Representations, Inscriptions, Descriptions and Learning: A Kaleidoscope of Windows[1]

James J. Kaput
Department of Mathematics
University of Massachusetts at Dartmouth

Introduction

My task is to reflect on the papers in the two Special Issues on "Representations and the Psychology of Mathematics Education," Journal of Mathematical Behavior Vol. 17, Numbers 1 and 2, especially to find common themes and opportunities for further progress. The papers were produced independently, but were based on and influenced by the discussions in the PME Working Group on Representations from 1990 to 1993. Gerald Goldin and Claude Janvier edited all the papers in the collection, and Goldin refers to several of them in his theoretical overview paper.

The content of this paper is determined by the others mainly through complementarity and a focus on issues not directly addressed by, and perspectives not taken by, the papers. These issues and perspectives have to do with the roles of detailed features of conventional notations and student notational productions in problem solving and learning - how do notations actually work in particular circumstances, especially in designed learning contexts? What do the decidedly cognitivist framework and language adopted in most of the papers prevent us from seeing or understanding? Finally, how can analyses of representational activity and the features of notations inform the design of learning environments within computational media, especially those that foster deliberate and intense representational activity? I do not pretend to answer such questions, but rather to raise them in relation to the collected papers. I also offer an extended example to help illustrate the beginnings of a notational analysis in the context of instructional design that might be considered complementary to the bulk of those provided in this collection.

Authors use terms in different ways, and so I will discuss terminology at the outset in a way that is intended to frame and clarify usage across the papers. I shall begin with a discussion of terminologies in relation to Goldin's goal of a "unified model" (Goldin, this issue), and then go on to point out how the same terms are used differently by Greer and Harel (1998), Even (1998), Cifarelli (this issue), Boulton-Lewis, Hall (1998), Owens & Clements (this issue), and Hitt (this issue). Larer in the paper, a closer look at the paper by Mesquita (this issue) leads me to a discussion of geometry in the computational medium and the impacts on the phenomena that Mesquita reports that result from a move to dynamic geometry. We also remark briefly on the status of mathematics as a language (Vergnaud, this issue).

Of course, the particularity of language use is one window on authors' assumptions, perspectives and intent. I will use that window just as you will use it to understand me. So we are already at our first reflexive duality, looking at and looking through the language window while simultaneously being distracted by our own reflection in it. And what do we see through this

[1] Work in this paper was supported by NSF Applications of Advanced Technology Program, Grant #RED 9619102 and Department of Education OERI grant # R305A60007. The views offered in the paper are those of the author and need not reflect those of the Foundation or the Department of Education.

warped (whorfed?) window? More windows, in a jumble, none straight or transparent, some reflecting into others, some translucent, some that seem to change when looked at, and some that seem to be treated as writing surfaces. A peculiar, but inviting kaleidoscope.

This area of study is notorious for its complexity and subtlety because it seems to connect to everything we want to know or study. And our work is especially sensitive to foundational assumptions about knowledge, mind, learning, language, development, and culture - assumptions that inevitably define ontologies, methodologies and explanatory objectives, sometimes explicitly, sometimes tacitly. It is also sensitive to point of view - to whether we use the language of researcher-observer, educator, or student. There is no neutral ground and no high ground from which a privileged perspective is possible. But we must start somewhere, and we will begin with basic issues of terminology and the assumptions and entailments of the ways the authors approach the basic issues. We will then offer an illustration of the kinds of instructional design issues that arise when representational activities occur in computational media.

Terminologies and Goldin's Goal of a "Unified Model"

The Abstract Correspondence Approach

Goldin's overview explicitly addresses terminology as he attempts a unified framework, which he describes as a "unified model." It is less a model than a very general framework for a way of talking about representational phenomena, problem solving, and (to a lesser extent) learning and development. In an aggressive effort to be broadly inclusive in his treatment, he begins where I did (Kaput, 1985), with an abstract "correspondence" characterization of representation adapted from Palmer (1977) that deliberately pays no attention to what kinds of things are involved in the correspondence - only that there be two entities that are taken, by an actor or an observer, to be in some referential relation to one another, one taken to "represent" the other. In the case of an actor, the referential connection may be experienced as a "standing for" or "corresponds to" relation between one part of her/his experience and another. Such a referential connection is hypothesized by an observer to be expressed in actions (including writings and utterances). For an observer, the referential connection may also include connections between an actor's hypothesized mental events and externally observable actions on physical material. Goldin also asserts as representational such cases of referential relationships as that between DNA and the biological material whose growth it controls. All such relationships, nonetheless, require an observer to be asserted into our collective world. Following Palmer Goldin suggests, as did I in earlier work and in our joint work (Goldin & Kaput, 1992, 1996), that there follows an obligation to say what is representing what and in what ways.

Such an abstract starting point enables us to talk about many different kinds of "representing" that in languages other than English often have different designations, as many have pointed out. But it also requires us to distinguish these different kinds of representing. However, by its explicitness regarding what is representing what and in what ways, it embodies biases towards a style of description and assumptions regarding what is knowable and in what ways that may not be universally shared. Moreover, its generality may be misleading in the sense that other approaches and perspectives may not fit this style of description. I will illustrate with specific examples shortly.

It is probably not accidental that Goldin and I share a background in abstract mathematics, where a premium is put on generality, and where the operation of abstracting away from content-based detail is as natural as walking. However, I now feel that much is to be gained by adopting alternative points of view while simultaneously exploiting the local conceptual stability of a correspondence perspective.

Problem Solving vs. Instructional Design

One other important but tacit feature of Goldin's approach is that it is born of a long-standing effort to understand problem solving independent of everyday classroom instruction. It is not rooted in instructional design. This fact has large implications regarding its applicability. Indeed, the papers, as a collection, are concerned more with problem solving rather than instruction. Of course, most mathematics instructors attempt to teach problem solving skill, and most use problems to teach mathematical content. A few of the papers take specific instructional sequences to teach particular content as their object of study - the papers by Even (functions), Boulton-Lewis, and Hall (arithmetic using manipulatives). But none addresses issues of representation in the context of a specific extended curriculum. But, while none addresses issues of representation in the context of a specific extended curriculum, this is the area where detailed analyses of representational strategies based on the work of the papers in this joint issue, coupled with long term instructional design, may have the biggest practical payoffs. I will try to illustrate this point via an extended example in the last major section of the paper.

