Real Options Valuation: a Monte Carlo Approach[1]

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Abstract

This paper provides a new approach for valuing a wide set of capital budgeting problems with many embedded real options dependent on many state variables and a related valuation algorithm based on Monte Carlo simulation.
The valuation approach decomposes of a complex real option problem with many options into a set of simple options, but taking into account deviations from value additivity due to interaction and strategical interdependence of the embedded real options, as noted by Trigeorgis (1993). The valuation approach presented in this paper is an alternative to the general switching approach for valuing complex option problems, as proposed by Kulatilaka and Trigeorgis (1994) and Kulatilaka (1995).
The related numerical algorithm is based on simulation along the lines of Longstaff and Schwartz (2001) and is extended in order to implement the decomposition approach.
We provide also an array of numerical results to show the convergence of the algorithm and a few real life capital budgeting problems, to see how they can be tackled with our approach.

JEL Classification: C15, C63, G13, G31.

1 Introduction

Traditional Monte Carlo simulation has been considered a powerful and flexible tool for capital budgeting for a very long time. It is a recommended methodology for capital budgeting decisions in many Corporate Finance textbooks. Actually, it permits to include a wide set of value drivers, it is flexible enough to cope with many real life situations and it does not suffer the "curse of dimensionality" affecting other numerical methods. Yet, as pointed out by many authors,[1] it seems not so suited to tackle capital budgeting problems with (potentially, many) real options.

Mason and Merton [34] first described a capital budgeting problem as a collection of real options, i.e. a set of opportunities managers (usually) have to deviate from a previously decided course of actions. Real options are capital budgeting decisions contingent on the value assumed by some relevant and well specified state variables. Projects involving individual real options have been evaluated since the early stage of development of the Real Options Theory (see e.g. Majd and Myers [35] and McDonald and Siegel [38, 39]).[2] Generally speaking, the numerical techniques for financial options can be successfully employed to evaluate single real options: as far as the mathematics of real option valuation is concerned, there would be no need of a Theory specifically devoted to individual real options. An exception is represented by Brennan and Schwartz [11] who evaluate the investment in a mine considering the compound effect of the flexibility to temporarily shut down and restart the operations and to abandon the project.[3] A widely accepted classification of simple real options is the one presented in Mason and Merton [34] (see also Amram and Kulatilaka [1] and Trigeorgis [45] for more details and references therein) and includes: the option to defer an investment decision, the option to partially or completely abandon operations,

[1] See for instance Trigeorgis [45, pp. 54-57].

[2] For a comprehensive bibliography on the subject, see Dixit and Pindyck [20] and Trigeorgis [45].

[3] This line of research, involving the option to switch from one operating mode to the others and with the possibility to reverse the action at some cost, has been followed up by other authors. Dixit [19] studied an investment problem with the flexibility of entry and exit the operations over time. Kulatilaka [25, 26] introduced a model to evaluate an investment project in a industrial plant firing two different types of fuel, endowed with the flexibility to switch from one fuel to the other according to the relative movements of their market prices. Kulatilaka and Trigeorgis [29] and Kulatilaka [27] (see also Trigeorgis [45, Ch. 5, pp. 171-201]) propose a general model of managerial flexibility based on the option to switch among properly defined operating modes. In this work we propose a different, alternative approach to model the general flexibility embedded in a capital budgeting problem.

the option to alter the scale of current operations, the options to switch the existing assets to an alternative use and many others. The valuation of these options can be easily done by employing the same techniques used for financial option pricing.[4]

Unfortunately, real life investment decisions usually present many options at once or, following Trigeorgis [43], an investment decision can be seen as a portfolio of interacting opportunities. The interactions among the contingent decisions make valuation harder. As a rule, the value of a portfolio of interacting options deviates from additivity and in some cases the difference with respect to the sum of the values of the individual real options considered in isolation can be significant. Hence, the problem of decomposing a complex investment project into a set of individual options quite often does not have a straightforward solution. This fact prevents the use of valuation techniques devoted to individual options, well known in Financial Option Theory and calls for a valuation approach specific for problems involving many real options.[5]

Kulatilaka and Trigeorgis [29] and Kulatilaka [27] (see also Trigeorgis [45, Ch. 5, pp. 171-201]) proposed a valuation approach for complex problems based on the general idea of switching among different "operating modes". In their approach, given an investment with many embedded options, at any time a decision can be made, there is an option to switch from the current "mode" to a different one. The switching cost of taking the action is the "strike price" of the option. This valuation method is based on the analogy between machines with many operating modes (and related switching costs) and a capital budgeting problem: the operating modes are decision that the management can make in a dynamic way. For instance, the usual wait-to-invest option (a call option on the present value of the cash flows from operations of a given investment project) can be described as an (irreversible) option to switch from the mode "wait to invest" to the mode "invest." According to this approach, a flexible capital budgeting problem can be seen as a complex compound switch option among several and properly defined "modes." The technique based on the general switching flexibility, joint with some discrete-time approximation of the continuous-time dynamics of the state variable (either binomial lattices or Markov chains), is widely applied to capital budgeting problems (examples are in [28, 29, 43, 45]). Besides other problem, mainly related to the computational efficiency of a numerical valuation procedure based on this approach and which we discuss below,

[4] A good and comprehensive reference on this is Trigeorgis [45].

[5] A notable exception is Geske [23].

the general option to switch has the following main drawback. As discussed in Brekke and Øksendal [7], an *optimal switching problem* is a special type of impulse control problem (see Bensoussan and Lions [4] for a reference). If we are to model the problem in a continuous-time setting, and we use some discrete-time numerical valuation approach to obtain a solution, first one has to prove the existence of a finite solution and next the convergence of the discrete-time (numerical) solution to the continuous-time one. (Of course, for a switching problem with a finite number of decision dates the solution always exists.) In Brekke and Øksendal [7] the proof of the existence of an optimal solution in a continuous-time setting is offered for a class of switching problems.[6] The same result for a general optimal switching problem is not available yet, at the best our knowledge. This means that, although the approach based on the general option to switch is flexible, one has to be very careful to apply this approach in a continuous-time setting, since a solution may not exist.

The main contribution of this work is a different way to map a complex real options problem into a set of simple options and a way to comply with the hierarchical structure of the options. Our approach always provides well defined problem with a finite solution also in a continuous-time setting. Lastly, even if the approach based on the general option to switch proves to be fruitful in a low-dimensional setting, it becomes computationally intractable if there are many (i.e., more than two) state variables. Since simulation methods requires a computational effort which is linear with respect to the dimension of the state space, in this paper we propose an alternative approach for valuing multi-options and multi-assets problems based on the simulation approach developed by Longstaff and Schwartz [32].

Usually, the real options embedded in a capital budgeting problem are American-type claims. This means that closed-form solutions are rarely available and some numerical methods must be employed. Many methods have been proposed for real option pricing purposes. Most of them are plain extensions of well known algorithms used to price financial options. Roughly, they can be divided into three main classes: finite difference methods and other approaches dealing directly with PDE's (first introduced by Brennan and Schwartz [10]), Monte Carlo simulation methods (introduced by Boyle [6]) and lattice methods first proposed by Cox, Ross and Rubinstein [18]. All these approaches have some flaws when applied to real options valuation. Finite difference are quite hard to implement if the project has many

[6] Basically, these are the same kind of problems described in Brennan and Schwartz.

interacting options. Bi- or trinomial lattices,[7] although very flexible for capital budgeting problems with many embedded options (see Trigeorgis [44] and [45, Ch. 10]), suffers the curse of dimensionality. Yet, real life capital budgeting problems usually involve multiple state variables (underlying assets, as long as options are concerned). This feature, assuming that the stochastic model of these variables are known,[8] makes real options even more difficult to evaluate. Multi-factor real options problems have been studied for instance by Triantis and Hodder [42], Cortazar and Schwartz [15], Geltner, Riddiuogh and Stojanovic [22], Cortazar, Schwartz and Salinas [16], Martzoukos and Trigeorgis [36], Brekke and Schieldrop [8] and others.

For all the above mentioned reasons, simulation seems to be the most suited numerical technique for real options. Unfortunately traditional Monte Carlo simulation (as introduced by Boyle [6]) is a forward-looking technique, whereas dynamic programming (to evaluate American options) applies backward recursion. Many approaches have been proposed to match simulation and dynamic programming: Bossaerts [5] proposes two moment estimators of optimal stopping time; Tilley [41] provides an algorithm in which simulated paths are bundled to estimate probability weights of the state space; Barraquand and Martineau [2] give a stratification method for pricing high-dimensional options, in the same spirit of Tilley's approach; Broadie and Glasserman [12] proposed an algorithm based on simulated trees with a small number of dates where early exercise is allowed.

A very promising approach has been presented by Longstaff and Schwartz [32]. This numerical method is based on Monte Carlo simulation and uses least squares linear regression to determine the optimal stopping time of the problem. This approach, called Least Squares Monte Carlo Approach (LSM), has the additional feature of being a very intuitive, pedagogically clear and flexible tool. We will provide an extension of this algorithm to evaluate complex investment projects with many interacting options and many state variables, along the lines of the proposed decomposition approach.

We illustrate our methodology in two steps. The first step is a valuation approach which decomposes a complex real options problems into a sequence

[7] In this class of algorithms we include also discrete-time and discrete state Markov chains. See for instance, Kulatilaka [27] for an application of Markov chains to real options valuation.

[8] One of the major problem in real options applications is the specification of the stochastic model and the correct estimation of the parameters of the model. Traditionally, some very simple models are used (e.g. geometric Brownian motion or mean-reverting processes) which are more suited to describe the price of traded assets. We will rely upon the usual assumptions, leaving the issue of the underlying process specification as a subject for future research.

of simple options and some way to comply with their interdependencies. Since the building blocks of the complex problem are simple real options, we can take benefit of the numerical methods usually employed for individual financial options. The second step is an extension of the simulation approach proposed by Longstaff and Schwartz [32] to multi-options problems.

