w : v_{i+1})$. It follows that $x_{i_j}$ divides $(w : v_{i+1})_{red}$. Let $w = \prod_{k=1}^n x_k^{c_k}$. Then (1) implies that $x_{i_j}$ divides $(w_{red} : (v_{i+1})_{red}) \prod_{k,\, c_k > b_k > 0} x_k$. However, $b_{i_j} = 0$, as we have seen in the proof of (a). Therefore, $x_{i_j}$ divides $(w_{red} : (v_{i+1})_{red})$. Since $\sqrt{I_i}$ is generated by the monomials $w_{red}$ where the monomials $w$ are the generators of $I_i$, condition (b) follows.

Proof of (vii): By assumption $S/I$ is level. This means that $S/I$ is Cohen-Macaulay and that all generators of the canonical module $\omega_{S/I}$ of $S/I$ have the same degree, say $g$. In this situation the $a$-invariant $a(S/I)$ of $S/I$ is just $-g$, see [1, Section 3.6]. Suppose $d = \dim S/I$; then $I$ has a graded minimal free resolution $\mathbb{F}$ of length $q = n - d - 1$ with $F_q = S^b(-c)$, Since $\omega_{S/I}$ may be represented as the cokernel of $F_{q-1}^* \to F_q^*$, which is dual of the map $F_q \to F_{q-1}$ with respect to $S(-n)$, it follows that $a(S/I) = c - n$.

For $i = 1, \ldots, n$ we set again

$$t_i = \max\{\nu_i(u) \colon u \in G(I)\}.$$

By Corollary 2.2, one has the upper bound $a(S/I) \leq \sum_{i=1}^n t_i - n$. Since we assume that $S/I$ has maximal $a$-invariant, the upper bound is reached. Let $I^p \subset T$ the complete polarization of $I$. This polarization requires precisely $t = \sum_{i=1}^n t_i - n$ 1-step polarizations. It follows that $S/I$ is obtained from $T/I^p$ as a residue class ring modulo a regular sequence of linear forms of length $t$. From the above description of the $a$-invariant we now conclude that $a(T/I^p) = a(S/I) - t = 0$. Let $\mathbb{G}$ be the multigraded minimal free resolution of the squarefree monomial ideal $I^p$. Since $\operatorname{proj\,dim} I^p = \operatorname{proj\,dim} I = q$, and since $a(T/I^p) = 0$, we see that $G_q = T(-m)^b$, where $m = n + t = \dim T$. This implies that $G_q$ as a multigraded module is isomorphic to $T(-e)^b$ where $e = (1, 1, \ldots, 1)$.

For $i = 1, \ldots, m$ let $e_i$ be the $i$th canonical basis vector of $\mathbb{Z}^m$. Then $e = \sum_{i=1}^m e_i$, and we may assume that $\deg x_i = e_i$ for $i = 1, \ldots, n$, while the new variables have the multidegrees $e_i$ with $i = n + 1, \ldots, m$. We define a new multigrading on $T$ and $T/I^p$: for an element $f$ of multidegree $a$ we set $\deg' f = \pi(a)$, where $\pi \colon \mathbb{Z}^m \to \mathbb{Z}^n$ is the projection onto the first $n$ components of $\mathbb{Z}^m$.

As above, let $N$ be the multiplicative set generated by the $t$ new variables which are needed to polarize $I$. Then $I^p T_N = \sqrt{I} T_N$, and localization with respect to $N$ preserves the new multigrading since $\deg' f = 0$ for all $f \in N$. Therefore $\mathbb{G}_N$ is, with respect to the new grading, a multigraded free $T_N$-resolution of $\sqrt{I}T_N$ with $(G_q)_N = T_N(-1,\ldots,-1)^b$ and $(-1,\ldots,-1) \in \mathbb{Z}^n$.