Internal vs. External Representations

Cognitivism and Dualism: Roads Not Takable

Goldin, and in a more tacit way, most of the other authors make a fundamental distinction between "internal representations" and "external representations." The former refer to hypothesized mental constructs and the latter to material notations of one kind or another. Several basic cognitivist/dualist assumptions are often, but not necessarily, wrapped up in this distinction, including the very idea of mental representation, which begs such questions as: What is it? What do we mean when we say it "represents" something? For whom? How? What is the difference between the experience of an internal representation and that of an external representation? And is an external representation a socially or a personally constituted system? I should note that Goldin, however, characterizes representation sufficiently abstractly so as to avoid being trapped in strict dualist framework. His characterization is broad enough as a way of describing or accounting for observations to include representations as descriptive of structures encoded physically in brains, neurons, DNA, etc.

Aside from Vergnaud, the papers in this collection do not take up such questions directly, although most work within this internal-external linguistic framework, with differing degrees of explicitness. Cifarrelli seems to use the word "representation" exclusively as mental representation, whereas Even uses the word to mean material representation. The other authors make the internal/external distinction at some level of explicitness, although the analysis by Vergnaud complicates the distinction by attending to the actions from which mental structures are constituted.

To illustrate the challenges of encompassing all points of view from within a Goldin-like framework, let us consider the phenomenon of *fusion* as examined by Nemirovsky & Monk (in press). While we often discuss symbol and referent as if they are experienced as independent entities, with some kind of specifiable connection between them - correspondence, association, indexical, etc. - Nemirovsky and Monk question whether this approach actually can account for the creative functionality of symbol use in the lived-in world identified by Werner & Kaplan (1962), and where symbols are frequently not experientially distinguished from referents. They point out that while identification of symbol and referent is treated by anthropologists as fetishism, animism and magic, and by psychologists as varieties of pathology, we treat regularly symbols in place of what we know they stand for, despite the fact that we know that the picture of the person is different from the person, or the drawn figure is not really a circle, the toy car is not a real car, the square we gestured in space with our hand is not a real square, etc. Nemirovsky and Monk (in press) note: “Fusion experiences are pervasive in everyday life and can adopt infinite forms, from discussing directions on a map to commenting on a photograph; from drawing a face to gesturing the shape of an object.” They go on to analyze in great detail the fusion experiences displayed in a student’s conversations with a researcher about the construction and meanings of certain graphs of motion. The fusions involved the curve in the graph with the paths taken by the objects in motion, with the curvature shapes of the graph with the speed of the objects, and so on. Nemirovsky and Monk stress that fusion is a functional way of using symbols and tools, and not *con*-fusion. It is a way of maintaining structure and orientation in time and in the space of actions and possibilities surrounding or activated by a symbol-rich experience.

While Nemirovsky and Monk draw on the classic Werner-Kaplan (1962) approach to understanding symbol formation, I suggest that an evolutionary psychological perspective might also be fruitful. In particular, Donald (1991) examines the evolution of representational capacity from early primates to modern humans, and in doing so, he identifies a stage before spoken language that he refers to as “mimetic” beginning about 1.5 million years ago and during which some major pre-human achievements occurred, including the use of fire for cooking, the development of sophisticated tools, migrations out of Africa, and so on. Mimetic culture involved an intentional decentering, a use of the body (dance and gesture) to stand for or to refer to something else. Gesture, facial expression, body orientation and movement continue to play an essential role in the deep organization of experience and the tacit support of conversation. I suspect that aspects of fusion have their roots in the mimetic dimension of human experience, although they are expressed in symbolic behavior in our modern, symbol-saturated, linguistically mediated culture.

Speculations aside, however, we need to acknowledge that the kind of analysis offered by Nemirovsky, Monk and others, provides insights to phenomena not easily reached from the correspondence view of representation and its dualist entailments. We will see further illustration of the limits of this view below, although, for simplicity’s sake, we will continue to use the language of the collection of papers to discuss them.

Systems and Structures vs. Loose, Isolated Notational Elements

Goldin takes pains (as do Goldin & Kaput, 1992, 1996) to argue that “external representations” occur in *systems*. Such systems, while they can be personal and idiosyncratic under certain circumstances, usually are cultural artifacts that cannot be separated from what is normally taken as “mathematical content.” Indeed, learning such systems and how to operate within them dominates school mathematics. Moreover, such systems are seldom used singly or in isolation from one another. Most mathematical activity involves multiple representation systems used in combination with one another, as Even illustrates so clearly. Several papers explicitly examine connections between external representation systems, including Boulton-Lewis, Hall, and Hitt. Greer and Harel deal with the question of how or whether students can learn to recognize common structures across different situations, situations that they would describe as “isomorphic.” Greer and Harel do not, however, explicate these connections in terms of detailed notational and situational particulars.

The Language Aspects of Mathematics

But what is such a thing as, say, the base-ten placeholder system? Is it internal or external, or neither? On one hand, it is a shared cultural artifact amounting to a language with referential function and, most especially, through the very special organizations of actions upon it, a powerful computational function. In another sense, it can appear concretely instantiated in a physical medium - paper, computer screen, whatever. But then, is it internal or external? And what does it represent (numbers?), for whom and under what circumstances? Is it a mathematical thing? That is, is it part of mathematics, or only a language used to represent and work with the *real* mathematical objects, whole numbers? One could ask similar questions about the Cartesian coordinate system. These kinds of questions arise when one begins with the internal/external distinction. On the other hand, as a way of framing discussion within a cognitivist perspective, it seems to have heuristic value.

Vergnaud argues that it is incorrect to think of mathematics *only* as a language. The italicized "only" in the previous sentence reflects my interpretation of his point. My view is that mathematics, as a means of organizing experience, rooted in schematically organizing action, must include the expressive features of language. This expressive side of mathematics is invigorated through the use of notation schemes instantiated within computational media (Shaffer & Kaput, submitted). While cognitive and social constructivist approaches to mathematics learning based on organizing students' actions into schema have provided credible outlines of instructional design regarding the *objects* of mathematical languages, the challenge of how to build students' expressive power employing mathematical notation schemes remains. Somehow, the problem of how to organize expressive acts in such a way as to produce expressive competence within conventional notation schemes and systems seems more difficult. For example, it seems easier to get a handle on students' invention of algorithms *within* the number system notation system than it is to develop means by which students can develop number systems. However, promising insights are developing from various quarters, including, for example, diSessa, et al. (1991), Hoyles and Noss (1998), and the Freudenthal Institute (Gravemeijer, et al., in press). My hunch is that the basic notation schemes are more fundamental and more complex cultural achievements than we may realize, and hence learning them as expressive tools requires more time and deeper expressive engagement than we have devoted to date. The placeholder number system, the algebraic systems (including variables), and the coordinate systems are remarkable and hard-won cultural achievements within which most other mathematics is constituted.