The paper is organized as follows. Section 2 introduces the suited environment for capital budgeting purposes. Instead of the usual CAPM economy, we develop our model in the more flexible equilibrium economy proposed by Cox, Ingersoll and Ross [17]. This permits to point out the risk factors and and the related premia, very useful if the capital budgeting problem depends on many (not necessary traded) factors, instead of a unique market factor as in CAPM. Section 3 presents a method to describe a wide class of capital budgeting problems using a small set of well defined building blocks, which are specifically designed both to comprise the largest possible number of actual options problems and to be easily solved by the simulation algorithm presented below. These small problems will be the building blocks of our valuation algorithm. Section 4 presents an extension of the LSM algorithm to numerically evaluate the building blocks of our algorithm. Section 5 provides a set of numerical examples to see how to apply our approach to real life capital budgeting problems and to show the efficiency of the numerical algorithm based on simulation.

2 The economy

2.1 State variables

Let there be given a Cox-Ingersoll-Ross economy with financial market and a representative agent (see [17]).

There are n state variables $X_1, X_2, \ldots, X_n$, for short denoted $X' = (X_1, X_2, \ldots, X_n)$.[9] The values of these variables is the only relevant information to make the capital budgeting decisions. These variables can be either prices of traded securities or observable values of non-traded assets (factors). The dynamics of the state variables, with respect to the objective probability measure follow the Markov processes

$$dX_i(t) = a_i(t, X(t))dt + b_i(t, X(t))dB(t) \quad \text{with } X_i(0) = x_i, \quad i = 1, \ldots, n$$

where $a_i : \mathbb{R}^n \mapsto \mathbb{R}$ and $b_i : \mathbb{R}^n \mapsto \mathbb{R}^n$ are such that the solution of the stochastic differential equations above exists and $dB(t)$ is the increment of a

[9] A prime denotes transposition.

standard n-dimensional Brownian motion, with $\mathbb{E}[dB_i(t)dB_j(t)] = 0$. With a matrix notation, the process is

$$dX(t) = a(t, X)dt + b(t, X)dB(t) \qquad \text{with } X(0) = x$$

where $a' = (a_1, \ldots, a_n)$ and

$$b(t, x) = \begin{pmatrix} b_1(t, x) \\ \vdots \\ b_n(t, x) \end{pmatrix}$$

is a positive definite $n \times n$ matrix with full rank for all t.

2.2 Financial market

There is a financial market where n non-redundant financial assets[10] are traded, i.e the financial market is *complete*. The prices of these assets, given by the processes $\{P^j(t, X_t)\}$, $j = 1, .., n, \ldots$, are contingent on the n state variables (if they are not the state variables themselves). Since there is no confusion, in the rest of the paper the dependence of the financial asset prices on the state variables will be often dropped. The dynamics of the asset prices are

$$\frac{dP^j(t)}{P^j(t)} = (\mu_j(t) - \delta_j(t))\, dt + \sigma_j(t)dB(t)$$

where δ_j is the payout rate,[11] μ_j is the total expected instantaneous rate of return and σ_j is a $1 \times n$ vector valued function.

An instantaneously riskless asset is available with instantaneous rate of return r.

The set of investments opportunities remains unchanged within the relevant time horizon.[12]

The market is assumed to be in equilibrium and the related asset pricing relation is

$$\mu_j(t) = r(t) + \sum_{i=1}^{n} \Psi_i(t, X) \frac{P^j_{X_i}(t)}{P^j(t)} \quad j = 1, \ldots, n \tag{2.1}$$

[10]We will use the definition "financial asset" in broader terms. Actually, also traded commodities are included in this set.

[11]If the j-th financial asset is a traded commodity, δ_j is the related convenience yield (see Brennan [9]).

[12]We make this assumption to avoid the difficulty arising from change in the production technology. To see how the change in the set of investment opportunities affect the asset pricing relation, see Cox, Ingersoll and Ross [17].

where $\Psi' = (\Psi_1, \ldots, \Psi_n)$ are the (factor) risk premia, $P^j_{X_i}$ is the derivative of asset j with respect to state variable X_i, and $r(t)$ is the equilibrium instantaneous riskless rate.

Accordingly, since the financial market is complete, there is a unique equilibrium risk-neutral probability measure. With respect to this probability measure, the dynamic of the state variables is

$$dX(t) = \hat{a}(t, X)dt + b(t, X)dB(t) \qquad \text{with } X(0) = x \tag{2.2}$$

where $\hat{a} = (a - \Psi)$ is the risk-adjusted drift.

To simplify our arguments, we will assume from now on that the riskless rate is non-stochastic and constant. The analysis would be the same, at the cost of more cumbersome formulas, if we assume a stochastic riskless rate.

2.3 Contingent claim valuation

If a (necessarily) redundant contingent claim, for instance an option on a traded asset (or on a factor), is given, with maturity T and payoff $\Pi(T, X_T)$, where Π is a known function,[13] we can evaluate the contingent claim with respect to the prices in the financial market.

Let $F(t, X_t)$ be the value of the claim at $t \leq T$, with $F(T, X_T) = \Pi(T, X_T)$. If the claim is European, i.e. it can be exercised only at T, the price at any time $t < T$ is

$$F(t, X_t) = e^{-r(T-t)}\mathbb{E}^*_t[\Pi(T, X_T)], \tag{2.3}$$

and if the claim is American, i.e. it can be exercised at any time before T, and is still available at t,

$$F(t, X_t) = \max_{\tau \in \mathcal{T}(t,T)} \left\{ e^{-r(\tau-t)}\mathbb{E}^*_t[\Pi(\tau, X_\tau)] \right\}, \tag{2.4}$$

where $\mathcal{T}(t, T)$ is the set of stopping times in $[t, T]$ with respect to the information generated by the state variables X and $\mathbb{E}^*_t[\cdot]$ is the expectation with respect to the unique risk neutral probability, conditional on the information available at t.[14] See Bensoussan [3] and Karatzas [24].

[13] We assume that Π has enough properties so that expectation and variance with respect to the relevant probability measure can be properly defined. Formally, $\Pi \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$, the space of square-integrable functions with respect to $\mathbb{Q}$, where Ω denotes the space of all possible states of the economy, $\mathcal{F}$ is the filtration generated by the state variables and $\mathbb{Q}$ is the equilibrium risk-neutral probability measure on $\mathcal{F}$.

[14] We will denote conditional expectation also with $\mathbb{E}^*_t[\cdot] = \mathbb{E}^*[\cdot \mid \mathcal{F}_t]$.

Interesting enough, equations (2.3) and (2.4) are simply net present values, with respect to the riskless rate, of certainty equivalent payoffs. This valuation paradigm will be used also for valuing capital budgeting projects (and the embedded real options).

3 A general valuation approach for capital budgeting

Real options are contingent claims on real investment projects. They are contingent on the state variables X of the reference economy. Since the financial market is complete, Fisher separation theorem holds and the valuation principle based on discounting certainty equivalent cash flows applies.

Simple (i.e. individual) real options have been extensively studied. We are interested in valuing investment opportunities on real assets when many interacting options are available. The individual real options embedded in a capital budgeting decision can interact in many ways. In what follow we present a simple and fairly general way to decompose a complex real options problem into a set of simple building blocks. Assuming that the simple options are well defined, we introduce a set of possible ways of interaction to properly capture the interdependencies among simple options. This methodology can be successfully applied to a large family of capital budgeting problems: basically it can be used for all problems which do not have the opportunity to reverse an already taken action. As an example of a reversible action, the above cited article by Brennan and Schwartz [11] provides a valuation approach for an investment opportunity in a mine with the flexibility to shut down and restart the operations at some costs.[15]

The idea underlying the approach we propose is very simple: *a capital budgeting problem is a set of simple options with some link among them. These links permits to comply with the hierarchy and interdependencies among the embedded real options.* Some interdependencies are as simple as the compoundedness. As an example, assume that the management has both the flexibility to wait for the best timing of an investment decision and the flexibility to abandon it, once the investment decision is made. In this case there are two simple options, a call and a put, on the same underlying variable, i.e. the present value of cash flows from operations. These options are compounded, since the option to abandon is available only after and if the option to invest is exercised. It is straightforward to extend this line of

[15] The extension of the simulation technique to these problems is the subject of future research.

reasoning to many compound options, like in sequential investment problems, where any step of the investment process provides the opportunity to do all subsequent steps.

On the other hand, there can be some complex relations among real options. For instance, assume that an investment decision provides the management with many alternative options and that the management can, at any time, exercise some (or one) of them, killing all the other options. It seems quite obvious that, with many options at hand, the management will exercise the most favorable and, if these are American options, there is also a timing option embedded in this decision.

Many capital budgeting problems can be decomposed into a sequence of options. Besides the compound option case, described above, we formalize the following two additional possible way to interact: sum of independent options and mutually exclusive options.

3.1 Independent options

In some capital budgeting problems, there can be many independent options available to the management. For instance, launching a new product in a country gives the opportunity to launch the product also in other countries. The investment opportunity in each country is an option itself and every such option can generally be exercised independently of the others. In this sense, the management has a *portfolio* of options.

If the options are independent, then their values are not influenced by the other options. Hence, the value of the portfolio of options is the sum of the values of the simple options. This is the case, described in Corporate Finance textbooks, of value additivity of investment projects. We want to stress that "independent" in this case means "strategically independent" and not "stochastically independent." Actually, the assets underlying the independent options may not be (and usually are not) stochastically independent.

Let there be given H options, with maturities T_h, payoffs $\Pi_h(,t,X_t)$ and values $F_h(t,X_t)$, $h=1,\ldots,H$. In what follows, since our approach is recursive, we will phrase it by choosing a generich step in the valuation procedure and assuming that the results of the previous steps are known. Hence, the $F_h(t,X_t)$, $h=1,\ldots,H$ have been already determined (together with the related optimal stopping times) in the previous steps.