Let $\mathbb{H}$ be the multigraded minimal free $S$-resolution of $\sqrt{I}$. Then $\mathbb{H}T_N$ is the minimal multigraded free $T_N$-resolution of $\sqrt{I}T_N$. A comparison with the (possibly non-minimal) graded free $T_N$-resolution $\mathbb{G}_N$ shows that $H_q$ is a direct summand of copies of $S(-1,\ldots,-1)$. Since $S/I$ and $S/\sqrt{I}$ are Cohen-Macaulay of the same dimension, we see that $q = \operatorname{proj\,dim} I = \operatorname{proj\,dim} \sqrt{I}$. Therefore all summands in the last step of the resolution $\mathbb{H}$ of $S/\sqrt{I}$ have the same shift. This show that $S/\sqrt{I}$ is level. □

**Remark 2.7.** In Theorem 2.6(i) (or (iv)), it suffices to require that $I$ is an arbitrary homogeneous (generalized) Cohen-Macaulay ideal whose radical $\sqrt{I}$ is a monomial ideal, i.e. we do not need to require that $I$ itself is a monomial ideal.

Indeed it is enough to prove that there is a surjective homomorphism $H^i_{\mathfrak{m}}(S/I) \longrightarrow H^i_{\mathfrak{m}}(S/\sqrt{I})$ for all $i$. The natural surjective map $S/I \longrightarrow S/\sqrt{I}$ induce for all $i$ commutative diagrams

$$\begin{array}{ccc} \operatorname{Ext}^i(S/\sqrt{I}, S) & \longrightarrow & \operatorname{Ext}^i(S/I, S) \\ \downarrow & & \downarrow \\ H^i_{\sqrt{I}}(S) & \longrightarrow & H^i_I(S). \end{array}$$

Since $H^i_{\sqrt{I}}(S) \cong H^i_I(S)$ and since $\operatorname{Ext}^i(S/\sqrt{I}, S) \longrightarrow H^i_{\sqrt{I}}(S)$ is an essential extension (see [12]), it follows that $\operatorname{Ext}^i_S(S/\sqrt{I}, S) \longrightarrow \operatorname{Ext}^i_S(S/I, S)$ is injective for all $i$. Hence the desired conclusion follows by local duality.

On the other hand, as for the Gorenstein property, we must assume that $I$ is a monomial ideal. For example, $I = (xy + yz, xz)$ is a complete intersection, hence, a Gorenstein ideal, while $\sqrt{I} = (xy, yz, xz)$ is not Gorenstein.

## 3. The inverse problem

The results of the previous section indicate the following question: for a subset $F \subset [n]$, let $P_F$ be the prime ideal generated by the $x_i$ with $i \in F$. The minimal prime ideals of a squarefree $I$ are all of this form, and since $I$ is a radical ideal it is the intersection of its minimal prime ideals, say, $I = \bigcap_{i=1}^r P_{F_i}$ with $F_i \subset [n]$.

Suppose $I$ is Cohen-Macaulay. For which exponents $a_{ij}$ is the ideal

$$J = \bigcap_{i=1}^{r} (x_j^{a_{ij}} : j \in F_i)$$

again Cohen-Macaulay?

Of course if we raise the $x_i$ uniformly to some power, say $x_i$ is replaced by $x_i^{a_i}$ everywhere in the intersection, then the resulting ideal $J$ is the image of the flat map $S \to S$ with $x_i \mapsto x_i^{a_i}$ for all $i$. Thus in this case $J$ will be Cohen-Macaulay, if $I$ is so. On the other hand, if we allow arbitrary exponents, the question seems to be quite delicate, and we do not know a general answer. However, if we require that for *all* choices of exponents the resulting ideal is again Cohen-Macaulay, a complete answer is possible.

We need a definition to state the next result. Let $L$ be a monomial ideal. Lyubeznik [7] defines the *size* of $L$ as follows: let $L=\bigcap_{j=1}^r Q_j$ be an irredundant primary decomposition of $L$, where the $Q_i$ are monomial ideals. Let $h$ be the height of $\sum_{j=1}^r Q_j$, and denote by $v$ the minimum number $t$ such that there exist $j_1,\ldots,j_t$ with $\sqrt{\sum_{i=1}^t Q_{j_i}}=\sqrt{\sum_{j=1}^r Q_j}$. Then $\operatorname{size} L=v+(n-h)-1$.

Since for monomial ideals the operations of forming sums and taking radicals can be exchanged, the numbers $v$ and $h$, and hence the size of $L$ depends only on the associated prime ideals of $L$.

We shall need the following result of Lyubeznik [7, Proposition 2]:

**Lemma 3.1.** *Let $L$ be a monomial ideal in $S$. Then* $\operatorname{depth} S/L \geq \operatorname{size} L$.

Now we can state the main result of this section.