Inscriptions vs. Notation Schemes vs. Notation (or Representation) Systems

I suspect that I am one of a group who, in the early-mid 1980's, introduced the phrase "multiple linked representations." As I noted in (Kaput, 1991), this perspective and language comes dangerously close to Platonism, and it is now clear that it is fundamentally cognitivist in spirit. The recent debate between cognitivists and situationists (Anderson, et. al, 1996; 1997; Greeno, 1997; Cobb & Bowers, in press) has exposed the incommensurability of these theoretical stances. This incommensurability is starkly revealed in the lack of connection between the papers in this collection and the work in the situationist and activity-theoretic traditions. A forthcoming book edited by Cobb, Yackel and McClain (in press), refers virtually not at all to the work described in this collection, and vice-versa. These two bodies of work seem to exist in parallel universes.

Another perspective on the representation problem is offered by certain researchers in the sociology of scientific knowledge, led by Latour (1987, 1993) (see Roth & McGinn, 1998, for an introductory review). For these researchers, the notion of *inscription* is central, an idea that I, influenced by Goodman (1976), approached via the construct of "representation scheme" in (Kaput, 1987) (I also used the phrase "notation scheme" interchangeably). One difference is that the word "inscription" is used by third-party observers to characterize marks in a physical medium apart from any reference to how they might be used, understood, or perceived, *and*, apart from any structure they might embody, from the third-party point of view. They are merely marks in a medium. "Representation scheme" was intended to refer to marks in a *structured* system, in principle machine-compilable, apart from their use in a representational way, standing for something else - similar to what Goldin and others in the collection refer to as an "external representation system." The set of all finite linearly ordered sequences of numeric characters serves as an example. "Notation *system*" (used interchangeably with "representation system" and even "symbol system") was intended to refer to a notation *scheme* used in a representational way. For example, the linearly ordered sequences of numeric characters used to represent permutations would constitute a notation *system*. The same notation scheme could be used in a different notation system, e.g., to represent whole numbers rather than permutations. Put differently, the scheme does not include a referent, and the system does. (In both these cases we are ignoring the action-structure that represents the composing of permutations and combining of numbers, respectively.) Since notation schemes are almost always discussed vis-à-vis their use to represent something else, we almost always frame discussions in terms of systems rather than schemes.

A second critical difference between inscription and notation scheme as theoretical constructs is that the former is used in a situationist explanatory framework and the latter in a cognitivist framework. Hence inscriptions are discussed relative to their production and active use, including how they serve as "boundary objects" shared between discourse communities, how they are moved from one place to another, how they are overlaid, modified, discussed, and so on, in a social context apart from their possible connections with hypothetical cognitive events.

The papers in this collection were conceived mostly before these ideas became current (which helps explain the lack of reference to their approaches and perspectives), and hence do not focus on the *detailed* features of student produced inscriptions as they may bear upon the learning and problem solving being studied. It should be noted that this style of analysis and the data on which it could be based are often not far from the analyses offered. Many papers include examples of student inscriptions, e.g., Boulton-Lewis, Greer and Harel, Cifarelli, Even, Owens & Clements. However, we seldom see how the particular features of those inscriptions influence the course of problem solving and learning. We are all believe that they do, and the authors show *that* they do, but none show*how* they do. This is a central question yet to be fully addressed.

Addressing this question will necessarily require tracking how these features evolve as students progress through problem solving or an instructional sequence. Cifarelli, near the end of his paper, acknowledges the dynamic nature of student productions in problem solving. It would be interesting to see this observation followed up in subsequent work, particularly in the style of Meira (1992), for example, where microanalysis of student productions is at the heart of the methodology.

Geometry in the Computational Medium

The papers in this collection, with the exception of those by Edwards, Hall and Even, focus on the mathematics of static inert media. The papers by Even and Hall (and to a less direct extent the paper by Hitt), focus on linked notations. However, certain papers point up explicit student difficulties with mathematics in static inert media that directly relate to the potentials of notations in dynamic interactive media.

Let us take a closer look at the paper by A. Lobo Mesquita which examines student difficulties with geometric figures. In particular, she examines in detail the consequences of the fact that in most school geometry situations a drawn figure is, by necessity, a single concrete inscription whereas it is intended to stand for an idealized geometric object in what Poincare called the "Geometric Space." The physical drawing resides in what Poincare called the "Representative Space," subject to interpretation by human sensory apparatus. This subtle connection between the particular and the general is understood only gradually by students as she illustrates.

However, it is exactly this connection that is addressed by computer-based geometry environments, first the Geometric Supposers, and then dynamic geometry environments such as Cabri and the Geometer's Sketchpad. Here a particular *construction* (no longer merely a drawing), is subject to variation under the constraints of its construction. This has two immediate and powerful consequences: (1) to expose the generality of the construction, since a given construction can usually have a continuum of instantiations as revealed by dragging any "free" point, and (2) the logico-geometric structure of the construction is made more explicit through the patterns of movements of the figure as it is varied under the given constraints, with constructed and consequent incidences, length-relations, symmetries, and so on, all preserved and subject to observation, inquiry, explanation and even proof. Since the examples and behaviors of the paper bear so directly on the two factors changed by dynamic geometry, it would be especially interesting if students' reported reactions to the examples offered in the paper were compared with student reactions to analogous figures constructed and manipulated in a dynamic geometry environment.

We now turn to an illustration of how the computational medium offers notational opportunity for instructional design within a curricular context.

Representation and Instructional Design in Computational Media: A Sample Analysis from Calculus

To illustrate the kinds of issues that arise when one approaches representational issues from an instructional design perspective, I will provide an example interspersed with discussion of basic issues raised in the various papers.