The possibility to exercise all the H options independently is itself an option. Since the H subsequent options are independent of each others, the value of the option to exercise them, denoted $G(t,X_t)$, is the sum of their values:

$$G(t, X_t) = \sum_{h=1}^{H} F_h(t, X_t). \tag{3.1}$$

According to this approach, we can describe also the case when there is some risk, specific to the investment project at hand, which affects the decisions to be taken on such project. More specifically, the availability of the H options above may depend on some event. This is the case, for instance, with R&D projects. If at the end of any stage of the process there is a test and this can be failed or succeeded, then there is an event with two possible outcomes. Usually, as a consequence of each of the two outcomes there are some options: in the event the test is successful, there is the option to follow up the R&D process; if the test is failed, there can be the option to abandon the investment.

In our framework, the technical uncertainty surrounding an investment project is different from the uncertainty driven from the state variables, X. Since it is specific to the investment project at hand, we consider the technical uncertainty stochastically independent of X. Hence, there is no premium for this risk according to our pricing relation in (2.1).

Usually, the probability of the technical event affecting the project is (assumed to be) known. If the event has H possible outcomes, $p_h > 0$ is the probability of the h-th one, $\sum_h p_h = 1$, and assuming that the technical uncertainty dissipates at $T' < T_h$, $h = 1, \ldots, H$, the option value is

$$G(t, X_t) = e^{-r(T'-t)} \sum_{h=1}^{H} p_h \mathbb{E}_t^* \left[F_h(T', X_{T'}) \right], \tag{3.2}$$

where the h-th subsequent option can be exercised, if American, in the interval $[T', T_h]$ and, if European, at T_h.

This model can be easily generalized to many sources of technical uncertainty and to the case the event can happen in a given time interval according to a continuous-time distribution (eg. a Poisson process).

3.2 Options on options

A real option can offer, when exercised, one more option. This happens in many staged investments in which each installment is an option on the subsequent stages. If this is the case, then the value of the previous claim depends also on the value of the subsequent one.

Let there be given H compounded real options, that is, the h-th option, besides its own payoff, gives the “right” to exercise the $(h + 1)$-th option,

$h = 0, \ldots, H-1$. The payoffs of the options, dependent on X_t, are denoted $\Pi_h(t, X_t)$. For definiteness, the maturities are $T_1 \leq T_2 \leq \ldots \leq T_H$. We will denote F_h the value of the h-th option and, as above, we assume that its value is already known. In what follows, if t is greater than T_h, then the value of the h-th option is zero, because that option is not available any longer: $F_h(t, X_t) \equiv 0$ if $t > T_h$. The value F_h depends also on F_{h+1}. If the h-th option is American, then its value at $t \leq T_h$ is

$$F_h(t, X_t) = \max_{\tau \in \mathcal{T}(t, T_h)} \left\{ e^{-r(\tau - t)} \mathbb{E}_t^* [\Pi_h(\tau, X_\tau) + F_{h+1}(\tau, X_\tau)] \right\}. \tag{3.3}$$

If the h-th option is European,

$$F_h(t, X_t) = e^{-r(T_h - t)} \mathbb{E}_t^* [\Pi_h(T_h, X_{T_h}) + F_{h+1}(T_h, X_{T_h})], \tag{3.4}$$

The above is true for $h = 1, \ldots, H-1$.

F_{h+1} can be the (already determined) value of an option, but also the value of many independent options offered at the same time as seen in Section 3.1, or the expected value of the options that will be available as soon as some technical uncertainty resolves, or the best out of a given set of options, as described below (Section 3.3).

By specifying the above expressions we can find out well know cases. If, $H = 2$, $\Pi_1 \equiv -K_1$, and Π_2 is either $\max\{P_t - K_2, 0\}$ or $\max\{K_2 - P, 0\}$, i.e. the subsequent option is an European call or put on a non-dividend paying asset whose price, $P_t = P(t, X_t)$, evolves according to a geometric Brownian motion, and strike K_2, and the previous option is a call on the second option, then closed form solutions are available from Geske [23]. Again, if $H = 2$, the first option is an European call or put and the second is an option to exchange one asset for another (i.e., $\Pi_2 = \max\left\{P^1 - P^2, 0\right\}$), the assets pay no dividend and their prices are geometric Brownian motions, then closed form formulas are available in Carr [13] (extending the results in Margrabe [33]). If the assets pay a continuous dividend, Martzoukos e Trigeorgis [36] provide extensions of the closed-form formulae in Geske [23] and Carr [13]. In all the other cases, closed-form formulas are not available. We will use the known available solutions as a benchmark for our numerical evaluation approach (see Table 9).

3.3 Mutually exclusive options

Let there be given H mutually exclusive real options. For the sake of definiteness, we may think of two opposite decisions regarding the same real asset (abandon/expansion, lease/sell, ...): once the decision is made, the other competing option expires.[16]

The payoff of these options are denoted Π_h, $h = 1, 2, \ldots, H$, and are contingent on the value of the state variables X. The options have maturities T_h, $h = 1, 2, \ldots, H$. With no loss of generality, we assume $T_1 \leq T_2 \leq \ldots \leq T_H$. As usual, let $F_h(t, X_t)$ be the value of the h-th real option. The management is asked to decide, within the time horizon T_H, for the best available option. We assume that the decision, once taken, is irreversible. In this sense, there is a timing option also in the choice of the best (out of H) option. Actually, since the decision is irreversible, the management may be interested in delaying the choice of the option to be exercised (and keeping the options open).

Let $G(t, X_t)$ be the value of the opportunity to choose the best one out of the H available options. Assume for the moment that at least one of the H options is an American-style claim. We define the control as a couple (τ, ζ), where τ is a stopping time in $\mathcal{T}(t, T_H)$ and ζ takes value in the set $\{1, 2, \ldots, H\}$. The value of the opportunity to select the best option is

$$G(t, X_t) = \max_{(\tau,\zeta)} \left\{ e^{-r(\tau - t)} \mathbb{E}_t^* [F_\zeta(\tau, X_\tau)] \right\} \tag{3.5}$$

If the options are all European-style claims, that is, the maturity T_h is the only date when the h-th option can be exercised, for all h, then the problem is still (3.5), but the stopping time is restricted to the set $\{T_1, \ldots, T_H\}$.

Although the opportunity to select the best option seems to depend on the value of the available options, F_h, $h = 1, \ldots, H$, the choice is not made until the time to exercise the most favorable option has come, because the decision about the option is irreversible. In other words, when the time to choose has come, the chosen option is exercised. This can be easily understood by considering that, if the decision is irreversible, it would be sub-optimal to choose an option and than to wait some time before exercising it, because in the meanwhile some other options might turn more valuable. Hence, if option $\zeta = h$ is chosen at time $\tau = t$, then $F_h(t, X_t) = \Pi_h(t, X_t)$ and as a consequence $G(t, X_t) = \Pi_h(t, X_t)$.

This framework encompasses some known results. If we restrict the maturities of the options, so that $T_1 = T_2 = \ldots = T_H$, assume that these

[16] Since we are describing the building blocks of real life capital budgeting projects, it may happen that, as a consequence of the choice of the best option, other options are made available which are embedded in the one selected. The value of these (compounded) options is included in F_ζ, the maximand of the control problem in (3.5). The value F_ζ is in this case given by equation (3.3) and is assumed to be known.

options are European, and put $\Pi_h(t, X_t) = P^h(X_t) - K$, where P^h is a geometric Brownian motion for all h, then the problem in (3.5) reduces to the well known option on the maximum on H assets with prices $P_h(X_t)$ dependent on the state variables X_t and strike K (see Stulz [40], Johnson [30], and Martzoukos e Trigeorgis [36] if the underlying assets pay a continuous dividend yield). As above, in all the other cases we have to resort to numerical evaluation. We will benchmark numerical results against closed-form formula solutions, if these are available (see Table 9).

4 A generalization of the Least Square Monte-Carlo approach

Since the seminal article by Boyle [6], simulation has been considered a suited numerical approach for option pricing as long as early exercise is not optimal (or allowed). On the other hand, there are American-type contingent claims for which early exercise can be the optimal strategy. In this case, the most natural approach to valuation has been considered stochastic control theory and dynamic programming. Unfortunately, simulation is forward looking and this feature seems to be in contrast with the backward looking nature of dynamic programming. Longstaff and Schwartz [32] provides a valuation algorithm, called Least Squares Monte Carlo (LSM) based on simulation but taking into account the possibility of early exercise. Their algorithm provides a way to determine the optimal stopping time of an American-like claim and then, by applying equation (2.4), to find the estimate of the claim. In what remains of this section, we first describe shortly the LSM and next we extend it in order to solve the basic relations among options needed to apply the decomposition approach presented in Section 3

4.1 Longstaff-Schwartz approach for simple options

Given the valuation problem in (2.4) for an American claim contingent on X and expiring at T, an approximation of the value is obtained by choosing an integer N so that the time span $[0, T]$ is divided into N intervals whose width is $\Delta t = T/N$. Next, the dynamics of the state variables is simulated by generating K paths of $\{X_t\}$. We will denote $X_t(\omega)$ the value of the process at time t along the ω-th simulated path and $\tau(\omega)$ the path-wise stopping time.

The goal of the algorithm is to find the optimal exercise time restricted to the set of dates

$$\{t_0 = 0, t_1 = \Delta t, \ldots, t_N = N\Delta t\}.$$

As usual, the optimal policy is obtained by backward dynamic programming: if at time t_n, along the path ω, the claim has not been exercised yet (i.e., the stopping time along the ω-th path, as determined in previous time steps of the algorithm, is greater that t_n), the optimal decision is made by comparing the payoff $\Pi(t_n, X_t(\omega))$ with $F(t, X_t(\omega))$, the (optimal) value function of problem (2.4). If $F(t, X_t(\omega)) = \Pi(t_n, X_t(\omega))$ then $\tau(\omega) = t_n$: the optimal stopping time along the ω-th path is updated.