**Theorem 3.2.** *Let $I\subset S=K[x_1,\ldots,x_n]$ be a Cohen-Macaulay squarefree monomial ideal, and write*

$$I=\bigcap_{i=1}^r P_{F_i},$$

*where the sets $F_i\subset [n]$ are pairwise distinct, and all have the same cardinality $c$.*

*For $i=1,\ldots,r$ and $j=1,\ldots,c$ we choose integers $a_{ij}\geq 1$, and set*

$$Q_{F_i}=(x_j^{a_{ij}}\colon j\in F_i) \quad \text{for} \quad i=1,\ldots,r.$$

*Then the following conditions are equivalent:*

(a) *for all choices of the integers $a_{ij}$ the ideal*

$$J=\bigcap_{i=1}^r Q_{F_i}$$

*is Cohen-Macaulay;*

(b) *for each subset $A\subset [r]$, the ideal $I_A=\bigcap_{i\in A} P_{F_i}$ is Cohen-Macaulay;*

(c) $\operatorname{height} P_{F_i}+P_{F_j}=c+1$ *for all $i\neq j$;*

(d) *for $r\geq 2$ either $|\bigcup_{i=1}^r F_i|=c+1$, or $|\bigcap_{i=1}^r F_i|=c-1$;*

(e) *after a suitable permutation of the elements of $[n]$ we either have*

$$F_i=\{1,\ldots,i-1,i+1,\ldots,c,c+1\} \quad \text{for} \quad i=1,\ldots,r,$$

*or*

$$F_i=\{1,\ldots,c-1,c-1+i\} \quad \text{for} \quad i=1,\ldots,r;$$

(f) $\operatorname{size} I=\dim S/I$;

(g) *$S/L$ is Cohen-Macaulay for any monomial ideal $L$ such that* $\operatorname{Ass} L=\operatorname{Ass} I$.

*Proof.* (a) $\Rightarrow$ (b): Let $Q_{F_i}=(x_j^2\colon j\in F_i)$ if $i\in A$, and $Q_{F_i}=P_{F_i}$ if $i\notin A$. By assumption, $J=\bigcap_{i=1}^r Q_{F_i}$ is Cohen-Macaulay. Hence the complete polarization $J^p$ of $J$ is again Cohen-Macaulay. We have $J^p=\bigcap_{i=1}^r Q_{F_i}^p$ with $Q_{F_i}^p=(x_jy_j\colon j\in F_i)$ if

$i \in A$, and $Q^p_{F_i} = P_{F_i}$ if $i \notin A$. Let $N$ be the multiplicative set generated by all the variables $x_i$. Then $J^p_N$ is Cohen-Macaulay, and hence

$$J^p_N = \bigcap_{i\in A} (y_j \colon j \in F_i).$$

This shows that $I_A = \bigcap_{i\in A} P_{F_i}$ is Cohen-Macaulay.

(b) $\Rightarrow$ (c): Consider the exact sequence

$$0 \longrightarrow S/(P_{F_i} \cap P_{F_j}) \longrightarrow S/P_{F_i} \oplus S/P_{F_j} \longrightarrow S/(P_{F_i} + P_{F_j}) \longrightarrow 0.$$

The rings $S/P_{_i}$ and $S/P_{_j}$ are Cohen-Macaulay of dimension $n-c$, while $S/(P_{F_i}+P_{F_j})$ is Cohen-Macaulay of dimension $n-d$ where $d$ is the height of $P_{F_i}+P_{F_j}$. The exact sequence yields that $S/(P_{F_i} \cap P_{F_j})$ is Cohen-Macaulay if and only if $d = c+1$.

Since by assumption $S/P_{F_i} \cap P_{F_j}$ is Cohen-Macaulay for all $i \neq j$, the assertion follows.

(c) $\Rightarrow$ (d): We must show: given a collection of subsets $F_1, \ldots, F_r \subset [n]$ with

(i) $|F_i| = c$ for all $i$;
(ii) $|F_i \cup F_j| = c+1$ for all $i \neq j$.

Then either $|\bigcup_{i=1}^r F_i| = c+1$, or $|\bigcap_{i=1}^r F_i| = c-1$.