Anchoring Student Learning in Concrete, Experientially Real Data: The Inadequacy of the "Big Three" Linked Representations

Hot, bi-directional links between pairs of the traditional "Big Three" (numerical, graphical and character-string) notations have dominated the attention of educators and researchers (see the paper by Even), and, indeed, have driven computer software design and calculator design in recent years. This has had the effect of enlarging the representational island and increasing students' ability to move around on it, but it has failed to connect to the students' experiential mainland. In the words of Anna Sfard in her plenary address (with Patrick Thompson) at the PME-NA Annual Meeting in October, 1994, "If the representations only represent each other, then the emperor is *only* clothes." A strong illustration of the kinds of learning that take place in the absence of anchoring in concrete experience is given by Schoenfeld, et al. (1994). These researchers provided an extremely detailed analysis of a single student's learning in a linked "Big Three" environment, supplemented by a version of a target game based on *Green Globs* which involves students defining functions whose graphs will pass through as many pre-given, randomly generated points on the plane as possible. The student's learning was characterized by fragility, instability and disconnectedness from other knowledge or sense-making capability. (This was a capable, mature college-age student who had taken several prior courses involving algebra.) The student's conception of function was "only clothes" in the sense that, for her, the representations were only referring to one another and not to any data associated with phenomena or situations grounded in her experience. By contrast, for the researchers the representations were physical embodiments of their own ideas of function. Other reports of student learning difficulty in multiple linked function environments indicate the inadequacy of linked representations and the strong need to provide experiential anchors for function representations.

Our current work in the SimCalc Project (Kaput, Roschelle & Stroup, in press) puts phenomena and situations at the center and treats the various representations of functions as means for understanding and reasoning about those phenomena and situations as reflected in Figure 1.

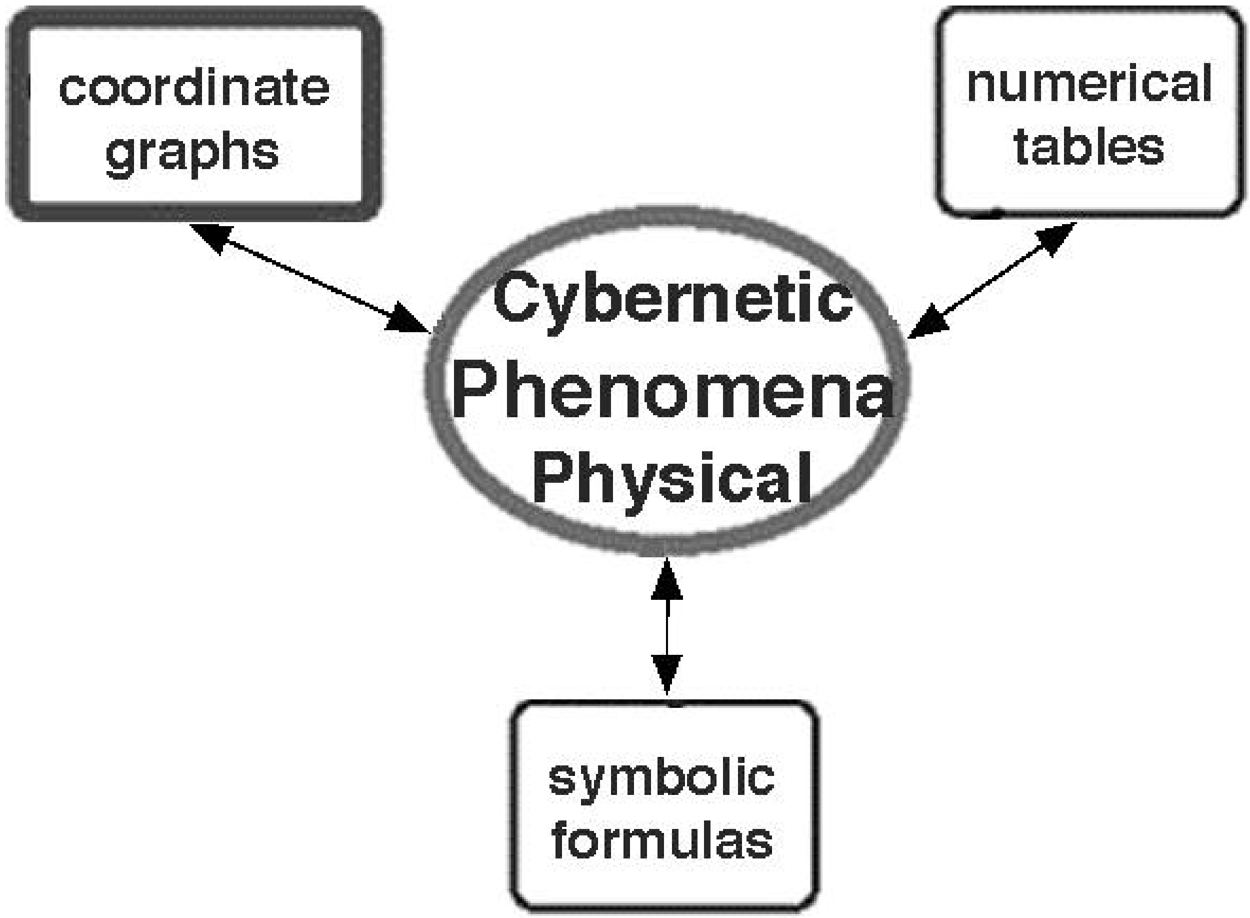

Figure 1. Putting Phenomena at the Center

Here we are explicit about physical and cybernetic phenomena experienced as separate from the person, but, perhaps generated in interaction with the person. Of course there are other kinds of phenomena of direct interest, such as notational phenomena - the behavior of notations as we interact with them. Or experienced kinesthetic phenomena, as occur in whole body motion or when a student moves an object using her hand.

The reader will notice the bidirectionality of the arrows in Figure 1. The next two sections discuss interpretations of these arrows.

From Representing to Creating and Controlling Situations or Phenomena

Historically, we have assumed that mathematics was to be used to represent aspects of situations or phenomena within the notational systems of mathematics (often referred to as "modeling") in order to reason about and make sense of those situations or phenomena, which were taken as given. This can be taken as our intended meaning of the outward-pointing arrows. A second interpretation of the outward pointing arrows is data-transfer, from the physical environment to a computational one through the use of measuring devices connected to a computational device which can display the data in some mathematical notation system. Our work has focused on the mathematics of motion, so in many cases, the notations describe velocities, positions, times, and combinations of these in graphs, equations and tables, with our emphasis on graphs indicated in the figure. However, such motion phenomena, and others such as fluid-flow, are not only modeled by the notations that describe them, they can be *controlled* by those notations. These phenomena can either be cybernetic, as with screen-objects whose movement is controlled by mathematical functions - represented as graphs, equations or tables - or physical, as with toy cars linked to a computer where their motion is controlled by mathematical functions defined on the computer. This can be taken as our intended meaning of the inward-pointing arrows. See (Nemirovsky, Kaput & Roschelle, 1998) for more details and concrete examples.