Unfortunately, $F(t, X_t)$ is not available at this step of the procedure. A way around this difficulty is offered by the Bellman equation of the optimal stopping problem in discrete time:

$$F(t_n, X_{t_n}) = \max\left\{\Pi(t_n, X_{t_n}), e^{-r(t_{n+1}-t_n)}\mathbb{E}^*_{t_n}\left[F(t_{n+1}, X_{t_{n+1}})\right]\right\}.$$

By this equation, we can determine the path-wise optimal policy, restricted to the given dates, by comparing the continuation value,

$$\Phi(t_n, X_{t_n}) = e^{-r(t_{n+1}-t_n)}\mathbb{E}^*\left[F(t_{n+1}, X_{t_{n+1}}) \mid \mathcal{F}_{t_n}\right] \tag{4.1}$$

with the payoff. So, the decision rule at time step t_n along the ω-th path is:

$$\text{if} \qquad \Phi(t_n, X_{t_n}(\omega)) \leq \Pi(t_n, X_{t_n}(\omega)) \quad \text{then} \quad \tau(\omega) = t_n. \tag{4.2}$$

At $t_n = T$, since the claim is expiring, $\Phi(t_n, X_{t_n}) = 0$, and the rule reduces to exercise the claim if the payoff is positive. At any t_n, the optimal stopping time is found by recursively applying the decision rule in (4.2), back from $t_n = T$ to t_n. If we have determined, at some previous step of this procedure, $\tau(\omega) > t_n$, and condition (4.2) holds true at the current step, then the stopping time along path ω is updated: $\tau(\omega) = t_n$. At $t_n = 0$, when the optimal stopping times along all paths are determined, the value of the American contingent claim is estimated by averaging the path-wise values:

$$F(0, x) = \frac{1}{K}\sum_{\omega=1}^{K} e^{-r\tau(\omega)}\Pi(\tau(\omega), X_{\tau(\omega)}(\omega)).$$

The problem boils down to one of finding the continuation value at (t, X_t), in order to apply the decision rule in (4.2). This is the point where LSM differs from all other approaches proposed to evaluate American-type contingent claim with simulation. The intuition behind LSM is the following: if at t the claim is still available, the continuation value is the expectation, conditional on the information available at that date, of future optimal payoffs from the contingent claim. To clarify the next steps, we slightly modify the previously introduced notation: let $\Pi(t, s, \tau, \omega)$ be the (non-necessarily positive) cash flow from the contingent claim optimally exercised at time s (with respect to the stopping time $\tau(\omega)$), conditional on not being exercised at $t < s$, along the ω-th path. Hence,

$$\Pi(t, s, \tau, \omega) = \begin{cases} \Pi(s, X_s(\omega)) & \text{if } s = \tau(\omega) \\ 0 & \text{if } s \neq \tau(\omega). \end{cases}$$

The dependence of this cash flow on t is due to the fact that, when we apply recursively the decision rule in (4.2), the stopping time along the ω-th path can change step by step.

The continuation value at t_n is the present value (with respect to the equilibrium risk neutral probability) of all future expected cash flows from the contingent claim

$$\Phi(t_n, X_{t_n}) = \mathbb{E}^*_{t_n}\left[\sum_{i=n+1}^{N} e^{-r(t_i - t_n)}\Pi(t_n, t_i, \tau, \cdot)\right]. \tag{4.3}$$

Since Φ is an element of a linear vector space,[17] then we can represent the continuation value as a linear combination:

$$\Phi(t, X_t) = \sum_{j=1}^{\infty} \phi_j(t) L_j(t, X_t)$$

where L_j is the j-th element in the orthonormal basis. In Longstaff and Schwartz [32] $L_j(t, X_t)$ are either Hermite, or Laguerre polynomials or also powers of X_t. If only $J < \infty$ elements in the basis are used to determine Φ, we obtain an approximation of the continuation value. Following Longstaff and Schwartz,

$$\Phi^J(t, X_t) = \sum_{j=1}^{J} \phi_j(t) L_j(t, X_t).$$

[17] Φ belongs to the Hilbert space $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$ and any Hilbert space has a countable orthonormal basis.

Now, $\phi_j(t)$ can be estimated by a linear least squares regression of $\Phi^J(t, X_t)$ onto the basis $\{L_j(t, X_t)\}$:[18]

$$\left\{\hat{\phi}_j(t_n)\right\}_{j=1}^{J} =$$
$$= \arg\min \left\| \sum_{j=1}^{J} \phi_j(t_n) L_j(t_n, X_{t_n}) - \sum_{i=n+1}^{N} e^{-r(t_i - t_n)} \Pi(t, t_i, \tau, \cdot) \right\|^2 .$$

The estimated continuation value,

$$\hat{\Phi}^J(t_n, X_{t_n}) = \sum_{j=1}^{J} \hat{\phi}_j(t_n) L_j(t, X_{t_n})$$

is then used to apply recursively the decision rule in (4.2).

Accuracy of the estimates of the value of the American contingent claim can be increased by increasing the number of time steps, N, the number of simulated paths, K, and the number of basis function, J. Actually, given N, the algorithm has been proved to converge to the actual value of the (corresponding Bermudan with N dates) claim if $J \to \infty$ and if $K \to \infty$ and the estimation errors are asymptotically normally distributed (see Clément, Lamberton and Protter [14]).

4.2 An extension to multi-option problems

Since we are interested in valuing capital budgeting projects with many embedded (American) options, we have to extend the LSM algorithm presented in Section 4.1 to the framework we have introduced in Section 3.

As far as the case with H independent options is concerned (see Section 3.1), the value of the option to exercise them, according to equation (3.1), is simply the sum of their values obtained with LSM algorithm. If there is a project-specific source of uncertainty, which is stochastically independent on the state variables, and which resolves at T', with $0 < T' < T_h$, $h = 1, \ldots, H$, according to notation introduced in Section 3.1 the relevant equation is (3.2). In this case, the valuation approach is slightly different from the one described above. At this step of the algorithm we have already found the values (and the related stopping times) of the subsequent options, F_h. Actually, since the subsequent options cannot be exercised in

[18] We denote by $\|\cdot\|$ the norm in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$.

the interval $[0, T']$, their value is

$$\begin{aligned} F_h(0,x) &= e^{-rT'}\mathbb{E}^*\left[F_h(T', X_{T'})\right] \\ &= \mathbb{E}_0^*\left[\sum_{i=1}^{N} e^{-rt_i}\Pi_h(0,t_i,\tau,\cdot)\right] \end{aligned}$$

where

$$\Pi_h(t,s,\tau,\omega) = \begin{cases} \Pi_h(s, X_s(\omega)) & \text{if } s = \tau_h(\omega) \\ 0 & \text{if } s \neq \tau_h(\omega), \end{cases}$$

where τ_h denotes the stopping time for option h, $h = 1, \ldots, H$. Note that $T' \leq \tau_h \leq T_h$ and that $\Pi_h(0, t_i, \tau, \omega)$ has been already found applying the LSM approach to the h-th option. To find G we just need to apply equation (3.2):

$$G(0,x) = \sum_{h=1}^{H} p_h F_h(0,x).$$

For the compound option case (see Section 3.2), the algorithm is the following. According to the recursive nature of the valuation problem, we assume that the path-wise stopping time for the (subsequent) $(h+1)$-th option has been already determined. We are to compute the path-wise stopping time for the h-th option. The Bellman equation for problem (3.3) is

$$\begin{aligned} &F_h(t_n, X_{t_n}) = \\ &\max\left\{\Pi_h(t_n, X_{t_n}) + F_{h+1}(t_n, X_{t_n}), e^{-r(t_{n+1}-t_n)}\mathbb{E}_{t_n}^*\left[F_h(t_{n+1}, X_{t_{n+1}})\right]\right\}. \end{aligned}$$

Hence, to find out the stopping time of option h, denoted $\tau_h(\omega)$, at time step t_n along the ω-th path, the decision rule is the following:

$$\text{if } \Phi_h(t_n, X_{t_n}(\omega)) \leq \Pi_h(t_n, X_{t_n}(\omega)) + F_{h+1}(t_n, X_{t_n}(\omega)) \text{ then } \tau_h(\omega) = t_n \tag{4.4}$$

where Φ_h is the continuation value from the Bellman equation (see equation (4.1)), Π_h is the payoff of the h-th option, F_{h+1} is the value of the $(h+1)$-th option and τ_h is the stopping time for option h. This decision rule replaces the one in (4.2).

To apply this rule we have to estimate the continuation value Φ_h and the value of the subsequent option, F_{h+1}. The former is found by extending the Longstaff and Schwartz idea. Note that

$$\Phi_h(t_n, X_{t_n}(\omega)) = \mathbb{E}_{t_n}^*\left[\sum_{i=n+1}^{N} e^{-r(t_i-t_n)} \sum_{\ell=h}^{H} \Pi_\ell(t_n, t_i, \tau, \cdot)\right]$$

On the other hand,

$$F_{h+1}(t_n, X_{t_n}(\omega)) = \mathbb{E}^*_{t_n}\left[\sum_{i=n}^{N} e^{-r(t_i-t_n)} \sum_{\ell=h+1}^{H} \Pi_\ell(t_n, t_i, \tau, \cdot)\right],$$

i.e., according to equation (4.3), the value of the $(h+1)$-th option is the present value of expected cash flow obtained from optimally exercising that option and all subsequent options, starting from the current date. It should be noted that, at this step, $\Pi_\ell(t_n, t_n, \tau, \omega)$ is known, $\ell = h+1, \ldots, H$.

In order to apply this rule, since the conditions in Longstaff and Schwartz [32] still apply,[19] Φ_h is approximated by Φ_h^J and this can be estimated by least squares regression of the discounted conditional cash flows from option h onto the basis $\{L_j, j = 1, \ldots, L\}$.

It should be noted that the above procedure encompasses also the case in which some of the real options are European. Actually, if option $(h+1)$ is European with maturity T_{h+1}, at $t_n < T_{h+1}$, $\Pi_{h+1}(t_n, t_i, \tau, \cdot) \equiv 0$ for all $t_i \neq T_h$.