Suppose this is not the case. Then, since $|F_1 \cap F_2| = c-1$ and $|F_1 \cup F_2| = c+1$, there exist integers $i$ and $j$ such that $F_1 \cap F_2 \not\subset F_i$, and $F_j \not\subset F_1 \cup F_2$. The conditions (i) and (ii) then imply that there exists an element $x \in F_1 \cap F_2$ such that $F_1 \cup F_2 \setminus \{x\} = F_i$, and an element $y \in F_j \setminus (F_1 \cup F_2)$ such that $F_j = \{y\} \cup (F_1 \cap F_2)$. It follows that $F_i \cup F_j = (F_1 \cup F_2) \cup \{y\}$. This contradicts (ii).

(d) $\Rightarrow$ (e): Assume that $|\bigcup_{i=1}^r F_i| = c+1$. After a suitable permutation of the elements of $[n]$ we may assume that $\bigcup_{i=1}^r F_i = \{1, \ldots, c+1\}$. Since $|F_i| = c$, there exists $j_i \in \{1, \ldots, c+1\}$ such that $F_i = \{1, \ldots, c+1\} \setminus \{j_i\}$. Since the sets $F_i$ are pairwise distinct it follows that $j_i \neq j_k$ for $i \neq k$. Thus after applying again suitable permutation we may assume that $j_i = i$ for $i = 1, \ldots, r$.

The second statement follows similarly.

(e) $\Rightarrow$ (f): In the first case, $v = 2$ and $h = (c+1)$, while in the second case, $v = r$ and $h = c-1+r$. Thus in both cases $\operatorname{size} I = n - c = \dim S/I$.

(f) $\Rightarrow$ (g): By Lemma 3.1 and the remark preceding the lemma, we have

$$\operatorname{depth} S/L \geq \operatorname{size} L = \operatorname{size} I = \dim S/I = \dim S/L.$$

Hence $S/L$ is Cohen-Macaulay.

Finally the implication (g) $\Rightarrow$ (a) is trivial. □

**Corollary 3.3.** *With notation as above, the following conditions are equivalent:*

(a) *$J$ is a Gorenstein ideal for all choices of the integers $a_{ij}$;*
(b) *$r = 1$ or $c = 1$.*

*Proof.* If $r = 1$ or $c = 1$, then $J$ is complete intersection for all choices of the integers $a_{ij}$. Thus (b) implies (a).

Conversely suppose condition (b) is not satisfied. We assume that $c > 1$, and have to show that $r = 1$. By Theorem 3.2 we have $|\bigcap_{i=1}^r F_i| = c-1$ or $|\bigcup_{i=1}^r F_i| = c+1$.

In the first case we may assume that $F_i = \{1,\ldots,c-1,i+c-1\}$ for $i = 1,\ldots r$. Assume $r > 1$, and let $Q_{F_1} = (x_1^2, x_2, \ldots, x_c)$ and $Q_{F_i} = P_{F_i}$ for $i \geq 2$. Then $J = \bigcap_{i=1}^r Q_{F_i} = (x_1^2, x_1x_2, \prod_{i=0}^{r-1} x_{c+i})$ is not Gorenstein, a contradiction.

In the second case suppose that $r \geq 3$. With the same argument as in the proof of Theorem 3.2 it follows that $I_A = \bigcap_{i\in A} P_{F_i}$ is a Gorenstein ideal for all subsets $A \subset [r]$. Therefore $P_{F_1} \cap P_{F_2} \cap P_{F_3}$ is Gorenstein. We may assume that $F_1 = \{1,2,\ldots,c\}$, $F_2 = \{2,3,\ldots,c+1\}$ and $F_3 = \{1,3,4,\ldots,c+1\}$. Then $P_{F_1} \cap P_{F_2} \cap P_{F_3} = (x_1x_2, x_1x_{c+1}, x_2x_{c+1}, x_3, \ldots, x_c)$ is not Gorenstein, a contradiction.

On the other hand, if $r = 2$, then $|\bigcap_{i=1}^r F_i| = c-1$, and we are again in the first case. Thus we must have that $r = 1$. □

**Remark 3.4.** From a view point of Stanley-Reisner rings, the ideal $I$ in the first case of condition (e) in Theorem 3.2 corresponds to an iterated cone of a 0-dimensional simplicial complex. In this case it is known that $S/I$ itself is Gorenstein if the corresponding 0-dimensional simplicial complex consists of at most 2 points, see [9, Theorem 5.1(e)]. The corollary also follows from this fact.

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