These kinds of affordances turn a fundamental representational relationship between mathematics and experience from one-way to bi-directional. This in turn supports a much tighter and more rapid interaction on which to base learning. Because the mathematical notation that controls a phenomenon also can be treated as a model of it, one can test a hypothetical model against expectations or predictions immediately - by "running" it. Note that the feedback structure often requires two

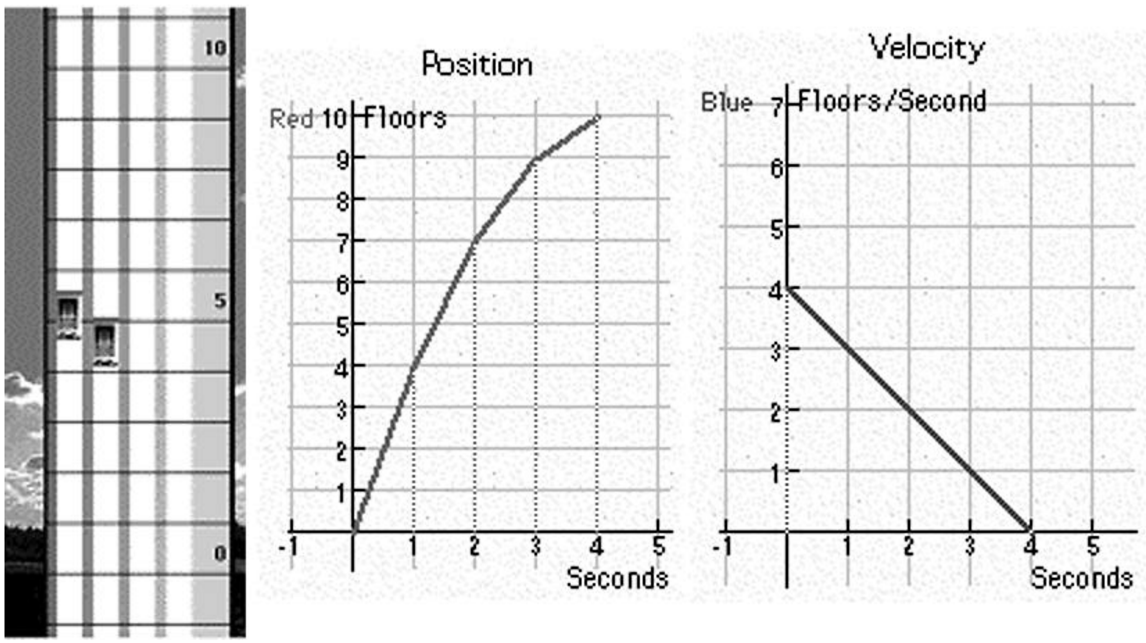

Figure 2: Matching Red's Position by Controlling Blue's Velocity

phenomena, P and P', where P is given as a target and P' is defined or controlled by the student attempting to match P in another system of description. In cases involving rate-totals connections, P may act as a referent phenomenon for either a rate or totals description, and the student inputs one of these and gets feedback in terms of the other. For example, (See Figure 2) suppose we are given a vertical motion P of a Red elevator (on the left) and its position description (a position vs. time graph, for instance as in Figure 2). Then the student controls the motion P' of a parallel Blue elevator (on Red's right) by constructing its velocity, say a graph (or even a formula) to match the motion P of Red with feedback available by watching both elevators run simultaneously. Of central importance is that the student's intentions can be made visible, explicit and testable through the phenomena that the student controls. By exercising our many representational options, we can address an enormous range of learning objectives.

From Multiple Linked *Representations* to Multiple Linked *Descriptions* of Experientially Real Situations or Phenomena

First we set the stage for the distinction between different representations of the "same" description and the "same" representation of "different" descriptions. Situations or phenomena admitting of quantitative analysis almost always have two kinds of quantitative descriptions, one describing the total amount of the quantity at hand with respect to some other quantity such as time, and the other describing its rate of change with respect to that other quantity. In the SimCalc Project we have taken the perspective that understanding the two-way relations between totals and rates descriptions of varying quantities (and the situations that they describe), is a fundamental aspect of quantitative reasoning. It is exactly this relationship that is at the heart of the Fundamental Theorem of Calculus, and indeed, at the heart of Calculus itself. Given the centrality of Calculus to our curriculum and to the mathematics, science and technology of western civilization, the connections between these two types of description is difficult to overestimate.

Now, these rates-totals connections can be instantiated in any of the different representation systems we have mentioned so far because both the rates and the totals descriptions are, in most cases of interest to us, functions. In Figure 2 we saw a case where the two descriptions were both instantiated in the same representation system, coordinate graphs, but mentioned that they could have been represented across different systems. Furthermore, we continue to take advantage of linked representations, so that we not only can connect graphs and formulas, we can cross-connect, for example, a rate graph to a totals formula. See Figure 3 for an illustration of the many possible linkages.