As far as the case with H mutually exclusively options is considered (see Section 3.3), at any time step we have to find the optimal control (τ, ζ), according to equation (3.5). The Bellman equation at time t_n is

$$\begin{aligned} &G(t_n, X_{t_n}) = \\ &\quad \max\left\{F_1(t_n, X_{t_n}), \ldots, F_H(t_n, X_{t_n}), e^{-r(t_{n+1}-t_n)}\mathbb{E}^*_{t_n}\left[G(t_{n+1}, X_{t_{n+1}})\right]\right\}. \end{aligned}$$

Hence, the decision rule, along the ω-th path is:

$$\text{if } \Phi(t_n, X_{t_n}(\omega)) \leq \max_h \{F_h(t_n, X_{t_n}(\omega))\} \text{ then } (\tau, \zeta)(\omega) = (t_n, \bar{h}) \tag{4.5}$$

where Φ is the continuation value according to the Bellman equation,

$$\bar{h} = \arg\max \{F_h(t_n, X_{t_n}(\omega))\},$$

and $(\tau, \zeta)(\omega) = (\tau(\omega), \zeta(\omega))$. In order to apply the decision rule in (4.5) we have to estimate $\Phi(t_n, X_{t_n})$ and $F_h(t_n, X_{t_n})$. To this aim, let

$$\Pi(t, s, \tau, \zeta, \omega) = \begin{cases} \Pi_h(t, s, \tau, \omega) & \text{if } h = \zeta(\omega) \\ 0 & \text{otherwise} \end{cases}$$

[19] Φ_h is in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$.

Since the continuation value of the option to select the best option out of H available options is the present value of the expected cash flows conditional on following the optimal exercise strategy, then

$$\Phi(t_n, X_{t_n}) = \mathbb{E}^*_{t_n}\left[\sum_{i=n+1}^{N} e^{-r(t_i-t_n)}\Pi(t_n, t_i, \tau, \zeta, \cdot)\right].$$

This can be approximated by Φ^J according to Longstaff and Schwartz [32], and Φ^J can be estimated by Least Squares regression of the discounted cash flows $\Pi(t_n, t_i, \tau, \zeta, \omega)$ onto the basis $\{L_j, j = 1, \ldots, J\}$. To apply the decision rule in (4.5), also F_h need to be estimated. Yet, since at this step, $\Pi_h(t_n, t_n, \tau, \omega)$ is known, $h = 1, \ldots, H$, and

$$F_h(t_n, X_{t_n}(\omega)) = \mathbb{E}^*_{t_n}\left[\sum_{i=n}^{N} e^{-r(t_i-t_n)}\Pi_h(t_n, t_i, \tau, \cdot)\right]$$

hence, we can apply (4.5) to find out the control $(\omega, \zeta)(\omega)$ at t_n.

All the above cases are plain extensions of LSM. Hence, the convergence results in Clément, Lamberton and Protter [14] still apply.

5 Applications

In this section we provide several numerical experiments to show how the approach presented in Sections 3 and 4 can be used to model and evaluate complex capital budgeting problems with many underlyings and many interacting real options. To illustrate the efficiency of the extended LSM approach, assuming that the underlying factors are geometric Brownian motions, we benchmark the estimates against the results obtained by applying the extended Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) and, if closed-form formula are available, against exact solutions.

There are two families of examples. The first set of numerical experiments are abstract situations whose main purpose is to illustrate the efficiency of the valuation approach when applied to the building blocks of our approach to model complex multi-options problems. We will present examples for the independent options case (Section 3.1), the compound option case (Section 3.2) and the case with mutually exclusively options (Section 3.3). For all examples we will show both the European and the

American cases. This example permits to see the influence of various parameter values on the accuracy of the numerical methods proposed in this work.

The second set of numerical experiments deals with a real option valuation problem presented in Lint and Pennings [31], involving four sources of uncertainty. We will see how different problems specifications can be easily modelled in our approach.

5.1 Warm up applications

Example 1. Let there be given an investment project which, after a first outlay, K_0, (for instance, R&D, infrastructure, etc.) can have two possible outcomes at time T'. Each outcome can give rise to a potential business. These business can be obtained by paying some additional capital expenditure.[20] The values of these business are the present values of the respective cash flows from operations. We assume that the values of the business are, under the risk neutral equilibrium probability, GBM's with dynamics

$$\frac{dV_t^i}{V_t^i} = \alpha_i dt + \sigma_i dB_t^i \qquad V_0^i = V^i$$

$i = 1, 2$, where $\alpha_i = r - \delta_i$, δ_i is an equilibrium shortfall rate of return,[21] r is the annualized continuously compounded risk-free interest rate, α_i and σ_i are given on an annual basis, and $\mathbb{E}[dB_t^1 dB_t^2] = \rho dt$. The decision on the business to be developed can be deferred until the technical uncertainty and the market uncertainty is dissipated. Hence, the following real options are embedded in the case at hand:

option to develop business V^1: by paying K_1 within $T_1 = 5$ years. The payoff (at maturity) of this option is $\Pi_1(t, V_t^1) = \max\{V_t^1 - K_1, 0\}$ and the value will be denoted F_1;

option to develop business V^2: the needed additional costs is K_2 and maturity is $T_2 = T_1 = 5$ years. The related payoff (at maturity) is $\Pi_2(t, V_t^2) = \max\{V_t^2 - K_2, 0\}$ and the value is F_2.

[20] For the sake of definiteness, one can think of an investment project in an undeveloped land with potential oil or gas reserves. The first outlay is given by the exploration costs. We assume that the geological tests can show, after a given time-period, that alternatively either gas or oil can be extracted. The outcomes of exploration are uncertain. The additional capital outlays are needed to build the facilities for extraction.

[21] For more details on the equilibrium shortfall rate of return for non traded real assets, see McDonald and Siegel [37].

For simplicity, we assume that the decision to spend the capital outlays $K_0 = 1$ is committed (i.e., not an option). We can see K_0 as the cost needed to obtain information about the feasibility of the business. The probabilities of the possible outcomes are p_i. The options to develop the business can be exercised in the interval $[T', T_i]$. Figure 1 provides a graphical representation of the problem.

The base case parameters are given in Table 1. The values of the investment project for several values of the parameters of the stochastic process of the first business (S, σ and δ) and for different maturities are presented in Table 2.[22] By inspection, we can see that the overall accuracy (as measured by Root Mean Square Error[23]) is fair. Keeping the parameters of one of the two businesses constant, we can see that the value of the project is increasing in the current value of one of the two underlyings, increasing with respect to volatility and to maturity, whereas it is decreasing in the convenience yield.

The second and the third examples can be considered two strategic alternatives of the same capital budgeting problem.

Example 2. Let there be given a real asset (a business) whose value V_t follows a GBM

$$\frac{dV_t}{V_t} = \alpha dt + \sigma dB_t, \qquad V_0 = V$$

under the equilibrium risk-neutral probability, where $\alpha = r - \delta$, δ is an equilibrium shortfall rate of return and r is (continuously compounded) annual riskless rate. The following options are available:

option to defer the investment: the payoff (at maturity) as if the option was in isolation is $\Pi_1(t, V_t) = \max\{e_1 V_t - I_1, 0\}$, that is, with a cost outlay I_1 we can get a given percentage e_1, $0 < e_1 < 1$ of the whole business. The maturity of the option is T_1 (years). As usual F_1 will denote the option value;

[22] We point out that, since the options to develop the two business are strategically independent, then the value of the project is not dependent on the correlation ρ between the two business.

[23] Root Mean Square Error (RMSE) is defined as usual:

$$RMSE = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(\frac{F_i - \hat{F}_i}{F_i}\right)^2},$$

where F_i is the (accurate) value of the i-th project value, $\hat{F}_i$ is the related estimate obtained by simulation and m is the number of cases (in the table).

option to expand: the payoff is $\Pi_2(t, V_t) = \max\{e_2 V_t - I_2, 0\}$ with $e_2 = 1 - e_1$; i.e., with a capital expenditure I_2 we can complete the investment in the business. The maturity is T_2 years, the value F_2.

The payoffs above are not the true ones for the problem at hand. Actually, since the expansion option is available only after the investment option has been exercised, then the payoffs (at maturity) of the investment option is $\max\{e_1 V_t - I_1 + F_2(t, V_t), 0\}$. A graphical representation of the problem is offered in Figure 2. We stress that, although the second option can be exercised in the interval $[0, T_2]$, the actual time interval for the second option is from the time the first option is exercised (a stopping time) to T_2. The base case parameters are in Table 3. The numerical results for a set of value of the parameters of the project are in Table 4. As far as accuracy and the dependence of the project value on parameters is concerned, what was said for Table 2 still holds true.

Example 3. Given the same real asset in Example 2, we are going to evaluate a different strategic alternative (see Figure 3):

option to defer the investment: by paying the outlay $I = I_1 + I_2$ we can get the whole business whose value is V and the opportunity is available until T_1 (years). Hence, the payoff of this option is (at maturity) $\Pi_1(t, V_t) = \max\{V_t - I, 0\}$ and its value is F_1;

option to contract the scale of the project: we can recover part of the capital outlay, $X = I_2$, by reducing the scale of the business by k percent. This option is available, after the option to invest has been exercised, until T_2. Hence, the payoff (at maturity) is $\Pi_2(t, V_t) = \max\{X - kV_t, 0\}$ and the value is F_2.

As in Example 2, since the option to defer gives rise, when exercised, to the option to reduce the scale of the project, then the actual payoff (at maturity) of the first option is $\max\{V_t - I + F_2(t, V_t), 0\}$.

From a strategic viewpoint, this example offers an alternative approach to the same investment opportunity showed in Example 2. Actually, in that case the approach was more conservative, because the second stage takes place only if the first step is successful. In this example, on the other hand, we can obtain the same real asset, but we can recover part of the sunk cost if the business turn to be less favorable than expected. Although at a first sight the two alternatives might seem basically the same, Example 3 has a larger "operating leverage" given the higher level of fixed costs. Hence, *coeteris paribus*, if the uncertainty surrounding the value of the business (i.e.,

σ) is higher, in principle, the second alternative should worth more, because the upside potential is increased by the "operating leverage." Actually, by inspection of Table 5, we can see that there is no clear domination of the second (high-leveraged) approach over the first one.

The values of the project, for a set of parameter values, are in Table 5. The usual comments on accuracy we did for the previous table still apply.

Example 4. On the same asset as is Example 2, let there be given the following opportunities:

option to defer the investment: by paying I_1 we can acquire e_1 percent of the asset. This option can be exercised within T_1 (years). The payoff (at maturity) is $\Pi_1(t, V_t) = \max\{e_1 V_t - I_1, 0\}$ and its value is F_1;

option to expand: we can get the remaining part ($e_2 = 1 - e_1$) of the business with an additional capital expenditure I_2 by time T_2. The related payoff (at maturity) is $\Pi_2(t, V_t) = \max\{e_2 V_t - I_2, 0\}$ and the value is F_2;

option to abandon: we can abandon the business ($k = e_1$), after the first investment, as an alternative to the option to expand, saving $X < I_1$, by year T_3. The payoff of this option (at maturity) is $\Pi_3(t, V_t) = \max\{X - kV_t, 0\}$ and the value is F_3.

Since the option to expand and the option to abandon are mutually exclusive, only one of them can be exercised. The option map of the investment project is given in Figure 4. The base case parameters are in Table 6. The results for a wide set of parameter values, are in Table 7.

5.2 Best of two product standards

The second set of examples is related to an actual case study drawn from Lint and Pennings [31] (see also Martzoukos and Trigeorgis [36]). This example permits to see a capital budgeting problem can be tackled according to the decomposition approach and also to compare the estimates obtained by simulation to exact solutions.

The case is the following. There is a firm which is considering the development of two product standards in consumer electronic industry in a given time horizon. There is a cost outlay to be paid upfront to obtain the options to invest in the two product standards. The standard that finally will prevail is uncertain at the date of the first outlay. If the firm invests in both technologies, it acquires an option on the best of two assets (product standards). Each underlying asset of this option is the market (present) value of the resulting cash flows if that standard prevails. Moreover, the underlying assets are correlated. The cost of introducing each standard is the strike price of the option. Also the strike prices for the two technologies are stochastic and correlated with the other state variables.

The underlying assets are V^i, the market value of i-th business (i.e., the value of cash flows obtained by product standard i), and C^i, the cost to introduce the standard i, $i = 1, 2$. These variables are assumed to follow correlated geometric Brownian motions with equilibrium rate of return shortfall and volatilities respectively δ_{V_i}, σ_{V_i}, δ_{C_i} and σ_{C_i}, $i = 1, 2$. Correlations are ρ_{ij}, $i \neq j$, $i, j = 1, \ldots, 4$.

Hence, the investment project has the following embedded options:

option to defer investment: by paying I we can acquire the option to choose the best of the two standards later on. This option can be exercised within T_0 (years). The payoff (at maturity), as if it was in isolation, is $\Pi_0 = \max\{-I, 0\}$ and its value is F_0;

options defer investment: with an additional capital expenditure C^h, we can get the value of the related product standard, V^h. The maturity of this option is T_h. The related payoff at maturity is $\Pi_h = \max\{V_t^h - C_t^h, 0\}$ and the value is F_h, $h = 1, 2$.

The structure of the problem is described by the option map in Figure 5. As usual, since the option to invest in both standards provide the opportunity to choose for the best, then the actual payoff (at maturity) of the first option is

$$\max\{G(t, V_t^1, V_t^2, C_t^1, C_t^2) - I, 0\}.$$

The base case parameters are in Table 8.

Besides the above described case, we have evaluated different versions of the investment problem by considering several features of the set of opportunities. In particular, we have evaluated also the impact of higher volatility, lower correlation, longer maturity and different investment scale on the option value.

To compare numerical results with exact solutions, we consider also the case of non-stochastic development costs for both technologies and $C_1 = C = C_2$. With this choice of parameters the problem has an analytic solution: if both the dividend yields are zero ($\delta_{V_1} = 0 = \delta_{V_2}$), then the model reduces to the European[24] option on the maximum of two risky assets and the solution formula has been provided by Stulz [40]; if at least one of the dividend yields is not zero, the extension of Stulz' formula for the European option on the maximum of two assets is in Martzoukos and Trigeorgis [36]. Moreover, if $V_0^2 = C_0^2 = 0$ (i.e., only one of the two standards is valuable), the options are European and $I = 0.1C_2$, then the problem reduces to the compound-exchange option studied by Carr [13] and a closed-form valuation formula is available. Again, if at least one of the dividend yield is not zero, the extension of Carr's formula is in Martzoukos and Trigeorgis [36]. Numerical results are presented in Table 9. As for the other examples, accuracy is fair in most of the cases.

[24] Note that, if both the dividend yields are zero, the American option and the European option are the same.

References

[1] Amram, M.; Kulatilaka, N. (1999): *Real Options. Managing Strategic Investment in an Uncertain World*, Harvard Business School Press, Boston - MA.

[2] Barraquand, J.; Martineau, D. (1995): Numerical Valuation of High Dimensional Multivariate American Securities, *Journal of Financial and Quantitative Analysis*, Vol. 30, pp. 383-405.

[3] Bensoussan, A. (1984): On the Theory of Option Pricing, *Acta ApplicandæMathematicæ*, Vol. 2, pp. 139-158.

[4] Bensoussan, A.; Lions, J.-L. (1984): *Impulse Control and Quasi-Variational Inequalities*, Gauthier-Villars, Paris.

[5] Bossaerts, P. (1989): Simulation Estimators of Optimal Early Exercise, *Working Paper, Carnegie-Mellon University.*

[6] Boyle, P. P. (1977): Options: a Monte Carlo Approach, *Journal of Financial Economics*, Vol. 4, pp. 323-338.

[7] Brekke, K. A.; Øksendal, B. (1994): Optimal Switching in an Economic Activity Under Uncertainty, *SIAM Journal of Control and Optimization*, Vol. 32, pp. 1021-1036.

[8] Brekke, K. A.; Schieldrop, B. (2000): Investment in Flexible Technologies under Uncertainty, in *Project Flexibility, Agency, and Competition*: M. J. Brennan and L. Trigeorgis (eds.), Oxford University Press, pp. 34-49.

[9] Brennan, M. J. (1991): The Price of Convenience and the Valuation of Commodity Contingent Claims, in *Stochastic Models, Applications to Resources, Environment and Investment Problems*, D. Lund e B. Øksendal (eds.), Elsevier Science Publishers, North Holland, Amsterdam - The Netherlands, pp. 33-71.

[10] Brennan, M.; Schwartz, E. (1977): The valuation of American Put Options, *Journal of Finance*, Vol. 32, pp. 449-462.

[11] Brennan, M.J.; Schwartz, E. (1985): Evaluating Natural Resource Investments, *Journal of Business*, Vol. 58, No. 2, pp. 135-157.

[12] Broadie, M.; Glasserman, P. (1997): Pricing American-style securities using simulation, *Journal of Economic Dynamics and Control*, Vol. 21, pp. 1323-1352.

[13] Carr. P. (1988): The Valuation of Sequential Exchange Opportunities, *Journal of Finance*, Vol. 43, pp. 1235-1256.

[14] Clément, E.; Lamberton, D.; Protter, P. (2001): An Analysis of the Longstaff-Schwartz algorithm for American Options, *working paper*.

[15] Cortazar, G.; Schwartz, E. S. (1994): The Valuation of Commodity Contingent Claims, *Journal of Derivatives*, Vol. 1, pp. 27-39.

[16] Cortazar, G.; Schwartz, E. S.; Salinas, M. (1998): Evaluating Environmental Investments: A Real Options Approach, *Management Science*, Vol. 44, pp. 1059-1070.

[17] Cox, J.C.; Ingersoll, J. E.; Ross, A.S. (1985): An Intertemporal General Equilibrium Model of Asset Prices, *Econometrica*, Vol. 53, pp. 363-384.

[18] Cox, J.C.; Ross, A.S.; Rubinstein, M. (1979): Option Pricing: a Simplified Approach, *Journal of Financial Economics*, Vol. 7, pp. 229-263.

[19] Dixit, A. (1989): Entry and Exit Decisions Under Uncertainty, *Journal of Political Economy*, Vol. 97, pp. 620-638.

[20] Dixit, A.K.; Pindyck, R.S. (1994): *Investment Under Uncertainty*, Princeton University Press, Princeton - NJ.

[21] Gamba, A.; Trigeorgis, L. (2001): A Log-transformed Binomial Lattice Extension for Multi-Dimensional Option Problems, presented at the *5th Annual Conference on Real Options, UCLA*.

[22] Geltner, D.; Riddiough, T.; Stojanovic, S. (1995): Insight on the Effect of Land Use Choice: The Perpetual Option on the Best of Two Underlying Assets, *Working Paper - Massachussets Institute of Technology*.

[23] Geske, R. (1979): The valuation of compound options, *Journal of Financial Economics*, Vol. 7, pp. 63-81.

[24] Katatzas, I. (1988): On the Pricing of American Options, *Applied Mathematics and Optimization*, Vol. 17, pp 37-60.

[25] Kulatilaka, N. (1988): Valuing the flexibility of Flexible Manufacturing Systems, *IEEE Transactions in Engineering Management*, Vol. 35, pp. 250-257.

[26] Kulatilaka, N. (1993): The Value of Flexibility: the Case of a Dual-Fuel Industrial Steam Boiler, *Financial Management*, Vol. 22, pp. 271-280.

[27] Kulatilaka, N. (1995): The Value of Flexibility: a General Model of Real Options, in *Real Options in Capital Investment: Models, Strategies, and Applications*, Lenos Trigeorgis (ed.), Praeger, Westport - CT, pp. 89-107.

[28] Kulatilaka, N. (1995): Operating Flexibilities in Capital Budgeting: Substitutability and Complementarity in Real Options, in *Real Options in Capital Investment: Models, Strategies, and Applications*, Lenos Trigeorgis (ed.), Praeger, Westport - CT, pp. 121-132.