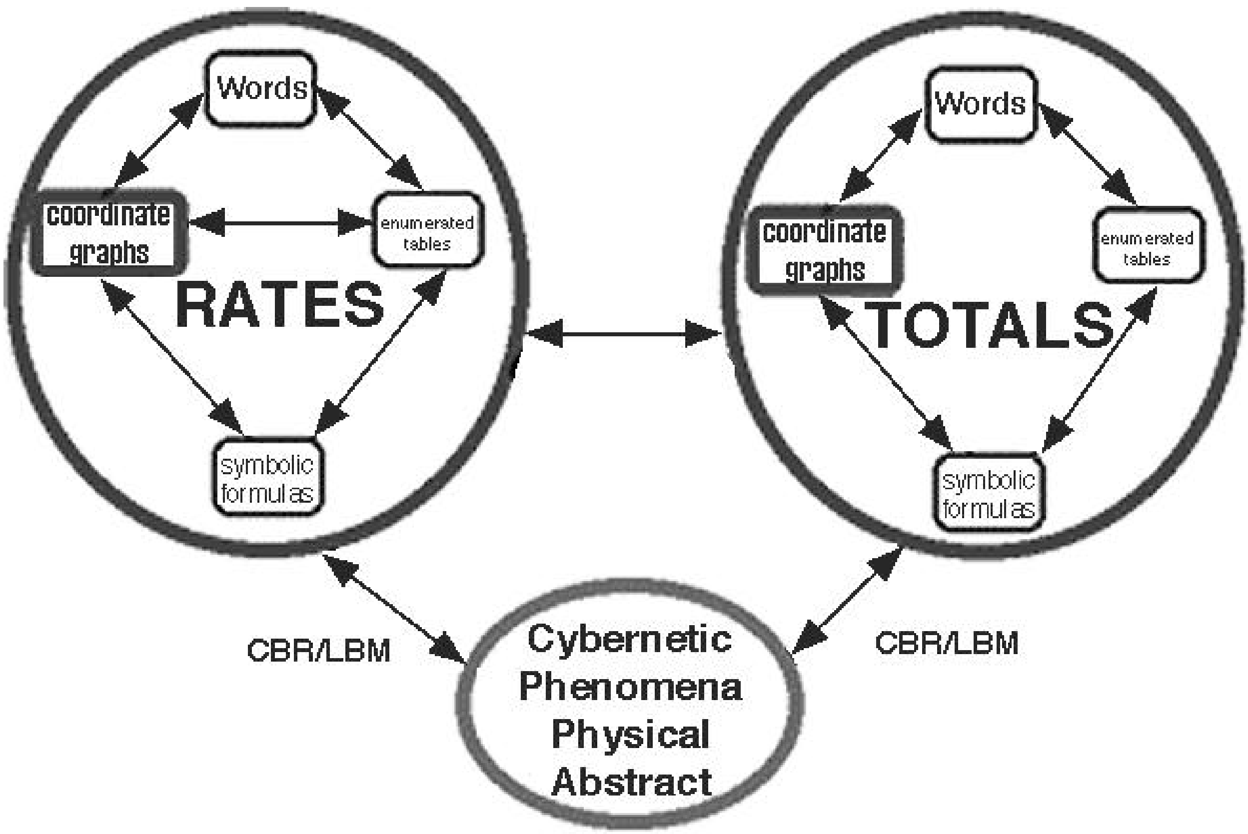

Figure 3: Linked Representations *AND* Linked Descriptions

Similar points could be made about different descriptions in other mathematical domains, such as statistical data analysis, probability, graph theory (especially rich representationally). In all these cases, differences at the level of descriptions runs structurally deeper than do the differences in representation systems, although it is of course the case that representations differ in their abilities to render that structure available to us. This description-representation issue is worthy of much further study, especially cross-domain study.

This new description-representation distinction adds another burden to our terminology and the need to be explicit. To speak only of a “model” or a “representation” without further elaboration is clearly inadequate given the many different interpretations possible.

The Invisible Effects of Static, Inert Media: Dominance of Character String Notation Schemes

Fish don’t know they are wet. Similarly, we have historically taken without question the static, inert media in which notation schemes have been instantiated and used in representational and computational ways. And just as fish have gills and fins rather than lungs and legs, our ways of doing mathematics have certain features that evolved due to the nature of the media in which they evolved. Thus, among notation systems, one type, the character string based systems, dominates the others. Character string systems are very compact, they can support intricate syntax for both reading and transformations and so can support complex computation and reasoning far beyond what could be achieved by a notationally naked mind, and they can be representationally neutral in the sense of being able to denote enormous varieties of referents without needing to share the visual features of what they are representing (e.g., the meaning of “big” does not require the word “big” to be big).

Changes from static inert to dynamic interactive medium affect the place of character string notation systems in mathematics in at least three basic ways. The first is by enabling character strings to “come alive” - to embody procedures, or algorithms, that can be autonomously executed in order to “do something,” carry out a computation, evaluate an expression, perform a translation or transformation, etc. - essentially to do anything a computer program can do. Second, the medium can support live linkages among notation systems. Third, it can support the instantiation of new notation schemes that can be linked with others, new or old. We now turn attention to this third, most promising affordance.

This deep notational bias is reflected in a deep cultural and curricular bias that has led to the bulk of school mathematics being centered on the teaching of reading and writing in character-based notation schemes, especially routines for calculation or reasdoning. The momentum of this notational bias from static inert media led to the design of most first generation electronic educational technology in mathematics to require character string input, e.g., keyboards and calculator keys. This character string orientation extends to underlying issues of legitimacy - what counts as significant mathematics is typically taken to be mathematics expressed in terms of character strings. While we have no reason to expect that the extraordinarily powerful of character string mathematics will ever or should ever disappear or be displaced, we should expect that it will be increasingly augmented by mathematics expressed in other notational styles, especially those that draw upon the new structurable flexibilities and visually rich notations of dynamic, graphic and direct-manipulation media. We see in the papers by Edwards, Even and Hitt (this issue), gradual movement away from exclusive dependence on character strings. Of course, the work by Janvier beginning in the 1970’s, pioneered the study of graphical notation systems. His posthumously published paper in this issue raises the question of how the subtle and pervasive semantics of time structures descriptions of phenomena, whether or not the key variables used to describe the phenomena are themselves temporal.

Historical Approaches: Globally Defined Functions Are Primary

Historical necessity pushed mathematics towards globally defined functions - a character string definition of a function or quantitative relationship was globally defined (over its natural domain) essentially by default - through the identification of the function or quantitative relationship with the character string. But, of course, the phenomena and situations that these are used to model do not usually have such “simply” definable characteristics over unbounded domains. Rather, they are defined over finite extent (domains) and often change their essential character within that finite extent. Much elaborate and beautiful mathematics has been produced to achieve the flexibility that closed form functions do not admit - infinite series approximations of various kinds, Fourier series, and so on - all of which require an apprenticeship in the algebraic world that most citizens today cannot afford and whose role in sense-making is fundamentally transformed by new technologies. (Note that their significance does not disappear, but is played out in different ways, e.g., by being embodied in tools that are used by people who may not need to master that style of mathematics.)

A Alternative: Piecewise Defined, Visually Editable Functions

In MathWorlds, an important feature is the ability to construct and modify functions that are connected to motion and other simulations through strictly graphical means - direct manipulation of piecewise defined functions (see also Yerushalmy, 1997). As indicated in Figure 4 below, one can create a piecewise constant velocity function that controls the motion of the Clown in the motion simulation by dragging the black dots associated with parts of the graphs.