[29] Kulatilaka, N.; Trigeorgis, L. (1994): The general flexibility to switch: Real Options revisited, *International Journal of Finance*, Vol. 6, pp. 778-798.

[30] Johnson, H. (1987): Options on the Maximum or the Minimum of Several Assets, *Journal of Financial and Quantitative Analysis*, Vol. 22.

[31] Lint, O.; Pennings, E. (1999): The option value of developing two product standards when the final standard is uncertain, *working paper*.

[32] Longstaff, F. A.; Schwartz, E. S. (2001): Valuing American Options by Simulation: a Simple Least-Squares Approach, *The Review of Financial Studies*, Vol. 14, N. 1, pp. 113-147.

[33] Margrabe, W. (1978): The Value of an Option to Exchange One Asset for Another, *Journal of Finance*, Vol. 33, pp. 177-186.

[34] Mason, S. P.; Merton, R. C. (1985): The role of contingent claims analysis in corporate finance, in *Recent advances in corporate finance*, E. I. Altman and M. G. Subrahmanyam (eds), Irwin, Homewood - IL, pp. 7-54.

[35] Majd, S; Myers, S. C. (1984): Calculating Abandonment Value Using Option Pricing Theory, *M.I.T. Sloan School of Management - Working Paper No. 1462-83*.

[36] Martzoukos, S. H.; Trigeorgis, L. (1999): General Multi-stage Capital Investment Problems With Multiple Uncertainties, *working paper, University of Cyprus.*

[37] McDonald, R.; Siegel, D. (1984): Option pricing when the underlying asset earns a below-equilibrium rate of return: a note, *Journal of Finance*, Vol. 39, No. 1, pp. 261-265.

[38] McDonald, R.; Siegel, D. (1985): Investments and the Valuation of Firms When There is an Option to Shut Down, *International Economic Review*, Vol. 26, No. 2, pp. 331-349.

[39] McDonald, R.; Siegel, D. (1986): The Value of Waiting to Invest, *Quarterly Journal of Economics*, Vol. 101, No. 4, pp. 707-727.

[40] Stulz, R. M. (1982): Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications, *Journal of Financial Economics*, Vol. 10, pp. 161-185.

[41] Tilley, J. A. (1993): Valuing American options in a path simulation model, *Transaction of the Society of Actuaries*, Vol. 45, pp. 83-104.

[42] Triantis, A. J.; Hodder, J. E. (1990): Valuing Flexibility as a Complex Option, *Journal of Finance*, Vol. 45, pp. 549-565.

[43] Trigeorgis, L. (1993): The nature of Option Interactions and the Valuation of Investments with Multiple Real Options, *Journal of Financial and Quantitative Analysis*, Vol. 28, pp. 1-20.

[44] Trigeorgis, L. (1991): A Log-Transformed Binomial Numerical Analysis Method for Valuing Complex Multi-Option Investments, *Journal of Financial and Quantitative Analysis*, Vol. 26, pp. 309-326.

[45] Trigeorgis, L. (1996): *Real Options: managerial flexibility and strategy in resource allocation*, MIT Press, Cambridge - MA.

A Exhibits

A.1 Example 1

Table 1: Example 1 - base case parameters

i	1	2
δ_i	δ	0.05
σ_i	σ	0.15
V_i	S	80
K_i	100	80
T_i	T	
ρ	0	
K_0	8	
T'	1	
$p_1 = p_2$	0.5	
r	0.05	

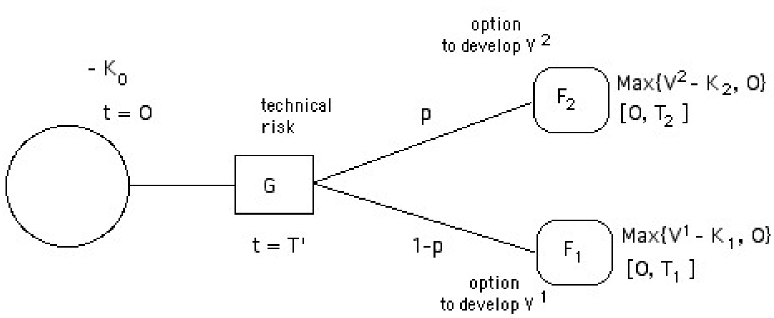

Figure 1: Example 1 - option map

Table 2: Example 1

				American				European			
S	σ (%)	δ (%)	T	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
100	20	1	4	7.302	7.224	0.118	- 0.011	7.051	7.020	0.102	- 0.004
100	20	1	5	9.202	9.116	0.135	- 0.009	8.844	8.828	0.110	-0.002
100	20	3	4	4.796	4.820	0.048	0.005	4.497	4.509	0.122	0.003
100	20	3	5	6.074	6.080	0.084	0.001	5.621	5.628	0.083	0.001
100	30	1	4	10.551	10.527	0.175	-0.002	10.300	10.306	0.153	0.001
100	30	1	5	12.656	12.570	0.122	-0.007	12.294	12.286	0.208	-0.001
100	30	3	4	8.157	8.171	0.109	0.002	7.746	7.781	0.161	0.004
100	30	3	5	9.701	9.749	0.155	0.005	9.072	9.098	0.160	0.003
90	20	1	4	4.036	3.985	0.087	-0.013	3.785	3.789	0.080	0.001
90	20	1	5	5.826	5.770	0.069	-0.010	5.468	5.499	0.079	0.006
90	20	3	4	2.086	2.072	0.049	-0.007	1.812	1.836	0.071	0.013
90	20	3	5	3.322	3.315	0.081	-0.002	2.914	2.926	0.076	0.004
90	30	1	4	7.268	7.213	0.151	-0.007	7.017	7.046	0.153	0.004
90	30	1	5	9.276	9.212	0.143	-0.007	8.916	8.873	0.201	-0.005
90	30	3	4	5.283	5.316	0.078	0.006	4.931	4.973	0.098	0.008
90	30	3	5	6.781	6.814	0.074	0.005	6.243	6.269	0.107	0.004
						RMSE	0.007			RMSE	0.005

"Lattice" is the value obtained by the Generalized Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) with $N = 600$ time steps.

"Sim" is the estimate obtained with the extended LSM approach proposed in this paper with $N = 50$ time steps, powers of the underlyings with $J = 8$ terms (and mixed terms up to the second power), and $K = 100\,000$ paths.

"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 20 times the LSM and then calculating the standard deviation of the estimate.

"RMSE" is the root means square error; i.e., the square root of the sum of the squares of the relative errors (rel.err) with respect to the value obtained with the lattice approach.

A.2 Example 2

Table 3: Example 2 - base case parameters

$I_1 = I_2$	80
e_1	0.5
V_0	S
T_2	T
T_1	$T - 2$
r	0.05

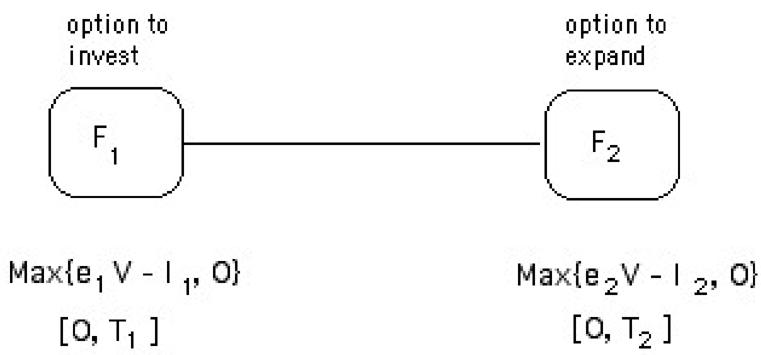

Figure 2: Example 2 - option map

Table 4: Example 2

				American				European			
S	σ (%)	δ (%)	T	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
100	20	3	4	1.364	1.363	0.037	-0.001	1.363	1.353	0.059	-0.007
100	20	3	5	2.857	2.843	0.074	-0.005	2.850	2.862	0.056	0.004
100	20	5	4	0.899	0.908	0.025	0.010	0.871	0.873	0.027	0.003
100	20	5	5	1.858	1.854	0.050	-0.002	1.786	1.792	0.044	0.004
100	30	3	4	5.197	5.218	0.101	0.004	5.170	5.165	0.124	-0.001
100	30	3	5	8.311	8.282	0.165	-0.003	8.241	8.142	0.223	-0.012
100	30	5	4	4.116	4.160	0.101	0.011	3.974	3.960	0.104	0.004
100	30	5	5	6.489	6.491	0.111	-0.000	6.193	6.217	0.117	0.004
110	20	3	4	2.781	2.771	0.072	-0.004	2.777	2.786	0.056	0.003
110	20	3	5	4.917	4.879	0.091	-0.008	4.903	4.894	0.094	-0.002
110	20	5	4	1.920	1.915	0.052	-0.003	1.857	1.857	0.039	-0.000
110	20	5	5	3.341	3.363	0.060	0.007	3.198	3.202	0.088	0.001
110	30	3	4	7.942	7.916	0.233	-0.003	7.896	7.890	0.114	-0.001
110	30	3	5	11.659	11.690	0.139	0.003	11.550	11.539	0.177	-0.001
110	30	5	4	6.419	6.461	0.097	0.007	6.184	6.192	0.128	0.001
110	30	5	5	9.280	9.356	0.149	0.008	8.825	8.824	0.197	-0.000
						RMSE	0.006			RMSE	0.004

“Lattice” is the value obtained by the Generalized Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) with $N = 10^4$ time steps.
“Sim” is the estimate obtained with the extended LSM approach proposed in this paper with $N = 50$ time steps, powers of the underlyings with $J = 8$ terms, and $K = 100\ 000$ paths.
“std.dev” is the standard deviation of the LSM estimate. It is obtained by iterating 20 times the LSM and then calculating the standard deviation of the estimate.
“RMSE” is the root means square error; i.e., the square root of the sum of the squares of the relative errors (rel.err) with respect to the value obtained with the lattice approach.

A.3 Example 3

Figure 3: Example 3 - option map.

Table 5: Example 3

				American				European			
S	σ (%)	δ (%)	T	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
100	20	3	4	1.198	1.194	0.029	-0.004	1.175	1.162	0.066	-0.011
100	20	3	5	2.627	2.610	0.041	-0.006	2.595	2.594	0.060	-0.000
100	20	5	4	0.913	0.926	0.029	0.015	0.895	0.902	0.024	0.008
100	20	5	5	1.898	1.909	0.049	0.006	1.850	1.849	0.046	-0.001
100	30	3	4	4.894	4.874	0.087	-0.004	4.835	4.821	0.087	-0.003
100	30	3	5	8.005	7.982	0.180	-0.003	7.931	7.995	0.090	0.008
100	30	5	4	4.198	4.218	0.051	0.005	4.110	4.117	0.075	0.002
100	30	5	5	6.641	6.635	0.150	-0.001	6.440	6.468	0.110	0.004
110	20	3	4	2.492	2.483	0.040	-0.004	2.452	2.463	0.055	0.005
110	20	3	5	4.576	4.556	0.081	-0.004	4.530	4.518	0.067	-0.003
110	20	5	4	1.955	1.985	0.035	0.015	1.917	1.925	0.044	0.004
110	20	5	5	3.416	3.417	0.051	-0.000	3.322	3.304	0.066	-0.005
110	30	3	4	7.541	7.540	0.092	-0.000	7.462	7.459	0.136	-0.000
110	30	3	5	11.287	11.229	0.159	-0.005	11.186	11.229	0.164	0.004
110	30	5	4	6.555	6.557	0.114	-0.000	6.410	6.443	0.088	0.005
110	30	5	5	9.505	9.492	0.105	-0.001	9.189	9.177	0.129	-0.001
						RMSE	0.006			RMSE	0.005

"Lattice" is the value obtained by the Generalized Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) with $N = 10^4$ time steps.
"Sim" is the estimate obtained with the extended LSM approach proposed in this paper with $N = 50$ time steps, powers of the underlyings with $J = 8$ terms, and $K = 100\ 000$ paths.
"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 20 times the LSM and then calculating the standard deviation of the estimate.
"RMSE" is the root means square error; i.e., the square root of the sum of the squares of the relative errors (rel.err) with respect to the value obtained with the lattice approach.

A.4 Example 4

Table 6: Example 4 - base case parameters

$I_1 = I_2$	50
X	30
e_1	0.5
k	0.5
V_0	S
T_1	$T - 2$
T_2	T
T_3	$T - 0.5$
r	0.05

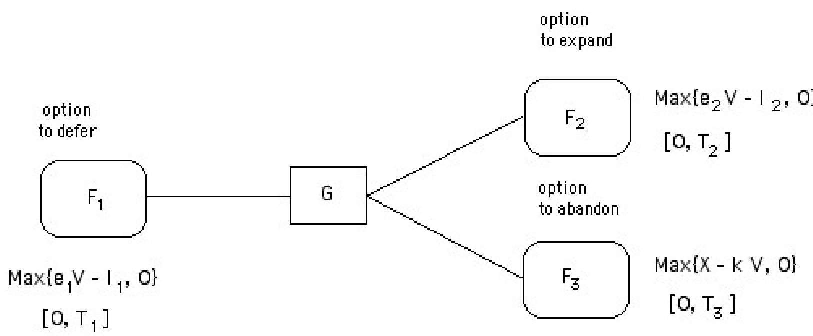

Figure 4: Example 4 - option map

Table 7: Example 4

				American				European			
S	σ (%)	δ (%)	T	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
100	20	10	3	6.533	6.625	0.034	0.014	5.094	5.240	0.060	0.029
100	20	10	5	8.447	8.440	0.029	-0.001	5.414	5.493	0.050	0.015
100	20	5	3	9.131	9.225	0.036	0.010	8.822	8.977	0.066	0.018
100	20	5	5	13.120	13.131	0.041	0.001	12.228	12.365	0.061	0.011
100	40	10	3	15.566	15.874	0.068	0.020	13.475	13.912	0.150	0.032
100	40	10	5	20.459	20.525	0.112	0.003	15.615	15.923	0.164	0.020
100	40	5	3	18.568	18.987	0.065	0.023	17.930	18.452	0.164	0.029
100	40	5	5	26.080	26.329	0.120	0.010	24.253	24.462	0.322	0.009
80	20	10	3	0.904	0.944	0.008	0.044	0.732	0.786	0.019	0.073
80	20	10	5	2.311	2.316	0.023	0.002	1.640	1.684	0.029	0.027
80	20	5	3	1.694	1.783	0.021	0.053	1.646	1.729	0.023	0.051
80	20	5	5	4.786	4.835	0.047	0.010	4.545	4.624	0.055	0.017
80	40	10	3	6.573	6.902	0.066	0.050	5.783	6.116	0.119	0.058
80	40	10	5	11.169	11.321	0.128	0.014	8.847	9.030	0.136	0.021
80	40	5	3	8.392	8.816	0.092	0.051	8.138	8.515	0.121	0.046
80	40	5	5	15.294	15.635	0.128	0.022	14.383	14.641	0.185	0.018
						RMSE	0.027			RMSE	0.034

"Lattice" is the value obtained by the Generalized Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) with $N = 10^4$ time steps.
"Sim" is the estimate obtained with the extended LSM approach proposed in this paper with $N = 50$ time steps, powers of the underlyings with $J = 8$ terms, and $K = 100\ 000$ paths.
"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 20 times the LSM and then calculating the standard deviation of the estimate.
"RMSE" is the root means square error; i.e., the square root of the sum of the squares of the relative errors (rel.err) with respect to the value obtained with the lattice approach.

A.5 Best of two product standards

Table 8: Best of two product standards - base case parameters

V_0^i	100	$i = 1, 2$
C_0^i	100	$i = 1, 2$
r	0.07	$i = 1, \ldots, 4$
δ_i	0.1	$i = 1, \ldots, 4$
σ_i	0.2	$i = 1, \ldots, 4$
ρ_{ij}	0.5	$i \neq j, i, j = 1, \ldots, 4$
T_0	0	
I	0	
T_i	2	$i = 1, 2$

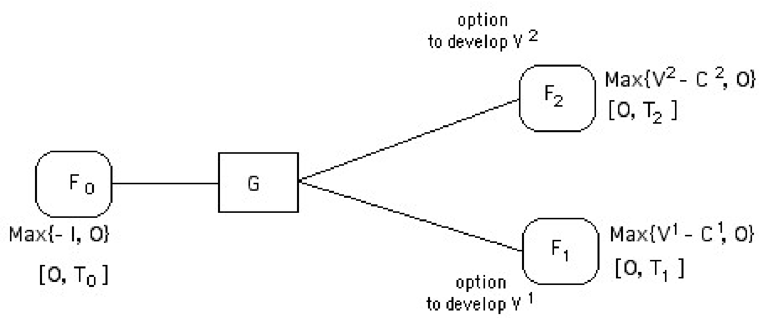

Figure 5: Best of two product standards - option map

Table 9: Example 5.2

	American				European			
	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
base case	16.427	16.605	0.070	0.011	15.707	16.173	0.110	0.030
$\rho_{ij} = 0$	23.286	23.356	0.123	0.003	22.261	22.670	0.178	0.018
$V_1 = C_1 = 90$	15.616	15.799	0.069	0.012	14.931	15.324	0.107	0.026
$T_1 = T_2 = 5$	21.567	21.747	0.105	0.008	18.402	18.853	0.205	0.024
$C_1 = 90$, $C_2 = 110$, $\sigma_{C_1} = .1$, $\sigma_{C_2} = .3$, $\rho_{ij} = 0$	24.876	24.306	0.113	-0.023	23.775	23.789	0.109	0.001
$C_1 = 90$, $C_2 = 110$, $\sigma_{C_1} = .3$, $\sigma_{C_2} = .1$, $\rho_{ij} = 0$	25.575	25.168	0.088	-0.016	24.238	24.215	0.150	-0.001
$\sigma_i = .3$	24.627	25.007	0.149	0.015	23.545	24.154	0.173	0.026
$I = 10$, $T_0 = 2$, $T_1 = T_2 = 3$	11.432	11.049	0.072	-0.034	10.154	10.223	0.094	0.007
$\sigma_{C_i} = 0$, $\delta_i = 0$	26.593	26.921	0.124	0.012	26.593	26.802	0.132	0.008
$\sigma_{C_i} = 0$	12.567	12.550	0.080	-0.001	11.403	11.705	0.171	0.027
$I = .1C_1$, $V_2 = C_2 = 0$, $T_0 = 2$, $T_1 = 3$	6.668	6.551	0.029	-0.017	5.705	5.645	0.077	-0.011
			RMSE	0.016			RMSE	0.019

"Lattice" is the value obtained by the Generalized Log-Transformed binomial lattice approach (Gamba and Trigeorgis [21]) with $N = 500$ time steps for the cases with 2 underlyings (below the double line) and with $N = 50$ steps for the cases with 4 underlyings (above the double line).

"Sim" is the estimate provided with the extended LSM approach with $N = 50$ time steps, power series with $J = 8$ terms (and mixed terms up to the second power) and simulating $K = 100\,000$ paths when the problem has 2 underlyings (below the double line) and $K = 50\,000$ when the problem has 4 underlyings (above the double line).

"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 10 times the LSM and then calculating the standard deviation of the estimate.

"RMSE" is the root means square error.

Exact solutions are given in the last three cases, as far as European options are concerned. When $\sigma_{C_i} = 0$, $\delta_i = 0$ (Stulz [40]), $F(0, x) = 26.608$. If $\sigma_{C_i} = 0$, $\delta_i = 0$ (Martzoukos and Trigeorgis [36]) $F(0, x) = 11.411$. If $I = .1C_1$, $V_2 = C_2 = 0$, $T_0 = 2$, $T_1 = 3$ (Carr [13]) $F(0, x) = 5.709$.