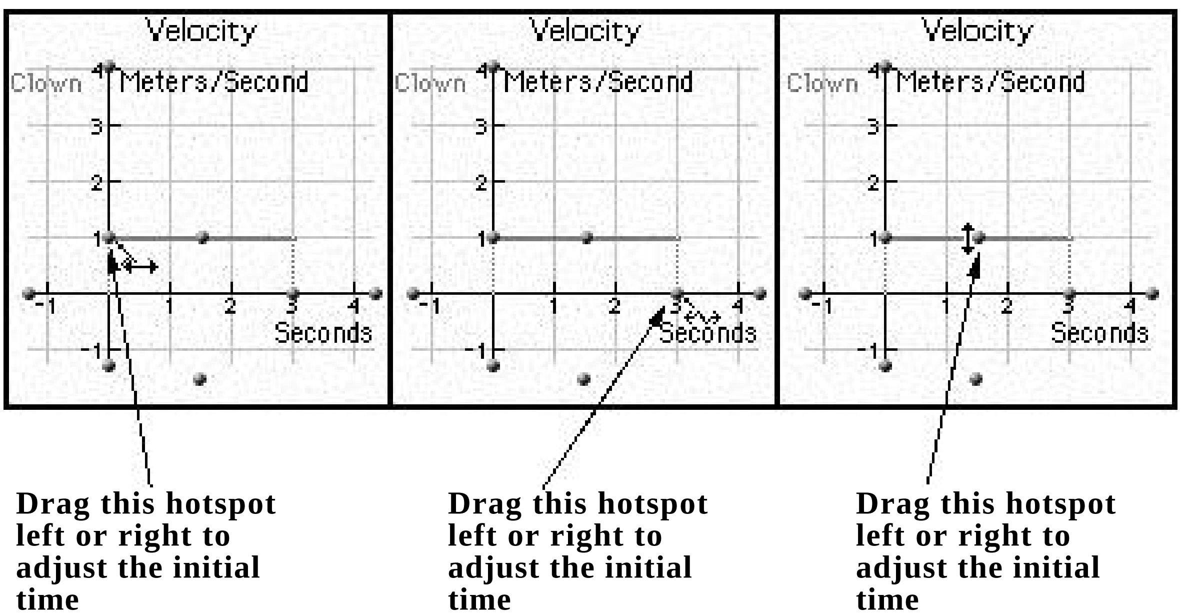

Figure 4: Visually Editable Velocity Graphs

An important point is that the historical power of algebra lay in its status as an action notation scheme, one that supports structured actions on well-formed character strings in support of mathematical reasoning and modeling. By contrast, in static inert media, coordinate graphs have typically been used as static notations - to be inspected, and analyzed as fixed inscriptions. In recent times, by instantiating coordinate graphs in dynamic interactive media as linked to algebraic representations, as noted above they become manipulable, but in global form. That is, one acts uniformly on the function over its entire domain - translating, stretching, reflecting, etc. (See Confrey, 1991) However, the approach illustrated in Figure 4 goes a step beyond this, in not requiring an algebraic interpretation or linkage. Instead, much in the spirit of Dynamic Geometry, the graphs are directly manipulable: pieces can be inserted and modified, etc. The manipulability of algebraic notation has been replaced by a new kind of manipulabilty.

Three sets of reasons lie behind our use in the SimCalc Project of piecewise defined functions: (1) from a modeling orientation, phenomena occur in segments of time, each of which has bounded extent, so the phenomenon usually is not regarded as being defined in a unitary fashion "forever;" (2) work by Nemirovsky and colleagues has established that people naturally tend to interpret graphs on the basis of interval-analyses (Nemirovsky, 1994, 1996; Nemirovsky, Tierney & Wright, in press; Tierney & Nemirovsky, 1995; Monk & Nemirovsky, 1994), wherein complicated graphs linked to phenomena are parsed one section or interval at a time according to the student's current experience of both the graph and whatever it is linked to; and (3) in order to achieve the variation needed to avoid the logical and psychological degeneracy of constant or linear functions while preserving computational tractability, we chose to begin with graphically editable piecewise constant and linear functions whose slope and area computations can be approached using simple arithmetic and geometry, temporarily avoiding the subtleties of limits and approximations (Kaput, Roschelle & Stroup, in press).

Curricular Implications: Moving Past the Algebra Bottleneck

In addition to the factors identified above, the weight of the historical tradition based in the extraordinary achievements of algebra-based mathematics in the hands of the masters who invented and used it (these constitute some of the greatest intellectual achievements of Western civilization), the assumption that the mathematics of change and variation, including the ideas and techniques of calculus, should be learned exclusively in the language of algebra - this assumption - has continued by default up to the current time. It is deeply embedded in our curriculum and our institutional structures (Kaput, 1997), and indeed, it is reflected in both our popular and intellectual cultures.

No one should presume to challenge the power of algebra, and indeed, much mathematics absolutely requires algebra, including most of classical mathematics. Rather, we can and should challenge its currently dominant place in the curriculum as a prerequisite for access to other important mathematics by students who will never need the specialized techniques that the algebra makes possible. On the other hand, all that we have learned about student learning suggests that students bring enormously rich resources to us that are intimately tied to their natural ways of being in and constructing their world. Our ongoing work provides strong evidence that these resources, combined with technologically supported learning environments that instantiate new notation systems and ways of acting upon them, offer dramatically new possibilities for mathematics learning.

Conclusions

We are entering a new era in the study of representation, where new issues need attention and new perspectives and analytical tools are becoming available. The same factors that are being felt across our society and our intellectual landscape are at work here: new technologies yield new media that alter the semiotic foundations of mathematics; situationist perspectives provide new ways to integrate, or supersede, traditional cognitivist and social-psychological perspective and methods; advances in the sociology of science provide fresh perspectives on the creation and communication of the inscriptions that move among us and that constitute an essential dimension of our world; and new obligations are opposed upon mathematics educators to render ever more mathematics learnable by ever more, and hence more diverse, people.

We are entering a new era in the study of representation, where new issues need attention and new perspectives and analytical tools are becoming available. The same factors that are being felt across our society and our intellectual landscape are at work here: new technologies yield new media that alter the semiotic foundations of mathematics; situationist perspectives provide new ways to integrate, or supercede, traditional cognitivist and social-psychological perspectives and methods; advances in the sociology of science provide fresh perspectives on the creation and communication of the inscriptions that move among us and constitute an essential dimension of our world; and new obligations are imposed upon mathematics educators to render ever more mathematics learnable by ever more, and hence more diverse, people.

I hope that this paper, as a small twist of the representational kaleidoscope, has provided a few indications of how the work in this collection can serve the needs of the new era. Of special interest is how we may extend this work to inform the design of instruction, especially in the contexts of school-implementable curricula, which is where we ultimately turn for a fair test of any of our endeavors.

References

Anderson, J. R., Reder, L. M., & Simon, H. A. (1996). Situated learning and education. Educational Researcher,25(4), 5-11.

Anderson, J. R., Reder, L. M., & Simon, H. A. (1997). Situative vs. cognitive perspective: Form vs. substance. Educational Researcher,26(1), 18-21.

Cobb, P., & Bowers, J. (in press). Cognitive and situated learning perspectives in theory and practice. To appear in Educational Researcher.

Cobb, P., Yackel, E., & McClain, K. (in press). Symbolizing and communicating in mathematics classrooms. Hillsdale, NJ: Erlbaum.

Confrey, J., & Smith, E. (1991). A framework for functions: Prototypes, multiple representations, and transformations. Paper presented at the Proceedings of the thirteenth annual meeting of Psychology of Mathematics Education-NA, Blacksburg, VA.

diSessa, A. A., Hammer, D., Sherin, B. & Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. Journal of Mathematical Behavior, 10, 117-160.

Donald, M. (1991). Origins of the modern mind: Three stages in the evolution of culture and cognition. Cambridge, MA: Harvard University Press.

Goodman, N. (1976). Languages of art. (Revised ed.). Amherst, MA: University of Massachusetts Press.

Goldin, G. A., & Kaput, J. (1992). Comments at Working Group on Representations, Annual Meeting of the International Group for the Psychology of Mathematics Education, Durham, New Hampshire.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (in press). Symbolizing, modeling, and instructional design. To appear in P. Cobb, E. Yackel, and K. McClain (Eds.), Symbolizing and communicating in mathematics classrooms. Hillsdale, NJ: Erlbaum.

Greeno, J. (1997). On claims that answer the wrong questions. Educational Researcher, 26(1), 5-17.

Hoyles, C., & Noss, R. (Eds.) (1998). Mathematics for a new millennium. London, England: Springer-Verlag.

Kaput, J. (1985). Representation and problem solving: Methodological issues related to modeling. In E. Silver (Ed.), Teaching and learning mathematical problem solving: Multiple research perspectives, (pp. 381-398). Hillsdale, NJ: Lawrence Erlbaum.

Kaput, J. (1987). Toward a theory of symbol use in mathematics. In C. Janvier (Ed.), Problems of representation in mathematics learning and problem solving, (pp. 159-196). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. (1991). Notations and representations as mediators of constructive processes. In E. v. Glasersfeld (Ed.), Constructivism and mathematics education, (pp. 53-74). Dordrecht, Netherlands: Kluwer.

Kaput, J. (1997). Rethinking calculus in terms of learning and thinking. The American Mathematics Monthly, 104 (8), 731-737.

Kaput, J., Roschelle, J., & Stroup, W. (in press). SimCalc: Accelerating Students' Engagement with the Math of Change. To appear in M. Jacobson & R. Kozma (Eds.) Learning the sciences of the 21st century: Research, design, and implementation of advanced technology learning environments. Hillsdale, NJ: Lawrence Erlbaum.

Latour, B. (1987). Science in action: How to follow scientists and engineers through society. Cambridge, MA: Harvard University Press.

Latour, B. (1993). La clef de Berlin et autres lecons d'un amateur de sciences [The key to Berlin and other lessons of a science lover]. Paris: Editions la Decouverte.

Meira, L. (1992). The microevolution of mathematical representations in children's activity. Paper presented at the Proceedings of the Sixteenth Annual Conference of the PME, Vol. 2, Durham, NH.

Monk, S., & Nemirovsky, R. (1994) The Case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, J. Kaput, & A. Schoenfeld (Eds.), Research in Collegiate Mathematics Education. Vol. 1. Providence, RI: American Mathematics Society, 139-168.

Nemirovsky, R. (1994). On ways of symbolizing: The case of Laura and velocity sign. The Journal of Mathematical Behavior, 13, 389-422

Nemirovsky, R. (1996). Mathematical narratives. In N. Bednarz, C. Kieran, & L. Lee (Eds.), Approaches to algebra: Perspectives for research and teaching, (p. 197-223). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Nemirovsky, R., Kaput, J., & Roschelle, J. (1998). Enlarging mathematical activity from modeling phenomena to generating phenomena. Paper presented at the Proceedings of the 22nd International Conference of the Psychology of Mathematics Education in Stellenbosch, South Africa.

Nemirovsky, R., & Monk, S. (in press). "If you look at it the other way..." An exploration into the nature of symbolizing. In Cobb, P., Yackel, E. and McClain, K. (Eds.) Symbolizing and communicating in mathematics classrooms. Hillsdale, NJ: Erlbaum.

Nemirovsky, R., Tierney, C., Wright, T. (in press). Body motion and graphing. Cognition and Instruction.

Palmer, S. E. (1977). Fundamental aspects of cognitive representation. In E. Rosch & B. B. Lloyd (Eds.), Cognition and categorization. Hillsdale, NJ: Lawrence Erlbaum Associates.

Roth, W-M., & McGinn, M. K. (1998). Inscriptions: Toward a theory of representing as social practice. Review of Education Research, 68(1), 35-59.

Schoenfeld, A., Smith, J., & Arcavi, A. (1994). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), Advances in instructional psychology, (Vol. 4). Hillsdale, NJ: Lawrence Erlbaum.

Shaffer, D., & Kaput, J. (submitted). Mathematics and virtual culture: An evolutionary perspective on technology and mathematics education. Educational Studies in Mathematics.

Tierney, C., & Nemirovsky, R. (1995). Children's Graphing of Changing Situations. Presented at the 1995 Annual Meeting of the American Educational Research Association. San Francisco, CA.

Werner, H., & Kaplan, B. (1962). Symbol formation. New York: Wiley.

Yerushalmy, M. (1997). Emergence of new schemes for solving algebra word problems: The impact of technology and the function approach. Paper presented at the Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education, Lahti, Finland.